# CHAPTER 1

## EXERCISES 1.2

**1.** The solution is $x = 2/3$.

**2.** The solution is $x = -5/14$.

**3.** Since $x^2 + 2x - 3 = (x+3)(x-1)$, solutions are $x = -3$ and $x = 1$.

**4.** Since $12x^2 + 11x - 5 = (3x-1)(4x+5)$, solutions are $x = 1/3$ and $x = -5/4$.

**5.** Since the discriminant $5^2 - 4(2)(10) = -55$ is negative, the equation has no real solutions.

**6.** Quadratic formula 1.5 gives $x = \dfrac{-10 \pm \sqrt{100 - 4(-4)(9)}}{-8} = \dfrac{-10 \pm \sqrt{244}}{-8} = \dfrac{5 \pm \sqrt{61}}{4}$.

**7.** Since $x^2 + 8x + 16 = (x+4)^2$, the solution is $x = -4$ with multiplicity 2.

**8.** Since $4x^2 - 36x + 81 = (2x-9)^2$, the solution is $x = 9/2$ with multiplicity 2.

**9.** Quadratic formula 1.5 gives $x = \dfrac{-5 \pm \sqrt{25 - 4(2)(-10)}}{4} = \dfrac{-5 \pm \sqrt{105}}{4}$.

**10.** Since the discriminant $(-8)^2 - 4(4)(9) = -80$ is negative, the equation has no real solutions.

**11.** Since $x^3 - 3x^2 + 3x - 1 = (x-1)^3$, the solution is $x = 1$ with multiplicity 3.

**12.** Possible rational solutions are $\pm 1$, $\pm 1/2$, $\pm 1/4$, $\pm 1/8$. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the cubic,

$$8x^3 + 12x^2 + 6x + 1 = (2x+1)(4x^2 + 4x + 1) = (2x+1)(2x+1)^2 = (2x+1)^3.$$

The only solution is $x = -1/2$ with multiplicity 3.

**13.** Possible rational solutions are $x = \pm 1, \pm 2, \pm 5, \pm 10$. We find that $x = 2$ is a solution. We factor $x - 2$ from the cubic,

$$x^3 - 2x^2 + 5x - 10 = (x-2)(x^2 + 5) = 0.$$

The other two solutions are complex.

**14.** Possible rational solutions are $x = \pm 1, \pm 3, \pm 9$, but it should also be clear that no positive value of $x$ can satisfy the equation. We find that $x = -1$ is a solution. We factor $x + 1$ from the cubic,

$$x^3 + 4x^2 + 12x + 9 = (x+1)(x^2 + 3x + 9) = 0.$$

Since the discriminant of the quadratic is negative, the other two solutions are complex.

**15.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 8$, $\pm 16$, $\pm 32$, $\pm 64$, but it should also be clear that no positive value of $x$ can satisfy the equation. We find that $x = -4$ is a solution. We factor $x + 4$ from the cubic,

$$x^3 + 12x^2 + 48x + 64 = (x+4)(x^2 + 8x + 16) = (x+4)(x+4)^2 = (x+4)^3.$$

The only solution is $x = -4$ with multiplicity 3.

**16.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 9$, $\pm 12$, $\pm 18$, $\pm 36$. We find that $x = -3$ is a solution. We factor $x + 3$ from the quartic,

$$x^4 + 7x^3 + 9x^2 - 21x - 36 = (x+3)(x^3 + 4x^2 - 3x - 12).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 12$. We find that $x = -4$ is a zero. We factor $x + 4$ from the cubic,

$$x^4 + 7x^3 + 9x^2 - 21x - 36 = (x+3)(x+4)(x^2 - 3).$$

The solutions are $x = -3, -4, \pm\sqrt{3}$.

**17.** Since $x^4 - 16 = (x^2+4)(x^2-4) = (x^2+4)(x+2)(x-2)$, the real solutions are $x = \pm 2$.

**18.** Possible rational solutions are $\pm 1$, $\pm 3$, $\pm 5$, $\pm 15$, $\pm 1/2$, $\pm 3/2$, $\pm 5/2$, $\pm 15/2$. We find that $x = -5$ is a solution. We factor $x + 5$ from the quartic,

$$2x^4 + 9x^3 - 6x^2 - 8x - 15 = (x + 5)(2x^3 - x^2 - x - 3).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 3$, $\pm 1/2$, $\pm 3/2$. We find that $x = 3/2$ is a zero. We factor $2x - 3$ from the cubic,

$$2x^4 + 9x^3 - 6x^2 - 8x - 15 = (x + 5)(2x - 3)(x^2 + x + 1).$$

Since the quadratic has a negative discriminant, the only real solutions are $x = -5$ and $x = 3/2$.

**19.** Possible rational solutions are $\pm 1$, $\pm 3$, $\pm 9$, $\pm 1/2$, $\pm 3/2$, $\pm 9/2$, $\pm 1/3$, $\pm 1/6$. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the quartic,

$$6x^4 + x^3 + 53x^2 + 9x - 9 = (2x + 1)(3x^3 - x^2 + 27x - 9).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 3$, $\pm 9$, $\pm 1/3$. We find that $x = 1/3$ is a zero. We factor $3x - 1$ from the cubic,

$$6x^4 + x^3 + 53x^2 + 9x - 9 = (2x + 1)(3x - 1)(x^2 + 9).$$

Since the quadratic has complex zeros, the only real solutions are $x = -1/2$ and $x = 1/3$.

**20.** No real numbers can satisfy this equation.

**21.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 9$, $\pm 12$, $\pm 18$, $\pm 24$, $\pm 36$, $\pm 72$. We find that $x = 24$ is a solution. We factor $x - 24$ from the cubic,

$$x^3 - 23x^2 - 21x - 72 = (x - 24)(x^2 + x + 3).$$

Since the quadratic has a negative discriminant, the only real solution is $x = 24$.

**22.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 9$, $\pm 12, \pm 16$, $\pm 18$, $\pm 24$, $\pm 32$, $\pm 36$, $\pm 48$, $\pm 64$, $\pm 72$, $\pm 96, \pm 144$, $\pm 192$, $\pm 192$, $\pm 288$, $\pm 576$. We find that $x = -4$ is a solution. We factor $x + 4$ from the quartic,

$$x^4 - 4x^3 - 44x^2 + 96x + 576 = (x + 4)(x^3 - 8x^2 - 12x + 144).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 6$, $\pm 8$, $\pm 9$, $\pm 12, \pm 16$, $\pm 18$, $\pm 24$, $\pm 36$, $\pm 48$, $\pm 72$, $\pm 144$. We find that $x = -4$ is a zero. We factor $x + 4$ from the cubic,

$$x^4 - 4x^3 - 44x^2 + 96x + 576 = (x + 4)(x + 4)(x^2 - 12x + 36) = (x + 4)^2(x - 6)^2.$$

Thus, $x = -4$ and $x = 6$ are solutions, each of multiplicity 2.

**23.** Since $3x^4 + x^3 + 5x^2 = x^2(3x^2 + x + 5)$, and the quadratic has a negative discriminant, the only real solution is $x = 0$ with multiplicity 2.

**24.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 5$, $\pm 10$, $\pm 20$, $\pm 1/2$, $\pm 5/2, \pm 1/3$, $\pm 2/3$, $\pm 4/3$, $\pm 5/3$, $\pm 10/3$, $\pm 20/3$, $\pm 1/6$, $\pm 5/6$. We find that $x = 5/6$ is a zero. We factor $6x - 5$ from the cubic,

$$6x^3 + x^2 + 19x - 20 = (6x - 5)(x^2 + x + 4).$$

Since the quadratic has a negative discriminant, $x = 5/6$ is the only real solution.

**25.** Possible rational zeros are $\pm 1$, $\pm 3$, $\pm 5$, $\pm 9$, $\pm 15$, $\pm 45$. We find that $x = -5$ is a solution. We factor $x + 5$ from the polynomial,

$$x^5 + 5x^4 - 9x - 45 = (x + 5)(x^4 - 9) = (x + 5)(x^2 + 3)(x^2 - 3).$$

Solutions are $x = -5$ and $x = \pm\sqrt{3}$.

**26.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 5$, $\pm 6$, $\pm 8$, $\pm 10, \pm 12$, $\pm 15$, $\pm 20$, $\pm 24$, $\pm 30$, $\pm 40$, $\pm 60$, $\pm 120$. We find that $x = 1$ is a solution. We factor $x - 1$ from the polynomial,

$$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = (x-1)(x^4 - 14x^3 + 71x^2 - 154x + 120).$$

We use the same set of rational possibilities for the quartic. We find that $x = 2$ is a zero. When we factor $x - 2$ from the quartic,

$$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = (x-1)(x-2)(x^3 - 12x^2 + 47x - 60).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, $\pm 5$, $\pm 6$, $\pm 10, \pm 12$, $\pm 15$, $\pm 20$, $\pm 30$, $\pm 60$. We find that $x = 3$ is a zero. We factor $x - 3$ from the cubic,

$$\begin{aligned} x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 &= (x-1)(x-2)(x-3)(x^2 - 9x + 20) \\ &= (x-1)(x-2)(x-3)(x-4)(x-5). \end{aligned}$$

Solutions are therefore $x = 1,\ 2,\ 3,\ 4,\ 5$.

**27.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 1/2$, $\pm 1/4$, but it should also be clear that no positive value of $x$ can satisfy the equation. We find that $x = -1/2$ is a solution. We factor $2x + 1$ from the quartic,

$$4x^4 + 4x^3 + 17x^2 + 16x + 4 = (2x+1)(2x^3 + x^2 + 8x + 4).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 1/2$, and once again we reject the positive values. We find that $x = -1/2$ is a zero. We factor $2x + 1$ from the cubic,

$$4x^4 + 4x^3 + 17x^2 + 16x + 4 = (2x+1)(2x+1)(x^2 + 4).$$

Thus, $x = -1/2$ is the only real solution and it has multiplicity 2.

**28.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 1/5$, $\pm 2/5$, $\pm 4/5$, $\pm 1/25$, $\pm 2/25$, $\pm 4/25$. We find that $x = 2/5$ is a solution. We factor $5x - 2$ from the quartic,

$$25x^4 - 120x^3 + 109x^2 - 36x + 4 = (5x-2)(5x^3 - 22x^2 + 13x - 2).$$

Possible rational zeros of the cubic are $\pm 1$, $\pm 2$, $\pm 1/5$, $\pm 2/5$. We find that $x = 2/5$ is a zero. We factor $5x - 2$ from the cubic,

$$25x^4 - 120x^3 + 109x^2 - 36x + 4 = (5x-2)(5x-2)(x^2 - 4x + 1).$$

Thus, $x = 2/5$ is a real solution with multiplicity 2 and the quadratic formula gives the remaining two solutions

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

**29.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$, $\pm 5$, $\pm 8$, $\pm 10$, $\pm 20$, $\pm 25, \pm 40$, $\pm 50$, $\pm 100$, $\pm 200$. We find that $x = 1$ is a zero. We factor $x - 1$ from the polynomial,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x^4 + 10x^3 + 57x^2 + 182x + 200).$$

We use the same rational numbers for the quartic, but reject the positive values. We find that $x = -2$ is a zero. We factor $x + 2$ from the quartic,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x+2)(x^3 + 8x^2 + 41x + 100).$$

For zeros of the cubic we use $-1$, $-2$, $-4$, $-5$, $-10$, $-20$, $-25$, $-50$, $-100$. We find that $x = -4$ is a zero. When we factor it out,

$$x^5 + 9x^4 + 47x^3 + 125x^2 + 18x - 200 = (x-1)(x+2)(x+4)(x^2 + 4x + 25).$$

Since the quadratic has a negative discriminant, the real solutions are $x = -4,\ -2,\ 1$.

**30.** Possible rational solutions are $\pm1$, $\pm2$, $\pm3$, $\pm4$, $\pm6$, $\pm8$, $\pm9$, $\pm12$, $\pm16$, $\pm18$, $\pm24$, $\pm27$, $\pm36$, $\pm48$, $\pm54$, $\pm72$, $\pm81$, $\pm108$, $\pm144$, $\pm162$, $\pm216$, $\pm324$, $\pm432$, $\pm648$, $\pm1296$. We find that $x = 3$ is a solution. When we factor $x - 3$ from the polynomial,

$$x^6 + 16x^4 - 81x^2 - 1296 = (x - 3)(x^5 + 3x^4 + 25x^3 + 75x^2 + 144x + 432).$$

When we note that no positive value can satisfy the fifth degree polynomial, possible rational zeros are $-1$, $-2$, $-3$, $-4$, $-6$, $-8$, $-9$, $-12$, $-16$, $-18$, $-24$, $-27$, $-36$, $-48$, $-54$, $-72$, $-108$, $-144$, $-216$, $-432$. We find that $x = -3$ is a zero. When we factor $x + 3$ from the polynomial,

$$x^6 + 16x^4 - 81x^2 - 1296 = (x - 3)(x + 3)(x^4 + 25x^2 + 144).$$

Since there can be no real zeros of the quartic, the real solutions are $x = \pm3$.

**31.** $2(x + 5)(x - 1)$

**32.** $2[x - (3 + \sqrt{65})/4][x - (3 - \sqrt{65})/4]$

**33.** $(x + 2)(x - 3)(x - 10)$

**34.** $24(x + 5/3)(x - 1/4)(x - 1/2)$

**35.** $(x + 1)(x - 1)^3$

**36.** $16(x - 1/2)^2(x + 1/2)^2$

**37.** One polynomial is $(x+1/3)(x-4/5)(x-3)(x-4)^3$. This polynomial could be multiplied by any constant.

**38.** According to the rational root theorem, possible rational zeros must be divisors of $a_0$ divided by divisors of 1. This means that possible rational zeros are divisors of $a_0$.

## EXERCISES 1.3

**1.** With formula 1.10, the distance is $\sqrt{2^2 + 1^2} = \sqrt{5}$.

**2.** With formula 1.10, the distance is $\sqrt{6^2 + (-3)^2} = 3\sqrt{5}$.

**3.** With formula 1.10, the distance is $\sqrt{(-2)^2 + (-6)^2} = 2\sqrt{10}$.

**4.** With formula 1.10, the distance is $\sqrt{(-7)^2 + (-3)^2} = \sqrt{58}$.

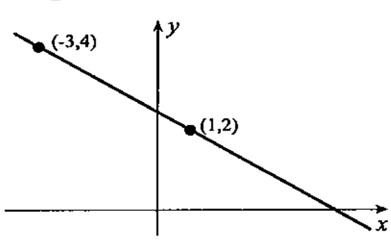
**5.** With formula 1.11, the midpoint is $\left(\dfrac{1+3}{2}, \dfrac{3+4}{2}\right) = \left(2, \dfrac{7}{2}\right)$.

**6.** With formula 1.11, the midpoint is $\left(\dfrac{-2+4}{2}, \dfrac{1-2}{2}\right) = \left(1, -\dfrac{1}{2}\right)$.

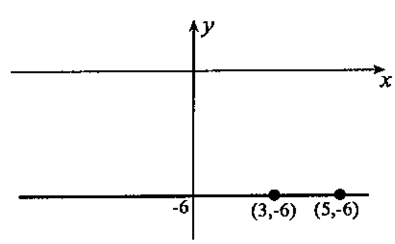
**7.** With formula 1.11, the midpoint is $\left(\dfrac{-1-3}{2}, \dfrac{-2-8}{2}\right) = (-2, -5)$.

**8.** With formula 1.11, the midpoint is $\left(\dfrac{3-4}{2}, \dfrac{2-1}{2}\right) = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

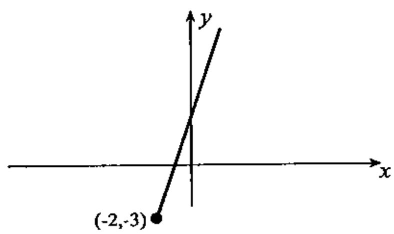
**9.** With slope $-2/4 = -1/2$, equation 1.13 gives
$y - 2 = -\dfrac{1}{2}(x - 1) \implies x + 2y = 5.$



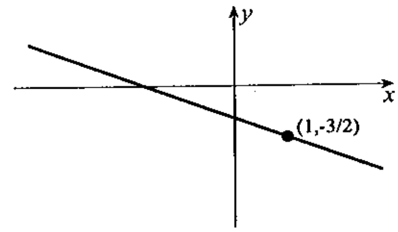

**10.** The line is horizontal.
Its equation is $y = -6$.



**11.** With formula 1.13, the equation is
$y + 3 = 3(x + 2) \implies y = 3x + 3.$



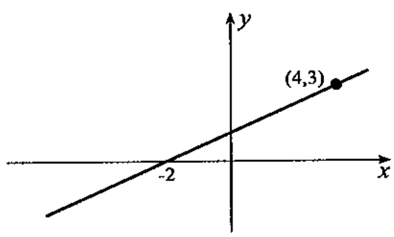

**12.** With formula 1.13, the equation is
$y + \frac{3}{2} = -\frac{1}{2}(x - 1) \implies x + 2y + 2 = 0.$



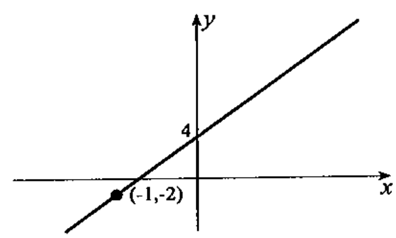

**13.** The equation of the $y$-axis is $x = 0$.

**14.** The equation of the $x$-axis is $y = 0$.

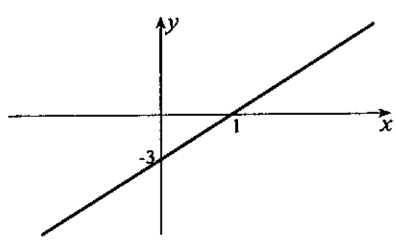
**15.** With $m = (3 - 0)/(4 + 2) = 1/2$, equation 1.13 gives $y = (1/2)(x + 2) \implies 2y = x + 2$.



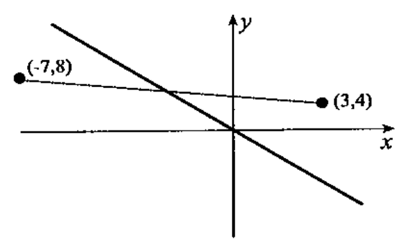

**16.** With $m = (4 + 2)/(0 + 1) = 6$, equation 1.13 gives $y - 4 = 6(x - 0) \implies y = 6x + 4$.




**17.** With slope $(0 + 3)/(1 - 0) = 3$, equation 1.13 gives $y - 0 = 3(x - 1) \implies y = 3x - 3$.




**18.** The midpoint of the line segment is
$$\left(\frac{3 - 7}{2}, \frac{4 + 8}{2}\right) = (-2, 6).$$
With $m = (6 - 0)/(-2 - 0) = -3$, equation 1.13 gives $y - 0 = -3(x - 0) \implies y = -3x$.




**19.** Since slopes $-1$ and $1$ are negative reciprocals, the lines are perpendicular.

**20.** Since slopes of both lines are $-1/3$, they are parallel.

**21.** Since slopes of both lines are $1/3$, they are parallel.

**22.** Since slopes $-2/3$ and $3/2$ are negative reciprocals, the lines are perpendicular.

**23.** Since slopes are $3$ and $-1/2$, the lines are neither parallel nor perpendicular.

**24.** Since slopes are $1$ and $-2/3$, the lines are neither parallel nor perpendicular.

**25.** The lines are perpendicular.

**26.** Since slopes are $-1$ and $3$, the lines are neither parallel nor perpendicular.

**27.** When we subtract the equations, $y + 2y = 0 + 3 \implies y = 1$. The point of intersection is $(-1, 1)$.

**28.** The point of intersection is $(1, 2)$.

**29.** When we subtract 3 times the second equation from the first, $4y + 18y = 6 - 9 \implies y = -3/22$. The point of intersection is $(24/11, -3/22)$.

**30.** When we substitute $y = 2x + 6$ into the second equation, $x = (2x + 6) + 4 \implies x = -10$. The point of intersection is $(-10, -14)$.

**31.** When we substitute $x = 2 - 2y/3$ into the second equation, $2(2 - 2y/3) - y/4 = 15 \Longrightarrow y = -132/19$. The point of intersection is $(126/19, -132/19)$.

**32.** If we multiply the first equation by 5 and add the result to the second equation, we obtain $73x = 37 \Longrightarrow x = 37/73$. The point of intersection is $(37/73, 153/146)$.

**33.** Formula 1.16 gives $\dfrac{|3+4-1|}{\sqrt{1+1}} = \dfrac{6}{\sqrt{2}}$.

**34.** Formula 1.16 gives $\dfrac{|1-6-3|}{\sqrt{1+4}} = \dfrac{8}{\sqrt{5}}$.

**35.** Formula 1.16 gives $\dfrac{|5-1-4|}{\sqrt{1+1}} = 0$.

**36.** Formula 1.16 gives $\dfrac{|3+1|}{\sqrt{1+1}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$.

**37.** The distance is 7.

**38.** Formula 1.16 gives $\dfrac{|60+4+3|}{\sqrt{225+4}} = \dfrac{67}{\sqrt{229}}$.

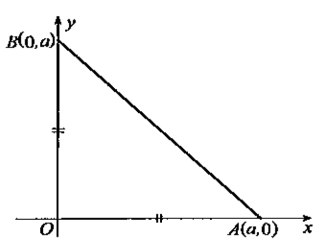
**39.** Since the point of intersection of the given lines is $(3, 1)$, and the slope of $x + 2y = 15$ is $-1/2$, the required equation is $y - 1 = -(1/2)(x - 3) \Longrightarrow x + 2y = 5$.

**40.** Since the slope of $x - y = 4$ is 1, and the point of intersection of $2x + 3y = 3$ and $x - y = 4$ is $(3, -1)$, the required equation is $y + 1 = -1(x - 3) \Longrightarrow x + y = 2$.

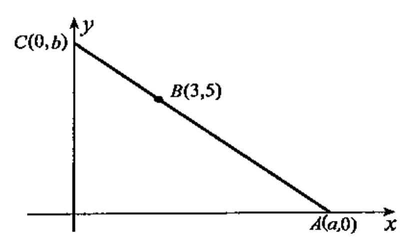
**41.** Since the slope of the line through $(1, 2)$ and $(-3, 0)$ is $2/4$, the required equation is $y - 6 = (1/2)(x - 5) \Longrightarrow 2y = x + 7$.

**42.** Since the slope of the line through $(-3, 4)$ and $(1, -2)$ is $(4 + 2)/(-3 - 1) = -3/2$, the equation of the required line is $y + 2 = (2/3)(x + 3) \Longrightarrow 2x = 3y$.

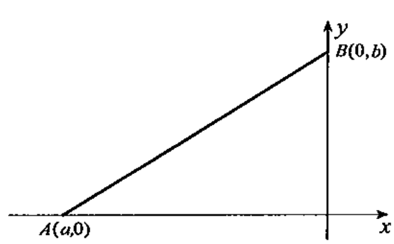
**43.** Since $\|OA\| = \|OB\|$, coordinates of $A$ and $B$ are $(a, 0)$ and $(0, a)$, respectively. Since the area of $\Delta OAB$ is 8, it follows that $(1/2)a^2 = 8 \Longrightarrow a = 4$. The equation of line $AB$ is $y = -(x - 4) \Longrightarrow x + y = 4$.



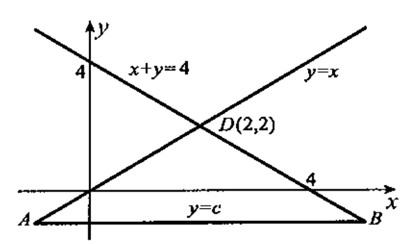

**44.** When we equate slopes of line segments $AC$ and $AB$, $-\dfrac{b}{a} = \dfrac{5}{3-a} \Longrightarrow b = \dfrac{5a}{a-3}$. Combine this with the fact that the area of $\Delta OAB = (1/2)ab = 30$, and we obtain $a = 6$ and $b = 10$. The equation of the line is $y = -(5/3)(x - 6) \Longrightarrow 5x + 3y = 30$.




**45.** Since the slope of line segment $AB$ is 2, and its length is 3, it follows that $-b/a = 2$ and $\sqrt{a^2 + b^2} = 3$. These can be solved for $a = -3/\sqrt{5}$ and $b = 6/\sqrt{5}$. The equation of $AB$ is $y = 2x + 6/\sqrt{5}$.

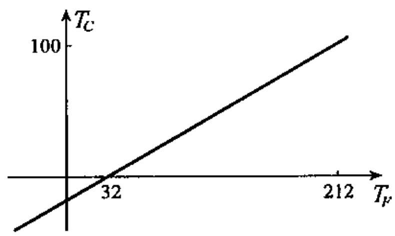


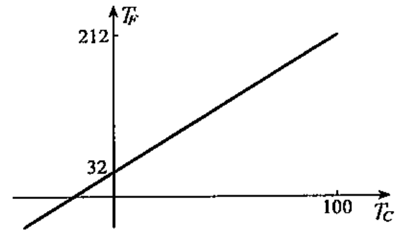

**46.** If the equation is $y = c$, then $A = (c, c)$ and $B = (4 - c, c)$. Since triangle $ABD$ has area 9, it follows that $9 = \frac{1}{2}(2 - c)(4 - 2c)$, solutions of which are 5, $-1$. The required equation is therefore $y = -1$.




**47.** (a) The conversion equation is $T_C = 5(T_F - 32)/9$.
(b) The conversion equation is $T_F = 9T_C/5 + 32$.
(c) They are one and the same line if we plot $T_F$ along the horizontal axis and $T_C$ along the vertical axis (left figure below). If we plot $T_F$ along the vertical axis and $T_C$ along the horizontal axis in part

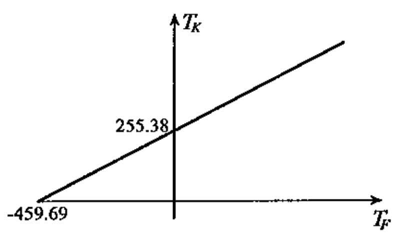
(b), we obtain the right figure below.

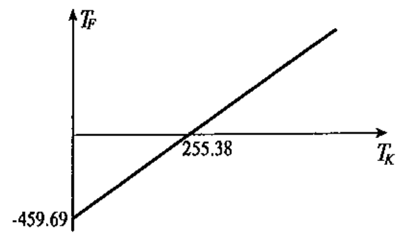



(d) If we set $T_F = T_C$ in the equation from part (a), we obtain $T_C = 5(T_C - 32)/9 \Longrightarrow 4T_C = -160 \Longrightarrow T_C = -40$.

**48.** (a) The conversion equation is $T_K = 5(T_F - 32)/9 + 273.16$.
(b) The conversion equation is $T_F = 9(T_K - 273.16)/5 + 32$.
(c) They are one and the same line if we plot $T_F$ along the horizontal axis and $T_K$ along the vertical axis (left figure below). If we plot $T_F$ along the vertical axis and $T_K$ along the horizontal axis in part (b), we obtain the right figure below.

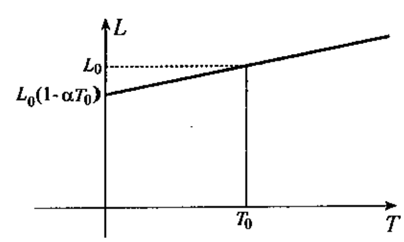



(d) If we set $T_F = T_K$ in the equation from part (a), we obtain $T_K = 5(T_K - 32)/9 + 273.16 \Longrightarrow 4T_K = 2298.44 \Longrightarrow T_K = 574.61$.

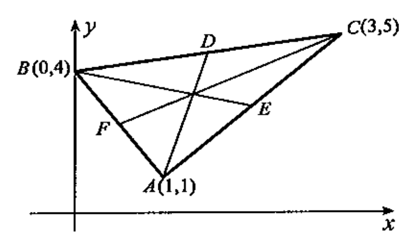
**49.** (a) If the temperature is changed to $T$, the change in the length of the bar is $\alpha L_0(T - T_0)$, and therefore its length is

$$L = L_0 + \alpha L_0(T - T_0) = L_0[1 + \alpha(T - T_0)].$$



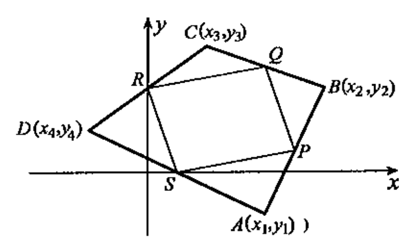

(b) The ends will be in contact when the rails have length $L = 10.003$ m. This occurs when $10.003 = 10[1 + 1.17 \times 10^{-5}(T - 20)] \Longrightarrow T = 45.6$.

**50.** Coordinates of $E$, the midpoint of side $AC$, are $(2, 3)$. Since the slope of $BE$ is $-1/2$, the equation of median $BE$ is $y - 4 = -(1/2)x \Longrightarrow x + 2y = 8$. Similarly, equations for medians $AD$ and $CF$ are $7x - y = 6$ and $y = x + 2$, respectively. Line segments $AD$ and $CF$ intersect in the point $(4/3, 10/3)$, and this point also satisfies $x + 2y = 8$. Thus, the three medians intersect at the point $(4/3, 10/3)$.




**51.** Coordinates of $P$, $Q$, $R$, and $S$ are $P((x_1 + x_2)/2, (y_1 + y_2)/2)$, $Q((x_2 + x_3)/2, (y_2 + y_3)/2)$, $R((x_3 + x_4)/2, (y_3 + y_4)/2)$, and $S((x_1 + x_4)/2, (y_1 + y_4)/2)$. Slopes of the line segments $PS$, $RQ$, $PQ$, and $RS$ are, respectively,

$$\frac{(y_2 - y_4)/2}{(x_2 - x_4)/2}, \quad \frac{(y_2 - y_4)/2}{(x_2 - x_4)/2}, \quad \frac{(y_3 - y_1)/2}{(x_3 - x_1)/2}, \quad \frac{(y_3 - y_1)/2}{(x_3 - x_1)/2}.$$




Since $PS$ and $RQ$ are parallel, as are $PQ$ and $RS$, $PQRS$ is a parallelogram.

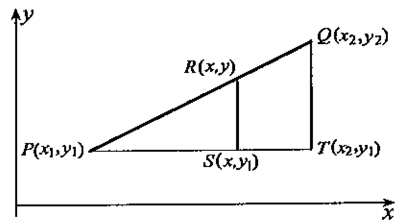
**52.** The midpoint of the line segment is $(1,-1)$, and its slope is $-3/2$. The equation of the perpendicular bisector is $y+1=(2/3)(x-1) \implies 2x=3y+5$.

**53.** If coordinates of the point are $(x,y)$, then $(x-1)^2+(y-2)^2=(x+1)^2+(y-4)^2$ and $(x-1)^2+(y-2)^2=(x+3)^2+(y-1)^2$. When expanded and simplified, these reduce to $x-y=-3$ and $8x+2y=-5$, the solution of which is $(-11/10, 19/10)$.

**54.** Suppose two nonvertical, parallel lines have slopes $m_1$ and $m_2$. Their equations can be expressed in the form $y=m_1x+b_1$ and $y=m_2x+b_2$. To find their point of intersection we set $m_1x+b_1=m_2x+b_2$. Since the lines do not intersect, there must be no solution of this equation. This happens only if $m_1=m_2$; that is, the lines have the same slope. Conversely, if two nonvertical lines have the same slope, their equations must be of the form $y=mx+b_1$ and $y=mx+b_2$, where $b_1 \neq b_2$. When we attempt to find their point of intersection by setting $mx+b_1=mx+b_2$, we find that $b_1=b_2$, a contradiction. In other words, the lines do not intersect.

**55.** Because triangles $PTQ$ and $PSR$ are similar, ratios of corresponding sides are equal,

$$\frac{\|PQ\|}{\|PR\|}=\frac{\|PT\|}{\|PS\|}=\frac{x_2-x_1}{x-x_1}.$$




If we subtract 1 from each side of this equation,

$$\frac{\|PQ\|}{\|PR\|}-1=\frac{x_2-x_1}{x-x_1}-1$$

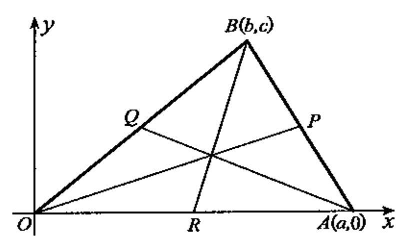
$$\implies \quad \frac{\|PQ\|-\|PR\|}{\|PR\|}=\frac{x_2-x}{x-x_1} \quad \implies \quad \frac{r_2}{r_1}=\frac{x_2-x}{x-x_1}.$$

Thus, $r_2x-r_2x_1=r_1x_2-r_1x \implies x=\dfrac{r_1x_2+r_2x_1}{r_1+r_2}$. A similar proof gives the corresponding formula for the $y$-coordinate of $R$.

