Instructor's Solutions Manual
to accompany

ANALYSIS
with an Introduction to Proof

5th Edition

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PEARSON

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This manual is intended to accompany the 5th edition of *Analysis with an Introduction to Proof* by Steven R. Lay (Pearson, 2013). It contains solutions to nearly every exercise in the text. Those exercises that have hints (or answers) in the back of the book are numbered in **bold** print, and the hints are included here for reference. While many of the proofs have been given in full detail, some of the more routine proofs are only outlines. For some of the problems, other approaches may be equally acceptable. This is particularly true for those problems requesting a counterexample. I have not tried to be exhaustive in discussing each exercise, but rather to be suggestive.

Let me remind you that the starred exercises are not necessarily the more difficult ones. They are the exercises that are used in some way in subsequent sections. There is a table on page 3 that indicates where starred exercises are used later. The following notations are used throughout this manual:

$\mathbb{N}$ = the set of natural numbers $\{1, 2, 3, 4, \ldots\}$
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\forall$ = "for every"
$\exists$ = "there exists"
$\ni$ = "such that"

I have tried to be accurate in the preparation of this manual. Undoubtedly, however, some mistakes will inadvertently slip by. I would appreciate having any errors in this manual or the text brought to my attention.

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Table of Starred Exercises

Note: The prefix P indicates a practice problem, the prefix E indicates an example, the prefix T refers to a theorem or corollary, and the absence of a prefix before a number indicates an exercise.

Starred Exercise	Later Use	Starred Exercise	Later use
2.1.26	T3.4.11	4.3.14	4.4.5
2.2.10	2.4.26	4.4.10	8.2.14
2.3.32	2.5.3	4.4.16	8.3.9
3.1.3	E7.1.7	4.4.17	T8.3.3
3.1.4	7.1.7	5.1.14	6.2.8
3.1.6	E8.1.1	5.1.16	T6.2.9
3.1.7	4.3.10, 4.3.15, E8.1.7, T9.2.9	5.1.18	5.2.14, 5.3.15
3.1.8	P8.1.3	5.1.19	5.2.17
3.1.24	4.1.7f, E5.3.7	5.2.10	T7.2.8
3.1.27	3.3.14	5.2.11	7.2.9b
3.1.30b	3.3.11, E4.1.11, 4.3.14	5.2.13	T5.3.5, T6.1.7, 7.1.13
3.2.6a	4.1.9a, T4.2.1, 6.2.23, 7.2.16, T9.2.9	5.2.16	9.2.15
3.2.6b	T6.3.8	5.3.13b	T6.2.8, T6.2.10
3.2.6c	T4.1.14	6.1.6	6. 2.14, 6.2.19
3.2.7	T8.2.5	6.1.8	7.3.13
3.3.7	T7.2.4, 7.2.3	6.1.17b	6.4.9
3.3.12	7.1.14, T7.2.4	6.2.8	T7.2.1
3.4.15	3.5.12, T4.3.12	6.3.13d	9.3.16
3.4.21	3.5.7	7.1.12	P7.2.5
3.5.8	9.2.15	7.1.13	7.2.5
3.6.12	5.5.9	7.1.16	7.2.17
4.1.6b	E4.2.2	7.2.9a	P7.3.7
4.1.7f	T4.2.7, 4.3.10, E8.1.7	7.2.11	T8.2.13
4.1.9a	5.2.10, 9.2.17	7.2.15	7.3.20
4.1.11	E4.3.4	7.2.20	E7.3.9
4.1.12	5.1.15	8.1.7	E8.2.6
4.1.13	5.1.13	8.1.8	8.2.13
4.1.15b	4.4.11, 4.4.18, 5.3.12	8.1.13a	9.3.8
4.1.16	5.1.15	8.2.12	9.2.7, 9.2.8
4.2.17	E6.4.3	8.2.14	T8.3.4
4.2.18	5.1.14, T9.1.10	9.1.15a	9.2.9

Analysis

with an Introduction to Proof

5th Edition

by Steven R. Lay

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Chapter 1 – Logic and Proof

Solutions to Exercises

Section 1.1 – Logical Connectives

1. (a) False: A statement may be false.
   (b) False: A statement cannot be both true and false.
   (c) True: See the comment after Practice 1.1.4.
   (d) False: See the comment before Example 1.1.3.
   (e) False: If the statement is false, then its negation is true.