**56.** No. Definition 1.1 defines parallelism only for different lines.

**57.** We choose a coordinate system with the origin at one vertex of the triangle, and the positive $x$-axis along one side. The coordinates of the vertices are then $O(0,0)$, $A(a,0)$, and $B(b,c)$. Using equation 1.11, coordinates of the midpoints of the sides are

$$P\left(\frac{a+b}{2},\frac{c}{2}\right), \quad Q\left(\frac{b}{2},\frac{c}{2}\right), \quad R\left(\frac{a}{2},0\right).$$



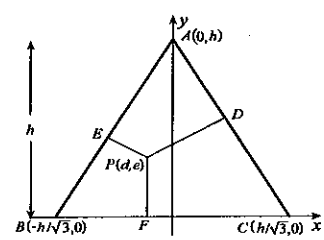

The sum of the squares of the lengths of the medians is

$$\begin{aligned}\|OP\|^2+\|AQ\|^2+\|BR\|^2 &= \left[\left(\frac{a+b}{2}\right)^2+\left(\frac{c}{2}\right)^2\right]+\left[\left(a-\frac{b}{2}\right)^2+\left(-\frac{c}{2}\right)^2\right]+\left[\left(b-\frac{a}{2}\right)^2+c^2\right]\\ &= \frac{3}{2}(a^2+b^2+c^2-ab).\end{aligned}$$

Three-quarters of the sum of the squares of the lengths of the sides is

$$\frac{3}{4}\left(\|OA\|^2+\|AB\|^2+\|OB\|^2\right)=\frac{3}{4}\left\{a^2+[(b-a)^2+c^2]+(b^2+c^2)\right\}=\frac{3}{2}(a^2+b^2+c^2-ab).$$

**58.** If we choose the coordinate system in the figure, equations of $AC$ and $AB$ are $y=-\sqrt{3}(x-h/\sqrt{3})$ $\implies \sqrt{3}x+y-h=0$ and $y=\sqrt{3}(x+h/\sqrt{3})$ $\implies \sqrt{3}x-y+h=0$. If $P(d,e)$ is any point interior to the triangle, we can use formula 1.16 to find the sum of the distances from $P$ to the three sides




$$\|PF\|+\|PD\|+\|PE\|$$

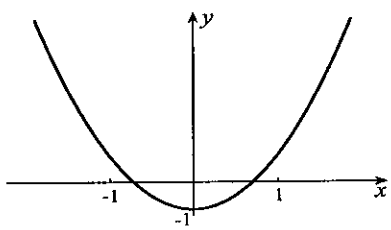
$$= e + \frac{|\sqrt{3}d + e - h|}{\sqrt{3+1}} + \frac{|\sqrt{3}d - e + h|}{\sqrt{3+1}} = e + \frac{1}{2}|\sqrt{3}d + e - h| + \frac{1}{2}|\sqrt{3}d - e + h|.$$

Because $P(d, e)$ is below and to the left of line $AC$, it follows that $\sqrt{3}d + e - h < 0$. Because $P$ is below and to the right of $AB$, $\sqrt{3}d - e + h > 0$. Hence,

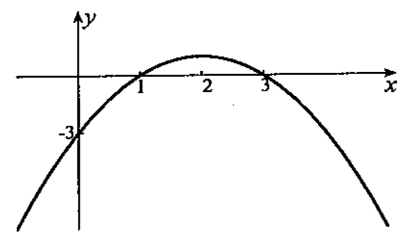
$$\|PF\| + \|PD\| + \|PE\| = e + \frac{1}{2}(-\sqrt{3}d - e + h) + \frac{1}{2}(\sqrt{3}d - e + h) = h.$$

## EXERCISES 1.4A

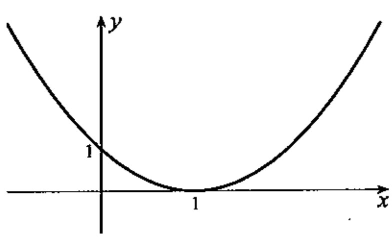
**1.** This is the parabola $y = 2x^2$ shifted 1 unit downward.



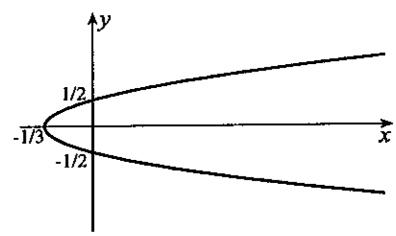

**2.** Factored in the form $y = -(x-3)(x-1)$, the $x$-intercepts of the parabola are 1 and 3. Its maximum is at $x = 2$.



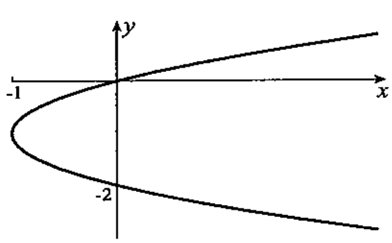

**3.** Factored in the form $y = (x-1)^2$, the parabola has its minimum at $(1, 0)$.



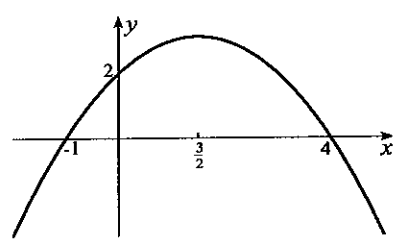

**4.** This is the parabola $x = 4y^2/3$ shifted 1/3 unit to the left.



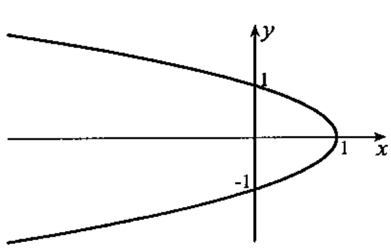

**5.** Factored in the form $x = y(2+y)$, the parabola opens to the right with $y$-intercepts 0 and $-2$.



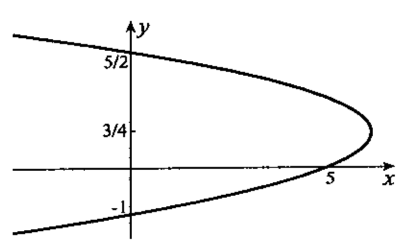

**6.** Factored in the form $y = -(x+1)(x-4)/2$, $x$-intercepts of the parabola are $-1$ and 4. Its maximum is at $x = 3/2$.



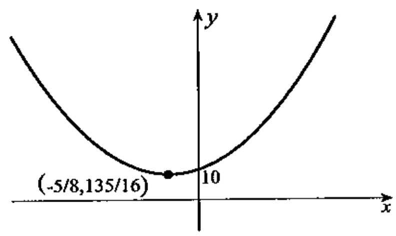

**7.** The parabola $x = 1 - y^2$ opens left with $x$-intercept 1 and $y$-intercepts $\pm 1$.



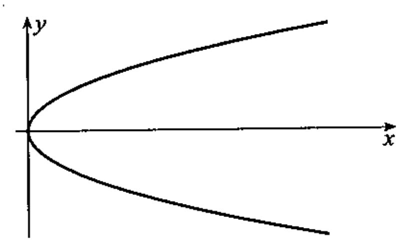

**8.** Factored in the form $x = -(2y-5)(y+1)$, $y$-intercepts of the parabola are $-1$ and $5/2$. Its maximum in the $x$-direction occurs for $y = 3/4$.



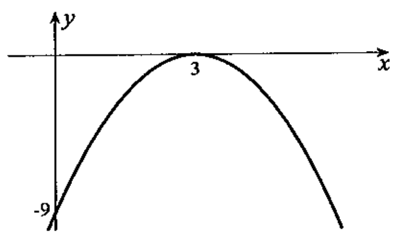
**9.** Expressed in the form $y = 4(x^2 + 5x/4) + 10$
$= 4(x + 5/8)^2 + 10 - 25/16$
$= 4(x + 5/8)^2 + 135/16$, the parabola
opens upward with minimum at $(-5/8, 135/16)$.



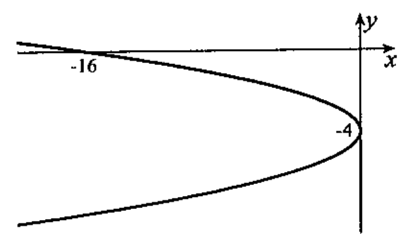

**10.** This parabola opens to the right.




**11.** Expressed in the form $y = -(x^2 - 6x + 9)$
$= -(x - 3)^2$, the parabola opens downward
from the point $(3, 0)$.




**12.** This parabola opens to the left
and touches the $y$-axis at $y = -4$.




**13.** (a) The $y$-intercept is $-5$, and $x$-intercepts are $\dfrac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$.
(b) The $x$-intercept is 4, and from $x = 4(y-1)^2$, the $y$-intercept is 1.

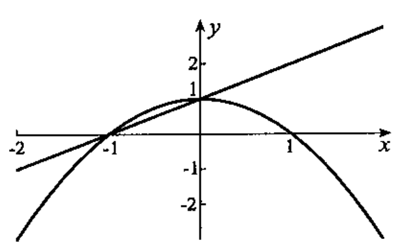
**14.** The equation must be of the form $y = ax^2 + 1$. Since $(2, 3)$ is on the parabola, $3 = 4a + 1 \implies a = 1/2$.

**15.** The equation must be of the form $x = 2 + ay^2$. Since $(0, 4)$ is on the parabola, $0 = 2 + 16a \implies a = -1/8$.

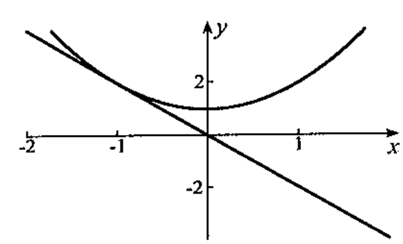
**16.** Since the parabola crosses the $x$-axis at $x = -1$ and $x = 3$, it must be of the form $y = a(x+1)(x-3)$. Since $(0, -1)$ is on the parabola, $-1 = a(1)(-3) \implies a = 1/3$.

**17.** Since the parabola touches the $x$-axis at $x = 1$, it must be of the form $y = a(x-1)^2$. Since $(0, 2)$ is on the parabola, $2 = a(-1)^2 \implies a = 2$.

**18.** We set $x + 1 = 1 - x^2 \implies 0 = x^2 + x$
$= x(x+1) \implies x = 0, -1$. Points of
intersection are $(-1, 0)$ and $(0, 1)$.

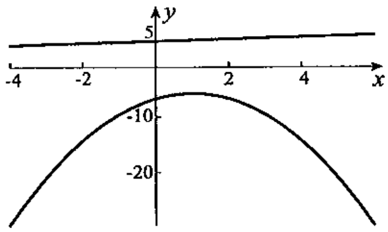


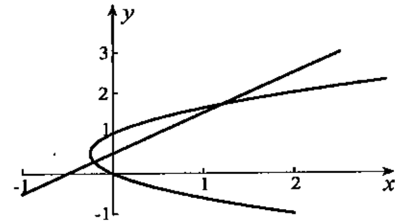

**19.** We set $-2x = 1 + x^2 \implies 0 = x^2 + 2x + 1$
$= (x+1)^2 \implies x = -1$. The only point
of intersection is $(-1, 2)$.




**20.** We set $2x - x^2 - 6 = 5 + x/5 \implies$
$5x^2 - 9x + 55 = 0$. With equation 1.5,
$x = \left(9 \pm \sqrt{81 - 4(5)(55)}\right)/10$. Since
$81 - 4(5)(55) < 0$, there are no points
of intersection.

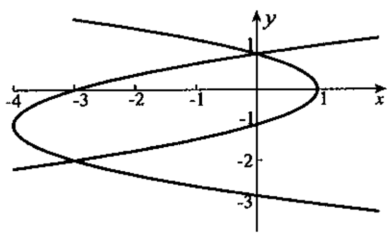
**21.** We set $y(y-1) = y - 1/2 \implies 2y^2 - 4y + 1 = 0$.
With equation 1.5, $y = (4 \pm \sqrt{16 - 8})/4$
$= (2 \pm \sqrt{2})/2$. Points of intersection are
$\left((1 \pm \sqrt{2})/2, (2 \pm \sqrt{2})/2\right)$.

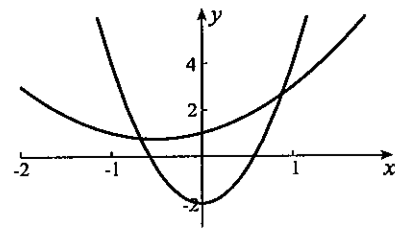




**22.** We set $-y^2+1=y^2+2y-3$
$\Longrightarrow 0=2y^2+2y-4=2(y+2)(y-1)$
$\Longrightarrow y=1,-2$. Points of intersection are $(0,1)$ and $(-3,-2)$.

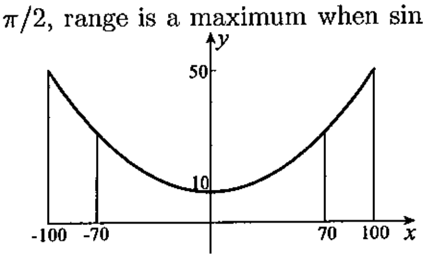
**23.** We set $6x^2-2=x^2+x+1 \Longrightarrow 5x^2-x-3=0$. Equation 1.5 gives $x=(1\pm\sqrt{1+60})/10$
$=(1\pm\sqrt{61})/10$. Points of intersection are $\left((1\pm\sqrt{61})/10,(43\pm 3\sqrt{61})/25\right)$.





**24.** The range of the shell is $R=(v^2/9.81)\sin 2\theta$. Since $0\le\theta\le\pi/2$, range is a maximum when $\sin 2\theta = 1 \Longrightarrow 2\theta=\pi/2 \Longrightarrow \theta=\pi/4$ radians.

**25.** With coordinates as shown, the equation of the parabola takes the form $y=ax^2+10$. Since the point $(100,50)$ is on the parabola, $50=(10\,000)a+10 \Longrightarrow a=1/250$. When $x=70$, we obtain $y=(70)^2/250+10=148/5$ m for the length of the supporting rod.




**26.** We set $5x=(x-2)^4+4 \Longrightarrow (x-2)^4-5x+4=0 \Longrightarrow x^4-8x^3+24x^2-37x+20=0$. Possible rational solutions are $\pm1$, $\pm2$, $\pm4$, $\pm5$, $\pm10$, $\pm20$. One solution is $x=1$, so that

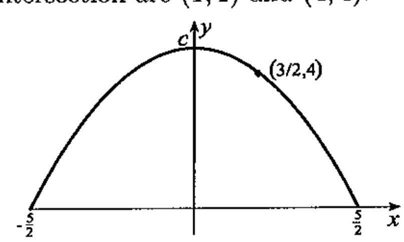
$$x^4-8x^3+24x^2-37x+20=(x-1)(x^3-7x^2+17x-20)=0.$$

We find that $x=4$ is a zero of the cubic, so that

$$x^4-8x^3+24x^2-37x+20=(x-1)(x-4)(x^2-3x+5).$$

Since $x^2-3x+5=0$ has no real solutions, the only points of intersection are $(1,1)$ and $(4,4)$.

**27.** With the coordinate system shown, the equation of the parabola takes the form $y=c(25/4-x^2)$. Since the point $(3/2,4)$ is on the parabola, $4=c(25/4-9/4) \Longrightarrow c=1$. The arch is therefore $25/4$ units high.



**28.** If the parabola is to pass through these points, then

$$2=a+b+c, \qquad 10=9a-3b+c, \qquad 4=9a+3b+c.$$

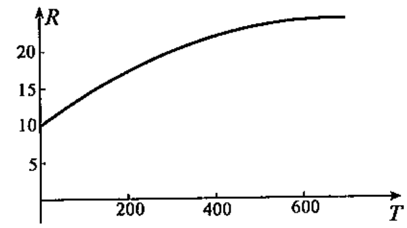
The solution of these equations is $a=1/2$, $b=-1$, $c=5/2$.

**29.** (a) Since resistances at temperatures 0°, 100°, and 700° are 10.000, 13.946, and 24.172,

$$10.000=R_0(1), \qquad 13.946=R_0(1+100a+10\,000b), \qquad 24.172=R_0(1+700a+490\,000b).$$

The solution of these equations is $R_0=10.000$,

$a = 0.004\,266\,2$, and $b = -3.202\,4 \times 10^{-6}$.
(b) The parabola is plotted to the right.

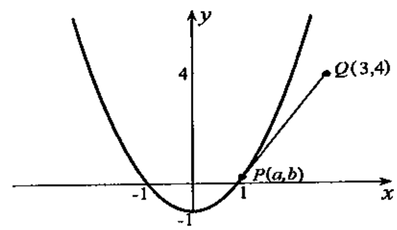



(c) The resistance is 20 ohms when

$$20 = 10(1 + aT + bT^2) \implies bT^2 + aT - 1 = 0.$$

Solutions of this equation are $T = \dfrac{-a \pm \sqrt{a^2 + 4b}}{2b}$.
Since only the positive solution is acceptable, $T = (-a + \sqrt{a^2 + 4b})/(2b) \approx 304$.

**30.** If $P(a, b)$ is the point at which the rope meets the parabola, the equation of the line $PQ$ is $y - 4 = [(b - 4)/(a - 3)](x - 3)$. The $x$-coordinates of points of intersection of this line with the parabola are given by the equation




$$x^2 - 1 = \left(\frac{b-4}{a-3}\right)(x - 3) + 4$$
$$\implies$$
$$x^2 - \left(\frac{b-4}{a-3}\right)x + 3\left(\frac{b-4}{a-3}\right) - 5 = 0.$$

Since the line is to meet the parabola in only one point, the discriminant of this quadratic must be zero

$$\left(\frac{b-4}{a-3}\right)^2 - 4\left[3\left(\frac{b-4}{a-3}\right) - 5\right] = 0 \quad \implies \quad (b-4)^2 - 12(b-4)(a-3) + 20(a-3)^2 = 0.$$

Since $P(a, b)$ is on the parabola, $b = a^2 - 1$, and when we substitute this into the above equation,

$$0 = (a^2 - 5)^2 - 12(a^2 - 5)(a - 3) + 20(a - 3)^2 = a^4 - 12a^3 + 46a^2 - 60a + 25.$$

Possible rational solutions of this equation are $\pm 1$, $\pm 5$, $\pm 25$. We find that $a = 1$ is a solution. When it is factored from the quartic,

$$0 = a^4 - 12a^3 + 46a^2 - 60a + 25 = (a - 1)(a^3 - 11a^2 + 35a - 25).$$

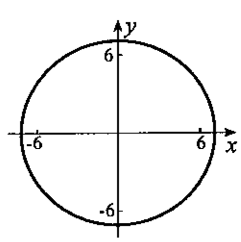
Once again $a = 1$ is a zero of the cubic, so that

$$0 = a^4 - 12a^3 + 46a^2 - 60a + 25 = (a - 1)(a - 1)(a^2 - 10a + 25) = (a - 1)^2(a - 5)^2.$$

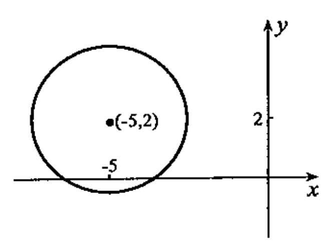
Clearly, $a = 5$ is inadmissible, and the required point is $(1, 0)$.

**EXERCISES 1.4B**

**1.** The circle is centred at the origin with radius $5\sqrt{2}$.

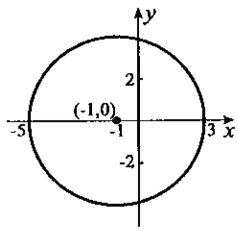


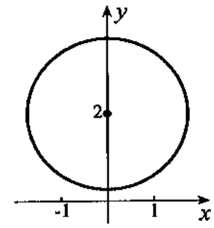

**2.** The centre of the circle is $(-5, 2)$ and its radius is $\sqrt{6}$.



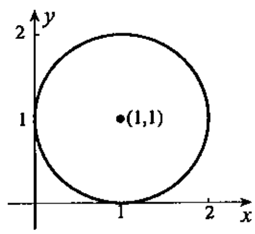

**3.** When we complete the square on the $x$-terms, $(x + 1)^2 + y^2 = 16$. The centre of the circle is $(-1, 0)$ and its radius is 4.

**4.** When we complete the square on the $y$-terms, $x^2 + (y - 2)^2 = 3$. The centre of the circle is $(0, 2)$ and its radius is $\sqrt{3}$.

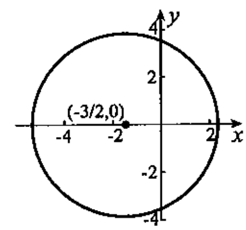


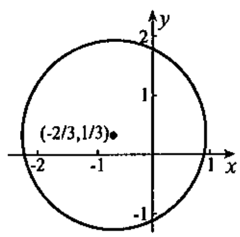

**5.** When we complete squares on $x$- and $y$-terms, $(x-1)^2+(y-1)^2=1$. The centre of the circle is $(1,1)$ and its radius is 1.



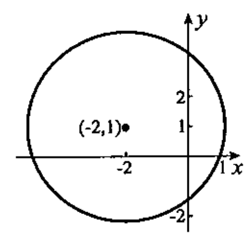

**6.** When we complete the square on the $x$-terms, $(x+3/2)^2+y^2=59/4$. The centre of the circle is $(-3/2,0)$ and its radius is $\sqrt{59}/2$.




**7.** When we complete squares on $x$- and $y$-terms, $(x+2/3)^2+(y-1/3)^2=23/9$. The centre of the circle is $(-2/3,1/3)$ and its radius is $\sqrt{23}/3$.




**8.** When we complete squares on $x$- and $y$-terms, $(x+2)^2+(y-1)^2=10$. The centre of the circle is $(-2,1)$ and its radius is $\sqrt{10}$.




**9.** When we complete squares on $x$- and $y$-terms, $(x-1)^2+(y-2)^2=0$. The only point satisfying this equation is $(1,2)$.

**10.** When we complete squares on $x$- and $y$-terms, $(x+3)^2+(y+3/2)^2=-35/4$. No point can satisfy this equation.

**11.** With centre $(0,0)$ and radius 2, the equation of the circle is $x^2+y^2=4$.

**12.** Since the centre is $(1,0)$ and its radius is 1, the equation of the circle is $(x-1)^2+y^2=1$.

**13.** Since the centre is $(3,4)$ and its radius is 2, the equation of the circle is $(x-3)^2+(y-4)^2=4$.

**14.** The figure indicates that the centre of the circle is $(3/2,-3/2)$. Its radius is then $\sqrt{(3/2)^2+(-3/2)^2}=3/\sqrt{2}$. The equation of the circle is therefore $(x-3/2)^2+(y+3/2)^2=9/2$.

**15.** If we take the equation of the circle in form 1.22, and substitute each of the points $(3,0)$, $(2,7)$, and $(-5,6)$,

$$\begin{aligned} 9+0+3f+e&=0, \\ 4+49+2f+7g+e&=0, \\ 25+36-5f+6g+e&=0, \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 3f+e&=-9, \\ 2f+7g+e&=-53, \\ -5f+6g+e&=-61. \end{aligned}$$

The solution of these equations is $f=2$, $g=-6$, and $e=-15$. The equation of the circle is $x^2+y^2+2x-6y-15=0$.

**16.** If we take the equation for the circle in form 1.22, and substitute each of the points $(1,3)$, $(5,1)$ and $(2,-2)$,

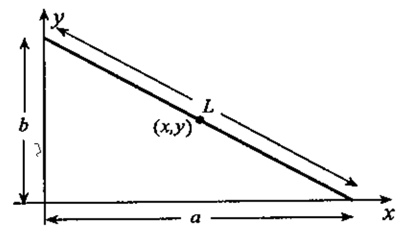
$$\begin{aligned} 1+9+f+3g+e=0, & \qquad & f+3g+e=-10,\\ 25+1+5f+g+e=0, & \quad\Longrightarrow\quad & 5f+g+e=-26,\\ 4+4+2f-2g+e=0, & \qquad & 2f-2g+e=-8. \end{aligned}$$

The solution of these equations is $f=-14/3$, $g=-4/3$, and $e=-4/3$. The equation of the circle is $x^2+y^2-14x/3-4y/3-4/3=0$ or $3x^2+3y^2-14x-4y-4=0$.

**17.** If $a$ and $b$ are as shown in the figure, coordinates of the midpoint of the ladder are $x=a/2$ and $y=b/2$. Since $a$ and $b$ always satisfy the equation $a^2+b^2=L^2$, it follows that

$$(2x)^2+(2y)^2=L^2 \quad\Longrightarrow\quad x^2+y^2=L^2/4.$$

Hence, the midpoint follows a quarter circle with centre at the foot of the wall and radius $L/2$.




**18.** If we take the equation of the circle in the form $(x-h)^2+(y-k)^2=r^2$, and substitute the two points

$$(3-h)^2+(4-k)^2=r^2, \qquad (1-h)^2+(-10-k)^2=r^2.$$

When these equations are subtracted, the result is

$$0=(3-h)^2-(1-h)^2+(4-k)^2-(-10-k)^2 \quad\Longrightarrow\quad h+7k=-19.$$

(a) When the centre of the circle is on the line $2x+3y+16=0$, we must also have $2h+3k+16=0$. When these equations are solved, $h=-5$ and $k=-2$. The radius of the circle is $r=\sqrt{(3+5)^2+(4+2)^2}=10$.
(b) When the centre is on the line $x+7y+19=0$, we must have $h+7k+19=0$. But this is the same equation obtained from the two points. In other words, there is an infinity of circles with centres on the line $x+7y+19=0$ passing through the points $(3,4)$ and $(1,-10)$. Any equation of the form $(x+7k+19)^2+(y-k)^2=r^2$, where $r^2=(3+7k+19)^2+(4-k)^2=50k^2+300k+500$.

**19.** We set $x^2+2x+(3x+2)^2=4 \Longrightarrow 0=10x^2+14x=2x(5x+7)$. Thus, $x=0$ and $x=-7/5$, and the points are $(0,2)$ and $(-7/5,-11/5)$.

**20.** We set $x^2+(1-2x)^2-4(1-2x)+1=0 \Longrightarrow 5x^2+4x-2=0$. Solutions are $x=(-4\pm\sqrt{16+40})/10=(-2\pm\sqrt{14})/5$. Intersection points are $\big((-2\pm\sqrt{14})/5,(9\mp2\sqrt{14})/5\big)$.

**21.** We set $x^2+(3x^2+4)^2=9 \Longrightarrow 9x^4+25x^2+7=0$. But this is impossible since $9x^4$ and $25x^2$ are both nonnegative. There are no points of intersection.

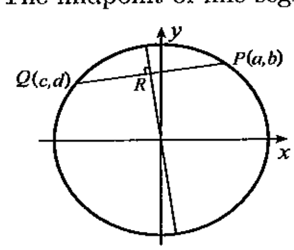
**22.** We set $(x+3)^2+16(x+1)=25 \Longrightarrow 0=x^2+22x=x(x+22)$. Thus, $x=0$ and $x=-22$. From $x=0$ we obtain the points $(0,\pm4)$, while $x=-22$ yields no points.

**23.** When we complete squares on $x$- and $y$-terms, $\left(x+\dfrac{f}{2}\right)^2+\left(y+\dfrac{g}{2}\right)^2=\dfrac{f^2}{4}+\dfrac{g^2}{4}-e=\dfrac{1}{4}(f^2+g^2-4e)$.
If $f^2+g^2-4e>0$, this equation represents a circle with centre $(-f/2,-g/2)$ and radius $\sqrt{f^2+g^2-4e}/2$. If $f^2+g^2-4e=0$, only the point $(-f/2,-g/2)$ satisfies the equation. If $f^2+g^2-4e<0$, no points can satisfy the equation.

**24.** If we choose a coordinate system with origin at the centre of the circle, the equation of the circle is $x^2+y^2=r^2$. Let $P(a,b)$ and $Q(c,d)$ be any two points on the circle. The midpoint of line segment $PQ$ has coordinates $\big((a+c)/2,(b+d)/2\big)$. Since the slope of the perpendicular bisector of $PQ$ is $-(c-a)/(d-b)$, the equation of the perpendicular bisector is

$$y-\frac{b+d}{2}=-\left(\frac{c-a}{d-b}\right)\left(x-\frac{a+c}{2}\right).$$

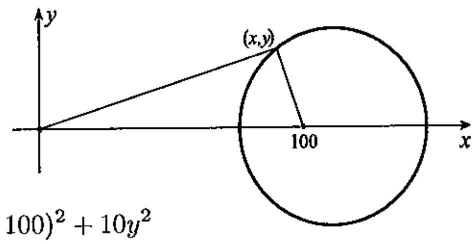
This line passes through the origin if and only if



$$-\frac{b+d}{2} = -\left(\frac{c-a}{d-b}\right)\left(-\frac{a+c}{2}\right) \iff -(b+d)(d-b) = (c-a)(a+c) \iff a^2+b^2 = c^2+d^2.$$

Since $P$ and $Q$ are on the circle, it follows that $a^2+b^2 = r^2$ and $c^2+d^2 = r^2$, and therefore $a^2+b^2 = c^2+d^2$.

**25.** If $I$ is the brightness of the source at $(100, 0)$, then the amount of light received at point $(x, y)$ from this light is $A_1 = \dfrac{kI}{(x-100)^2+y^2}$, where $k$ is a constant of proportionality. The amount of light received from the source at the origin is $A_2 = \dfrac{k(10I)}{x^2+y^2}$.




For $A_1 = A_2$,

$$\frac{kI}{(x-100)^2+y^2} = \frac{10kI}{x^2+y^2} \implies x^2+y^2 = 10(x-100)^2+10y^2$$

$$\implies 9x^2+9y^2-2000x+10^5 = 0 \implies \left(x-\frac{1000}{9}\right)^2 + y^2 = \frac{-10^5}{9} + \frac{10^6}{81} = \frac{10^5}{81}.$$

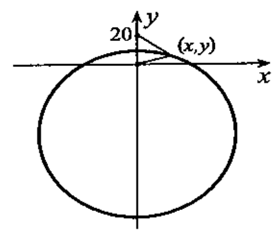
We have a circle centred at $(1000/9, 0)$ and radius $100\sqrt{10}/9$.

**26.** If $L$ is the loudness of the speaker at $(0, 20)$, then the amount of sound received at point $(x, y)$ from this speaker is

$$A_1 = \frac{kL}{x^2+(y-20)^2},$$

where $k$ is a constant of proportionality. The amount of sound received from the speaker at the origin is

$$A_2 = \frac{k(0.7L)}{x^2+y^2}.$$




For $A_1 = A_2$, $\dfrac{kL}{x^2+(y-20)^2} = \dfrac{7kL}{10(x^2+y^2)} \implies 10(x^2+y^2) = 7[x^2+(y-20)^2]$. This gives

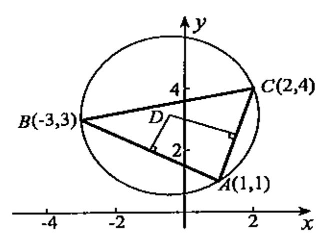
$$3x^2+3y^2+280y = 2800 \implies x^2 + \left(y+\frac{140}{3}\right)^2 = \frac{2800}{3} + \frac{140^2}{9} = \frac{28\,000}{9}.$$

We have a circle centred at $(0, -140/3)$ and radius $20\sqrt{70}/3$.

**27.** (a) The centre of the circle is the intersection of the perpendicular bisectors of $AB$ and $AC$ (see Exercise 24). These perpendicular bisectors have equations

$$y-2 = -\frac{4}{-2}(x+1), \quad y-\frac{5}{2} = -\frac{1}{3}\left(x-\frac{3}{2}\right).$$

The solution of these equations is $D(-3/7, 22/7)$. The radius of the circumcircle is therefore




$$\|AD\| = \sqrt{(10/7)^2+(-15/7)^2} = 5\sqrt{13}/7.$$

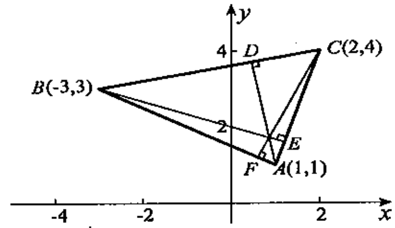
The equation of the circumcircle is $\left(x+\dfrac{3}{7}\right)^2 + \left(y-\dfrac{22}{7}\right)^2 = \dfrac{325}{49}$.

(b) We take the equation of the circle in the form $(x-h)^2+(y-k)^2 = r^2$. Since $(1, 1)$, $(-3, 3)$ and $(2, 4)$ are points on the circle,

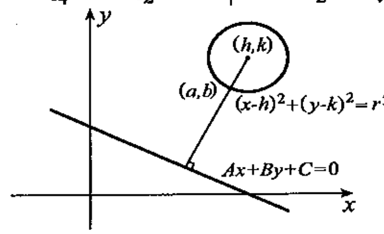
$$(1-h)^2+(1-k)^2 = r^2, \qquad (-3-h)^2+(3-k)^2 = r^2, \qquad (2-h)^2+(4-k)^2 = r^2.$$

The solution of this system is $h = -3/7$, $k = 22/7$, and $r = 5\sqrt{13}/7$, giving the same equation as in part (a).

**28.** Since the slope of $BC$ is $1/5$, the equation of altitude $AD$ is $y-1=-5(x-1) \Longrightarrow 5x+y=6$. Since the slope of $AC$ is 3, the equation of altitude $BE$ is $y-3=-(1/3)(x+3) \Longrightarrow x+3y=6$. The point of intersection of these altitudes is $(6/7,12/7)$. The equation of altitude $CF$ is $y-4=2(x-2)$, and the point $(6/7,12/7)$ satisfies this equation. Hence, the altitudes intersect at $(6/7,12/7)$.




**29.** The shortest distance occurs for the point $(a,b)$ on the circle where the line joining $(h,k)$ and $(a,b)$ is perpendicular to the line. In terms of slopes, this condition requires $(b-k)/(a-h)=B/A \Longrightarrow b=k+B(a-h)/A$. Since $(a,b)$ is on the circle, $(a-h)^2+(b-k)^2=r^2$. These equations imply that



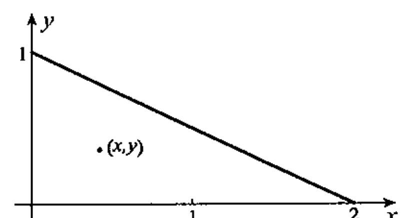

$$(a-h)^2+\left[k+\frac{B}{A}(a-h)-k\right]^2=r^2 \Longrightarrow (a-h)^2\left(1+\frac{B^2}{A^2}\right)=r^2 \Longrightarrow a=h\pm\frac{rA}{\sqrt{A^2+B^2}}.$$

Corresponding $y$-coordinates are $b=k\pm\dfrac{rB}{\sqrt{A^2+B^2}}$. The diagram indicates why there are two points. Using distance formula 1.16, the minimum distance is the smaller of the numbers

$$\frac{|A(h\pm rA/\sqrt{A^2+B^2})+B(k\pm rB/\sqrt{A^2+B^2})+C|}{\sqrt{A^2+B^2}}=\frac{|(Ah+Bk+C)\pm r(A^2+B^2)/\sqrt{A^2+B^2}|}{\sqrt{A^2+B^2}}$$
$$=\frac{|(Ah+Bk+C)\pm r\sqrt{A^2+B^2}|}{\sqrt{A^2+B^2}}.$$

**30.** If $(x,y)$ is the incentre, then it must be equidistant from the three sides of the triangle, $y=0$, $x=0$, and $x+2y-2=0$. When we equate these distances, using formula 1.16, the result is

$$|y|=|x|=\frac{|x+2y-2|}{\sqrt{1^2+2^2}}=\frac{|x+2y-2|}{\sqrt{5}}.$$




The equation $|y|=|x|$ implies that $y=\pm x$. We combine this with the two possibilities $x+2y-2>0$ and $x+2y-2<0$, for four possible cases. With $y=x$ and $x+2y-2<0$, we obtain $x=-(x+2x-2)/\sqrt{5} \Longrightarrow x=(3-\sqrt{5})/2$. This gives the incentre $\left((3-\sqrt{5})/2,(3-\sqrt{5})/2\right)$. With $y=x$ and $x+2y-2>0$, we obtain $x=(x+2x-2)/\sqrt{5} \Longrightarrow x=(3+\sqrt{5})/2$. This gives the point $\left((3+\sqrt{5})/2,(3+\sqrt{5})/2\right)$. With $y=-x$, we obtain the two additional points $\left((1+\sqrt{5})/2,-(1+\sqrt{5})/2\right)$ and $\left((1-\sqrt{5})/2,(\sqrt{5}-1)/2\right)$. Three circles can be drawn to touch the sides of the triangle when they are extended. The additional three points are the centres of these circles, but they are outside the triangle.

**31.** The amount of sound received at point $(x,y)$ from the source at $(x_1,y_1)$ is $A_1=kI_1/[(x-x_1)^2+(y-y_1)^2]$, where $k$ is a constant of proportionality. The amount of sound received from the source at $(x_2,y_2)$ is $A_2=kI_2/[(x-x_2)^2+(y-y_2)^2]$. For $A_1=A_2$,

$$\frac{kI_1}{(x-x_1)^2+(y-y_1)^2}=\frac{kI_2}{(x-x_2)^2+(y-y_2)^2}.$$

If we set $\alpha=I_1/I_2$, then

$$\alpha[(x-x_2)^2+(y-y_2)^2]=(x-x_1)^2+(y-y_1)^2$$

$$\Longrightarrow \quad (\alpha-1)x^2+(\alpha-1)y^2+2(x_1-\alpha x_2)x+2(y_1-\alpha y_2)y=x_1^2-\alpha x_2^2+y_1^2-\alpha y_2^2$$

$$\Longrightarrow \quad \left(x+\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(y+\frac{y_1-\alpha y_2}{\alpha-1}\right)^2=\frac{x_1^2-\alpha x_2^2}{\alpha-1}+\frac{y_1^2-\alpha y_2^2}{\alpha-1}+\left(\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(\frac{y_1-\alpha y_2}{\alpha-1}\right)^2.$$

This is a circle if the right side is positive. Consider

$$\begin{aligned}\frac{x_1^2-\alpha x_2^2}{\alpha-1}+\left(\frac{x_1-\alpha x_2}{\alpha-1}\right)^2 &= \frac{1}{(\alpha-1)^2}[(\alpha-1)(x_1^2-\alpha x_2^2)+(x_1-\alpha x_2)^2]\\ &= \frac{1}{(\alpha-1)^2}[\alpha(x_1^2-\alpha x_2^2)-(x_1^2-\alpha x_2^2)+(x_1^2-2\alpha x_1x_2+\alpha^2x_2^2)]\\ &= \frac{\alpha}{(\alpha-1)^2}(x_1-x_2)^2.\end{aligned}$$

With a similar result for the $y$-terms, we can write that

$$\left(x+\frac{x_1-\alpha x_2}{\alpha-1}\right)^2+\left(y+\frac{y_1-\alpha y_2}{\alpha-1}\right)^2=\frac{\alpha}{(\alpha-1)^2}[(x_1-x_2)^2+(y_1-y_2)^2].$$

We have a circle centre $\left(\dfrac{x_1-\alpha x_2}{1-\alpha},\dfrac{y_1-\alpha y_2}{1-\alpha}\right)$ and radius $\sqrt{\dfrac{\alpha}{(\alpha-1)^2}[(x_1-x_2)^2+(y_1-y_2)^2]}$. The equation of the line through $(x_1,y_1)$ and $(x_2,y_2)$ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \Longrightarrow \quad y=\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1.$$

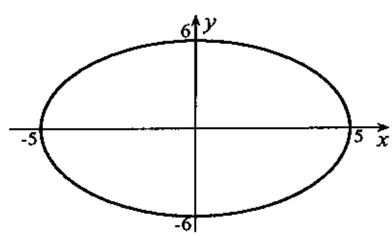
If we substitute $x=(x_1-\alpha x_2)/(1-\alpha)$ into the right side, we obtain

$$\begin{aligned}y &= \frac{y_2-y_1}{x_2-x_1}\left(\frac{x_1-\alpha x_2}{1-\alpha}-x_1\right)+y_1=\frac{y_2-y_1}{x_2-x_1}\left(\frac{x_1-\alpha x_2-x_1+\alpha x_1}{1-\alpha}\right)+y_1\\ &= \frac{y_2-y_1}{x_2-x_1}\left[\frac{\alpha(x_1-x_2)}{1-\alpha}\right]+y_1=(y_1-y_2)\left(\frac{\alpha}{1-\alpha}\right)+y_1\\ &= \frac{\alpha y_1-\alpha y_2+y_1-\alpha y_1}{1-\alpha}=\frac{y_1-\alpha y_2}{1-\alpha},\end{aligned}$$

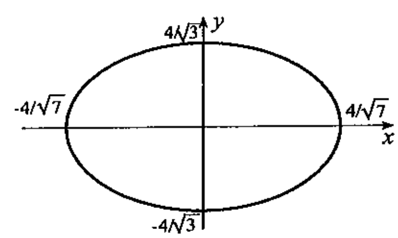
and this is the $y$-coordinate of the centre of the circle. Hence, the centre of the circle lies on the line through $(x_1,y_1)$ and $(x_2,y_2)$.

## EXERCISES 1.4C

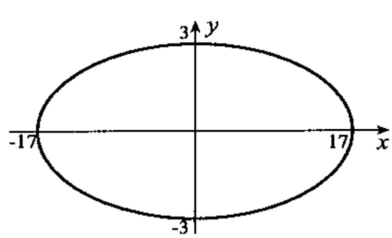
**1.** The ellipse is centred at the origin. The $x$- and $y$-intercepts are $\pm 5$ and $\pm 6$.



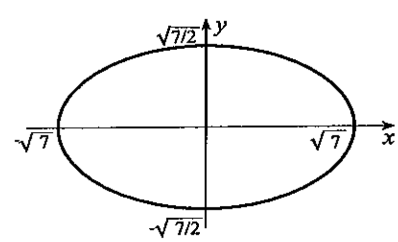

**2.** The ellipse is centred at the origin. Its $x$- and $y$-intercepts are $x=\pm 4/\sqrt{7}$ and $y=\pm 4/\sqrt{3}$.



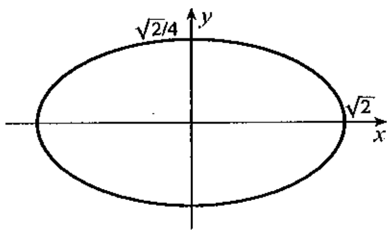

**3.** The ellipse is centred at the origin. Its $x$- and $y$-intercepts are $x=\pm 17$ and $y=\pm 3$.



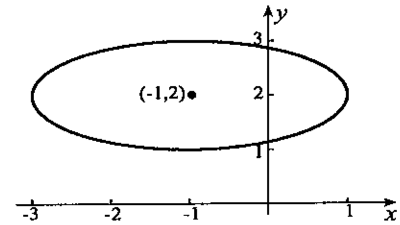

**4.** The ellipse is centred at the origin. Its $x$- and $y$-intercepts are $x=\pm\sqrt{7}$ and $y=\pm\sqrt{7/2}$.



**5.** The ellipse is centred at the origin. Its $x$- and $y$-intercepts are $x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}/4$.



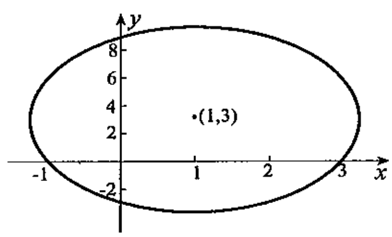

**6.** When we complete squares on $x$- and $y$-terms, $(x+1)^2/4 + (y-2)^2 = 1$. The centre of the ellipse is $(-1, 2)$. It intersects the line $y = 2$ when $x = -3$ and $x = 1$, and intersects the line $x = -1$ when $y = 1$ and $y = 3$.




**7.** When we complete squares on $x$- and $y$-terms,

$$\frac{(x-1)^2}{44/9} + \frac{(y-3)^2}{44} = 1.$$

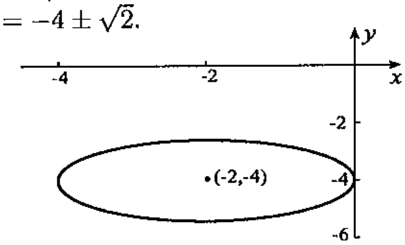
The centre of the ellipse is $(1, 3)$.
It intersects the line $y = 3$ when $x = 1 \pm 2\sqrt{11}/3$, and intersects the line $x = 1$ when $y = 3 \pm 2\sqrt{11}$.




**8.** When we complete squares on $x$- and $y$-terms,

$$\frac{(x+2)^2}{4} + \frac{(y+4)^2}{2} = 1.$$

The centre of the ellipse is $(-2, -4)$.
It intersects the line $y = -4$ when $x = -4$ and $x = 0$, and intersects the line $x = -2$ when $y = -4 \pm \sqrt{2}$.




**9.** If an ellipse of form $x^2/a^2 + y^2/b^2 = 1$ is to pass through the points $(-2, 4)$ and $(3, 1)$, then

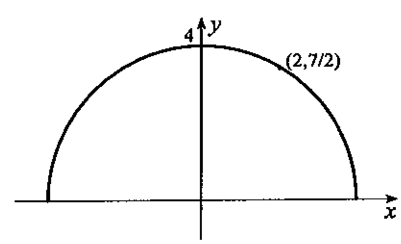
$$\frac{4}{a^2} + \frac{16}{b^2} = 1, \qquad \frac{9}{a^2} + \frac{1}{b^2} = 1.$$

These can be solved for $a^2 = 28/3$ and $b^2 = 28$.

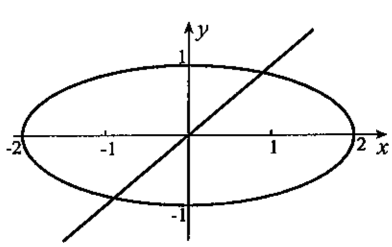
**10.** With the coordinate system shown, the equation of the ellipse must be of the form $x^2/a^2 + y^2/b^2 = 1$. Since $(0, 4)$ and $(2, 7/2)$ are points on the ellipse,

$$\frac{16}{b^2} = 1, \qquad \frac{4}{a^2} + \frac{49/4}{b^2} = 1.$$

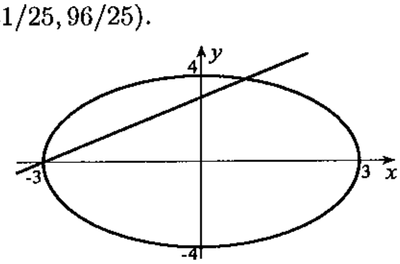
These imply that $a^2 = 256/15$ and $b^2 = 16$. The width of the arch is therefore $2a = 32/\sqrt{15}$.



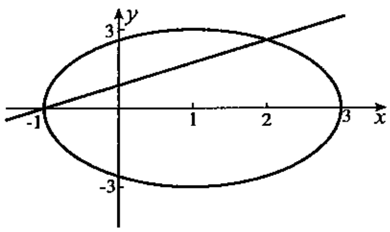

**11.** We set $x^2 + 4x^2 = 4 \implies x = \pm 2/\sqrt{5}$. Points of intersection are therefore $(\pm 2/\sqrt{5}, \pm 2/\sqrt{5})$.



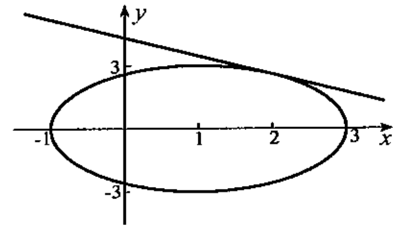

**12.** If we substitute $y = x + 3$ into the equation of the ellipse, $16x^2 + 9(x+3)^2 = 144$, from which $0 = 25x^2 + 54x - 63 = (x+3)(25x - 21)$. Points of intersection of the curves are $(-3, 0)$ and $(21/25, 96/25)$.



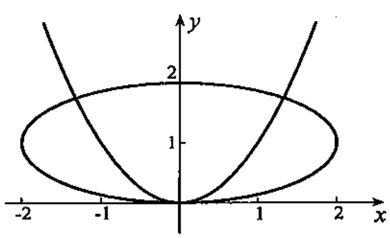
**13.** If we substitute $y = \sqrt{3}(x+1)/2$ into the equation of the ellipse, $9x^2 - 18x + 4\left[\frac{\sqrt{3}(x+1)}{2}\right]^2 = 27$, from which $0 = 12x^2 - 12x - 24 = 12(x-2)(x+1)$. Points of intersesction are therefore $(2, 3\sqrt{3}/2)$ and $(-1, 0)$.



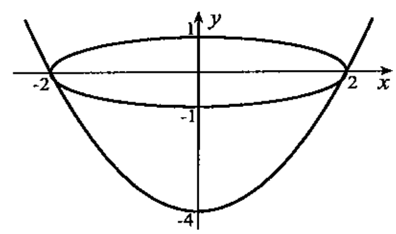

**14.** If we put $y = \sqrt{3}(5-x)/2$ into the equation of the ellipse, $9x^2 - 18x + 4\left[\frac{\sqrt{3}}{2}(5-x)\right]^2 = 27$, from which $0 = 12x^2 - 48x + 48 = 12(x-2)^2$. The line and ellipse therefore touch at the point $(2, 3\sqrt{3}/2)$.




**15.** We set $0 = x^2 + 4x^4 - 8x^2 = x^2(4x^2 - 7)$. Points of intersection are $(0,0)$ and $(\pm\sqrt{7}/2, 7/4)$.



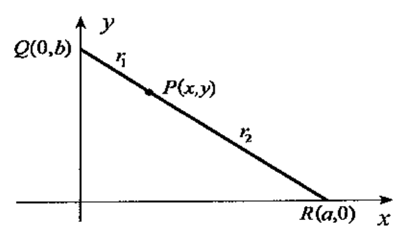

**16.** If we substitute $y = x^2 - 4$ into the equation of the ellipse, $x^2 + 4(x^2-4)^2 = 4$, from which $0 = 4x^4 - 31x^2 + 60 = (x^2-4)(4x^2-15)$. Solutions are $x = \pm 2$, and $\pm\sqrt{15}/2$. Points intersection are $(\pm\sqrt{15}/2, -1/4)$ and $(\pm 2, 0)$.




**17.** If $a$ and $b$ are as shown in the figure to the right, then $L^2 = a^2 + b^2$. Let $P(x, y)$ be the coordinates of the point on the ladder such that the ratio of the lengths $\|PQ\|$ and $\|PR\|$, is $r_1/r_2$. According to Exercise 55 in Section 1.3,

$$x = \frac{r_1 a}{r_1 + r_2}, \quad y = \frac{r_2 b}{r_1 + r_2}.$$

If we solve these for $a$ and $b$ and substitute

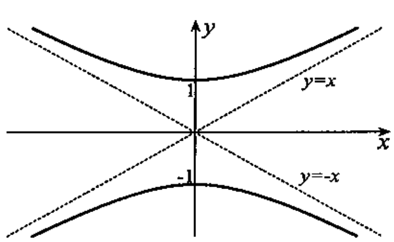



into the equation involving $L$, we obtain the equation of an ellipse

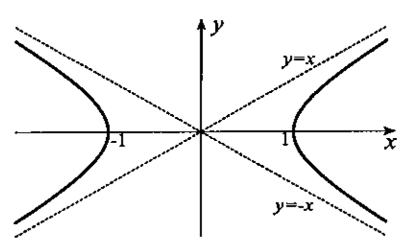
$$\left(\frac{r_1 + r_2}{r_1}x\right)^2 + \left(\frac{r_1 + r_2}{r_2}y\right)^2 = L^2 \quad \Longrightarrow \quad \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = \frac{L^2}{(r_1 + r_2)^2}.$$

## EXERCISES 1.4D

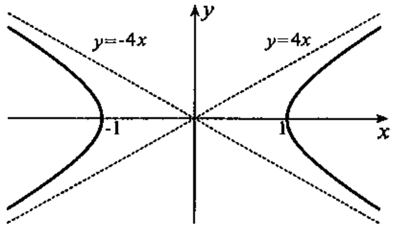
**1.** Asymptotes for the hyperbola are $y = \pm x$, intersecting at the origin. It intersects the $y$-axis at $y = \pm 1$.



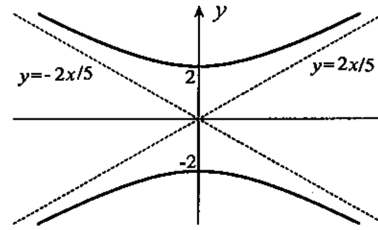

**2.** Asymptotes for the hyperbola are $y = \pm x$, intersecting at the origin. The hyperbola intersects the $x$-axis at $x = \pm 1$.



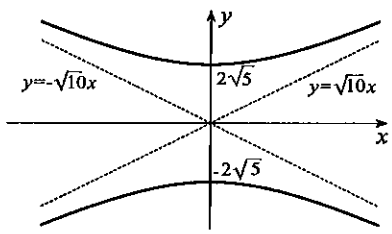
**3.** Asymptotes for the hyperbola are $y = \pm 4x$, intersecting at the origin. It intersects the $x$-axis at $x = \pm 1$.



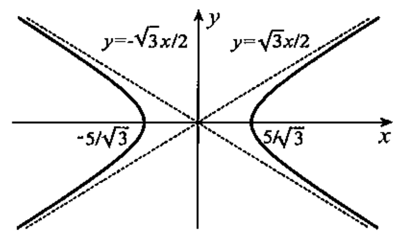

**4.** Asymptotes for the hyperbola are $y = \pm 2x/5$, intersecting at the origin. The hyperbola intersects the $y$-axis at $y = \pm 2$.



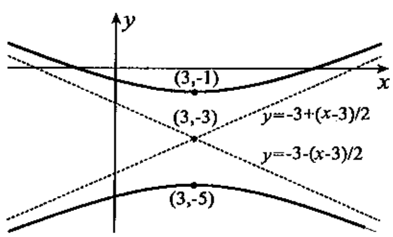

**5.** Asymptotes for the hyperbola are $y = \pm\sqrt{10}x$, intersecting at the origin. It intersects the $y$-axis at $y = \pm 2\sqrt{5}$.



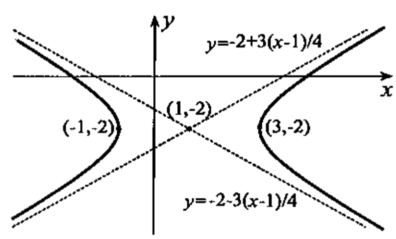

**6.** Asymptotes for the hyperbola are $y = \pm\sqrt{3}x/2$, intersecting at the origin. The hyperbola intersects the $x$-axis at $x = \pm 5/\sqrt{3}$.



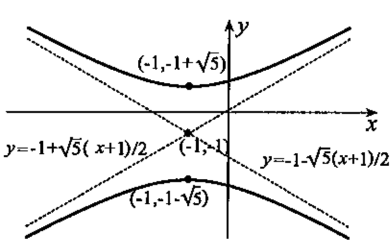

**7.** When we complete squares on $x$- and $y$-terms, the equation becomes $\dfrac{(y+3)^2}{4} - \dfrac{(x-3)^2}{16} = 1$. Asymptotes are $y = -3 \pm (x-3)/2$ intersecting in the point $(3, -3)$. The hyperbola cuts the line $x = 3$ when $y = -1$ and $y = -5$.



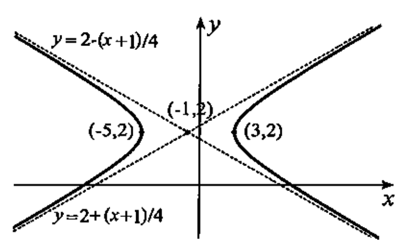

**8.** When we complete squares on $x$- and $y$-terms, the equation becomes $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{9/4} = 1$. Asymptotes are $y = -2 \pm 3(x-1)/4$ intersecting in the point $(1, -2)$. The hyperbola cuts the line $y = -2$ when $x = -1$ and $x = 3$.




**9.** When we complete squares on $x$- and $y$-terms, the equation becomes $\dfrac{(y+1)^2}{5} - \dfrac{(x+1)^2}{4} = 1$. Asymptotes are $y = -1 \pm \sqrt{5}(x+1)/2$ intersecting in the point $(-1, -1)$. The hyperbola cuts the line $x = -1$ when $y = -1 \pm \sqrt{5}$.




**10.** When we complete squares on $x$- and $y$-terms, the equation becomes $\dfrac{(x+1)^2}{16} - (y-2)^2 = 1$. Asymptotes are $y = 2 \pm (x+1)/4$ intersecting in the point $(-1, 2)$. The hyperbola cuts the line $y = 2$ when $x = -5$ and $x = 3$.

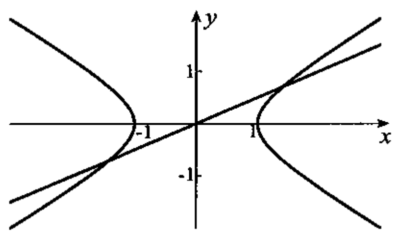


**11.** If we take the equation of the hyperbola in the form $x^2/a^2 - y^2/b^2 = 1$ with asymptotes $y = \pm bx/a$, then $b/a = 4$. Since the point $(1,2)$ is on the hyperbola,

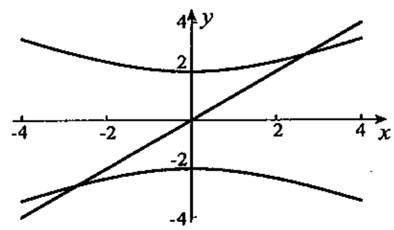
$$\frac{1}{a^2} - \frac{4}{b^2} = 1 \quad \Longrightarrow \quad \frac{1}{a^2} - \frac{4}{16a^2} = 1 \quad \Longrightarrow \quad a^2 = \frac{3}{4}.$$

Thus, $b^2 = 16a^2 = 12$, and the equation of the hyperbola is $16x^2 - y^2 = 12$.

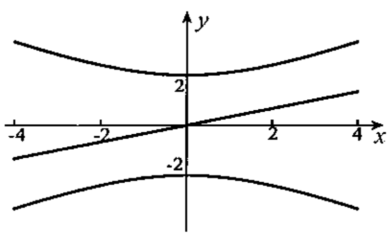
**12.** If we substitute $x = 2y$ into the equation of the hyperbola, $(2y)^2 - 2y^2 = 1 \Longrightarrow y = \pm 1/\sqrt{2}$. Points of intersection are therefore $\pm(\sqrt{2}, 1/\sqrt{2})$.



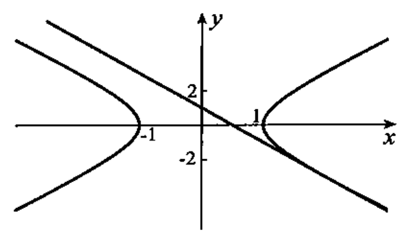

**13.** We set $9x^2 - 4x^2 = 36 \Longrightarrow x = \pm 6/\sqrt{5}$. Points of intersection are therefore $\pm(6/\sqrt{5}, 6/\sqrt{5})$.



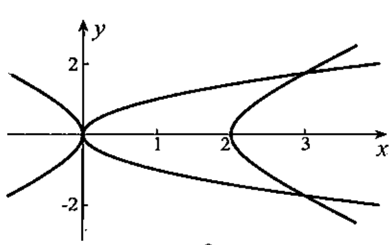

**14.** The figure suggests that the line and hyperbola do not intersect. This can be verified algebraically. If we substitute $x = 3y$ into the equation of the hyperbola, $36 = 9y^2 - 4(3y)^2 = -27y^2$, an impossibility.



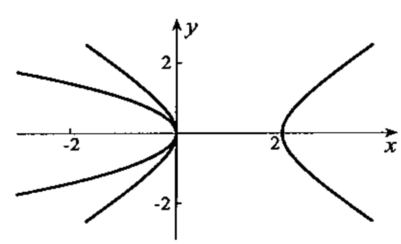

**15.** If we substitute $y = 1 - 2x$ into the equation of the hyperbola, $3x^2 - (1 - 2x)^2 = 3$ from which $0 = -x^2 + 4x - 4 = -(x - 2)^2$. The only point of intersection is $(2, -3)$.




**16.** If we substitute $x = y^2$ into the equation of the hyperbola, $0 = (y^2)^2 - 2(y^2) - y^2 = y^2(y^2 - 3)$. Thus, $y = 0, \pm\sqrt{3}$, and these give the points of intersection $(0, 0)$ and $(3, \pm\sqrt{3})$.




**17.** If we substitute $x = -y^2$ into the equation of the hyperbola, $0 = (-y^2)^2 - 2(-y^2) - y^2 = y^2(y^2 + 1)$. The only point of intersection is $(0, 0)$.

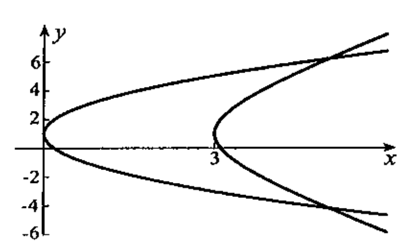



**18.** If we substitute $(y - 1)^2 = 27x/5$ into the equation of the hyperbola, $36 = 9(x - 1)^2 - 4\left(\dfrac{27x}{5}\right)$.

This equation reduces to

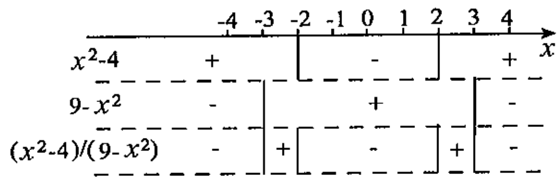
$$0 = 5x^2 - 22x - 15 = (x - 5)(5x + 3).$$

Thus, $x = 5$ or $x = -3/5$. Since $x$ cannot be negative (the equation of the parabola demands this), the solution $x = 5$ leads to the two points $(5, 1 \pm 3\sqrt{3})$.

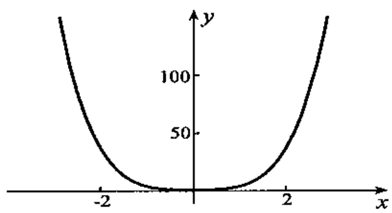


## EXERCISES 1.5

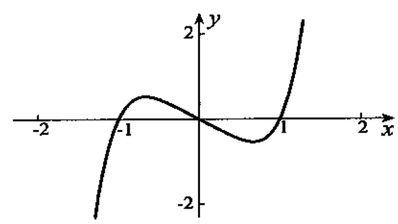
1. We require $9 - x^2 \geq 0 \implies x^2 \leq 9 \implies -3 \leq x \leq 3$.
2. Since $x - 2$ cannot be 0, the function is defined for all $x \neq 2$.
3. The function is defined for all $x \neq 0$.
4. Since $x^2 - 4$ must be positive, $x$ must be greater than 2 or less than $-2$. Hence, $|x| > 2$.
5. Since $4 - x^2$ must be positive, $x^2$ must be less than 4. This requires $-2 < x < 2$. We must also eliminate $x = 0$.
6. The first two lines of the diagram indicate when the expressions $x^2 - 4$ and $9 - x^2$ are positive and negative. The third line combines these to give the sign of $(x^2 - 4)/(9 - x^2)$. It indicates that $x$ must be in one of the intervals $-3 < x \leq -2$ or $2 \leq x < 3$; that is, $2 \leq |x| < 3$.



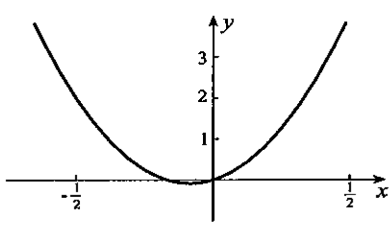

7. Since $-x^2 + 6x - 9 = -(x^2 - 6x + 9) = -(x - 3)^2$, the function is only defined for $x = 3$.
8. Since $x^3 - x^2 = x^2(x - 1) \geq 0$ for $x = 0$ and $x \geq 1$, the largest domain consists of the point $x = 0$ and the interval $x \geq 1$.
9. Since $f(-x) = 1 + (-x)^2 + 2(-x)^4$
$= 1 + x^2 + 2x^4 = f(x)$, the function is even.



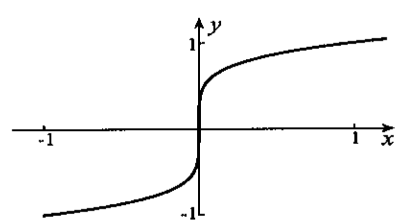

10. Since $f(-x) = (-x)^5 - (-x) = -x^5 + x$
$= -(x^5 - x) = -f(x)$, the function is odd.



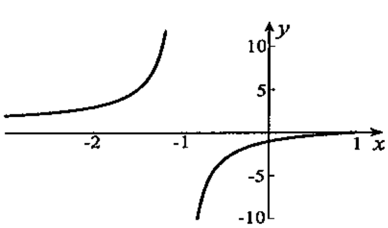

11. Since $f(-x) = 12(-x)^2 + 2(-x) = 12x^2 - 2x$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



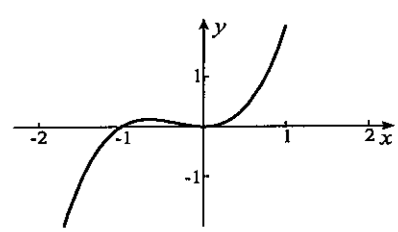

12. Since $f(-x) = (-x)^{1/5} = -x^{1/5} = -f(x)$, the function is odd.



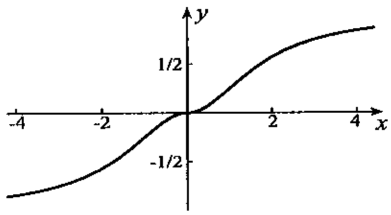

13. Since $f(-x) = \dfrac{-x - 1}{-x + 1} = \dfrac{x + 1}{x - 1}$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



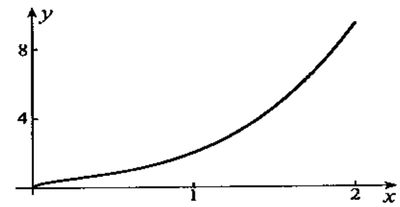

14. Since $f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2$, and this is neither $f(x)$ nor $-f(x)$, the function is neither even nor odd.