2. (a) False: p is the antecedent.
   (b) True: Practice 1.1.6(a).
   (c) False: See the paragraph before Practice 1.1.5.
   (d) False: "p whenever q" is "if q, then p".
   (e) False: The negation of $p \Rightarrow q$ is $p \wedge \sim q$.

3. **Answers in Book:** (a) The 3×3 identity matrix is not singular.
   (b) The function $f(x) = \sin x$ is not bounded on $\mathbb{R}$.
   (c) The function f is not linear or the function g is not linear.
   (d) Six is not prime and seven is not odd.
   (e) x is in D and $f(x) \geq 5$.

(f) (a_n) is monotone and bounded, but (a_n) is not convergent.
(g) f is injective, and S is not finite and not denumerable.

4. (a) The function $f(x) = x^2 - 9$ is not continuous at $x = 3$.
(b) The relation R is not reflexive and not symmetric.
(c) Four and nine are not relatively prime.
(d) x is not in A and x is in B.
(e) $x < 7$ and $f(x)$ is in C.
(f) (a_n) is convergent, but (a_n) is not monotone or not bounded.
(g) f is continuous and A is open, but $f^{-1}(A)$ is not open.

5. Answers in book: (a) Antecedent: M is singular; consequent: M has a zero eigenvalue.
(b) Antecedent: linearity; consequent: continuity.
(c) Antecedent: a sequence is Cauchy; consequent: it is bounded.
(d) Antecedent: $y > 5$; consequent: $x < 3$.

6. (a) Antecedent: it is Cauchy; consequent: a sequence is convergent.
(b) Antecedent: boundedness; consequent: convergence.
(c) Antecedent: orthogonality; consequent: invertability.
(d) Antecedent: K is closed and bounded; consequent: K is compact.

7 and 8 are routine.

9. Answers in book: (a) $T \wedge T$ is T. (b) $F \vee T$ is T. (c) $F \vee F$ is F. (d) $T \Rightarrow T$ is T. (e) $F \Rightarrow F$ is T.
(f) $T \Rightarrow F$ is F. (g) $(T \wedge F) \Rightarrow T$ is T. (h) $(T \vee F) \Rightarrow F$ is F. (i) $(T \wedge F) \Rightarrow F$ is T. (j) $\sim(F \vee T)$ is F.

10. (a) $T \wedge F$ is F. (b) $F \vee F$ is F. (c) $F \vee T$ is T. (d) $T \Rightarrow F$ is F. (e) $F \Rightarrow F$ is T. (f) $F \Rightarrow T$ is T.
(g) $(F \vee T) \Rightarrow F$ is F. (h) $(T \Rightarrow F) \Rightarrow T$ is T. (i) $(T \wedge T) \Rightarrow F$ is F. (j) $\sim(F \wedge T)$ is T.

11. Answers in book: (a) $p \wedge \sim q$; (b) $(p \vee q) \wedge \sim(p \wedge q)$; (c) $\sim q \Rightarrow p$; (d) $\sim p \Rightarrow q$; (e) $p \Leftrightarrow \sim q$.

12. (a) $n \wedge \sim m$; (b) $\sim m \wedge \sim n$ or $\sim(m \vee n)$; (c) $n \Rightarrow m$; (d) $m \Rightarrow \sim n$; (e) $\sim(m \wedge n)$.

13. (a) and (b) are routine. (c) $p \wedge q$.

14. These truth tables are all straightforward. Note that the tables for (c) through (f) have 8 rows because there are 3 letters and therefore $2^3 = 8$ possible combinations of T and F.

Section 1.2 - Quantifiers

1. (a) True: See the comment before Example 1.2.1.
(b) False: The negation of a universal statement is an existential statement.
(c) True: See the comment before Example 1.2.1.

2. (a) False: It means there exists at least one.
(b) True: Example 1.2.1.
(c) True: See the comment after Practice 1.2.4.

3. (a) No pencils are red.
(b) Some chair does not have four legs.
(c) Someone on the basketball team is over 6 feet 4 inches tall.
(d) $\forall\, x > 2,\ f(x) \neq 7$.

(e) $\exists\, x$ in $A \ni \forall\, y > 2$, $f(y) \le 0$ or $f(y) \ge f(x)$.
(f) $\exists\, x \ni x > 3$ and $\forall\, \varepsilon > 0$, $x^2 \le 9 + \varepsilon$.