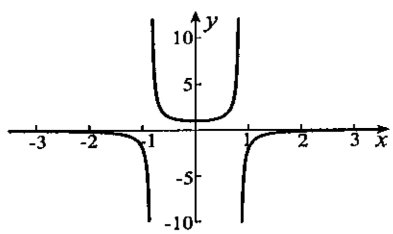
**15.** Since $f(-x) = \dfrac{(-x)|-x|}{3+(-x)^2} = -\dfrac{x|x|}{3+x^2} = -f(x)$, the function is odd.



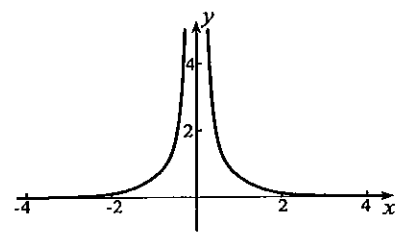

**16.** Since the function is not defined for $x < 0$, it cannot be even or odd.




**17.** Since $f(-x) = \dfrac{(-x)^2+1}{1-2(-x)^4} = \dfrac{x^2+1}{1-2x^4} = f(x)$, the function is even.




**18.** Since $f(-x) = \dfrac{(-x)^2+\sqrt{(-x)^4+1}}{(-x)^6+3(-x)^2} = \dfrac{x^2+\sqrt{x^4+1}}{x^6+3x^2} = f(x)$, the function is even.




**19.** Even and odd parts of this function are $f_e(x) = 3x^2$ and $f_o(x) = x^3 - 2x$.

**20.** Even and odd parts of this function are

$$f_e(x) = \frac{1}{2}\left(\frac{x-2}{x+5} + \frac{-x-2}{-x+5}\right) = \frac{x^2+10}{x^2-25}, \qquad f_o(x) = \frac{1}{2}\left(\frac{x-2}{x+5} - \frac{-x-2}{-x+5}\right) = \frac{-7x}{x^2-25}.$$

**21.** Since this function is even, its even part is itself and its odd part is zero.

**22.** Since this function is odd, its odd part is itself and its even part is zero.

**23.** Even and odd parts of this function are

$$f_e(x) = \frac{1}{2}\left(\frac{2x}{3+5x} + \frac{-2x}{3-5x}\right) = \frac{-10x^2}{9-25x^2}, \qquad f_o(x) = \frac{1}{2}\left(\frac{2x}{3+5x} - \frac{-2x}{3-5x}\right) = \frac{6x}{9-25x^2}.$$

**24.** Since this function is only defined for $x \geq 1$, it does not have even and odd parts.

**25.** When we set $x = 0$ in equation 1.30b, $f(0) = -f(0) \implies f(0) = 0$.

**26.** (a) Let $f(x)$ and $g(x)$ be any two even functions and set $h(x) = f(x)g(x)$. Then,

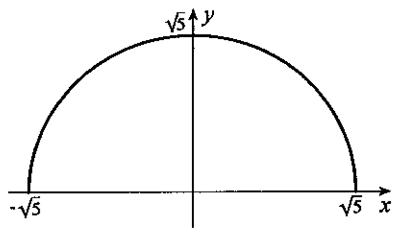
$$h(-x) = f(-x)g(-x) = [f(x)][g(x)] = f(x)g(x) = h(x).$$

Thus, $h(x)$ is an even function. The proof for two odd functions is similar.
(b) When $f(x)$ is even and $g(x)$ is odd, and $h(x) = f(x)g(x)$,

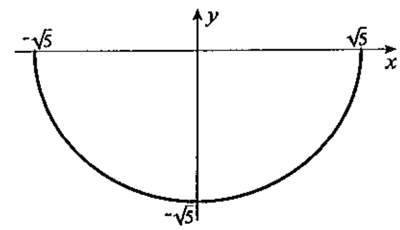
$$h(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -h(x).$$

Thus, $h(x)$ is an odd function.

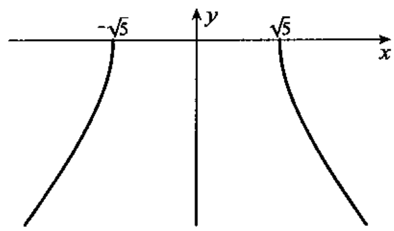
**27.** This is a semicircle.



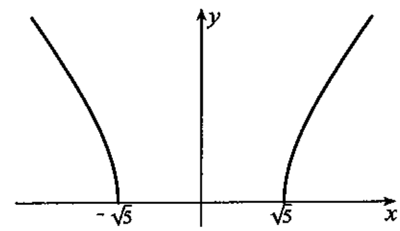

**28.** This is a semicircle.



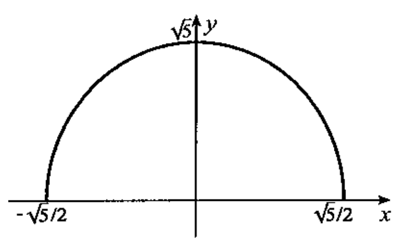

**29.** This is the lower half of the hyperbola $x^2 - y^2 = 5$.



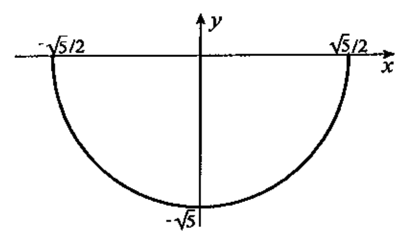

**30.** This is the upper half of the hyperbola $x^2 - y^2 = 5$.



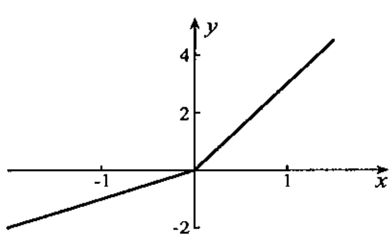

**31.** This is the upper half of the ellipse $4x^2 + y^2 = 5$.



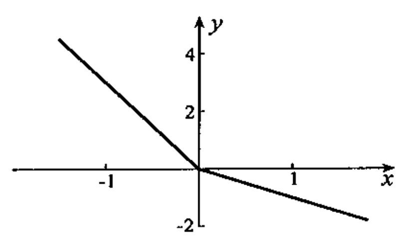

**32.** This is the lower half of the ellipse $4x^2 + y^2 = 5$.



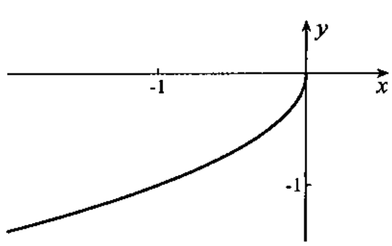

**33.** When $x \geq 0$, $y = x + 2x = 3x$; and when $x < 0$, $y = -x + 2x = x$.



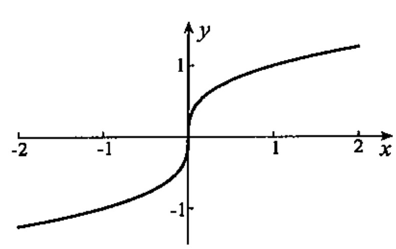

**34.** When $x \geq 0$, $y = x - 2x = -x$; and when $x < 0$, $y = -x - 2x = -3x$.



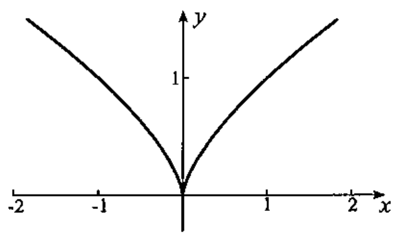

**35.** This is the lower half of the parabola $x = -y^2$



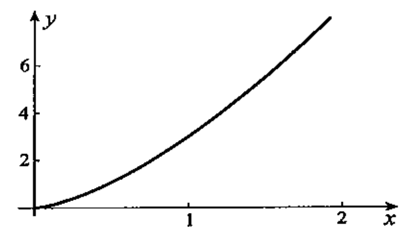

**36.** We rewrite the equation of the curve in the form $x = y^3$.



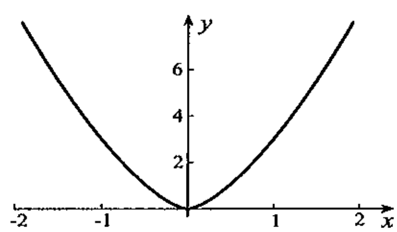
**37.** The curve is shown below.



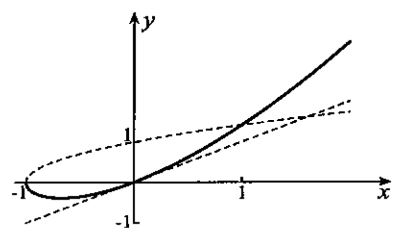

**38.** This function is only defined for $x \geq 0$.



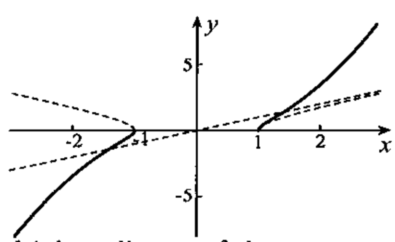

**39.** The curve is shown below.



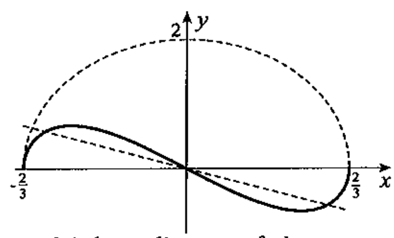

**40.** We multiply ordinates of the curves $y = x$ and $y = \sqrt{x+1}$, the top half of a parabola



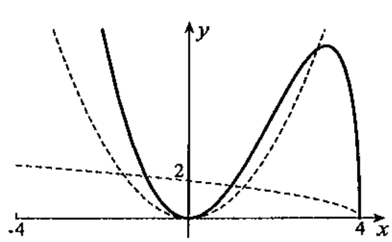

**41.** We multiply ordinates of the curves $y = x$ and $y = \sqrt{x^2 - 1}$, the top half of a hyperbola.



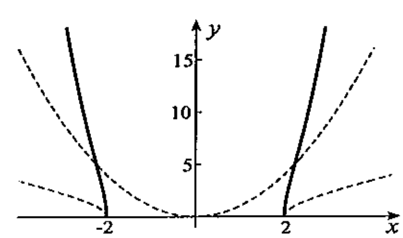

**42.** We multiply ordinates of the curves $y = -x$ and $y = \sqrt{4 - 9x^2}$, the top half of an ellipse.



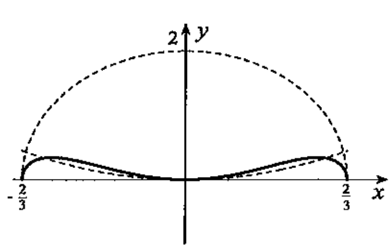

**43.** We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{4 - x}$, the top half of a parabola.



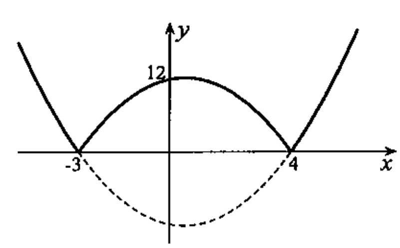

**44.** We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{x^2 - 4}$, the top half of a hyperbola.



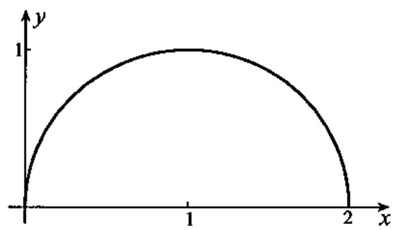

**45.** We multiply ordinates of the curves $y = x^2$ and $y = \sqrt{4 - 9x^2}$, the top half of an ellipse.



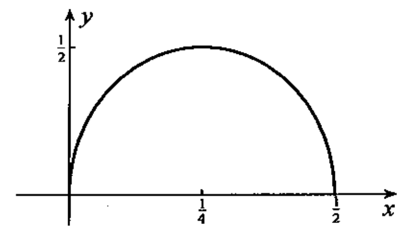

**46.** We draw the parabola $y = x^2 - x - 12$ $= (x - 4)(x + 3)$ and turn that part below the $x$-axis, upside down.



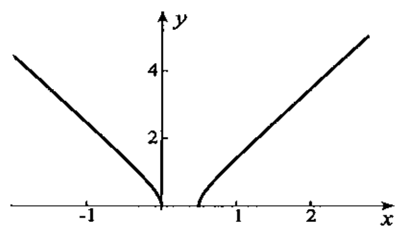
**47.** This is the top half of the circle
$$(x-1)^2+y^2=1.$$




**48.** This is the top half of the ellipse
$$16(x-1/4)^2+4y^2=1.$$

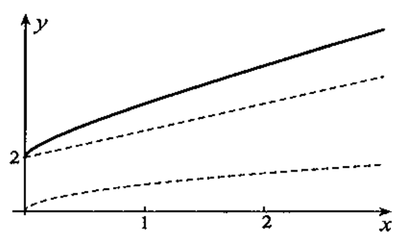



**49.** This is the top half of the hyperbola
$$16(x-1/4)^2-4y^2=1.$$

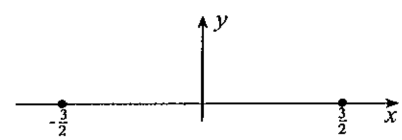



**50.** If we set $y=\sqrt{2x-x^2-4}$, square the equation, and complete the square on the $x$-terms, the result is $(x-1)^2+y^2=-3$. No real values of $x$ and $y$ can satisfy this equation.

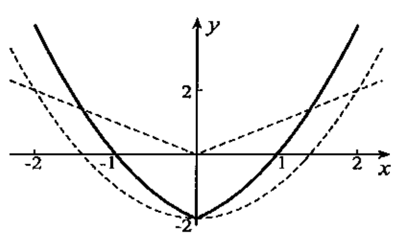
**51.** Addition of ordinates of the curves $y=x+2$ and $y=\sqrt{x}$ gives the curve.



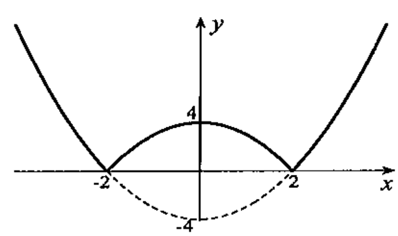

**52.** The square root $\sqrt{9-4x^2}$ is defined only for $|x|\le 3/2$, whereas the square root $\sqrt{4x^2-9}$ is defined for $|x|\ge 3/2$. The only points common to these intervals are $x=\pm 3/2$. The graph therefore consists of the two points $(\pm 3/2, 0)$



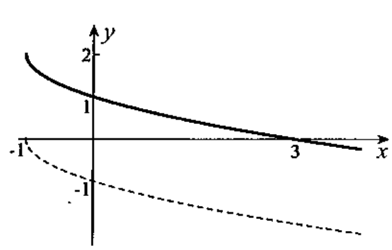

**53.** Addition of ordinates of the curves $y=x^2-2$ and $y=|x|$ gives the curve.

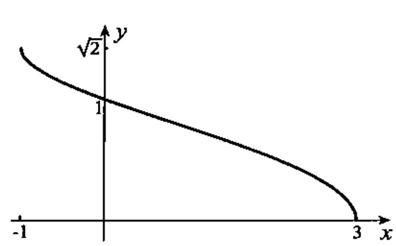



**54.** Since $\sqrt{(x^2-4)}=|x^2-4|$, we draw the parabola $y=x^2-4$, and then turn that part of the parabola between $\pm 2$ upside down.



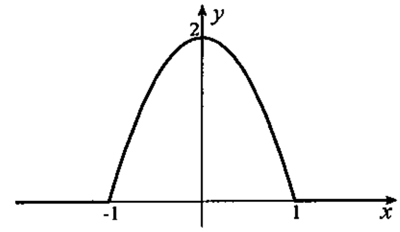

**55.** We first draw the lower half of the parabola $x=y^2-1$, shift it upwards 2 units (left figure below), and then take square roots of ordinates (right figure below).



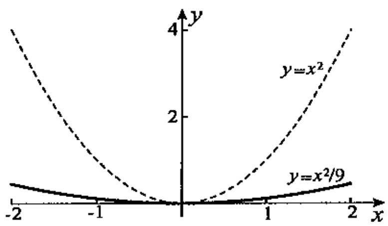
**56.** Since $\sqrt{(x^2-1)^2}-(x^2-1)=|x^2-1|-(x^2-1)$, it follows that

$$f(x)=\begin{cases}(x^2-1)-(x^2-1) & |x|>1\\ -(x^2-1)-(x^2-1) & |x|\le 1\end{cases}$$
$$=\begin{cases}0 & |x|>1\\ 2(1-x^2) & |x|\le 1.\end{cases}$$

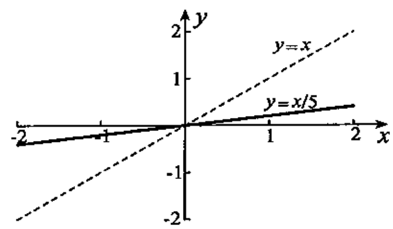



**57.** Every horizontal line that intersects the curve does so exactly once.

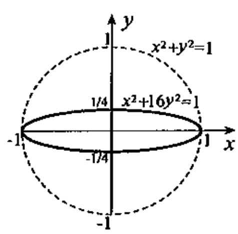
**58.** $y=x^2/9$ is a stretch of $y=x^2$ by a factor of 3 in the $x$-direction. It is also a compression by a factor of 9 in the $y$-direction.



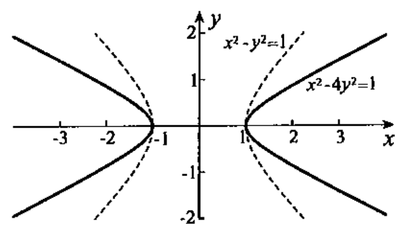

**59.** $y=x/5$ is a stretch of $y=x$ by a factor of 5 in the $x$-direction. It is also a compression of $y=x$ by a factor of 5 in the $y$-direction.



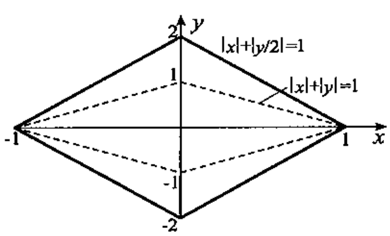

**60.** $x^2+16y^2=1$ is a compression of $x^2+y^2=1$ by a factor of 4 in the $y$-direction.



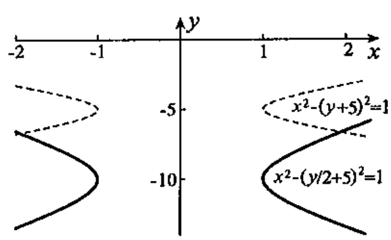

**61.** $x^2-4y^2=1$ is a compression of $x^2-y^2=1$ by a factor of 2 in the $y$-direction.



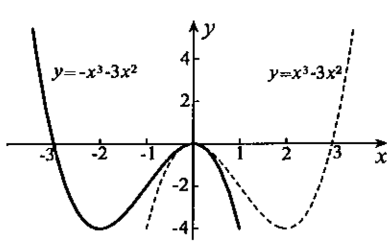

**62.** $|x|+|y/2|=1$ is a stretch of $|x|+|y|=1$ by a factor of 2 in the $y$-direction.



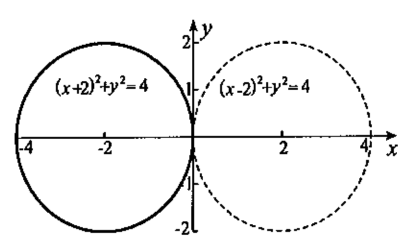

**63.** $x^2-(y/2+5)^2=1$ is a stretch of $x^2-(y+5)^2=1$ by a factor of 2 in the $y$-direction.



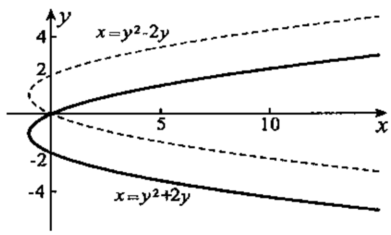

**64.** $y=-x^3-3x^2$ is $y=x^3-3x^2$ reflected in the $y$-axis.



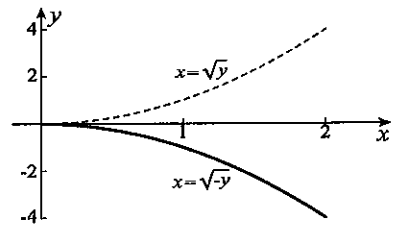

**65.** $(x+2)^2+y^2=4$ is $(x-2)^2+y^2=4$ reflected in the $y$-axis.



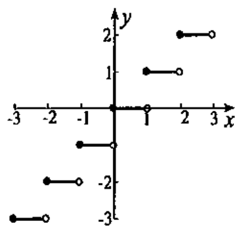
**66.** $x = y^2 + 2y$ is $x = y^2 - 2y$ reflected in the $x$-axis.



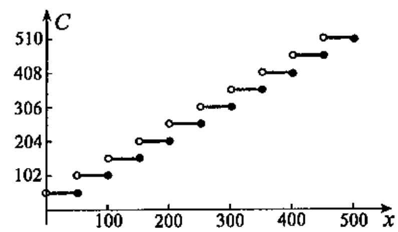

**67.** $x = \sqrt{-y}$ is $x = \sqrt{y}$ reflected in the $x$-aixs.




**68.** (a)



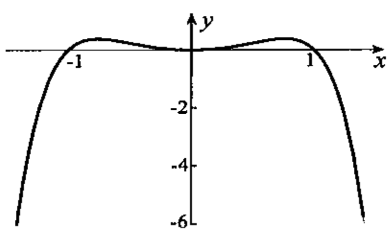
(b)



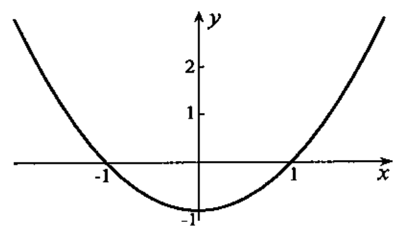

(c) Let $C(x)$ be the cost for mailing an item of $x$ grams $(0 \leq x \leq 500)$. When $x/50$ is an integer, $C(x) = 51(x/50)$. When $x/50$ is not an integer,

$$C(x) = 51\left(1 + \text{integer part of } \frac{x}{50}\right) = 51\left(1 + \lfloor x/50 \rfloor\right) = 51\lfloor 1 + x/50 \rfloor.$$

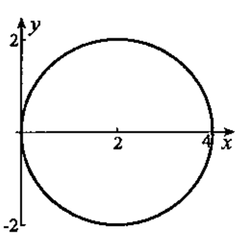
**69.** This defines a function.



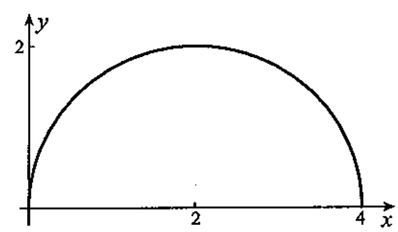
**70.** This defines a function.



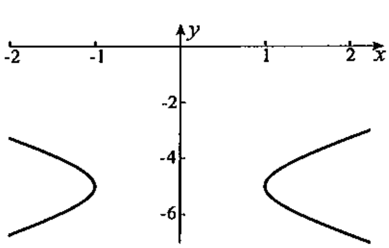
**71.** This does not define a function.



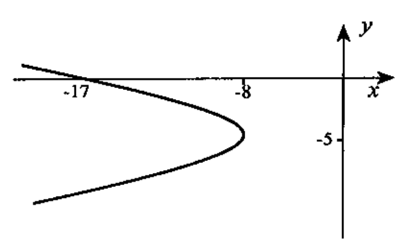
**72.** This defines a function.



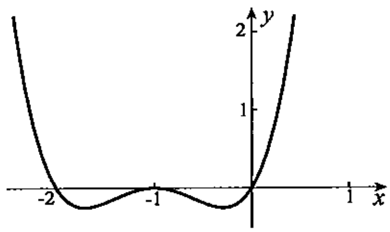
**73.** This does not define a function.



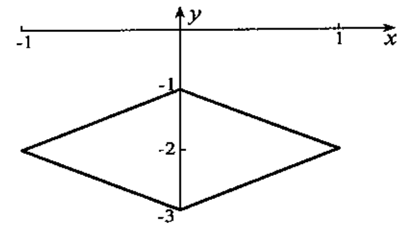
**74.** This does not define a function.



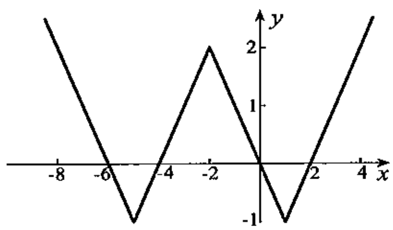
**75.** This defines a function.



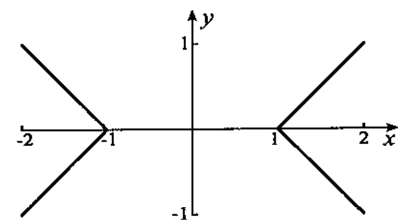

**76.** This does not define a function.



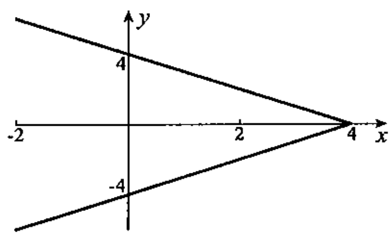

**77.** This defines a function.



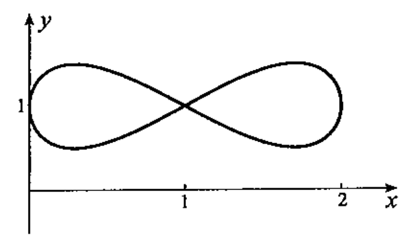

**78.** This does not define a function.



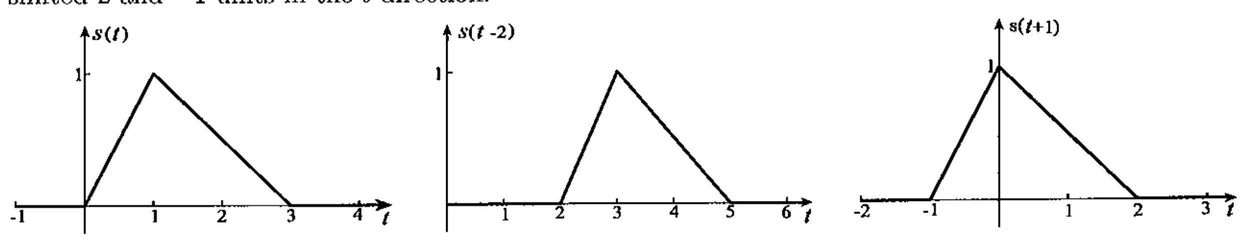

**79.** This does not define a function.




**80.** This does not define a function.

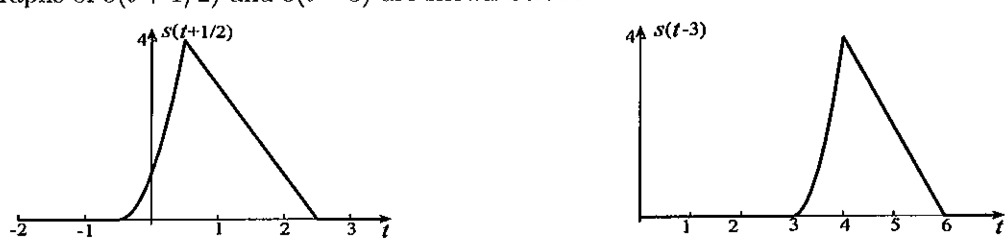



**81.** Graphs of $s(t-2)$ and $s(t+1)$ in the middle and right figures below are those of $s(t)$ in the left figure shifted 2 and $-1$ units in the $t$-direction.




**82.** (a) Since the equation of the parabola is $s = 4t^2$ and that of the line is $s = -2(t-3)$, the algebraic definition of the signal is $s(t) = \begin{cases} 0, & t < 0 \\ 4t^2, & 0 \le t \le 1 \\ 2(3-t), & 1 < t \le 3 \\ 0, & t > 3 \end{cases}$.

(b) Graphs of $s(t+1/2)$ and $s(t-3)$ are shown below.



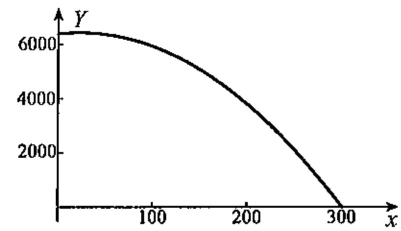

(c) The algebraic representations can be obtained by replacing $t$ by $t+1/2$ and $t-3$ in $s(t)$,

$$s(t+1/2)=\begin{cases}0, & t+1/2<0\\ 4(t+1/2)^2, & 0\le t+1/2\le 1\\ 2(3-t-1/2), & 1<t+1/2\le 3\\ 0, & t+1/2>3\end{cases}=\begin{cases}0, & t<-1/2\\ 4(t+1/2)^2, & -1/2\le t\le 1/2\\ 5-2t, & 1/2<t\le 5/2\\ 0, & t>5/2\end{cases};$$

$$s(t-3)=\begin{cases}0, & t-3<0\\ 4(t-3)^2, & 0\le t-3\le 1\\ 2(3-t+3), & 1<t-3\le 3\\ 0, & t-3>3\end{cases}=\begin{cases}0, & t<3\\ 4(t-3)^2, & 3\le t\le 4\\ 2(6-t), & 4<t\le 6\\ 0, & t>6\end{cases}.$$

**83.** If $x$ additional trees are planted, the number of trees is $255+x$, and the yield per tree is $25-x/12$. The total yield for this number of additional trees is

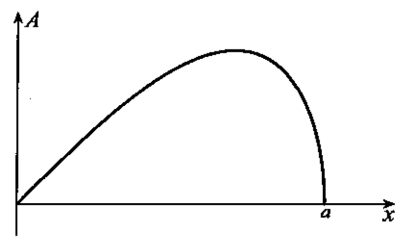
$$Y=f(x)=(255+x)\left(25-\frac{x}{12}\right)=\frac{1}{12}(255+x)(300-x).$$



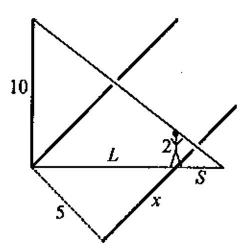

To keep $Y$ nonnegative, and not cut down any trees, we restrict the domain of the function to $0\le x\le 300$. The maximum on this parabola is halfway between its intercepts; that is, at $x=(300-255)/2=45/2$. Thus, 22 more trees should be planted. (We do not suggest 23 trees, since an extra tree would have to be purchased.)

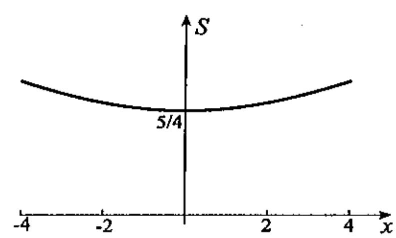
**84.** The area of the rectangle shown is $A=4xy$. When we solve the equation of the ellipse for the positive value of $y$, the result is $y=(b/a)\sqrt{a^2-x^2}$. The area of the rectangle can therefore be expressed in the form

$$A=f(x)=\frac{4bx}{a}\sqrt{a^2-x^2},\quad 0\le x\le a.$$



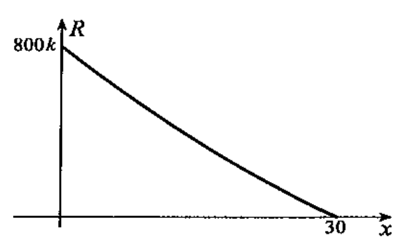

**85.** From similar vertical triangles in the left figure below, $\dfrac{10}{2}=\dfrac{L+S}{S}$, and this equation can be solved for $L=4S$. Since $L^2=x^2+25$, it follows that $16S^2=x^2+25$, or, $S=f(x)=\dfrac{\sqrt{x^2+25}}{4}$. To draw a graph of this function, we return to the equation $16S^2-x^2=25$. The graph is the upper half of this hyperbola (right figure below).



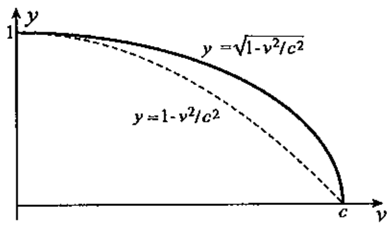

**86.** The amounts of A and B used to produce an amount $x$ of C are $2x/3$ and $x/3$, respectively. It follows that

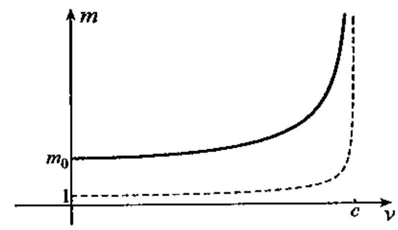
$$R=k\left(20-\frac{2x}{3}\right)\left(40-\frac{x}{3}\right)=\frac{2k}{9}(30-x)(120-x),\quad 0\le x\le 30.$$




The rate is a maximum at $x=0$.

**87.** To draw a graph of this function we begin with a graph of the parabola $y = 1 - v^2/c^2$ in the left figure below. It is shown only for nonnegative $v$ since speed is never negative. To draw $y = \sqrt{1 - v^2/c^2}$ in the same figure, we take square root of ordinates on the parabola. To obtain $y = 1/\sqrt{1 - v^2/c^2}$ in the right figure, we divide ordinates of $y = \sqrt{1 - v^2/c^2}$ into 1. The final graph is a result of multiplying all ordinates by the constant value $m_0$.



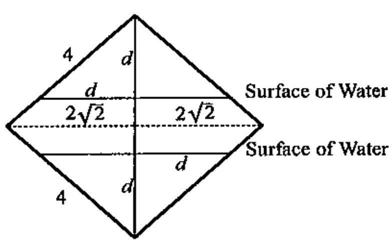



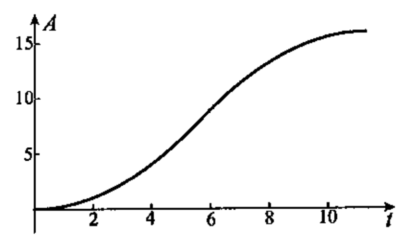

The graph suggests that as the speed of an object approaches the speed of light, its mass becomes indefinitely large. Consequently, it is impossible to accelerate an object with positive rest mass to the speed of light.

**88.** The depth $d$ of the lowest corner of the square below the surface of the water is given by $d = t/2$. For $0 \le t \le 4\sqrt{2}$ (when the lower half of the square is being submerged), the area submerged at time $t$ is $A = d^2 = (t/2)^2 = t^2/4$ (left figure below). For $4\sqrt{2} < t \le 8\sqrt{2}$ (when the top half is being submerged),

$$A = 16 - \left(4\sqrt{2} - \frac{t}{2}\right)^2 = -16 + 4\sqrt{2}t - \frac{t^2}{4}.$$

A graph of the function $A(t)$ is shown in the right figure below.



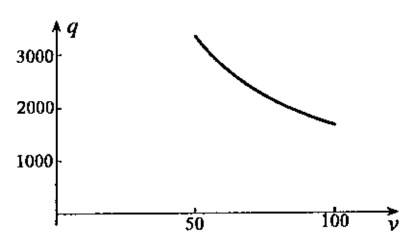

**89.** If the distance between cars is $d = 3v^2/500$, then the number of cars per kilometre is

$$\frac{1}{d}(1000) = \frac{500}{3v^2}(1000) = \frac{500\,000}{3v^2}.$$

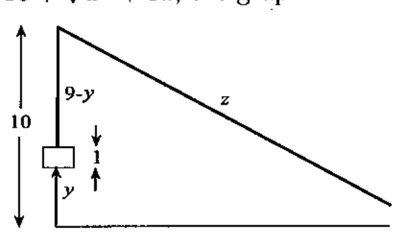
It follows that the number of cars passing any given point per hour is

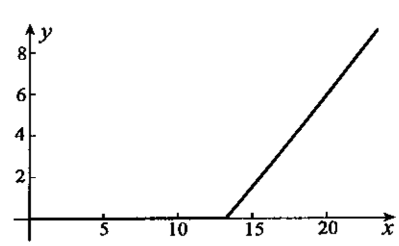
$$q = \left(\frac{500\,000}{3v^2}\right)v = \frac{500\,000}{3v}.$$

The graph indicates that $q$ is maximized for $v = 50$.

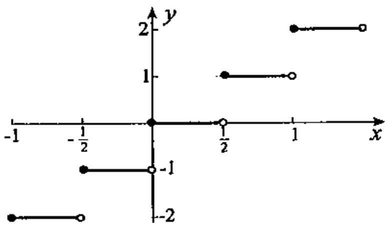



**90.** The rope becomes taut when $x = \sqrt{256 - 81} = 5\sqrt{7}$ and the box reaches the pulley when $x = \sqrt{625 - 81} = 4\sqrt{34}$. Between these values of $x$, the left figure below indicates that $z^2 = x^2 + 81$ and $z + (9 - y) = 25$. When we eliminate $z$ between these equations, the result is $x^2 + 81 = (16 + y)^2 \Longrightarrow y = -16 + \sqrt{x^2 + 81}$, the graph of which is shown in the right figure below.

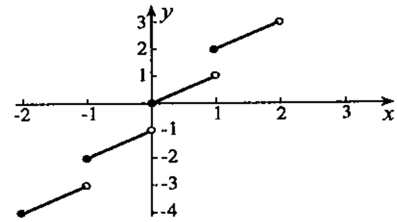


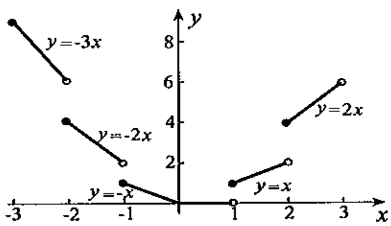
**91.**



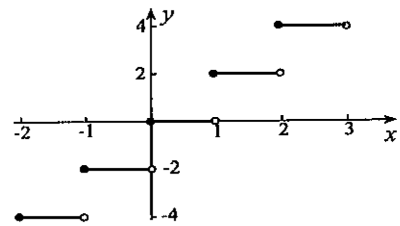

**92.**




**93.**




**94.**




**95.** To disprove this statement, it is sufficient to give an example for which it is invalid. Suppose that $f(x) = x$, $g(x) = x$, and $x = 1.8$. Then

$$\lfloor f(1.8) + g(1.8) \rfloor = \lfloor 1.8 + 1.8 \rfloor = \lfloor 3.6 \rfloor = 3,$$

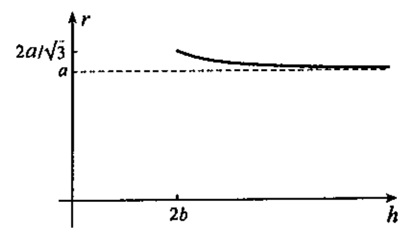
whereas

$$\lfloor f(1.8) \rfloor + \lfloor g(1.8) \rfloor = \lfloor 1.8 \rfloor + \lfloor 1.8 \rfloor = 1 + 1 = 2.$$

**96.** If we solve the equation for $r$ in terms of $h$, the result is $r = \dfrac{ah}{\sqrt{h^2 - b^2}}$. When $h = 2b$, $r = 2a/\sqrt{3}$. By writing the function in the form

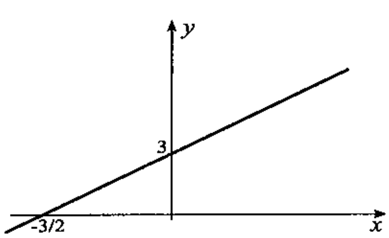
$$r = \frac{ah}{\sqrt{h^2 - b^2}} = \frac{a}{\sqrt{1 - \frac{b^2}{h^2}}},$$

we see that as $h$ increases, $r$ decreases, and for large $h$, $r$ is very close to $a$.

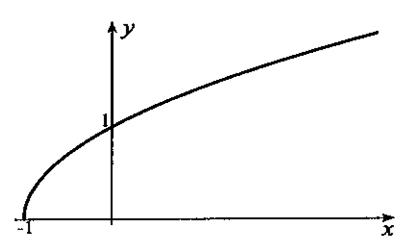



## EXERCISES 1.6

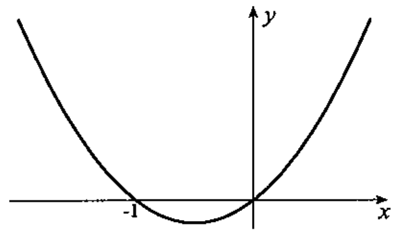
**1.** Since the function is increasing for all $x$, it has an inverse. When we solve $y = 2x + 3$ for $x$, we obtain $x = (y - 3)/2$, and therefore the inverse function is $f^{-1}(x) = (x - 3)/2$.



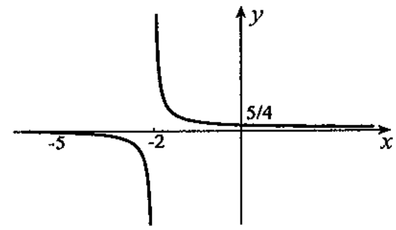

**2.** Since the function is increasing for $x \geq -1$, it has an inverse. When we solve $y = \sqrt{x + 1}$, for $x$, we obtain $x = y^2 - 1$, and therefore the inverse function is $f^{-1}(x) = x^2 - 1$, $x \geq 0$.



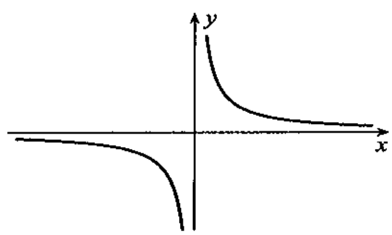
**3.** Since horizontal lines $y = c > -1/4$ intersect the graph in two points, there is no inverse function.



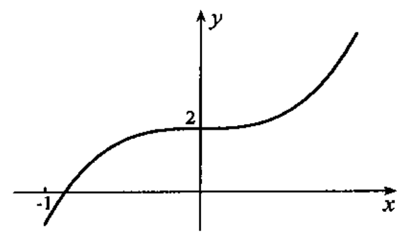

**4.** Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we solve $y = (x+5)/(2x+4)$ for $x$, we obtain $x = (5-4y)/(2y-1)$. The inverse function is $f^{-1}(x) = (5-4x)/(2x-1)$.



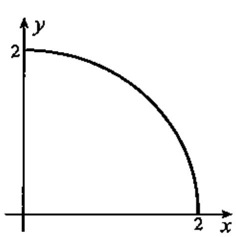

**5.** Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we solve $y = 1/x$ for $x$, we obtain $x = 1/y$. The inverse function is $f^{-1}(x) = 1/x$.



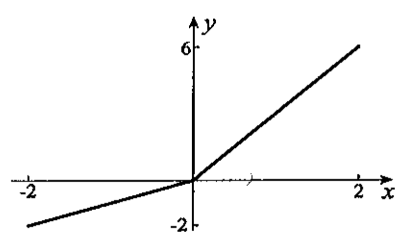

**6.** Since $f(x)$ is increasing for all $x$, the function has an inverse. Because $x = [(y-2)/3]^{1/3}$, the inverse function is $f^{-1}(x) = [(x-2)/3]^{1/3}$.



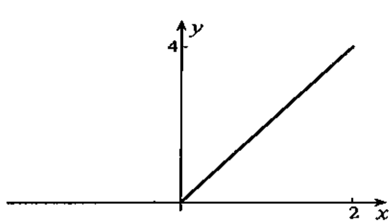

**7.** Since the function is decreasing for $0 \le x \le 2$, it has an inverse. When we solve $y = \sqrt{4-x^2}$ for $x$, we obtain $x = \sqrt{4-y^2}$. The inverse function is $f^{-1}(x) = \sqrt{4-x^2}$.



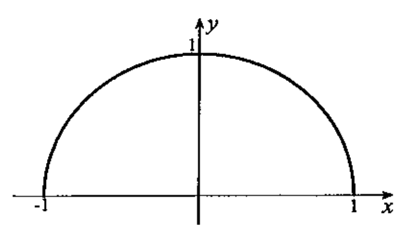

**8.** Because $f(x)$ is increasing for all $x$, the function has an inverse. Graphically, the inverse function is $f^{-1}(x) = x$ for $x < 0$ and $x/3$ for $x \ge 0$.



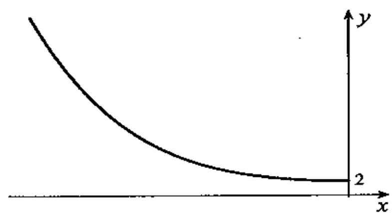

**9.** Since the line $y = 0$ intersects the graph in an infinity of points, there is no inverse function.



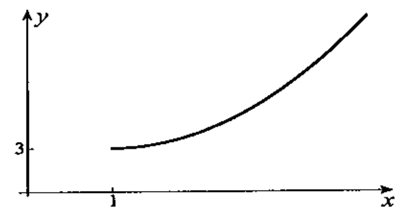

**10.** Since horizontal lines $y = c$, $0 \le c < 1$, intersect the graph in two points, there is no inverse function.



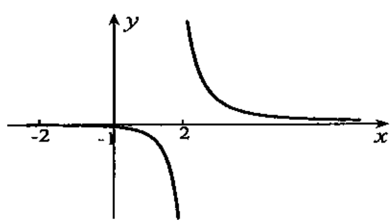
**11.** Since the function is decreasing for $x \le 0$, the function has an inverse. If we set $y = x^4 + 2x^2 + 2$, then $(x^2)^2 + 2x^2 + 2 - y = 0$, from which $x^2 = \left[-2 \pm \sqrt{4 - 4(2-y)}\right]/2$ $= -1 \pm \sqrt{y-1}$. Since $x^2$ must be positive, we choose the positive root, in which case $x = -\sqrt{-1 + \sqrt{y-1}}$. The inverse function is $f^{-1}(x) = -\sqrt{-1+\sqrt{x-1}}, \quad x \ge 2$.



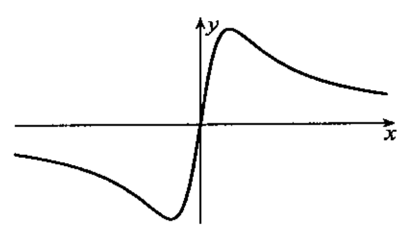

**12.** Because $f(x)$ is increasing for $x \ge 1$, it has an inverse function. If we write $x^2 - 2x + (4 - y) = 0$, then $x = \left[2 \pm \sqrt{4 - 4(4-y)}\right]/2 = 1 \pm \sqrt{y-3}$. Since $x > 1$, we must choose the positive sign, and the inverse function is $f^{-1}(x) = 1 + \sqrt{x-3}$, $x \ge 3$.



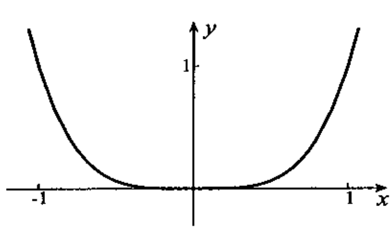

**13.** Since horizontal lines intersect the graph of $f(x)$ in only one point, the function has an inverse. When we set $y = [(x+2)/(x-2)]^3$, then $(x+2)/(x-2) = y^{1/3}$, or, $x + 2 = y^{1/3}(x-2)$, from which $x = 2(y^{1/3}+1)/(y^{1/3}-1)$. The inverse function is $f^{-1}(x) = 2(x^{1/3}+1)/(x^{1/3}-1)$.



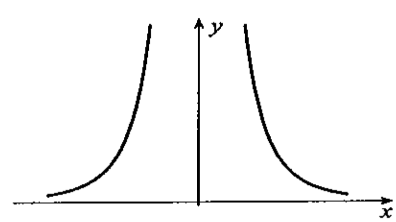

**14.** Since almost all horizontal lines that intersect the graph do so twice, there is no inverse function.



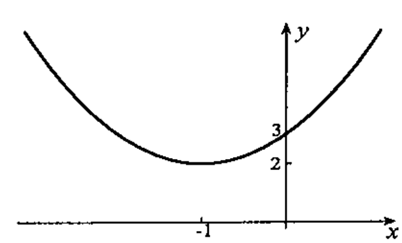

**15.** Since horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. For $x \ge 0$, the inverse function is $f^{-1}(x) = x^{1/4}$, and for $x \le 0$, the inverse is $f^{-1}(x) = -x^{1/4}$.



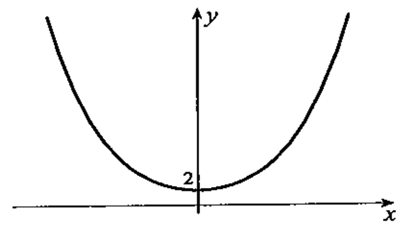

**16.** Since horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x < 0$ and $x > 0$, it does have inverses. For $x > 0$, the inverse function is $f^{-1}(x) = 1/x^{1/4}$, and for $x < 0$, the inverse is $f^{-1}(x) = -1/x^{1/4}$.



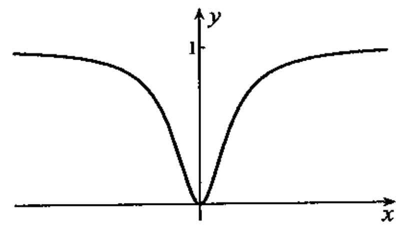

**17.** Since horizontal lines $y = c > 2$ intersect the graph twice, the function does not have an inverse. On the intervals $x \le -1$ and $x \ge -1$, it does have inverses. We set $y = x^2 + 2x + 3$, or, $x^2 + 2x + (3 - y) = 0$, and solve for $x = \left[-2 \pm \sqrt{4 - 4(3-y)}\right]/2 = -1 \pm \sqrt{y-2}$. For $x \le -1$, the inverse function is $f^{-1}(x) = -1 - \sqrt{x-2}$, and when $x \ge -1$, the inverse function is $f^{-1}(x) = -1 + \sqrt{x-2}$.



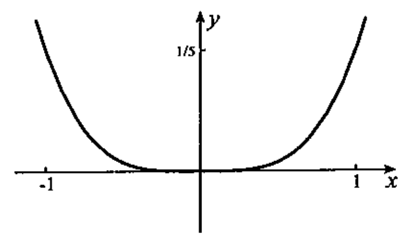
**18.** Because horizontal lines $y = c > 2$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^4 + 4x^2 + 2 = (x^2+2)^2 - 2$, and solve for $x^2 + 2 = \pm\sqrt{y+2}$, but only the positive result is acceptable. Hence $x^2 + 2 = \sqrt{y+2}$, and therefore $x = \pm\sqrt{\sqrt{y+2} - 2}$. For $x \ge 0$, the inverse function is $f^{-1}(x) = \sqrt{\sqrt{x+2} - 2}$, and for $x \le 0$, the inverse is $f^{-1}(x) = -\sqrt{\sqrt{x+2} - 2}$.




**19.** Because horizontal lines between $y = 0$ and $y = 1$ intersect the graph twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^2/(x^2+4)$, from which $x^2 = 4y/(1-y)$. Square roots give $x = \pm 2\sqrt{y/(1-y)}$. For $x \le 0$, the inverse function is $f^{-1}(x) = -2\sqrt{x/(1-x)}$, and for $x \ge 0$, the inverse is $f^{-1}(x) = 2\sqrt{x/(1-x)}$.

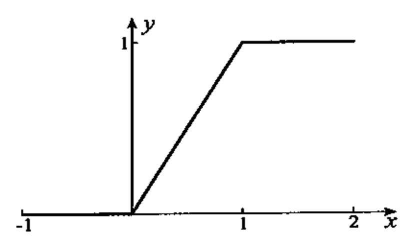



**20.** Because horizontal lines $y = c > 0$ intersect the graph of the function twice, the function does not have an inverse. On the intervals $x \le 0$ and $x \ge 0$, it does have inverses. To find them, we set $y = x^4/(x^2+4)$, from which $x^4 - yx^2 - 4y = 0$. This quadratic in $x^2$ has solutions $x^2 = (y \pm \sqrt{y^2 + 16y})/2$, but only the positive root is acceptable. Thus, $x = \pm\sqrt{(y + \sqrt{y^2+16y})/2}$. The inverse function for $x \ge 0$ is $f^{-1}(x) = \sqrt{(x + \sqrt{x^2+16x})/2}$, and for $x \le 0$ the inverse is $f^{-1}(x) = -\sqrt{(x + \sqrt{x^2+16x})/2}$.

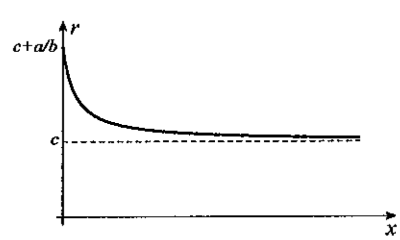


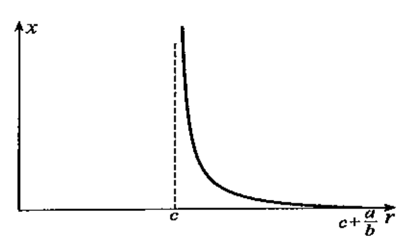

**21.** An example is the function $f(x) = 1$. On no interval is $f(x)$ one-to-one.

**22.** The function $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$ (shown to the right) is one-to-one on $0 \le x \le 1$, but on no other interval.

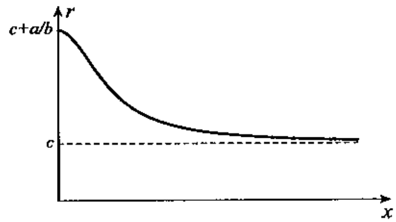


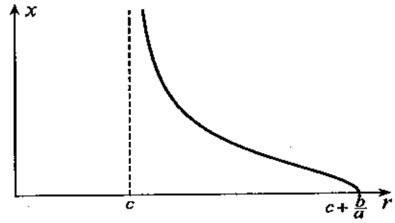

**23.** (a) From the given equation, $a/(x+b) = r - c$, from which $x + b = a/(r-c)$. The inverse function is $f^{-1}(r) = a/(r-c) - b$. The function and its inverse are shown below.



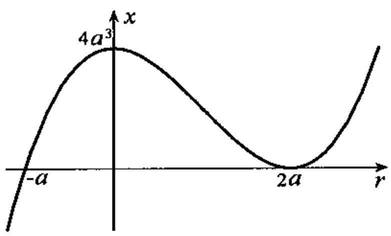



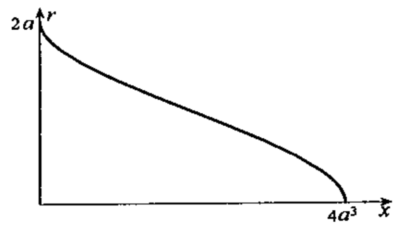

(b) From the given equation, $a/(x^2+b) = r - c$, from which $x^2 + b = a/(r-c)$, or, $x = \pm\sqrt{a/(r-c) - b}$. The inverse function is $f^{-1}(r) = \sqrt{a/(r-c) - b}$. The function and its inverse are shown below.




**24.** The demand function factors as $f(r) = (r+a)(r-2a)^2$, and therefore its graph must be somewhat as shown in the left figure below. It is decreasing from $r = 0$ to $r = 2a$ for the following reasons. First $f(0) = 4a^3$. Secondly, when written in the form $f(r) = 4a^3 - r^2(3a - r)$, we see that because $r^2(3a - r) > 0$ for $0 < r < 2a$, we must have $f(r) < 4a^3$ on this interval. The function therefore has an inverse function for $0 < r < 2a$. The domain of the inverse function is the range of $f(r)$, namely, $0 < x < 4a^3$. We can sketch its graph (right figure below) by reflecting the graph of $x = f(r)$ in the line $x = r$.




**25.** The graph shows that $f(x)$ has an inverse. To solve $y = x^2/(1+x)^2$ for $x$, we set

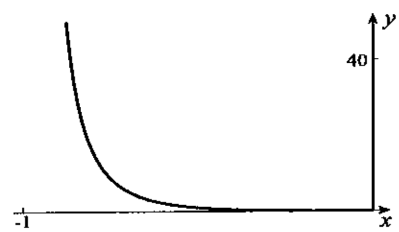
$$x^2 = (1+x)^2 y = (1 + 2x + x^2)y$$

$\Longrightarrow$

$$(y-1)x^2 + (2y)x + y = 0.$$

Solutions of this quadratic equation are

$$x = \frac{-2y \pm \sqrt{4y^2 - 4y(y-1)}}{2(y-1)} = \frac{-y \pm \sqrt{y}}{y-1}.$$




Since $x$ must be negative for all $y > 0$, we choose $x = \dfrac{-y + \sqrt{y}}{y-1} = \dfrac{\sqrt{y}(1 - \sqrt{y})}{(\sqrt{y}+1)(\sqrt{y}-1)} = \dfrac{-\sqrt{y}}{\sqrt{y}+1}$. The inverse function is therefore $f^{-1}(x) = \dfrac{-\sqrt{x}}{\sqrt{x}+1}$.

## EXERCISES 1.7

In questions 1–10 we multiply the degree measure by $\pi/180$ to find the radian measure of the angle. The answers are:

| | | | | |
|---|---|---|---|---|
| **1.** $\pi/6$ | **2.** $\pi/3$ | **3.** $3\pi/4$ | **4.** $-\pi/2$ | **5.** $-5\pi/3$ |
| **6.** $17\pi/4$ | **7.** $2\pi/5$ | **8.** $-32\pi/45$ | **9.** $107\pi/60$ | **10.** $-213\pi/180$ |

In questions 11–20 we multiply the radian measure by $180/\pi$ to find the degree measure of the angle. The answers are:

| | | | | |
|---|---|---|---|---|
| **11.** $60^\circ$ | **12.** $-225^\circ$ | **13.** $270^\circ$ | **14.** $1440^\circ$ | **15.** $-150^\circ$ |
| **16.** $180/\pi^\circ$ | **17.** $-540/\pi^\circ$ | **18.** $450/\pi^\circ$ | **19.** $-206.3^\circ$ | **20.** $1980/\pi^\circ$ |

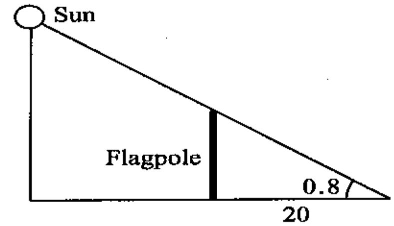
**21.** In each case, we divide the length of the arc by the radius 4 of the circle. The angles are (a) 1/2 radian (b) 7/4 radians (c) 4/5 radian.

**22.** Since the height from the top of the transit to the top of the building is $30\tan 1.30$, the height of the building is $2 + 30\tan(1.30) = 110.1$ m.

**23.** The height of the smaller building is $100\tan(3/5) = 68.4$ m. Since the vertical distance from the top of the smaller building to the top of the taller building is $100\tan(11/10)$, the height of the taller building is $68.4 + 100\tan(11/10) = 2.65 \times 10^2$ m.

**24.** From the figure to the right, the height of the flagpole is

$20 \tan 0.8 = 20.6$ m.




**25.** The cosine law gives the length $a$ of the third side of the triangle,

$a^2 = 2^2 + 3^2 - 2(2)(3)\cos(\pi/3) = 7 \implies a = \sqrt{7}.$

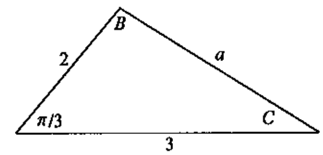
The sine law gives the angles,

$$\frac{\sin B}{3} = \frac{\sin(\pi/3)}{a} \quad \Longrightarrow \quad \sin B = \frac{3(\sqrt{3}/2)}{\sqrt{7}},$$

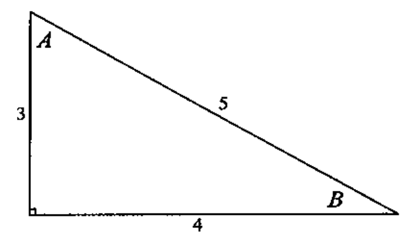
from which $B = 1.38$ radians;

$$\frac{\sin C}{2} = \frac{\sin(\pi/3)}{a} \quad \Longrightarrow \quad \sin C = \frac{2(\sqrt{3}/2)}{\sqrt{7}},$$

from which $C = 0.714$ radians.



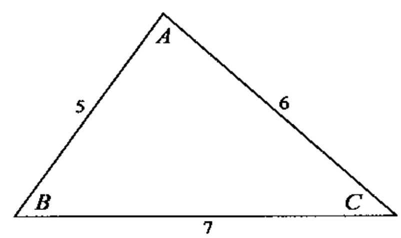

**26.** This is a right angle-angled triangle. Since $\sin A = 4/5$, it follows that $A = 0.927$ radians. From $\sin B = 3/5$, we obtain $B = 0.644$ radians.




**27.** The cosine law for the triangle gives

$$7^2 = 5^2 + 6^2 - 2(5)(6)\cos A \implies \cos A = \frac{61 - 49}{60} = \frac{1}{5},$$

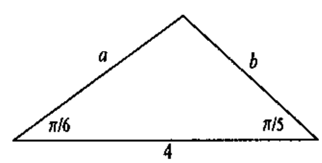
from which $A = 1.37$ radians. A similar calculation gives $B = 0.997$ radians. Then $C = \pi - A - B = 0.775$ radians.




**28.** The remaining angle is $\pi - \pi/6 - \pi/5 = 19\pi/30$ radians. The sine law gives the remaining two sides,

$$\frac{a}{\sin(\pi/5)} = \frac{4}{\sin(19\pi/30)} \implies a = \frac{4\sin(\pi/5)}{\sin(19\pi/30)} = 2.57.$$

$$\frac{b}{\sin(\pi/6)} = \frac{4}{\sin(19\pi/30)} \implies b = \frac{4\sin(\pi/6)}{\sin(19\pi/30)} = 2.19.$$




**29.** To prove 1.44a we proceed as follows (the proof of 1.44b is similar):

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$
$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

**30.** To verify 1.45a, we set $B = A$ in 1.43a,

$$\sin(A + A) = \sin A\cos A + \cos A\sin A \quad \Longrightarrow \quad \sin 2A = 2\sin A\cos A.$$

To verify 1.46a, set $B = A$ in 1.43c. Then use the fact that $\sin^2 A + \cos^2 A = 1$ to derive 1.46b,c. To verify 1.47, set $B = A$ in 1.44.

**31.** To derive 1.48b, we add 1.43a,b,

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B \quad\Longrightarrow\quad \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)].$$

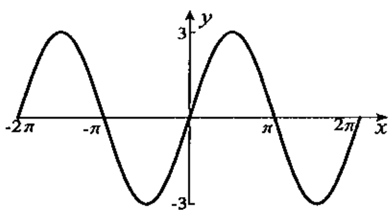
Verifications of 1.48a,c are similar.

**32.** If we set $X = A+B$ and $Y = A-B$, and solve for $A$ and $B$, results are $A = (X+Y)/2$ and $B = (X-Y)/2$. If we substitute these into 1.48b,

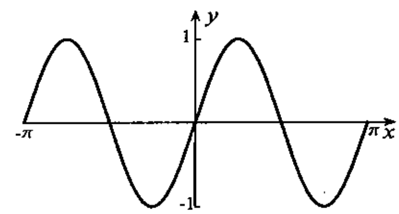
$$\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right) = \frac{1}{2}(\sin X+\sin Y) \quad\Longrightarrow\quad \sin X+\sin Y = 2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right).$$

Proofs of 1.49b,c,d are similar.

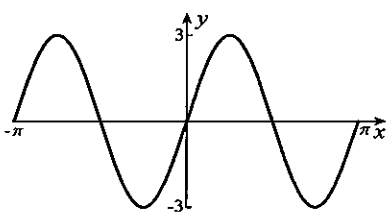
**33.** The amplitude is 3.