4. (a) Someone does not like Robert.
(b) No students work part-time.
(c) Some square matrices are triangular.
(d) $\forall\, x$ in B, $f(x) \le k$.
(e) $\exists\, x \ni x > 5$ and $3 \le f(x) \le 7$.
(f) $\exists\, x$ in $A \ni \forall\, y$ in B, $f(y) \le f(x)$.

5. Hints in book: The True/False part of the answers.
(a) True. Let $x = 3$. (b) True. 4 is less than 7 and anything smaller than 4 will also be less than 7.
(c) True. Let $x = \sqrt{5}$. (d) False. Choose $x \ne \pm\sqrt{5}$ such as $x = 2$.
(e) True. Let $x = 1$, or any other real number.
(f) True. The square of a real number cannot be negative.
(g) True. Let $x = 1$, or any real number other than 0. (h) False. Let $x = 0$.

6. (a) True. Let $x = 5$. (b) False. Let $x = 3$. (c) True. Choose $x \ne \pm\sqrt{3}$ such as $x = 2$.
(d) False. Let $x = \sqrt{3}$. (e) False. The square of a real number cannot be negative.
(f) False. Let $x = 1$, or any other real number. (g) True. Let $x = 1$, or any other real number.
(h) True. $x - x = x + (-x)$ and a number plus its additive inverse is zero.

7. **Answers in book: (a)** You can use (ii) to prove (a) is true. **(b)** You can use (i) to prove (b) is true.
Additional answers: (c) You can use (ii) to prove (c) is false. (d) You can use (i) to prove (d) is false.

8. The best answer is (c).

9. Hints in book: The True/False part of the answers.
(a) False. For example, let $x = 2$ and $y = 1$. Then $x > y$.
(b) True. For example, let $x = 2$ and $y = 3$. Then $x \le y$.
(c) True. Given any x, let $y = x + 1$. Then $x \le y$.
(d) False. Given any y, let $x = y + 1$. Then $x > y$.

10. (a) True. Given any x, let $y = 0$.
(b) False. Let $x = 0$. Then for all y we have $xy = 0 \ne 1$.
(c) False. Let $y = 0$. Then for all x we have $xy = 0 \ne 1$.
(d) True. Given any x, let $y = 1$. Then $xy = x$.

11. Hints in book: The True/False part of the answers.
(a) True. Let $x = 0$. Then given any y, let $z = y$. (A similar argument works for any x.)
(b) False. Given any x and any y, let $z = x + y + 1$.
(c) True. Let $z = y - x$.
(d) False. Let $x = 0$ and $y = 1$. (It is a true statement for $x \ne 0$.)
(e) True. Let $x \le 0$.
(f) True. Take $z \le y$. This makes "$z > y$" false so that the implication is true. Or, choose $z > x + y$.

12. (a) True. Given x and y, let $z = x + y$.
(b) False. Let $x = 0$. Then given any y, let $z = y + 1$.
(c) True. Let $x = 1$. Then given any y, let $z = y$. (Any $x \ne 0$ will work.)
(d) False. Let $x = 1$ and $y = 0$. (Any $x \ne 0$ will work.)
(e) False. Let $x = 2$. Given any y, let $z = y + 1$. Then "$z > y$" is true, but "$z > x + y$" is false.
(f) True. Given any x and y, either choose $z > x + y$ or $z \le y$.

13. Answer in book: (a) $\forall\, x,\ f(-x) = f(x)$; (b) $\exists\, x \ni f(-x) \neq f(x)$.

14. (a) $\exists\ k > 0 \ni \forall\, x, f(x+k) = f(x)$. (b) $\forall\ k > 0, \exists\, x \ni f(x+k) \neq f(x)$.

15. Answer in book: (a) $\forall\, x$ and y, $x \le y \Rightarrow f(x) \le f(y)$. (b) $\exists\, x$ and $y \ni x \le y$ and $f(x) > f(y)$.

16. (a) $\forall\, x$ and y, $x < y \Rightarrow f(x) > f(y)$. (b) $\exists\, x$ and $y \ni x < y$ and $f(x) \le f(y)$.

17. Answer in book: (a) $\forall\, x$ and y, $f(x) = f(y) \Rightarrow x = y$. (b) $\exists$ x and $y \ni f(x) = f(y)$ and $x \neq y$.

18. (a) $\forall\, y$ in $B\ \exists\, x$ in $A \ni f(x) = y$. (b) $\exists\, y$ in $B \ni \forall\, x$ in A, $f(x) \neq y$.

19. Answer in book: (a) $\forall\ \varepsilon > 0, \exists\ \delta > 0 \ni \forall\, x \in D, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
(b) $\exists\ \varepsilon > 0 \ni \forall\ \delta > 0, \exists\, x \in D \ni |x - c| < \delta$ and $|f(x) - f(c)| \ge \varepsilon$.

20. (a) $\forall\ \varepsilon > 0\ \exists\ \delta > 0 \ni \forall\, x$ and y in S, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.
(b) $\exists\ \varepsilon > 0 \ni \forall\ \delta > 0, \exists\, x$ and y in $S \ni |x - y| < \delta$ and $|f(x) - f(y)| \ge \varepsilon$.

21. Answer in book: (a) $\forall\ \varepsilon > 0, \exists\ \delta > 0 \ni \forall\, x \in D,\ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
(b) $\exists\ \varepsilon > 0 \ni \forall\ \delta > 0, \exists\, x \in D \ni 0 < |x - c| < \delta$ and $|f(x) - L| \ge \varepsilon$.

22. Answers will vary.

Section 1.3 – Techniques of Proof: I

1. (a) False: p is the hypothesis.
(b) False: The contrapositive is $\sim q \Rightarrow \sim p$.
(c) False: The inverse is $\sim p \Rightarrow \sim q$.
(d) False: $p(n)$ must be true for <u>all</u> n.
(e) True: Example 1.3.1.

2. (a) True: See the comment after Practice 1.3.4.
(b) False: It's called a contradiction.
(c) True: See the comment after Practice 1.3.8.
(d) True: See the end of Example 1.3.1.
(e) False: Must show $p(n)$ is true for all n.

3. Answers in book: (a) If all violets are not blue, then some roses are not red.
(b) If A is invertible, then there is no nontrivial solution to $A\mathbf{x} = \mathbf{0}$.
(c) If $f(C)$ is not connected, then either f is not continuous or C is not connected.

4. (a) If some violets are blue, then all roses are red.
(b) If A is not invertible, then there exists a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.
(c) If $f(C)$ is connected, then f is continuous and C is connected.

5. (a) If some roses are not red, then no violets are blue.
(b) If $A\mathbf{x} = \mathbf{0}$ has no nontrivial solutions, then A is invertible.
(c) If f is not continuous or C is not connected, then $f(C)$ is not connected.

6. For some of these, other answers are possible.
(a) Let $x = -4$.
(b) Let $n = -2$.
(c) If $0 < x < 1$, then $x^3 < x^2$. In particular, $(1/2)^3 = 1/8 < 1/4 = (1/2)^2$.
(d) An equilateral triangle.
(e) $n = 40$ or $n = 41$.
(f) 2 is prime, but not odd.
(g) 101, 103, etc.
(h) $3^5 + 2 = 245$ is not prime.
(i) Let $n = 5$ or any odd greater than 3.
(j) Let $x = 2$ and $y = 18$.
(k) Let $x = 0$.
(l) The reciprocal of 1 is not less than 1.
(m) Let $x = 0$.
(n) Let $x = -1$.

7. (a) Suppose $p = 2k + 1$ and $q = 2r + 1$ for integers k and r. Then $p + q = (2k + 1) + (2r + 1) = 2(k + r + 1)$, so $p + q$ is even.
   (b) Suppose $p = 2k + 1$ and $q = 2r + 1$ for integers k and r. Then $pq = (2k + 1)(2r + 1) = 4kr + 2k + 2r + 1 = 2(2kr + k + r) + 1$, so pq is odd.
   (c) Here are two approaches. The first mimics part (a) and the second uses parts (a) and (b).
   Proof 1: Suppose $p = 2k + 1$ and $q = 2r + 1$ for integers k and r. Then $p + 3q = (2k + 1) + 3(2r + 1) = 2(k + 3r + 2)$, so $p + 3q$ is even.
   Proof 2: Suppose p and q are both odd. By part (b), $3q$ is odd since 3 is odd. So by part (a), $p + 3q$ is even.
   (d) Suppose $p = 2k + 1$ and $q = 2r$ for integers k and r. Then $p + q = (2k + 1) + 2r = 2(k + r) + 1$, so $p + q$ is odd.
   (e) Suppose $p = 2k$ and $q = 2r$ for integers k and r. Then $p + q = 2k + 2r = 2(k + r)$, so $p + q$ is even.
   (f) Suppose $p = 2k$, then $pq = 2(kq)$, so pq is even. A similar argument applies when q is even.
   (g) This is the contrapositive of part (f).
   (h) Hint in book: look at the contrapositive.
   Proof: To prove the contrapositive, suppose $p = 2k + 1$. Then $p^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so p^2 is odd.
   (i) To prove the contrapositive, suppose $p = 2k$. Then $p^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so p^2 is even.