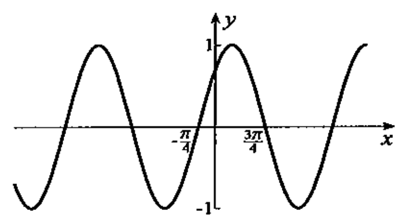

**34.** The period is $\pi$.



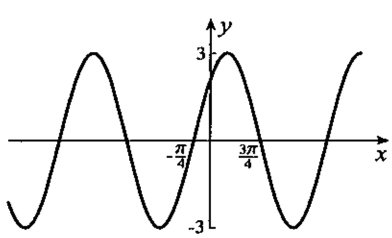

**35.** The amplitude is 3 and the period is $\pi$.



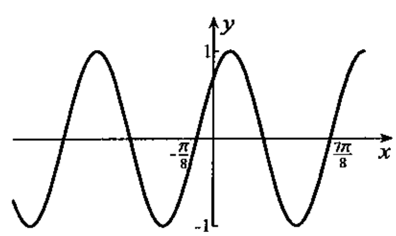

**36.** The sine curve is shifted $\pi/4$ units to the left.



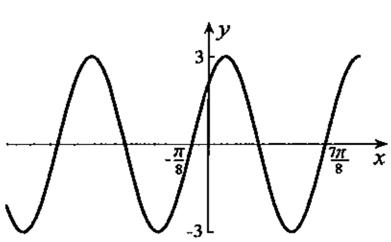

**37.** The amplitude is 3 and the curve is shifted $\pi/4$ units to the left.



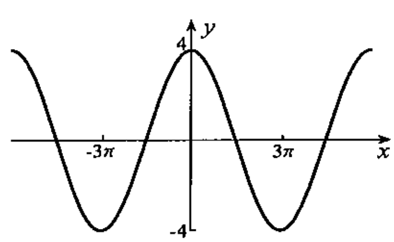

**38.** The period is $\pi$ and the curve is shifted $\pi/8$ units to the left.



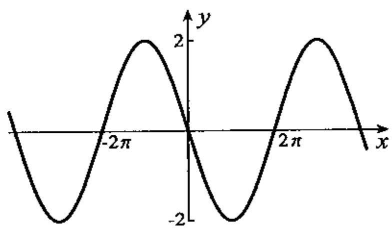

**39.** The period is $\pi$, the amplitude is 3, and the curve is shifted $\pi/8$ units to the left.



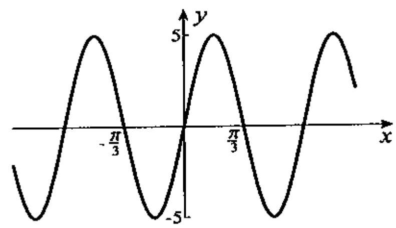

**40.** The amplitude is 4 and the period is $6\pi$.



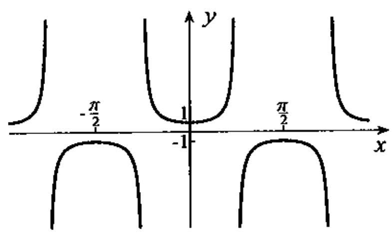
**41.** The amplitude is 2, the period is $4\pi$, and the curve is shifted $2\pi$ units to the right.



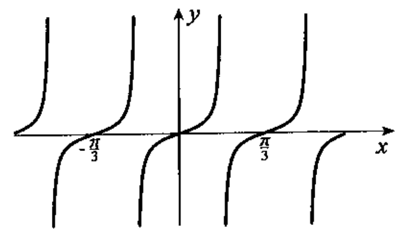

**42.** The amplitude is 5, the period is $2\pi/3$, and the curve is shifted $\pi/6$ units to the right.



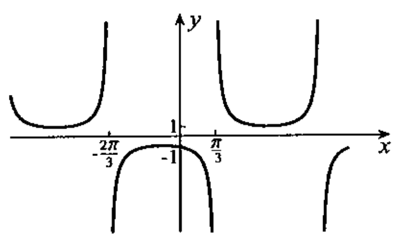

**43.** The period is $\pi$.



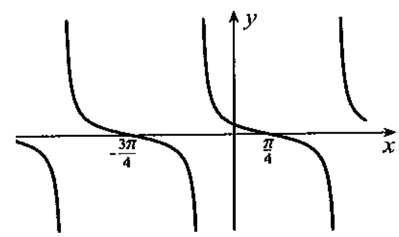

**44.** The period is $\pi/3$.



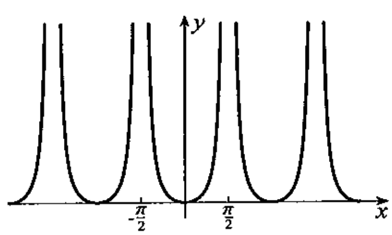

**45.** The cosecant curve is shifted $\pi/3$ units to the right.



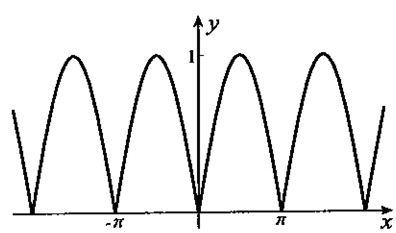

**46.** The cotangent curve is shifted $\pi/4$ units to the left.



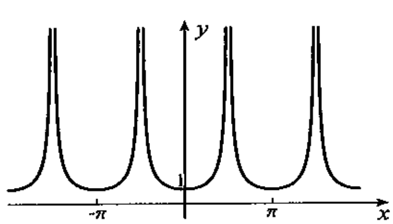

**47.** We square ordinates of the tangent curve.



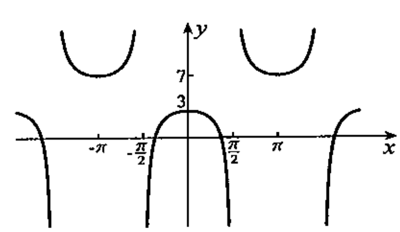

**48.** Since $f(x) = |\sin x|$, we invert that part of the sine curve below the $x$-axis.



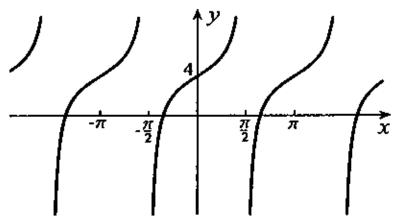

**49.** Since $f(x) = |\sec x|$, we invert that part of the secant curve below the $x$-axis.



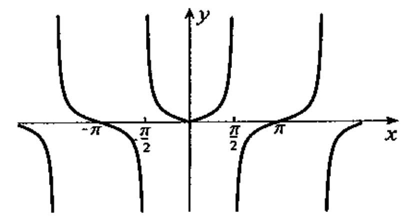

**50.** We invert the secant curve, double ordinates, and shift the curve vertically 5 units.



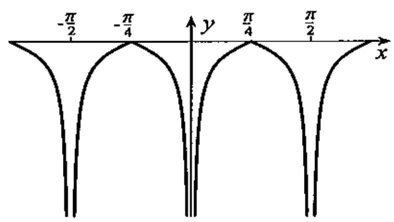
**51.** We double ordinates of the tangent curve and shift vertically 4 units.



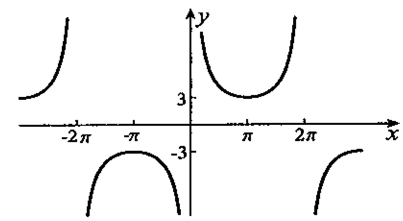

**52.** We reflect that part of the tangent curve to the right of the $y$-axis in the $y$-axis.




**53.** We invert that part of the cotangent curve above the $x$-axis and change the period.




**54.** The period is $4\pi$ and ordinates are multiplied by 3.




**55.** (a) $R = 42.69$ m (b) $R = 42.78$ m (c) For $R = 42.78$ when $\theta = \pi/4$ and $h = 2$, we must have

$$42.78 = \frac{v^2}{9.81\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2} + \frac{2(9.81)(2)}{v^2}}\right) = \frac{v^2}{9.81(2)}\left(1 + \sqrt{1 + \frac{8(9.81)}{v^2}}\right).$$

Therefore, $\dfrac{2(9.81)(42.78)}{v^2} - 1 = \sqrt{1 + \dfrac{8(9.81)}{v^2}}$. Squaring gives

$$1 - \frac{4(9.81)(42.78)}{v^2} + \frac{4(9.81)^2(42.78)^2}{v^4} = 1 + \frac{8(9.81)}{v^2},$$

from which

$$-4(9.81)(42.78)v^2 + 4(9.81)^2(42.78)^2 = 8(9.81)v^2 \quad \Longrightarrow \quad v = \sqrt{\frac{4(9.81)^2(42.78)^2}{4(9.81)(42.78) + 8(9.81)}} = 20.02 \text{ m/s}.$$

**56.** One angle satisfying the equation is $x = \pi/3$ radians. All solutions can be expressed in the form $x = \pi/3 + 2n\pi, 2\pi/3 + 2n\pi$, where $n$ is an integer. Following the lead of Example 1.38, these angles can be combined in the form

$$\frac{\pi}{2} \pm \frac{\pi}{6} + 2n\pi = \frac{(4n+1)\pi}{2} \pm \frac{\pi}{6}, \quad n \text{ an integer.}$$

**57.** There are no solutions of this equation.

**58.** One angle satisfying the equation is $x = 2\pi/3$ radians. All solutions can be expressed in the form $x = \pm 2\pi/3 + 2n\pi$, where $n$ is an integer.

**59.** One angle satisfying the equation is $x = \pi/6$ radians. All solutions can be expressed in the form $x = \pi/6 + n\pi$, where $n$ is an integer.

**60.** If we divide by $\cos x$, then $\tan x = 1$. One solution of this equation is $x = \pi/4$ radians. All solutions can be expressed in the form $x = \pi/4 + n\pi$, where $n$ is an integer.

**61.** This equation implies that $\cos x = \pm 1/\sqrt{2}$. One solution of $\cos x = 1/\sqrt{2}$ is $\pi/4$ radians and one solution of $\cos x = -1/\sqrt{2}$ is $3\pi/4$ radians. All solutions can be expressed in the form $x = \pi/4 + n\pi/2$, where $n$ is an integer.

**62.** One solution of this equation for $2x$ is $2x = 3\pi/4$. All solutions can be expressed in the form

$$2x = \pm\frac{3\pi}{4} + 2n\pi \qquad \Longrightarrow \qquad x = \pm\frac{3\pi}{8} + n\pi, \quad n \text{ an integer.}$$

**63.** One solution of this equation for $3x$ is $3x = -\pi/3$. All solutions can be expressed in the form

$$3x = -\frac{\pi}{3} + n\pi \qquad \Longrightarrow \qquad x = -\frac{\pi}{9} + \frac{n\pi}{3}, \quad n \text{ an integer.}$$

**64.** One solution of the equation $\sin 3x = 1/2$ for $3x$ is $3x = \pi/6$. All solutions can be expressed in the form

$$3x = \frac{\pi}{2} \pm \frac{\pi}{3} + 2n\pi \qquad \Longrightarrow \qquad x = \frac{\pi}{6} \pm \frac{\pi}{9} + \frac{2n\pi}{3}, \quad n \text{ an integer.}$$

**65.** One solution of this equation for $4x$ is $4x = 3\pi/4$. All solutions can be expressed in the form

$$4x = \pm\frac{3\pi}{4} + 2n\pi \qquad \Longrightarrow \qquad x = \pm\frac{3\pi}{16} + \frac{n\pi}{2}, \quad n \text{ an integer.}$$

**66.** If $\sin 2x = \sin x$, then $2\sin x \cos x = \sin x \Longrightarrow (2\cos x - 1)\sin x = 0$, and therefore either $\sin x = 0$ or $\cos x = 1/2$. Solutions of the former are $x = n\pi$, where $n$ is an integer, and solutions of the latter are $x = \pm\pi/3 + 2n\pi$, where $n$ is an integer.

**67.** This a quadratic equation in $\sin x$ that can be factored $(\sin x - 2)(\sin x + 1) = 0$. Either $\sin x = 2$ or $\sin x = -1$. The first of these is impossible, and solutions of the second are $x = -\pi/2 + 2n\pi$, where $n$ is an integer.

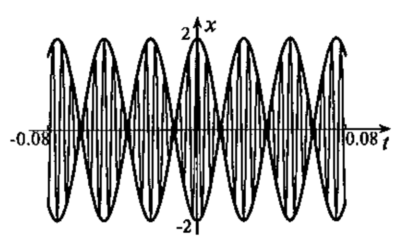
**68.** This equation implies that $\cot x = \pm 1/\sqrt{3}$. One solution of $\cot x = 1/\sqrt{3}$ is $\pi/3$ radians and one solution of $\cot x = -1/\sqrt{3}$ is $-\pi/3$ radians. All solutions can be expressed in the form $x = \pm\pi/3 + n\pi$, where $n$ is an integer.

**69.** If we square the equation, $\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 \Longrightarrow \sin x \cos x = 0$. Solutions of this equation are $x = n\pi$ and $x = (2n+1)\pi/2$, where $n$ is an integer. But only $x = 2n\pi$ and $x = (4n+1)\pi/2$ satisfy the original equation.

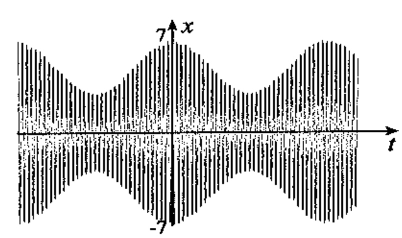
**70.** (a) Using identity 1.49c,

$$x(t) = \cos(440\pi t) + \cos(360\pi t) = 2\cos(400\pi t)\cos(40\pi t).$$

(b)




**71.** (a)

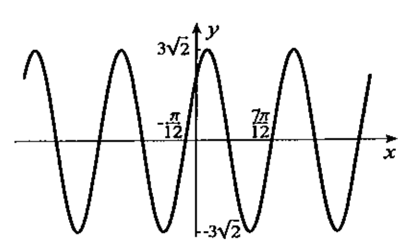



(b) The graph in part (a) indicates minimum and maximum amplitudes of 3 and 7.

**72.** If we set $f(x) = 3\sin 3x + 3\cos 3x = A\sin(3x + \phi)$, and expand the right side,

$$3\sin 3x + 3\cos 3x = A(\sin 3x \cos\phi + \cos 3x \sin\phi).$$

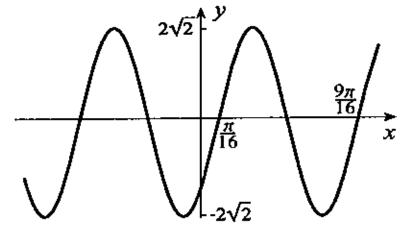
This will be true for all $x$ if we set $A\cos\phi = 3$ and $A\sin\phi = 3$. When these are squared and added, $3^2 + 3^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 3\sqrt{2}$, then $\sin\phi = 1/\sqrt{2}$ and $\cos\phi = 1/\sqrt{2}$. These are satisfied by $\phi = \pi/4$. The amplitude is $3\sqrt{2}$, the period is $2\pi/3$, and the phase shift is $-\pi/12$.



**73.** If we set $f(x) = 2\sin 4x - 2\cos 4x = A\sin(4x + \phi)$, and expand the right side,

$$2\sin 4x - 2\cos 4x = A(\sin 4x\,\cos\phi + \cos 4x\,\sin\phi).$$

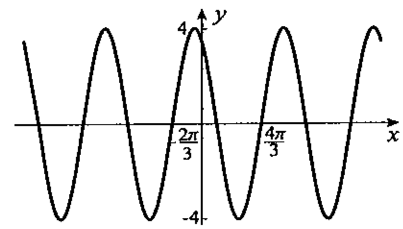
This will be true for all $x$ if we set $A\cos\phi = 2$ and $A\sin\phi = -2$. When these are squared and added, $2^2 + (-2)^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 2\sqrt{2}$, then $\sin\phi = -1/\sqrt{2}$ and $\cos\phi = 1/\sqrt{2}$. These are satisfied by $\phi = -\pi/4$. The amplitude is $2\sqrt{2}$, the period is $\pi/2$, and the phase shift is $\pi/16$.




**74.** If we set $f(x) = -2\sin x + 2\sqrt{3}\cos x = A\sin(x + \phi)$, and expand the right side,

$$-2\sin x + 2\sqrt{3}\cos x = A(\sin x\,\cos\phi + \cos x\,\sin\phi).$$

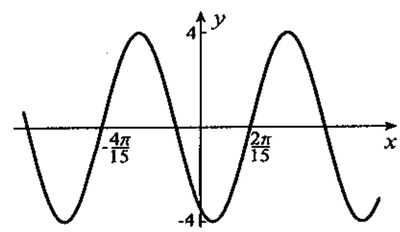
This will be true for all $x$ if we set $A\cos\phi = -2$ and $A\sin\phi = 2\sqrt{3}$. When these are squared and added, $(-2)^2 + (2\sqrt{3})^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 4$, then $\sin\phi = \sqrt{3}/2$ and $\cos\phi = -1/2$. These are satisfied by $\phi = 2\pi/3$. The amplitude is 4, the period is $2\pi$, and the phase shift is $-2\pi/3$.



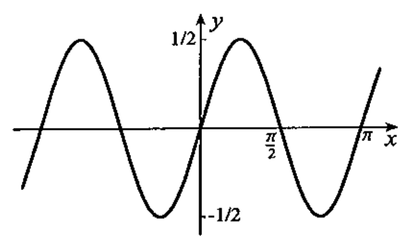

**75.** If we set $f(x) = -2\sin 5x - 2\sqrt{3}\cos 5x = A\sin(5x + \phi)$, and expand the right side,

$$-2\sin 5x - 2\sqrt{3}\cos 5x = A(\sin 5x\,\cos\phi + \cos 5x\,\sin\phi).$$

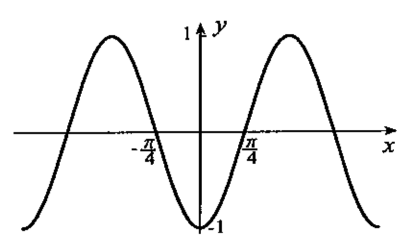
This will be true for all $x$ if we set $A\cos\phi = -2$ and $A\sin\phi = -2\sqrt{3}$. When these are squared and added, $(-2)^2 + (-2\sqrt{3})^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = 4$, then $\sin\phi = -\sqrt{3}/2$ and $\cos\phi = -1/2$. These are satisfied by $\phi = -2\pi/3$. The amplitude is 4, the period is $2\pi/5$, and the phase shift is $2\pi/15$.




**76.** This is simply done using equation 1.45, $f(x) = (1/2)\sin 2x$. The amplitude is $1/2$, the period is $\pi$, and the phase shift is 0.




**77.** This can be done using equation 1.46a, $f(x) = -\cos 2x = -\sin(\pi/2 - 2x) = \sin(2x - \pi/2)$. The amplitude is 1, the period is $\pi$, and the phase shift is $\pi/4$.




**78.** We expand $\cos 3x$,

$$\begin{aligned}\cos 3x &= \cos(2x + x) = \cos 2x\,\cos x - \sin 2x\,\sin x\\ &= (2\cos^2 x - 1)\cos x - (2\sin x\,\cos x)\sin x = 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x)\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

**79.** We expand $\sin 4x$,

$$\sin 4x = 2\sin 2x\,\cos 2x = 2(2\sin x\,\cos x)(2\cos^2 x - 1) = 8\cos^3 x\,\sin x - 4\cos x\,\sin x.$$

**80.** We expand $\tan 3x$,

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x\ \tan x} = \frac{\dfrac{2\ \tan x}{1 - \tan^2 x} + \tan x}{1 - \dfrac{2\ \tan^2 x}{1 - \tan^2 x}}$$

$$= \frac{2\ \tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\ \tan^2 x} = \frac{3 \tan x - \tan^3 x}{1 - 3\ \tan^2 x}.$$

**81.** We use double-angle formulas on the right side,

$$\frac{\sin x}{1 + \cos x} = \frac{2\ \sin(x/2)\ \cos(x/2)}{1 + [2\ \cos^2(x/2) - 1]} = \frac{\sin(x/2)}{\cos(x/2)} = \tan(x/2).$$

**82.** We expand the right side, $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{\tan x + \tan(\pi/4)}{1 - \tan x\ \tan(\pi/4)} = \dfrac{\tan x + 1}{1 - \tan x}.$

**83.** If we set $A \cos\omega x + B \sin\omega x = R \sin(\omega x + \phi)$, and expand the right side using identity 1.43a,

$$A \cos\omega x + B \sin\omega x = R(\sin\omega x \cos\phi + \cos\omega x \sin\phi).$$

This is satisfied if we set $A = R \sin\phi$ and $B = R \cos\phi$. When these are squared and added, $A^2 + B^2 = R^2$, from which we choose $R = \sqrt{A^2 + B^2}$. With this, $\phi$ is defined by $\sin\phi = A/\sqrt{A^2 + B^2}$ and $\cos\phi = B/\sqrt{A^2 + B^2}$.

**84.** No. The two equations define the quadrant for $\phi$, but the equation $\tan\phi = A/B$ does not.

**85.** If we set $2 \sin 2x \cos 2x = \cos 2x$, then either $\cos 2x = 0$ or $2 \sin 2x = 1$. The first implies that $2x = (2n + 1)\pi/2 \implies x = (2n+1)\pi/4$, where $n$ is an integer. The second implies that $2x = \pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$, where $n$ is an integer. The only solutions between 0 and 2 are $\pi/12$, $\pi/4$, and $5\pi/12$.

**86.** If we use identity 1.49c, $0 = \cos x + \cos 3x = 2 \cos 2x \cos x$. Hence, $\cos 2x = 0$ or $\cos x = 0$. Solutions of the first are defined by $2x = (2n + 1)\pi/2 \implies x = (2n + 1)\pi/4$, where $n$ is an integer. Solutions of $\cos x = 0$ are $(2n + 1)\pi/2$. The only solutions between 0 and 2 are $\pi/4$ and $\pi/2$.

**87.** If we use identity 1.49b on the left side of $\sin 4x - \sin 2x = \cos 3x$, then $\cos 3x = 2 \cos 3x \sin x$. This implies that $\cos 3x = 0$ or $\sin x = 1/2$. The first gives $3x = (2n + 1)\pi/2 \implies x = (2n + 1)\pi/6$, where $n$ is an integer. The second gives $x = \pi/6 + 2n\pi, 5\pi/6 + 2n\pi$. Since the solutions $\pi/6 + 2n\pi$ are contained in the set $(2n + 1)\pi/6$, the full set of solutions is $x = (2n + 1)\pi/6$, $(12n + 5)\pi/6$. The only solutions between 0 and 2 are $\pi/6$ and $\pi/2$.

**88.** If we square the equation,

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 3 \sin^2 x \cos^2 x \implies 3 \sin^2 x \cos^2 x - 2 \sin x \cos x - 1 = 0$$
$$\implies (3 \sin x \cos x + 1)(\sin x \cos x - 1) = 0 \implies \sin 2x = -2/3 \quad \text{or} \quad \sin 2x = 2.$$

The second of these is impossible. From the first,

$$2x = \begin{cases} -0.7297 + 2n\pi \\ -2.412 + 2n\pi \end{cases} \implies x = \begin{cases} -0.365 + n\pi \\ -1.21 + n\pi \end{cases},$$

where $n$ is an integer. Only $x = -0.365 + (2n + 1)\pi$, $-1.21 + 2n\pi$ satisfy the original equation. None of the solutions are between 0 and 2.

**89.** If $A$, $B$, and $C$ are the angles in a triangle, then $A + B + C = \pi$. If we take tangents of both sides of this equation,

$$\frac{\tan(A + B) + \tan C}{1 - \tan(A + B)\ \tan C} = 0 \implies \tan(A + B) + \tan C = 0.$$

If we expand $\tan(A + B)$,

$$\frac{\tan A + \tan B}{1 - \tan A\ \tan B} = -\tan C \implies \tan A + \tan B = -\tan C + \tan A\ \tan B\ \tan C,$$

and this gives the required result.

**EXERCISES 1.8**

**1.** $\text{Tan}^{-1}(-1/3) = -0.322$

**2.** $\text{Sin}^{-1}(1/4) = 0.253$

**3.** $\text{Sec}^{-1}(\sqrt{3}) = 0.955$

**4.** $\text{Csc}^{-1}(-2/\sqrt{3}) = -2\pi/3$

**5.** $\text{Cot}^{-1}(1) = \pi/4$

**6.** $\text{Cos}^{-1}(3/2)$ does not exist since the domain of $\text{Cos}^{-1}x$ is $-1 \le x \le 1$.

**7.** $\text{Sin}^{-1}(\pi/2)$ does not exist since the domain of $\text{Sin}^{-1}x$ is $-1 \le x \le 1$.

**8.** $\text{Tan}^{-1}(-1) = -\pi/4$

**9.** $\sin(\text{Tan}^{-1}\sqrt{3}) = \sin(\pi/3) = \sqrt{3}/2$

**10.** $\tan\left(\text{Sin}^{-1}3\right)$ does not exist since the domain of $\text{Sin}^{-1}x$ is $-1 \le x \le 1$.

**11.** $\text{Sin}^{-1}[\tan(1/6)] = \text{Sin}^{-1}0.16823 = 0.169$

**12.** $\text{Tan}^{-1}[\sin(1/6)] = \text{Tan}^{-1}[0.165\,896] = 0.164$

**13.** $\sec[\text{Cos}^{-1}(1/2)] = \sec(\pi/3) = 2$

**14.** $\text{Sin}^{-1}[\sin(3\pi/4)] = \text{Sin}^{-1}[1/\sqrt{2}] = \pi/4$

**15.** $\sin[\text{Sin}^{-1}(1/\sqrt{2})] = 1/\sqrt{2}$

**16.** $\text{Sin}^{-1}\left[\cos\left(\text{Sec}^{-1}(-\sqrt{2})\right)\right] = \text{Sin}^{-1}\left[\cos(-3\pi/4)\right] = \text{Sin}^{-1}\left[-1/\sqrt{2}\right] = -\pi/4$

**17.** Since one solution is $x = \text{Sin}^{-1}(1/3) = 0.340$, all solutions are $0.340 + 2n\pi$ and $\pi - 0.340 + 2n\pi$, where $n$ is an integer. They can also be represented more compactly in the form

$$\frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.340\right) + 2n\pi = (4n+1)\pi/2 \pm 1.23.$$

**18.** From the solution $x = \text{Tan}^{-1}(-1.2) = -0.876$, we obtain $x = n\pi - 0.876$, where $n$ is an integer.

**19.** One solution for $2x$ is $2x = \text{Cos}^{-1}(1/3) = 1.23$. All solutions are given by

$$2x = \pm 1.23 + 2n\pi \qquad \Longrightarrow \qquad x = \pm 0.615 + n\pi, \quad n \text{ an integer.}$$

**20.** One solution of $\cot 4x = -2.2$ for $4x$ is $4x = \text{Cot}^{-1}(-2.2) = 2.715$. All solutions are given by

$$4x = 2.715 + n\pi \qquad \Longrightarrow \qquad x = .679 + \frac{n\pi}{4}, \quad n \text{ an integer.}$$

**21.** One solution of $\sin(1-x) = 0.7$ for $1-x$ is $1 - x = \text{Sin}^{-1}(0.7) = 0.7754$. All solutions are given by

$$1 - x = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.7754\right) + 2n\pi \qquad \Longrightarrow \qquad x = 1 - \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.7754\right) - 2n\pi = -0.571 \pm 0.795 - 2n\pi,$$

where $n$ is an integer.

**22.** One solution of $\tan 3x = -3.2/3$ for $3x$ is $3x = \text{Tan}^{-1}(-3.2/3) = -0.8176$. All solutions are given by

$$3x = -0.8176 + n\pi \qquad \Longrightarrow \qquad x = -0.273 + \frac{n\pi}{3}, \quad n \text{ an integer.}$$

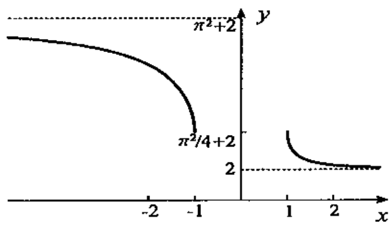
**23.** When we set $0 = 4\sin^2 x - 2(1 - \sin^2 x) - 1 = 6\sin^2 x - 3 = 3(2\sin^2 x - 1)$, it follows that $\sin x = \pm 1/\sqrt{2}$. Solutions are $x = \pm\pi/4 + n\pi = (4n \pm 1)\pi/4$, where $n$ is an integer.

**24.** Since $1 = 4\sin^2 x + 2(1 - \sin^2 x)$, we require $2\sin^2 x = -1$, an impossibility.

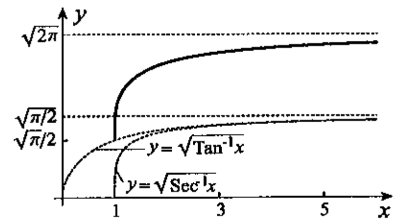
**25.** From this quadratic, $\cos x = \left(3 \pm \sqrt{9-4}\right)/2 = \left(3 \pm \sqrt{5}\right)/2$. We must choose $\cos x = \left(3 - \sqrt{5}\right)/2$. From the solution $x = \text{Cos}^{-1}[(3-\sqrt{5})/2] = 1.179$, all solutions are $x = \pm 1.179 + 2n\pi$, where $n$ is an integer.

**26.** This quadratic equation in $\sin x$ has solutions $\sin x = \left(3 \pm \sqrt{9+20}\right)/2 = \left(3 \pm \sqrt{29}\right)/2$. Because neither of these numbers is between $-1$ and $1$, there are no solutions to the equation.

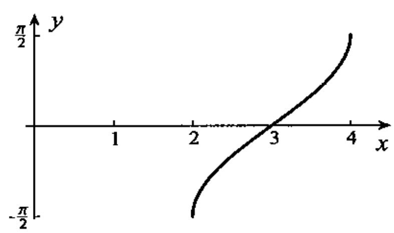
**27.** We square ordinates of $y = \mathrm{Csc}^{-1}x$, and then shift vertically 2 units.



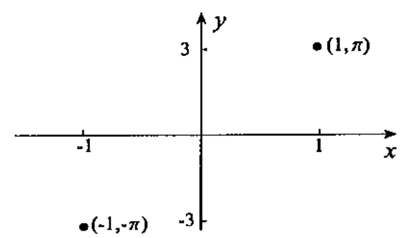

**28.** We add ordinates of $y = \sqrt{\mathrm{Tan}^{-1}x}$ and $y = \sqrt{\mathrm{Sec}^{-1}x}$.




**29.** We shift $y = \mathrm{Sin}^{-1}x$ to the right 3 units.




**30.** Since the domain of $\mathrm{Sin}^{-1}x$ is $|x| \le 1$, and that of $\mathrm{Csc}^{-1}x$ is $|x| \ge 1$, the function is defined only for $x = \pm 1$. Its graph is therefore two points.




**31.** Using equation 1.59, $\tan\phi = 1 \Longrightarrow \phi = \pi/4$ radians.

**32.** Using equation 1.59, $\tan\phi = -1/2 \Longrightarrow \phi = 2.68$ radians.

**33.** Using equation 1.59, $\tan\phi = 3/2 \Longrightarrow \phi = 0.983$ radians.

**34.** Using equation 1.59, $\tan\phi = 3 \Longrightarrow \phi = 1.25$ radians.

**35.** Since the line is vertical, $\phi = \pi/2$ radians.

**36.** Since the line is horizontal, $\phi = 0$.

**37.** Since slopes of the lines are $-1$ and 1, the lines are perpendicular.

**38.** Since slopes of both lines are $-1/3$, the lines are parallel.

**39.** Since slopes of both lines are $1/3$, the lines are parallel.

**40.** Since slopes of the lines are $-2/3$ and $3/2$, the lines are perpendicular.

**41.** Since slopes of the lines are 3 and $-1/2$, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{3+1/2}{1+3(-1/2)}\right| = 1.43 \text{ radians.}$$

**42.** Since slopes of the lines are 1 and $-2/3$, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{1+2/3}{1+(-2/3)}\right| = 1.37 \text{ radians.}$$

**43.** The lines are perpendicular.

**44.** Since slopes of the lines are $-1$ and 3, formula 1.60 gives

$$\theta = \mathrm{Tan}^{-1}\left|\frac{-1-3}{1-3}\right| = 1.11 \text{ radians.}$$

**45.** The sine law applied to triangle $OAB$ gives $(\sin\phi)/l = (\sin\theta)/L \Longrightarrow \sin\phi = (l/L)\sin\theta$. Thus, $\phi = \mathrm{Sin}^{-1}[(l/L)\sin\theta]$.

**46.** Since $0 = \tan^2 x(\sin x + 1) - 3(\sin x + 1) = (\tan^2 x - 3)(\sin x + 1)$, it follows that $\tan x = \pm\sqrt{3}$ or $\sin x = -1$. The solutions of these equations are $x = \pm\pi/3 + n\pi$ and $x = -\pi/2 + 2n\pi$, where $n$ is an integer, but only $x = n\pi \pm \pi/3$ satisfy the original equation.

**47.** If we square the equation, $\sin^2 x + 2\sin x \cos x + \cos^2 x = 1 \implies \sin x \cos x = 0$. Solutions of this equation are $x = n\pi$ and $x = (2n+1)\pi/2$, where $n$ is an integer. But only $x = 2n\pi$ and $x = (4n+1)\pi/2$ satisfy the original equation.

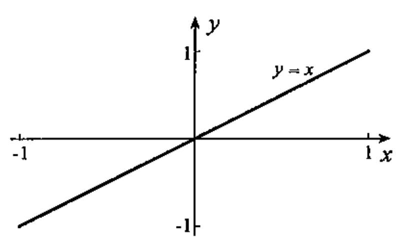
**48.** This equation implies that $\sin x = \pm 3\pi/4 + 2n\pi$, where $n$ is an integer. But for no $n$ are these values between between $\pm 1$. Hence, there are no solutions.

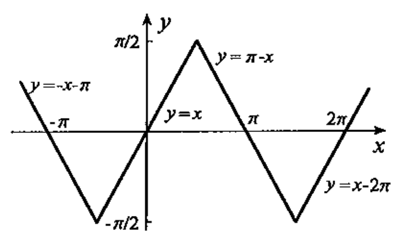
**49.** This equation implies that $\mathrm{Sin}^{-1}x = 2n\pi \pm \pi/3$, where $n$ is an integer. Since values of $\mathrm{Sin}^{-1}x$ must lie between $-\pi/2$ and $\pi/2$, $n$ must be zero. Thus, $x = \pm\sqrt{3}/2$.

**50.** This equation implies that $\tan x = \pm 3\pi/4 + 2n\pi$. Therefore, $x = \mathrm{Tan}^{-1}(2n\pi \pm 3\pi/4) + m\pi$ where $m$ and $n$ are integers.

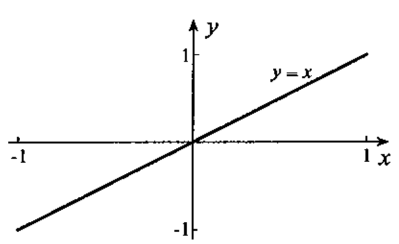
**51.** From the equation, $\tan(x^2 + 4) = \cos(2\pi - 5) = \cos 5$. This implies that $x^2 + 4 = n\pi + \mathrm{Tan}^{-1}(\cos 5)$, from which $x = \pm\sqrt{n\pi + \mathrm{Tan}^{-1}(\cos 5) - 4} = \pm\sqrt{n\pi - 3.724}$, where $n \geq 2$ is an integer.

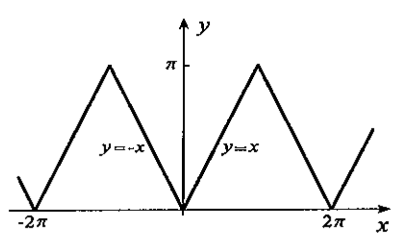
**52.** (a) Since $\mathrm{Sin}^{-1}x$ is defined only for $-1 \leq x \leq 1$, and on this interval, the sine function is the inverse of $\mathrm{Sin}^{-1}x$, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $-\pi/2 \leq x \leq \pi/2$, $\mathrm{Sin}^{-1}x$ is the inverse of $\sin x$ and therefore $f(x) = x$ on this interval. For $\pi/2 \leq x \leq 3\pi/2$, $f(x) = \pi - x$. These define $f(x)$ on the interval $-\pi/2 \leq x \leq 3\pi/2$ of length $2\pi$. Since $\sin x$ is $2\pi$-periodic, so also is $f(x)$, and the graph is shown to the right below.




**53.** (a) Since $\mathrm{Cos}^{-1}x$ is defined only for $-1 \leq x \leq 1$, and on this interval, the cosine function is the inverse of $\mathrm{Cos}^{-1}x$, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $0 \leq x \leq \pi$, $\mathrm{Cos}^{-1}x$ is the inverse of $\cos x$, and therefore $f(x) = x$ on this interval. For $-\pi \leq x \leq 0$, $f(x) = -x$. These define $f(x)$ on the interval $-\pi \leq x \leq \pi$ of length $2\pi$. Since $\cos x$ is $2\pi$-periodic, so also is $f(x)$, and the graph is shown to the right below.




**54.** If we expand $R\sin(2x + \phi)$ according to 1.43a and equate it to $f(x)$, we obtain

$$R\sin 2x\cos\phi + R\cos 2x\sin\phi = 4\sin 2x + \cos 2x.$$

This equation is satisfied for all $x$ if $R$ and $\phi$ satisfy $R\cos\phi = 4$ and $R\sin\phi = 1$. When these are squared and added, the result is $R^2 = 17$. Consequently, $R = \sqrt{17}$, and

$$\cos\phi = \frac{4}{\sqrt{17}}, \qquad \sin\phi = \frac{1}{\sqrt{17}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.245$ radians. Thus, $f(x)$ can be expressed in the form $\sqrt{17}\sin(2x + 0.245)$.

**55.** If we expand $R\cos(3x+\phi)$ according to 1.43c and equate it to $f(x)$, we obtain

$$R\cos 3x\cos\phi - R\sin 3x\sin\phi = -2\sin 3x + 4\cos 3x.$$

This equation is satisfied for all $x$ if $R$ and $\phi$ satisfy $R\cos\phi = 4$ and $R\sin\phi = 2$. When these are squared and added, the result is $R^2 = 20$. Consequently, $R = 2\sqrt{5}$, and

$$\cos\phi = \frac{2}{\sqrt{5}}, \qquad \sin\phi = \frac{1}{\sqrt{5}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.464$ radians. Thus, $f(x)$ can be expressed in the form $2\sqrt{5}\cos(3x + 0.464)$.

**56.** If we expand $R\sin(2x+\phi)$ according to 1.43a and equate it to $f(x)$, we obtain

$$R\sin 2x\cos\phi + R\cos 2x\sin\phi = -2\sin 2x + 4\cos 2x.$$

This equation is satisfied for all $x$ if $R$ and $\phi$ satisfy $R\cos\phi = -2$ and $R\sin\phi = 4$. When these are squared and added, the result is $R^2 = 20$. Consequently, $R = 2\sqrt{5}$, and

$$\cos\phi = \frac{-1}{\sqrt{5}}, \qquad \sin\phi = \frac{2}{\sqrt{5}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 2.03$ radians. Thus, $f(x)$ can be expressed in the form $2\sqrt{5}\sin(2x + 2.03)$.

**57.** If we expand $R\cos(3x+\phi)$ according to 1.43c and equate it to $f(x)$, we obtain

$$R\cos 3x\cos\phi - R\sin 3x\sin\phi = -4\sin 3x + 5\cos 3x.$$

This equation is satisfied for all $x$ if $R$ and $\phi$ satisfy $R\cos\phi = 5$ and $R\sin\phi = 4$. When these are squared and added, the result is $R^2 = 41$. Consequently, $R = \sqrt{41}$, and

$$\cos\phi = \frac{5}{\sqrt{41}}, \qquad \sin\phi = \frac{4}{\sqrt{41}}.$$

The only angle in the range $0 < \phi < \pi$ satisfying these is $\phi = 0.675$ radians. Thus, $f(x)$ can be expressed in the form $\sqrt{41}\cos(3x + 0.675)$.

**58.** We set
$$\begin{aligned} x(t) &= A\sin(\omega t + \phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t) \\ &= 4[\cos\omega t\cos(2\pi/3) - \sin\omega t\sin(2\pi/3)] + 3[\sin\omega t\cos(\pi/3) + \cos\omega t\sin(\pi/3)] \\ &= \left(-2\sqrt{3} + \frac{3}{2}\right)\sin\omega t + \left(-2 + \frac{3\sqrt{3}}{2}\right)\cos\omega t. \end{aligned}$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = \dfrac{3}{2} - 2\sqrt{3}$ and $A\sin\phi = \dfrac{3\sqrt{3}}{2} - 2$. When these are squared and added, the result is

$$A^2 = \left(\frac{3}{2} - 2\sqrt{3}\right)^2 + \left(\frac{3\sqrt{3}}{2} - 2\right)^2 = 25 - 12\sqrt{3} \implies A = \sqrt{25 - 12\sqrt{3}}.$$

Hence, $\cos\phi = \dfrac{3/2 - 2\sqrt{3}}{\sqrt{25 - 12\sqrt{3}}}$ and $\sin\phi = \dfrac{3\sqrt{3}/2 - 2}{\sqrt{25 - 12\sqrt{3}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 2.846$ radians.

**59.** We set
$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t) \\ &= 4[\cos\omega t\cos(2\pi/3) - \sin\omega t\sin(2\pi/3)] + 3[\sin\omega t\cos(\pi/3) + \cos\omega t\sin(\pi/3)] \\ &= \left(-2\sqrt{3} + \frac{3}{2}\right)\sin\omega t + \left(-2 + \frac{3\sqrt{3}}{2}\right)\cos\omega t. \end{aligned}$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = \frac{3\sqrt{3}}{2} - 2$ and $-A\sin\phi = \frac{3}{2} - 2\sqrt{3}$. When these are squared and added, the result is

$$A^2 = \left(\frac{3\sqrt{3}}{2} - 2\right)^2 + \left(\frac{3}{2} - 2\sqrt{3}\right)^2 = 25 - 12\sqrt{3} \implies A = \sqrt{25 - 12\sqrt{3}}.$$

Hence, $\cos\phi = \frac{3\sqrt{3}/2 - 2}{\sqrt{25 - 12\sqrt{3}}}$ and $\sin\phi = \frac{2\sqrt{3} - 3/2}{\sqrt{25 - 12\sqrt{3}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 1.275$ radians.

**60.** We set $x(t) = A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t)$

$$= 2(\sin\omega t\cos 4 + \cos\omega t\sin 4) + 3(\sin\omega t\cos 1 + \cos\omega t\sin 1)$$
$$= (2\cos 4 + 3\cos 1)\sin\omega t + (2\sin 4 + 3\sin 1)\cos\omega t.$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = 2\sin 4 + 3\sin 1$ and $-A\sin\phi = 2\cos 4 + 3\cos 1$. When these are squared and added, the result is

$$A^2 = (2\sin 4 + 3\sin 1)^2 + (2\cos 4 + 3\cos 1)^2 \implies A = \sqrt{13 + 12\cos 3}.$$

Hence, $\cos\phi = \frac{2\sin 4 + 3\sin 1}{\sqrt{13 + 12\cos 3}}$ and $\sin\phi = -\frac{2\cos 4 + 3\cos 1}{\sqrt{13 + 12\cos 3}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = -0.301$ radians.

**61.** We set $x(t) = A\sin(\omega t + \phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t)$

$$= 2(\sin\omega t\cos 4 + \cos\omega t\sin 4) + 3(\sin\omega t\cos 1 + \cos\omega t\sin 1)$$
$$= (2\cos 4 + 3\cos 1)\sin\omega t + (2\sin 4 + 3\sin 1)\cos\omega t.$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = 2\cos 4 + 3\cos 1$ and $A\sin\phi = 2\sin 4 + 3\sin 1$. When these are squared and added, the result is

$$A^2 = (2\cos 4 + 3\cos 1)^2 + (2\sin 4 + 3\sin 1)^2 \implies A = \sqrt{13 + 12\cos 3}.$$

Hence, $\cos\phi = \frac{2\cos 4 + 3\cos 1}{\sqrt{13 + 12\cos 3}}$ and $\sin\phi = \frac{2\sin 4 + 3\sin 1}{\sqrt{13 + 12\cos 3}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 1.270$ radians.

**62.** We set $x(t) = A\sin(\omega t + \phi) = A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) = f(t) + g(t) + h(t)$

$$= 5\sin\omega t + 4[\cos\omega t\cos(\pi/3) - \sin\omega t\sin(\pi/3)] + 2[\sin\omega t\cos(\pi/4) + \cos\omega t\sin(\pi/4)]$$
$$= (5 - 2\sqrt{3} + \sqrt{2})\sin\omega t + (2 + \sqrt{2})\cos\omega t.$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = 5 - 2\sqrt{3} + \sqrt{2}$ and $A\sin\phi = 2 + \sqrt{2}$. When these are squared and added, the result is

$$A^2 = (5 - 2\sqrt{3} + \sqrt{2})^2 + (2 + \sqrt{2})^2 \implies A = \sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}.$$

Hence, $\cos\phi = \frac{5 - 2\sqrt{3} + \sqrt{2}}{\sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}}$ and $\sin\phi = \frac{2 + \sqrt{2}}{\sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = 0.858$ radians.

**63.** We set $x(t) = A\cos(\omega t + \phi) = A(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = f(t) + g(t) + h(t)$

$$= 5\sin\omega t + 4[\cos\omega t\cos(\pi/3) - \sin\omega t\sin(\pi/3)] + 2[\sin\omega t\cos(\pi/4) + \cos\omega t\sin(\pi/4)]$$
$$= (5 - 2\sqrt{3} + \sqrt{2})\sin\omega t + (2 + \sqrt{2})\cos\omega t.$$

This will be true if we choose $A$ and $\phi$ to satisfy $A\cos\phi = 2 + \sqrt{2}$ and $-A\sin\phi = 5 - 2\sqrt{3} + \sqrt{2}$. When these are squared and added, the result is

$$A^2 = (2 + \sqrt{2})^2 + (5 - 2\sqrt{3} + \sqrt{2})^2 \implies A = \sqrt{45 + 14\sqrt{2} - 20\sqrt{3} - 4\sqrt{6}}.$$

Hence, $\cos\phi = \dfrac{2+\sqrt{2}}{\sqrt{45+14\sqrt{2}-20\sqrt{3}-4\sqrt{6}}}$ and $\sin\phi = -\dfrac{5-2\sqrt{3}+\sqrt{2}}{\sqrt{45+14\sqrt{2}-20\sqrt{3}-4\sqrt{6}}}$. The only angle in the interval $-\pi < \phi < \pi$ satisfying these is $\phi = -0.713$ radians.

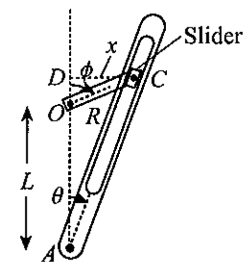
**64.** If we let $x$ be the distance from slider $C$ to line $OA$, then

$x = R\sin\phi$ and $x = (L + R\cos\phi)\tan\theta$.

Equating these gives

$R\sin\phi = (L + R\cos\phi)\tan\theta$,

from which $\tan\theta = \dfrac{R\sin\phi}{L+R\cos\phi}$.




Thus, $\theta = \operatorname{Tan}^{-1}\left(\dfrac{R\sin\phi}{L+R\cos\phi}\right)$.

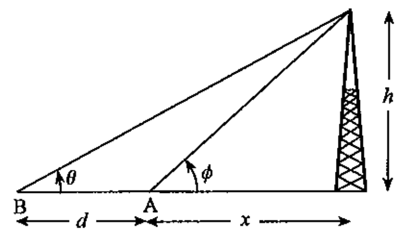
**65.** If $x$ is the distance from $A$ to the foot of the tower, then

$\tan\theta = \dfrac{h}{d+x}$ and $\tan\phi = \dfrac{h}{x}$.

From these,

$x = h\cot\theta - d$ and $x = h\cot\phi$.

Equating these gives




$$h\cot\theta - d = h\cot\phi \quad\Longrightarrow\quad \cot\theta = \cot\phi + \frac{d}{h}.$$

Since $\theta$ is an acute angle, we can write that

$$\theta = \operatorname{Cot}^{-1}\left(\cot\phi + \frac{d}{h}\right).$$

**66.** The equation

$$\theta = \theta_0\cos\omega t + \frac{v_0}{\omega L}\sin\omega t = R\sin(\omega t + \phi) = R\sin\omega t\cos\phi + R\cos\omega t\sin\phi$$

is satisfied if $R$ and $\phi$ satisfy

$$R\cos\phi = \frac{v_0}{\omega L} \qquad\text{and}\qquad R\sin\phi = \theta_0.$$

When these are squared and added, $R^2 = \theta_0^2 + v_0^2/(\omega^2L^2)$. If we choose $R = \sqrt{\theta_0^2 + v_0^2/(\omega^2L^2)}$, then

$$\cos\phi = \frac{v_0}{\omega L\sqrt{\theta_0^2 + v_0^2/(\omega^2L^2)}} = \frac{v_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}, \qquad \sin\phi = \frac{\theta_0}{\sqrt{\theta_0^2 + v_0^2/(\omega^2L^2)}} = \frac{\omega L\theta_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}.$$

Because $v_0 > 0$, it follows that $\cos\phi > 0$, and we may take $-\pi/2 < \phi < \pi/2$. Since $\sin\phi$ has the same sign as $\theta_0$, angle $\phi$ is in $0 < \phi < \pi/2$ when $\theta_0 > 0$, and is in $-\pi/2 < \phi < 0$ when $\theta_0 < 0$. Now

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\omega L\theta_0}{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}\,\frac{\sqrt{v_0^2+\omega^2L^2\theta_0^2}}{v_0} = \frac{\omega L\theta_0}{v_0}.$$

We can write $\phi = \operatorname{Tan}^{-1}(\omega L\theta_0/v_0)$ since principal values are between $-\pi/2$ and 0 when $\theta_0 < 0$, and between 0 and $\pi/2$ when $\theta_0 > 0$.

**67.** When $v_0 < 0$, then $\cos\phi < 0$ and $\phi$ is an angle in the second or third quadrant. If $\theta_0 > 0$, then $\sin\phi > 0$, and $\phi$ is in the first or second quadrant. Hence, $\phi$ must be in the second quadrant. Since $\tan\phi < 0$, the formula for $\phi$ is $\phi = \pi + \operatorname{Tan}^{-1}(\omega L\theta_0/v_0)$. On the other hand, if $\theta_0 < 0$, then $\sin\phi < 0$, and $\phi$ is in the third or fourth quadrant. Hence, $\phi$ must be in the third quadrant. Since $\tan\phi > 0$, the formula for $\phi$ is $\phi = -\pi + \operatorname{Tan}^{-1}(\omega L\theta_0/v_0)$.

**68.** The equation

$$y = y_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t = R\sin(\omega t + \phi) = R\sin\omega t\cos\phi + R\cos\omega t\sin\phi$$

is satisfied if $R$ and $\phi$ satisfy

$$R\cos\phi = \frac{v_0}{\omega}, \qquad R\sin\phi = y_0.$$

When these are squared and added, $R^2 = y_0^2 + v_0^2/\omega^2$. If we choose $R = \sqrt{y_0^2 + v_0^2/\omega^2}$, then

$$\cos\phi = \frac{v_0}{\omega\sqrt{y_0^2 + v_0^2/\omega^2}} = \frac{v_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}, \qquad \sin\phi = \frac{y_0}{\sqrt{y_0^2 + v_0^2/\omega^2}} = \frac{\omega y_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}.$$

Because $v_0 > 0$, it follows that $\cos\phi > 0$, and we may take $-\pi/2 < \phi < \pi/2$. Since $\sin\phi$ has the same sign as $y_0$, angle $\phi$ is in $0 < \phi < \pi/2$ when $y_0 > 0$, and is in $-\pi/2 < \phi < 0$ when $y_0 < 0$. Now

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\omega y_0}{\sqrt{v_0^2 + \omega^2 y_0^2}}\,\frac{\sqrt{v_0^2 + \omega^2 y_0^2}}{v_0} = \frac{\omega y_0}{v_0}.$$

We can write $\phi = \text{Tan}^{-1}(\omega y_0/v_0)$ since principal values are between $-\pi/2$ and 0 when $y_0 < 0$, and between 0 and $\pi/2$ when $y_0 > 0$.

**69.** When $v_0 < 0$, then $\cos\phi < 0$ and $\phi$ is an angle in the second or third quadrant. If $y_0 > 0$, then $\sin\phi > 0$, and $\phi$ is in the first or second quadrant. Hence, $\phi$ must be in the second quadrant. Since $\tan\phi < 0$, the formula for $\phi$ is $\phi = \pi + \text{Tan}^{-1}(\omega y_0/v_0)$. On the other hand, if $y_0 < 0$, then $\sin\phi < 0$, and $\phi$ is in the third or fourth quadrant. Hence, $\phi$ must be in the third quadrant. Since $\tan\phi > 0$, the formula for $\phi$ is $\phi = -\pi + \text{Tan}^{-1}(\omega y_0/v_0)$.

**70.** If we expand $A\cos(\omega t - \phi)$ and equate it to $f(x)$, we obtain

$$A\cos\omega t\cos\phi + A\sin\omega t\sin\phi = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\left[R\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t\right].$$

This equation is satisfied for all $t$ if $A$ and $\phi$ satisfy

$$A\cos\phi = \frac{E_0 R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad A\sin\phi = \frac{E_0\left(\omega L - \frac{1}{\omega C}\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

When these are squared and added, the result is $A^2 = E_0^2$. Consequently, $A = E_0$, and

$$\cos\phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}, \qquad \sin\phi = \frac{\omega L - \frac{1}{\omega C}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Because $R > 0$, it follows that $\cos\phi > 0$, and this is consistent with the demand that $-\pi/2 < \phi < \pi/2$. We could use the equation in $\sin\phi$ to define $\phi$, or we can also write that $\tan\phi = [\omega L - 1/(\omega C)]/R$ so that $\phi = \text{Tan}^{-1}\{[\omega L - 1/(\omega C)]/R\}$.

**71.** If $5\cos\omega t = A(\cos\omega t\cos\pi/6 + \sin\omega t\sin\pi/6) + 5(\cos\omega t\cos\phi - \sin\omega t\sin\phi)$, then

$$5 = \frac{\sqrt{3}A}{2} + 5\cos\phi \qquad \text{and} \qquad 0 = \frac{A}{2} - 5\sin\phi.$$

These imply that $(5\cos\phi)^2 + (5\sin\phi)^2 = (5 - \sqrt{3}A/2)^2 + (A/2)^2$, from which $25 = 25 - 5\sqrt{3}A + 3A^2/4 + A^2/4 \Longrightarrow A = 0$ or $A = 5\sqrt{3}$. Since $A$ must be positive, we choose $A = 5\sqrt{3}$, in which case

$$\cos\phi = -\frac{1}{2} \qquad \text{and} \qquad \sin\phi = \frac{\sqrt{3}}{2}.$$

These require $\phi = 2\pi/3 + 2n\pi$, where $n$ is an integer.

**72.** If $5\cos\omega t = A(\cos\omega t\cos 1 - \sin\omega t\sin 1) + 5(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$, then

$$5 = A\cos 1 + 5\sin\phi \qquad \text{and} \qquad 0 = -A\sin 1 + 5\cos\phi.$$

These imply that $(5\sin\phi)^2 + (5\cos\phi)^2 = (5 - A\cos 1)^2 + (A\sin 1)^2$, from which $25 = 25 - 10A\cos 1 + A^2 \Longrightarrow A = 0$ or $A = 10\cos 1$. Since $A$ must be positive, we choose $A = 10\cos 1$, in which case

$$\sin\phi = 1 - 2\cos^2 1 = -\cos 2 \qquad \text{and} \qquad \cos\phi = 2\sin 1\cos 1 = \sin 2.$$

From the second of these, we may write $\cos\phi = \cos(\pi/2 - 2)$. We conclude that $\phi = \pm(\pi/2 - 2) + 2n\pi$, where $n$ is an integer. But, $\sin(\pi/2 - 2 + 2n\pi) = \sin(\pi/2 - 2) = \cos 2$, which is not true. Hence, we must take $\phi = -(\pi/2 - 2) + 2n\pi = 2 + (4n - 1)\pi/2$.

**73.** When $x \geq 1$, we set $y = \mathrm{Csc}^{-1}x$, in which case $0 < y \leq \pi/2$. It follows that $x = \csc y$, and

$$\frac{1}{x} = \frac{1}{\csc y} = \sin y.$$

If we apply the inverse sine function to both sides of this equation, the result is

$$\mathrm{Sin}^{-1}\left(\frac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y) = y,$$

because $y$ is in the principal value range of the inverse sine function. Hence, when $x \geq 1$,

$$\mathrm{Csc}^{-1}x = \mathrm{Sin}^{-1}\left(\frac{1}{x}\right).$$

When $x \leq -1$, we again set $y = \mathrm{Csc}^{-1}x$, and obtain $\mathrm{Sin}^{-1}\left(\frac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y)$.

But in this case the right side is not equal to $y$, because $-\pi < y \leq -\pi/2$. To remedy this, we note that when $-\pi < y \leq -\pi/2$, we may write $\sin y = \sin(-\pi - y)$. Since $-\pi - y$ is in the principal range for the inverse sine function $(-\pi/2 \leq -\pi - y < 0)$, it follows that

$$\mathrm{Sin}^{-1}\left(\frac{1}{x}\right) = \mathrm{Sin}^{-1}(\sin y) = \mathrm{Sin}^{-1}[\sin(-\pi - y)] = -\pi - y = -\pi - \mathrm{Csc}^{-1}x.$$

Thus, $\mathrm{Csc}^{-1}x = -\pi - \mathrm{Sin}^{-1}(1/x)$.

**74.** When $0 \leq x \leq 1$, we set $y = \mathrm{Sin}^{-1}x$, in which case $0 \leq y \leq \pi/2$. It follows that $x = \sin y$, and because $\sin^2 y + \cos^2 y = 1$, we have

$$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}.$$

Since $y$ is an angle in the first quadrant, its cosine must be nonnegative, and therefore $\cos y = \sqrt{1 - x^2}$. When we apply the inverse cosine function to both sides of this equation, we obtain $\mathrm{Cos}^{-1}(\cos y) = y = \mathrm{Cos}^{-1}\sqrt{1 - x^2}$. When $-1 \leq x < 0$, we continue to set $y = \mathrm{Sin}^{-1}x$, and once again obtain $\cos y = \sqrt{1 - x^2}$, because $-\pi/2 \leq y < 0$. Application of the inverse cosine function gives $\mathrm{Cos}^{-1}(\cos y) = \mathrm{Cos}^{-1}\sqrt{1 - x^2}$, but the left side is not equal to $y$ because $y$ is not in the principal value range of the inverse cosine function. This is easily adjusted by noting that with $-\pi/2 \leq y < 0$, we have $\cos y = \cos(-y)$. Hence,

$$\mathrm{Cos}^{-1}(\cos y) = \mathrm{Cos}^{-1}[\cos(-y)] = -y = \mathrm{Cos}^{-1}\sqrt{1 - x^2};$$

that is, $y = -\mathrm{Cos}^{-1}\sqrt{1 - x^2}$.

**75.** When $x \geq 1$, we set $y = \mathrm{Sec}^{-1}x$, in which case $0 \leq y < \pi/2$. It follows that $x = \sec y$, and

$$\frac{1}{x} = \frac{1}{\sec y} = \cos y.$$

If we apply the inverse cosine function to both sides of this equation, the result is

$$\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y) = y,$$

because $y$ is in the principal value range of the inverse cosine function. Hence, when $x \geq 1$,

$$\text{Sec}^{-1}x = \text{Cos}^{-1}\left(\frac{1}{x}\right).$$

When $x \leq -1$, we again set $y = \text{Sec}^{-1}x$, and obtain $\quad \text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y)$.
But in this case the right side is not equal to $y$, because $-\pi \leq y < -\pi/2$. To remedy this, we note that when $-\pi \leq y < -\pi/2$, we may write $\cos y = \cos(-y)$. Since $-y$ is in the principal range for the inverse cosine function ($\pi/2 < -y \leq \pi$), it follows that

$$\text{Cos}^{-1}\left(\frac{1}{x}\right) = \text{Cos}^{-1}(\cos y) = \text{Cos}^{-1}[\cos(-y)] = -y = -\text{Sec}^{-1}x;$$

that is, $\text{Sec}^{-1}x = -\text{Cos}^{-1}(1/x)$.

**76.** When $x > 0$, we set $y = \text{Cot}^{-1}x$, in which case $0 < y < \pi/2$. It follows that $x = \cot y$, and

$$\frac{1}{x} = \frac{1}{\cot y} = \tan y.$$

If we apply the inverse tangent function to both sides of this equation, the result is

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y) = y,$$

because $y$ is in the principal value range of the inverse tangent function. Hence, when $x > 0$, we can say that $\text{Cot}^{-1}x = \text{Tan}^{-1}(1/x)$.

When $x < 0$, we again set $y = \text{Cot}^{-1}x$, and obtain

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y).$$

But in this case the right side is not equal to $y$, because $\pi/2 < y < \pi$. To remedy this, we note that $\tan y = \tan(y - \pi)$, and when $\pi/2 < y < \pi$, $y - \pi$ is in the principal value range for the inverse tangent function. It follows that

$$\text{Tan}^{-1}\left(\frac{1}{x}\right) = \text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}[\tan(y - \pi)] = y - \pi = \text{Cot}^{-1}x - \pi.$$

Thus, $\text{Cot}^{-1}x = \pi + \text{Tan}^{-1}(1/x)$.

**77.** When $x \geq 1$, we set $y = \text{Sec}^{-1}x$, in which case $0 \leq y < \pi/2$. It follows that $x = \sec y$, and because $1 + \tan^2 y = \sec^2 y$, we have $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$. Since $y$ is an angle in the first quadrant, its tangent is positive, and therefore $\tan y = \sqrt{x^2 - 1}$. When we apply the inverse tangent function to both sides of this equation, we obtain $\text{Tan}^{-1}(\tan y) = y = \text{Tan}^{-1}\sqrt{x^2 - 1}$; that is, $\text{Sec}^{-1}x = \text{Tan}^{-1}\sqrt{x^2 - 1}$.

When $x \leq -1$, we again set $y = \text{Sec}^{-1}x$, and obtain $\tan y = \sqrt{x^2 - 1}$, because $-\pi \leq y < -\pi/2$. Application of the inverse tangent function gives $\text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}\sqrt{x^2 - 1}$, but the left side is not equal to $y$ because $y$ is not in the principal value range of the inverse tangent function. This is easily adjusted by noting that $\tan y = \tan(\pi + y)$, and when $-\pi \leq y < -\pi/2$, $\pi + y$ is in the principal value range of the inverse tangent function. Hence $\text{Tan}^{-1}(\tan y) = \text{Tan}^{-1}[\tan(\pi + y)] = \pi + y = \text{Tan}^{-1}\sqrt{x^2 - 1}$; that is, $y = -\pi + \text{Tan}^{-1}\sqrt{x^2 - 1}$.

**78.** When $x \geq 1$, we set $y = \text{Csc}^{-1}x$, in which case $0 < y \leq \pi/2$. It follows that $x = \csc y$, and because $1 + \cot^2 y = \csc^2 y$, we have $\cot y = \pm\sqrt{\csc^2 y - 1} = \pm\sqrt{x^2 - 1}$. Since $y$ is an angle in the first quadrant, its cotangent is positive, and therefore $\cot y = \sqrt{x^2 - 1}$. When we apply the inverse cotangent function to both sides of this equation, we obtain $\mathrm{Cot}^{-1}(\cot y) = y = \mathrm{Cot}^{-1}\sqrt{x^2-1}$; that is, $\mathrm{Csc}^{-1}x = \mathrm{Cot}^{-1}\sqrt{x^2-1}$.

When $x \le -1$, we again set $y = \mathrm{Csc}^{-1}x$, and obtain $\cot y = \sqrt{x^2-1}$, because $-\pi < y \le -\pi/2$. Application of the inverse cotangent function gives $\mathrm{Cot}^{-1}(\cot y) = \mathrm{Cot}^{-1}\sqrt{x^2-1}$, but the left side is not equal to $y$ because $y$ is not in the principal value range of the inverse cotangent function. This is easily adjusted by noting that $\cot y = \cot(\pi + y)$, and when $-\pi < y \le -\pi/2$, $\pi + y$ is in the principal value range of the inverse cotangent function. Hence $\mathrm{Cot}^{-1}(\cot y) = \mathrm{Cot}^{-1}[\cot(\pi+y)] = \pi + y = \mathrm{Cot}^{-1}\sqrt{x^2-1}$; that is, $y = -\pi + \mathrm{Cot}^{-1}\sqrt{x^2-1}$.

**79.** If we set $y = 2\,\mathrm{Tan}^{-1}\sqrt{\dfrac{1+x}{1-x}}$, then $\dfrac{1+x}{1-x} = \tan^2(y/2)$. When we solve this equation for $x$, the result is

$$x = \frac{\tan^2(y/2)-1}{\tan^2(y/2)+1} = \frac{\tan^2(y/2)-1}{\sec^2(y/2)} = \sin^2(y/2) - \cos^2(y/2) = -\cos y.$$

Because $\cos y = -x$, it follows that $y = \mathrm{Cos}^{-1}(-x) = \pi - \mathrm{Cos}^{-1}x$, and the proof is complete.

## EXERCISES 1.9

**1.** If $\log_{10}(2+x) = -1$, then $2 + x = 10^{-1} \Longrightarrow x = -2 + 1/10 = -19/10$.

**2.** If $10^{3x} = 5$, then $3x = \log_{10} 5 \Longrightarrow x = (1/3)\log_{10} 5$.

**3.** If $\log_{10}(x^2+2x+1) = 1$, then $(x+1)^2 = 10^1 \Longrightarrow x = -1 \pm \sqrt{10}$.

**4.** If $\ln(x^2+2x+10) = 1$, then $x^2 + 2x + 10 = e^1 \Longrightarrow x = \left[-2 \pm \sqrt{4 - 4(10-e)}\right]/2$. These are not real.

**5.** If $10^{5-x^2} = 100 = 10^2$, then $5 - x^2 = 2 \Longrightarrow x = \pm\sqrt{3}$.

**6.** If $10^{1-x^2} = 100 = 10^2$, then $1 - x^2 = 2$. This equation does not have real solutions.

**7.** We write $1 = \log_{10}[(x-3)x]$, and take exponentials, $10 = (x-3)x \Longrightarrow 0 = x^2 - 3x - 10 = (x-5)(x+2) \Longrightarrow x = 5, -2$. Only $x = 5$ satisfies the original equation.