8. Suppose $f(x_1) = f(x_2)$. That is, $4x_1 + 7 = 4x_2 + 7$. Then $4x_1 = 4x_2$, so $x_1 = x_2$.

9. Answers in book:

(a)	$r \Rightarrow \sim s$	hypothesis
	$\sim s \Rightarrow \sim t$	contrapositive of hypothesis: 1.3.12(c)
	$r \Rightarrow \sim t$	by 1.3.12(l)
(b)	$\sim t \Rightarrow (\sim r \vee \sim s)$	contrapositive of hypothesis: 1.3.12(c)
	$\sim r \vee \sim s$	by 1.3.12(h)
	$\sim s$	by 1.3.12(j)
(c)	$r \vee \sim r$	by 1.3.12(d)
	$\sim s \Rightarrow \sim v$	contrapositive of hypothesis 4 [1.3.12(c)]
	$r \Rightarrow \sim v$	hypothesis 1 and 1.3.12(l)
	$\sim r \Rightarrow u$	hypotheses 2 and 3 and 1.3.12(l)
	$\sim v \vee u$	by 1.3.12(o)

10.

(a)	$\sim r$	hypothesis
	$\sim r \Rightarrow (r \vee \sim s)$	contrapositive of hypothesis: 1.3.12(c)
	$r \vee \sim s$	by 1.3.12(h)
	$\sim s$	by lines 1 and 3, and 1.3.12(j)
(b)	$\sim t$	hypothesis
	$\sim t \Rightarrow (\sim r \wedge \sim s)$	contrapositive of hypothesis: 1.3.12(c)
	$\sim r \wedge \sim s$	by lines 1 and 2, and 1.3.12(h)
	$\sim s$	by line 3 and 1.3.12(k)
(c)	$s \Rightarrow r$	contrapositive of hypothesis: 1.3.12(c)
	$t \Rightarrow u$	hypothesis
	$s \vee t$	hypothesis
	$r \vee u$	by 1.3.12(o)

11. Let p: The basketball center is healthy. q: The point guard is hot.
r: The team will win. s: The fans are happy.
t: The coach is a millionaire. u: The college will balance the budget.
The hypotheses are $(p \vee q) \Rightarrow (r \wedge s)$ and $(s \vee t) \Rightarrow u$. The conclusion is $p \Rightarrow u$.

Proof:	$p \Rightarrow (p \vee q)$	from the contrapositive of 1.3.12(k)
	$(p \vee q) \Rightarrow (r \wedge s)$	hypothesis
	$(r \wedge s) \Rightarrow s$	by 1.3.12(k)
	$s \Rightarrow (s \vee t)$	from the contrapositive of 1.3.12(k)
	$(s \vee t) \Rightarrow u$	hypothesis
	$p \Rightarrow u$	by 1.3.12(m)

Section 1.4 – Techniques of Proof: II

1. (a) True: See the comment before Example 1.4.1.
(b) False: Indirect proofs avoid this.
(c) False: Only the "relevant" steps need to be included.

2. (a) True: See the comment before Practice 1.4.2.
(b) False: The left side of the tautology should be $[(p \wedge \sim q) \Rightarrow c]$.
(c) True: See the comment after Practice 1.4.8.

3. Given any $\varepsilon > 0$, let $\delta = \varepsilon/3$. Then δ is also positive and whenever $2 - \delta < x < 2 + \delta$, we have $2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$ so that $6 - \varepsilon < 3x < 6 + \varepsilon$ and $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$, as required.

4. Given any $\varepsilon > 0$, let $\delta = \varepsilon/5$. Then δ is also positive and whenever $1 - \delta < x < 1 + \delta$, we have $1 - \frac{\varepsilon}{5} < x < 1 + \frac{\varepsilon}{5}$ so that $5 - \varepsilon < 5x < 5 + \varepsilon$ and $-2 - \varepsilon < 5x - 7 < -2 + \varepsilon$. Now multiply by -1 and reverse the inequalities: $2 + \varepsilon > 7 - 5x > 2 - \varepsilon$. This is equivalent to $2 - \varepsilon < 7 - 5x < 2 + \varepsilon$.

5. Let $x = 1$. Then for any real number y, we have $xy = y$.

6. Let $x = 0$. Then for any real number y, we have $xy = x$ because $xy = 0$.

7. Given any integer n, we have $n^2 + n^3 = n^2(1 + n)$. If n is even, then n^2 is even. If n is odd, then $1 + n$ is even. In either case, their product is even. [This uses Exercise 1.3.7(f).]

8. If n is odd, then $n = 2m + 1$ for some integer m, so $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 4m(m + 1) + 1$. If m is even, then $m = 2p$ for some integer p. But then $n^2 = 4(2p)(m + 1) + 1 = 8[p(m + 1)] + 1$. So $n^2 = 8k + 1$, where k is the integer $p(m + 1)$. On the other hand, if m is odd, then $m + 1$ is even and $m + 1 = 2q$ for some integer q. But then, $n^2 = 4m(2q) + 1 = 8(mq) + 1$. In this case, $n^2 = 8k + 1$, where k is the integer mq. In either case, $n^2 = 8k + 1$ for some integer k.

9. **Answer in book:** Let $n = -2$. Then $n^2 + \frac{3}{2}n = 4 + \left(\frac{3}{2}\right)(-2) = 4 - 3 = 1$, as required. The integer *is* unique.

10. Existence follows from Exercise 3. It is *not* unique. $x = -2$ or $x = 1/2$.

11. **Answer in book:** Let x be a real number greater than 5 and let $y = 3x/(5 - x)$. Then $5 - x < 0$ and $2x > 0$, so $y < 0$.

Furthermore, $\frac{5y}{y+3} = \frac{5\left(\frac{3x}{5-x}\right)}{\left(\frac{3x}{5-x}\right)+3} = \frac{15x}{3x + 3(5-x)} = \frac{15x}{15} = x$, as required.

12. Solve the quadratic $y^2 - 6xy + 9 = 0$ to obtain $y = 3x \pm 3\sqrt{x^2 - 1}$. Call one solution y and the other z. The proof is organized like Exercise 11.

13. Answer in book: Suppose that $x^2 + x - 6 \geq 0$ and $x > -3$. It follows that $(x - 2)(x + 3) \geq 0$ and, since $x + 3 > 0$, it must be that $x - 2 \geq 0$. That is, $x \geq 2$.

14. Suppose $\frac{x}{x-2} \leq 3$ and $x > 2$. Then $x - 2 > 0$, so we can multiply both sides of the first inequality by $x - 2$ to obtain $x \leq 3(x - 2)$. Thus $x \leq 3x - 6$. That is, $x \geq 3$. If $x = 2$, then $\frac{x}{x-2}$ is not defined.

15. Hint in book: Suppose $\log_2 7$ is rational and find a contradiction.

Proof: Suppose $\log_2 7 = a/b$, where a and b are integers. We may assume that $a > 0$ and $b > 0$. We have $2^{a/b} = 7$, which implies $2^a = 7^b$. But the number 2^a is even and the number 7^b is odd, a contradiction. Thus $\log_2 7$ must be irrational.

16. Suppose $|x + 1| \leq 3$ and consider the two cases: $x \geq -1$ and $x < -1$. If $x \geq -1$, then $x + 1 \geq 0$ so that $|x + 1| = x + 1$. This implies $x + 1 \leq 3$ and $x \leq 2$. So in this case we have $-1 \leq x \leq 2$.

On the other hand, if $x < -1$, then $x + 1 < 0$ and $|x + 1| = -(x + 1)$. This leads to $-(x + 1) \leq 3$, $x + 1 \geq -3$, and $x \geq -4$. In this case we have $-4 \leq x < -1$.

Combining the two cases, we get $-4 \leq x \leq 2$.

17. (a) This proves the converse, which is not a valid proof of the original implication.
(b) This is a valid proof using the contrapositive.

18. (a) This is a valid proof in the form of Example 1.3.12 (p): $[p \Rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \sim q) \Rightarrow r]$.
(b) This proves the converse, which is not a valid proof of the original implication.

19. (a) If $x = \frac{p}{q}$ and $y = \frac{m}{n}$, then $x + y = \frac{p}{q} + \frac{m}{n} = \frac{pn + qm}{qn}$, so $x + y$ is rational.
(b) If $x = \frac{p}{q}$ and $y = \frac{m}{n}$, then $xy = \frac{pm}{qn}$, so xy is rational.
(c) Hint in book: Use a proof by contradiction.

Proof: Suppose $x = \frac{p}{q}$, y is irrational, and suppose $x + y = \frac{r}{s}$. Then

$$y = (x + y) - x = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{qs},$$

so that y is rational, a contradiction. Thus $x + y$ must be irrational.

20. (a) False. For example, let $x = \sqrt{2}$ and $y = -\sqrt{2}$. Then $x + y = 0$.
(b) True. This is the contrapositive of Exercise 7(a).
(c) False. For example, let $x = y = \sqrt{2}$. Then $xy = 2$.
(d) True. This is the contrapositive of Exercise 13(b).

21. (a) Let $x = 0$ and $y = \sqrt{2}$. Then $xy = 0$.
(b) $y \neq \frac{xy}{x}$ when $x = 0$.
(c) If $x \neq 0$, then the conclusion is true.

22. It is false as stated, and a counterexample is $x = -\sqrt{2}$. Since x is negative, it's square root is not real, and hence not irrational. (Every irrational number is a real number.) If $x \geq 0$, then the result is true, and can be established by looking at the contrapositive. (In order to use the contrapositive, you have to know that the negation of "$\sqrt{x}$ is irrational" is "$\sqrt{x}$ is rational." This is only true if you know that $\sqrt{x}$ is a real number.)

23. Hint in book: Find a counterexample.
Solution: $6^2 + 8^2 = 10^2$ or $(-2)^2 + 0^2 = 2^2$.

24. If a, b, c are consecutive odd integers, then $a = 2k + 1$, $b = 2k + 3$, and $c = 2k + 5$ for some integer k. Suppose $a^2 + b^2 = c^2$. Then $(2k + 1)^2 + (2k + 3)^2 = (2k + 5)^2$. Whence $4k^2 - 4k - 15 = 0$ and $k = 5/2$ or $k = -3/2$. This contradicts k being an integer.

25. It is true. We may label the numbers as n, $n + 1$, $n + 2$, $n + 3$, and $n + 4$. We have the sum

$$S = n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10 = 5(n + 2),$$

so that S is divisible by 5.

26. It is true. We may label the numbers as n, $n + 1$, $n + 2$, and $n + 3$. We have the sum

$$S = n + (n + 1) + (n + 2) + (n + 3) = 4n + 6 = 2(2n + 3).$$

Since $2n + 3$ is odd, S is not divisible by 4.

27. Hint in book: Consider two cases depending on whether n is odd or even.
Proof: If n is odd, then n^2 and $3n$ are both odd, so $n^2 + 3n$ is even. Thus $n^2 + 3n + 8$ is even. If n is even, then so are n^2, $3n$, and $n^2 + 3n + 8$. (This uses Exercise 1.3.7.)

28. False. Let $n = 1$. Then $n^2 + 4n + 8 = 13$. Any other odd n will also work.

29. Hint in book: Let $z = \sqrt{2}^{\sqrt{2}}$. If z is rational, we are done. If z is irrational, look at $z^{\sqrt{2}}$.

Solution: We have $z^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{(\sqrt{2}\cdot\sqrt{2})} = \left(\sqrt{2}\right)^2 = 2.$

30. Counterexample: let $x = 1/3$ and $y = -8$. Then $(1/3)^{-8} = 3^8$ is a positive integer and $(-8)^{1/3}$ is a negative integer.

31. Suppose $x > 0$. Then $(x + 1)^2 = x^2 + 2x + 1 > x^2 + 1$, so the first inequality holds. For the second inequality, consider the difference $2(x^2 + 1) - (x + 1)^2$. We have

$$2(x^2 + 1) - (x + 1)^2 = 2x^2 + 2 - x^2 - 2x - 1 = x^2 - 2x + 1 = (x - 1)^2 \geq 0,$$

for all x, so the second inequality also holds.