**8.** We write $1 = \log_{10}[(3-x)x]$, and take exponentials, $10 = (3-x)x \Longrightarrow 0 = x^2 - 3x + 10$. This equation has no real solutions.

**9.** If we take exponentials, $x(x-3) = 10 \Longrightarrow 0 = x^2 - 3x - 10 = (x-5)(x+2) \Longrightarrow x = 5, -2$.

**10.** We write $\log_{10}[x^2(x-1)] = 2 \Longrightarrow x^2(x-1) = 10^2 = 100 \Longrightarrow 0 = x^3 - x^2 - 100 = (x-5)(x^2+4x+20) \Longrightarrow x = 5$.

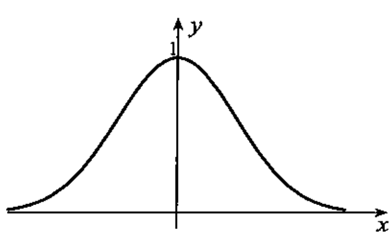
**11.** We write $\log_a[x(x+2)] = 2 \Longrightarrow x(x+2) = a^2 \Longrightarrow x^2 + 2x - a^2 = 0$. Solutions of this quadratic equation are $x = (-2 \pm \sqrt{4+4a^2})/2 = -1 \pm \sqrt{1+a^2}$. Since $x$ must be positive, only $x = -1 + \sqrt{1+a^2}$ is acceptable.

**12.** We take exponentials to obtain $x(x+2) = a^2 \Longrightarrow x^2 + 2x - a^2 = 0$. Solutions of this quadratic equation are $x = (-2 \pm \sqrt{4+4a^2})/2 = -1 \pm \sqrt{1+a^2}$.

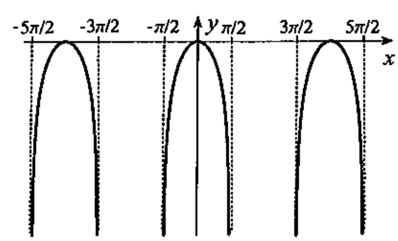
**13.** Taking exponentials gives $\log_{10}\left(\dfrac{x+3}{200x}\right) + 4 = 10^{-1} = 1/10$. Taking exponentials again gives

$$\frac{x+3}{200x} = 10^{-39/10} \Longrightarrow x + 3 = 200(10^{-39/10})x \Longrightarrow x = \frac{3}{200(10^{-39/10}) - 1}.$$

**14.**




**15.**



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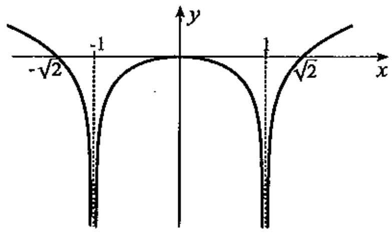


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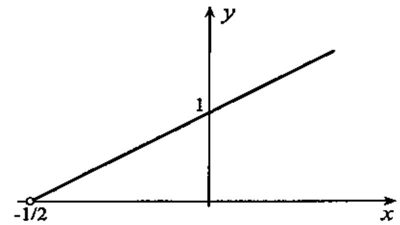


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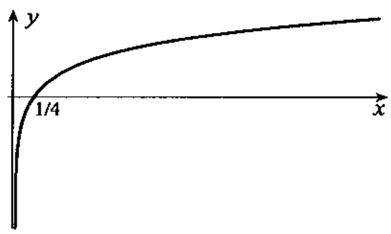


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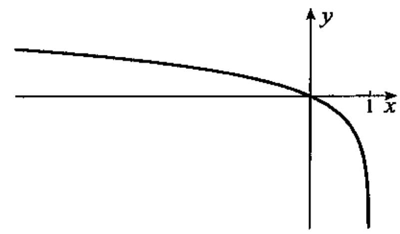


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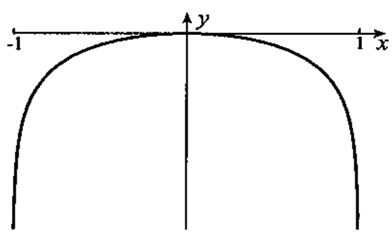


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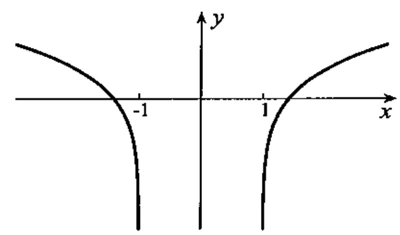


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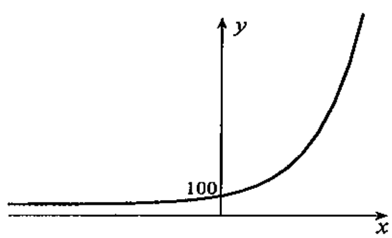


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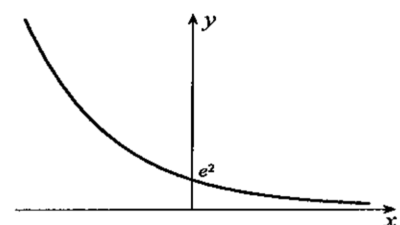


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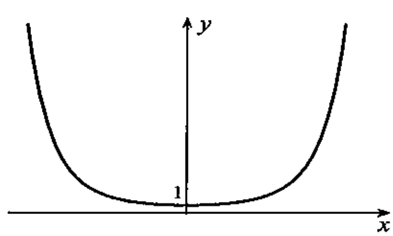


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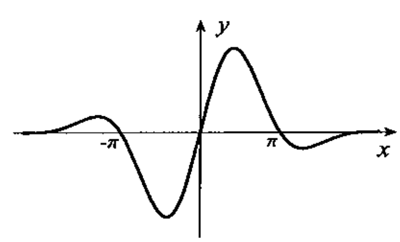


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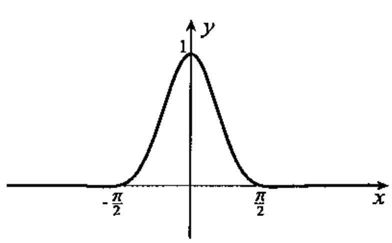


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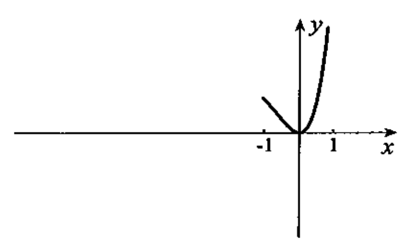
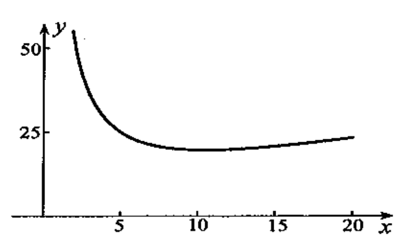



28. The graph shows that the minimum is between 10 and 11. Values $y(10) = 19.562$ and $y(11) = 19.557$ indicate that the minimum occurs for $x = 11$.



**29.** $f(-x) = \dfrac{1 - e^{-1/(-x)}}{1 + e^{-1/(-x)}} = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$. If we divide numerator and denominator by $e^{1/x}$, we obtain $f(-x) = \dfrac{e^{-1/x} - 1}{e^{-1/x} + 1} = -\left(\dfrac{1 - e^{-1/x}}{1 + e^{-1/x}}\right) = -f(x)$. Thus, $f(x)$ is an odd function.

**30.** The graph of $f(x)$ is symmetric about the $y$-axis, whereas the graph of $g(x)$ exists only for $x > 0$. They are identical to the right of the $y$-axis.

**31.** (a) The amount after the first year is $A_0(1.035)$. The amount after the second year is $A_0(1.035)^2$. Continuation leads to the formula $A_0(1.035)^t$ for the amount of timber after $t$ years.
(b) Timber doubles when $2A_0 = A_0(1.035)^t$. If we divide by $A_0$ and take logarithms to any base, say 10, $\log_{10} 2 = t \log_{10} 1.035 \implies t = \log_{10} 2/\log_{10} 1.035 = 20.1$ years.

**32.** After the first year its value is $20\,000(3/4)$. After two years, the value is $20\,000(3/4)^2$, and after $t$ years, it is $20\,000(3/4)^t$.

**33.** If we take $56^{\text{th}}$ roots of both sides of the equation,

$$10^{-6/56} = 1 - 2.08 \times 10^{-6}y \implies y = \frac{1 - 10^{-3/28}}{2.08 \times 10^{-6}} = 1.05 \times 10^5.$$

**34.** If $y$ is the logarithm of $x$ to base $a$, $y = \log_a x$, then $x = a^y$. It follows that $x = (1/a)^{-y}$, and this implies that $-y$ is the logarithm of $x$ to base $1/a$.

**35.** If we set $z = \log_a(x_1/x_2)$, then $a^z = \dfrac{x_1}{x_2} = \dfrac{a^{\log_a x_1}}{a^{\log_a x_2}} = a^{\log_a x_1 - \log_a x_2}$. Thus, $\log_a x_1 - \log_a x_2 = z = \log_a(x_1/x_2)$. If we set $z = \log_a x_1^{x_2}$, then $a^z = x_1^{x_2} = (a^{\log_a x_1})^{x_2} = a^{x_2 \log_a x_1}$. Thus, $x_2 \log_a x_1 = z = \log_a x_1^{x_2}$.

**36.** No. Both $x_1$ and $x_2$ must be positive.

**37.** (a) If we exponentiate both sides of $R = \log_{10}(I/I_0)$, we obtain $I/I_0 = 10^R \implies I = I_0\,10^R$.
(b) Richter scale readings are $\log_{10}(1.20 \times 10^6) = 6.08$ and $\log_{10}(6.20 \times 10^4) = 4.79$.

**38.** (a) After one interest period the accumulated value is $P[1 + i/(100n)]$. After two interest periods, it is $P[1 + i/(100n)]^2$. Continuing, the accumulated value after $t$ years, or $nt$ interest periods is $P[1 + i/(100n)]^{nt}$.
(b) When $A = 2P$, $i = 8$ and $n = 2$,

$$2P = P\left(1 + \frac{8}{200}\right)^{2t} = P\left(\frac{26}{25}\right)^{2t} \implies 2t \log_{10}(26/25) = \log_{10} 2 \implies t = \frac{\log_{10} 2}{2\log_{10}(26/25)} = 8.84.$$

Thus, money doubles in 9 years.
(c) If we write that $A = P\left(1 + \dfrac{i}{100n}\right)^{nt} = P\left[\left(1 + \dfrac{i}{100n}\right)^{100n/i}\right]^{it/100}$, and note that $\left(1 + \dfrac{i}{100n}\right)^{100n/i}$ gets closer and closer to $e$ as $n$ gets larger and larger, we conclude that as $n$ gets larger and larger, $A$ approaches $P\,e^{it/100}$.
(d) The accumulated value is $P = 1000\,e^{6(10)/100} = 1822.12$. For the accumulated value at 6% compounded only once each year, we obtain $1000(1.06)^{10} = 1790.85$.

**39.** (a) If the voltage is $V_0/e$ at time $\tau$, then

$$\frac{V_0}{e} = V_0 e^{-\tau/(RC)}.$$

Division by $V_0$ and logarithms give

$$-1 = -\frac{\tau}{RC} \implies \tau = RC.$$

(b) The voltage at time $t + \tau$ is $V_0 e^{-(t+\tau)/(RC)} = V_0 e^{-t/(RC)} e^{-\tau/(RC)} = V e^{-\tau/\tau} = V/e$.

**40.** (a) If the current is $i_0/e$ at time $\tau$, then

$$\frac{i_0}{e} = i_0 e^{-R\tau/L}.$$

Division by $i_0$ and logarithms give

$$-1 = -\frac{R\tau}{L} \implies \tau = \frac{L}{R}.$$

(b) The current at time $t+\tau$ is $i_0 e^{-R(t+\tau)/L} = i_0 e^{-Rt/L} e^{-R\tau/L} = ie^{-\tau/\tau} = i/e$.

**41.** If we multiply by $a^{2x}$, we obtain a quadratic equation in $a^{2x}$, namely, $0 = 3(a^{2x})^2 - 10(a^{2x}) + 3 = (a^{2x} - 3)(3a^{2x} - 1)$. Consequently, $a^{2x} = 3$ or $a^{2x} = 1/3$, and these give $x = \pm(1/2)\log_a 3$.

**42.** If we write $2^x + 2^{2x} = 2^{3x}$, and divide by $2^x$, then

$$1 + 2^x = 2^{2x} \implies (2^x)^2 - 2^x - 1 = 0 \implies 2^x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $2^x$ must be positive, we choose only $(1+\sqrt{5})/2$, and take logarithms,

$$x \log_{10} 2 = \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \implies x = \frac{\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)}{\log_{10} 2} = 0.694.$$

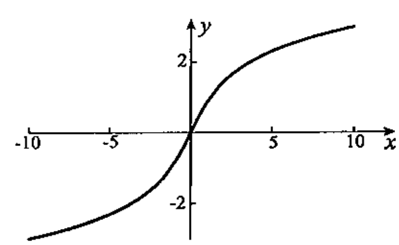
**43.** If we take logarithms to base 10,

$$(x+4)\log_{10} 3 = (x-1)\log_{10} 7 \implies x = \frac{4\log_{10} 3 + \log_{10} 7}{\log_{10} 7 - \log_{10} 3} = 7.48.$$

**44.** If we set $y = \log_x 2$, then $2 = x^y$. But, then $y = \log_{2x} 8$ also, and this implies that $8 = (2x)^y = 2^y x^y = 2^y(2) = 2^{y+1}$. Consequently, $y + 1 = 3$ or $y = 2$. Thus, $2 = x^2$, and since $x$ must be positive, $x = \sqrt{2}$.

**45.** Since

$$\begin{aligned} f(-x) &= \ln(-x + \sqrt{(-x)^2+1}) \\ &= \ln\left[\left(\sqrt{x^2+1} - x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right] \\ &= \ln\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right) \\ &= -\ln(\sqrt{x^2+1}+x) = -f(x), \end{aligned}$$




the function is odd. A plot is shown to the right.

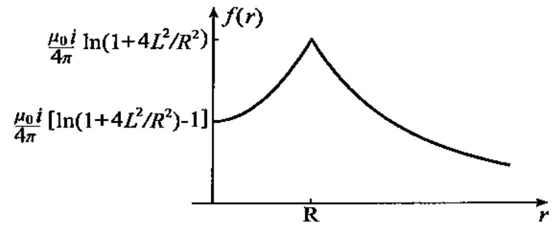
**46.** Repair costs for the second year are $50(1.2)$, for the third year, $50(1.2)^2$, and so on. If $R(t)$ represents repair costs for $t$ years, then $R(t) = 50 + 50(1.2) + \cdots + 50(1.2)^{t-1}$. If we multiply this by 1.2, then $1.2R(t) = 50(1.2) + 50(1.2)^2 + \cdots + 50(1.2)^t$, and

$$1.2R(t) - R(t) = 50(1.2)^t - 50 \implies R(t) = \frac{50(1.2)^t - 50}{1.2 - 1} = 250[(1.2)^t - 1].$$

Thus, the average yearly cost asociated with owning the car for $t$ years is

$$C(t) = \frac{1}{t}\left\{20\,000\left[1 - \left(\frac{3}{4}\right)^t\right] + 250\left[\left(\frac{6}{5}\right)^t - 1\right]\right\}.$$

**47.** (a) For $0 \le r \le R$, the graph is a parabola. For $r > R$, the graph decreases as $r$ increases, and gets closer and closer to the $r$-axis.




(b) For $f(0) = f(r)$,

$$\frac{\mu_0 i}{4\pi}\left[\ln\left(1+\frac{4L^2}{R^2}\right)-1\right] = \frac{\mu_0 i}{4\pi}\ln\left(1+\frac{4L^2}{r^2}\right).$$

This implies that

$$\frac{1}{e}\left(1+\frac{4L^2}{R^2}\right) = 1+\frac{4L^2}{r^2} \implies \frac{4L^2}{r^2} = \frac{1}{e}\left(1+\frac{4L^2}{R^2}\right)-1 \implies r = \frac{2LR}{\sqrt{(R^2+4L^2)/e-R^2}}.$$

**48.** If we multiply by $e^{2x}$,

$$(e^{2x})^2 - 2y(e^{2x}) - 1 = 0 \implies e^{2x} = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}.$$

Since $e^{2x}$ must be positive, $e^{2x} = y + \sqrt{y^2+1} \implies x = (1/2)\ln(y+\sqrt{y^2+1})$.

**49.** If we multiply by $e^x$, $\quad (e^x)^2 - 2y(e^x) + 1 = 0 \implies e^x = \dfrac{2y \pm \sqrt{4y^2-4}}{2} = y \pm \sqrt{y^2-1}$.

Thus, $x = \ln(y \pm \sqrt{y^2-1})$.

**50.** If we cross multiply,

$$y(e^x+e^{-x}) = e^x - e^{-x} \implies e^x(1-y) = e^{-x}(1+y) \implies e^{2x} = \frac{1+y}{1-y} \implies x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right).$$

## EXERCISES 1.10

**1.** $3\cosh 1 = \dfrac{3}{2}(e+e^{-1}) = 4.63$

**2.** $\sinh(\pi/2) = \dfrac{e^{\pi/2}-e^{-\pi/2}}{2} = 2.30$

**3.** $\tanh\sqrt{1-\sin 3} = \dfrac{e^{\sqrt{1-\sin 3}} - e^{-\sqrt{1-\sin 3}}}{e^{\sqrt{1-\sin 3}} + e^{-\sqrt{1-\sin 3}}} = 0.729$

**4.** $\mathrm{Sin}^{-1}(\mathrm{sech}10) = \mathrm{Sin}^{-1}\left(\dfrac{2}{e^{10}+e^{-10}}\right) = 9.08 \times 10^{-5}$

**5.** Since $2\,\mathrm{csch}1 = \dfrac{4}{e-e^{-1}} = 1.70 > 1$, there is no value for $\mathrm{Cos}^{-1}(2\,\mathrm{csch}1)$.

**6.** $\coth(\sinh 5) = \dfrac{e^{\sinh 5}+e^{-\sinh 5}}{e^{\sinh 5}-e^{-\sinh 5}} = 1.00$

**7.** $\sqrt{\ln|\sinh(-3)|} = \sqrt{\ln|(e^{-3}-e^{3})/2|} = 1.52$

**8.** $\mathrm{sech}[\sec(\pi/3)] = \mathrm{sech}2 = \dfrac{2}{e^2+e^{-2}} = 0.266$

**9.** $e^{-2\cosh e} = e^{-(e^e+e^{-e})} = 2.45 \times 10^{-7}$

**10.** $\sinh[\mathrm{Cot}^{-1}(-3\pi/10)] = \sinh 2.3266 = \dfrac{e^{2.3266}-e^{-2.3266}}{2} = 5.07$

**11.** We verify representatives of these identities:

$$\tanh(A+B) = \frac{\sinh(A+B)}{\cosh(A+B)} = \frac{\sinh A\cosh B + \cosh A\sinh B}{\cosh A\cosh B + \sinh A\sinh B}$$

$$= \frac{\dfrac{\sinh A\cosh B}{\cosh A\cosh B} + \dfrac{\cosh A\sinh B}{\cosh A\cosh B}}{\dfrac{\cosh A\cosh B}{\cosh A\cosh B} + \dfrac{\sinh A\sinh B}{\cosh A\cosh B}} = \frac{\tanh A + \tanh B}{1+\tanh A\tanh B}$$

$$\sinh 2A = \frac{e^{2A}-e^{-2A}}{2} = 2\left(\frac{e^A-e^{-A}}{2}\right)\left(\frac{e^A+e^{-A}}{2}\right) = 2\sinh A\cosh A$$

$$\cosh 2A = \frac{e^{2A}+e^{-2A}}{2} = \left(\frac{e^A+e^{-A}}{2}\right)^2 + \left(\frac{e^A-e^{-A}}{2}\right)^2 = \cosh^2 A + \sinh^2 A$$

Adding the two equations in 1.77a gives $\sinh(A+B)+\sinh(A-B) = 2\sinh A\cosh B$, and this is 1.77j. If we set $X = A+B$ and $Y = A-B$, then $A = (X+Y)/2$ and $B = (X-Y)/2$. Substitution of these into 1.77j gives

$$\sinh\left(\frac{X+Y}{2}\right)\cosh\left(\frac{X-Y}{2}\right) = \frac{1}{2}\sinh X + \frac{1}{2}\sinh Y.$$

This is 1.77l with $X$ and $Y$ replacing $A$ and $B$.

**12.** If we write the equations in the form

$$A(\cos kL - \cosh kL) = -B(\sin kL - \sinh kL), \quad A(\cos kL + \cosh kL) = -B(\sin kL + \sinh kL),$$

and divide one by the other,

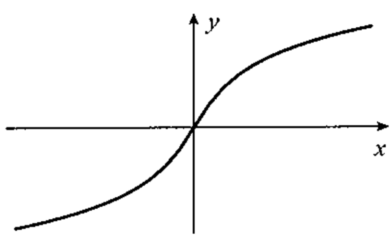
$$\frac{\cos kL - \cosh kL}{\cos kL + \cosh kL} = \frac{\sin kL - \sinh kL}{\sin kL + \sinh kL}.$$

Hence,

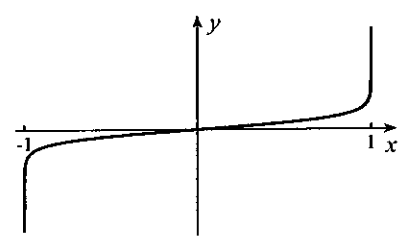
$$(\cos kL - \cosh kL)(\sin kL + \sinh kL) = (\cos kL + \cosh kL)(\sin kL - \sinh kL)$$

or, $2\cos kL \sinh kL = 2\sin kL \cosh kL$. Division by $2\cos kL\cosh kL$ gives $\tanh kL = \tan kL$.

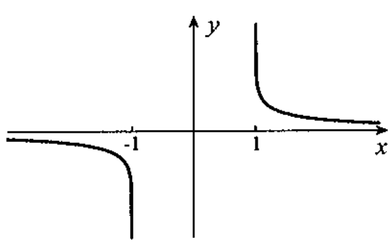
**13.** (a) $f(x) = \text{Sinh}^{-1}x$



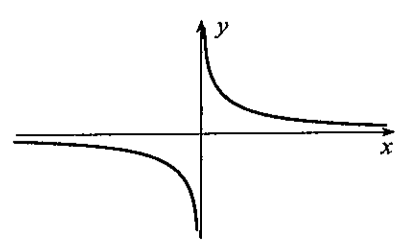

$f(x) = \text{Tanh}^{-1}x$




$f(x) = \text{Coth}^{-1}x$

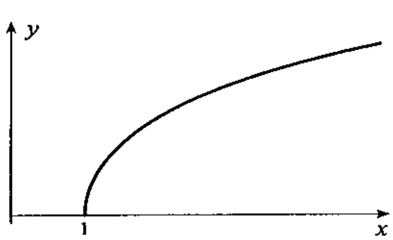



$f(x) = \text{Csch}^{-1}x$

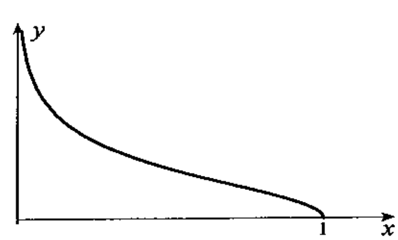



(b) These curves do not pass the horizontal line test for existence of inverse functions.

$f(x) = \text{Cosh}^{-1}x$




$f(x) = \text{Sech}^{-1}x$




(c) If we set $y = \text{Sinh}^{-1}x$, then $x = \sinh y = (e^y - e^{-y})/2$. Multiplication by $e^y$ gives

$$(e^y)^2 - 2x(e^y) - 1 = 0 \quad \Longrightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y$ must be positive, we must take $e^y = x + \sqrt{x^2+1} \Longrightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$. Derivations of the other two results are similar.

**14.** (a) Since $\ln\left(\dfrac{\sqrt{mg/\beta} - v}{\sqrt{mg/\beta} + v}\right) = -2\sqrt{\dfrac{mg}{\beta}}\left(\dfrac{\beta t}{m}\right) = -2\sqrt{\dfrac{\beta g}{m}}t$, exponentiation gives $\dfrac{\sqrt{mg/\beta} - v}{\sqrt{mg/\beta} + v} = e^{-2\sqrt{\beta g/mt}}$, and therefore $\sqrt{mg/\beta} - v = (\sqrt{mg/\beta} + v)e^{-2\sqrt{\beta g/mt}}$. When we solve this equation for $v$, the result is

$$v = \frac{\sqrt{mg/\beta} - \sqrt{mg/\beta}e^{-2\sqrt{\beta g/mt}}}{1 + e^{-2\sqrt{\beta g/mt}}} = \sqrt{\frac{mg}{\beta}}\left(\frac{1 - e^{-2\sqrt{\beta g/mt}}}{1 + e^{-2\sqrt{\beta g/mt}}}\right).$$

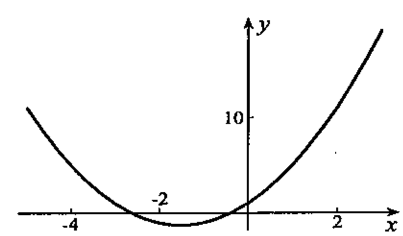
Multiplication of numerator and denominator by $e^{\sqrt{\beta g/mt}}$ gives

$$v = \sqrt{\frac{mg}{\beta}}\left(\frac{e^{\sqrt{\beta g/mt}} - e^{-\sqrt{\beta g/mt}}}{e^{\sqrt{\beta g/mt}} + e^{-\sqrt{\beta g/mt}}}\right) = \sqrt{\frac{mg}{\beta}} \tanh\left(\sqrt{\frac{\beta g}{m}}t\right).$$

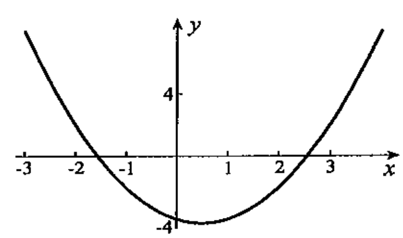
(b) Since the hyperbolic tangent function gets closer and closer to 1 as its argument gets large, it follows that the limiting velocity is $\sqrt{mg/\beta}$.

## EXERCISES 1.11

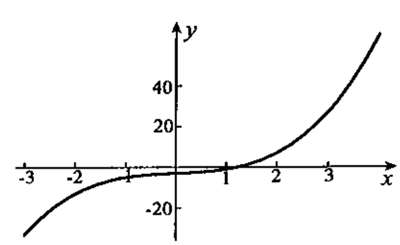
**1.** The plot shows two roots. My electronic device gives $-2.618\,034\,0$ and $-0.381\,966\,0$ for roots of $f(x) = x^2 + 3x + 1 = 0$. To verify that $-2.618\,034$ is accurate to six decimals, we calculate $f(-2.618\,033\,5) = -1.1 \times 10^{-6}$ and $f(-2.618\,034\,5) = 1.1 \times 10^{-6}$. A similar calculation verifies the accuracy of $-0.381\,966$.



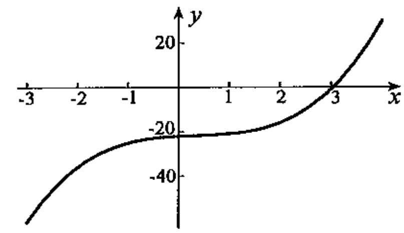

**2.** The plot shows two roots. My electronic device gives $-1.561\,552\,8$ and $2.561\,552\,8$ for roots of $f(x) = x^2 - x - 4 = 0$. To verify that $2.561\,553$ is accurate to six decimals, we calculate $f(2.561\,552\,5) = -1.3 \times 10^{-6}$ and $f(2.561\,553\,5) = 2.8 \times 10^{-6}$. A similar calculation verifies the accuracy of $-1.561\,553$.



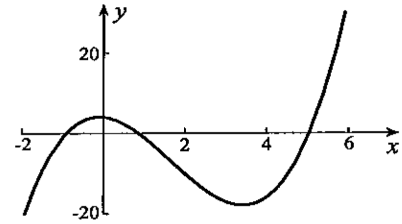

**3.** The plot shows one root. My electronic device gives $1.213\,411\,7$ for the root of $f(x) = x^3 + x - 3 = 0$. To verify that $1.213\,412$ is accurate to six decimals, we calculate $f(1.213\,411\,5) = -8.8 \times 10^{-7}$ and $f(1.213\,412\,5) = 4.5 \times 10^{-6}$.



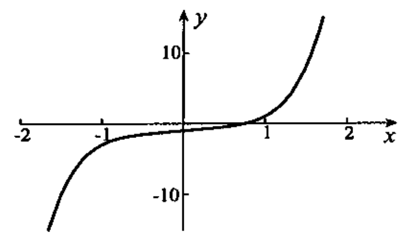
**4.** The plot shows one root. My electronic device gives 3.044 723 1 for the root of $f(x) = x^3 - x^2 + x - 22 = 0$. To verify that 3.044 723 is accurate to six decimals, we calculate $f(3.044\,722\,5) = -1.5 \times 10^{-5}$ and $f(3.044\,723\,5) = 8.0 \times 10^{-6}$.



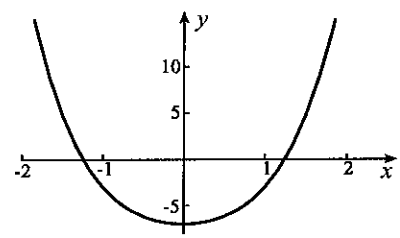

**5.** The plot shows three roots. My electronic device gives −0.911 503 3, 0.870 538 7 and 5.040 964 6 for the roots of $f(x) = x^3 - 5x^2 - x + 4 = 0$. To verify that −0.911 503 is accurate to six decimals, we calculate $f(-0.911\,503\,5) = -2.0 \times 10^{-6}$ and $f(-0.911\,502\,5) = 8.6 \times 10^{-6}$. The other roots are 0.870 539 and 5.040 965.



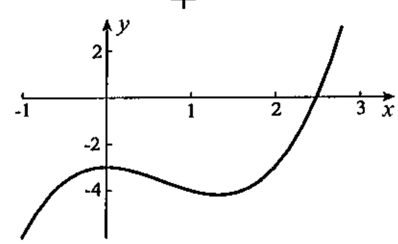

**6.** The plot shows one root. My electronic device gives 0.754 877 7 for the root of $f(x) = x^5 + x - 1 = 0$. To verify that 0.754 878 is accurate to six decimals, we calculate $f(0.754\,877\,5) = -4.4 \times 10^{-7}$ and $f(0.754\,878\,5) = 2.2 \times 10^{-6}$.



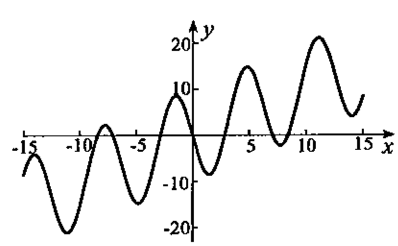

**7.** The plot shows two roots. My electronic device gives ±1.241 523 8 for the roots of $f(x) = x^4 + 3x^2 - 7 = 0$. To verify that 1.241 524 is accurate to six decimals, we calculate $f(1.241\,523\,5) = -4.0 \times 10^{-6}$ and $f(1.241\,524\,5) = 1.1 \times 10^{-5}$. Symmetry verifies the other root.



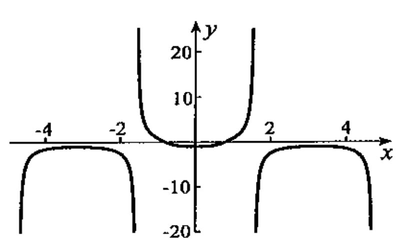

**8.** We rewrite the equation in the form $f(x) = x^3 - 2x^2 - 3 = 0$, the graph of which is shown to the right. My electronic calculator gives the root 2.485 584 0. To verify that 2.485 584 is accurate to six decimals, we calculate $f(2.485\,583\,5) = -4.3 \times 10^{-6}$ and $f(2.485\,584\,5) = 4.3 \times 10^{-6}$.



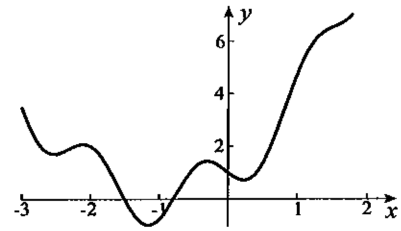

**9.** The plot shows seven roots, one of which is $x = 0$. My electronic device gives the other six as ±2.852 341 9, ±7.068 174 4, and ±8.423 204 0. To verify that 2.852 342 is a root of $f(x) = x - 10 \sin x$ accurate to six decimals, we evaluate $f(2.852\,341\,5) = -4.2 \times 10^{-6}$ and $f(2.852\,342\,5) = 6.4 \times 10^{-6}$. Verification that ±7.068 174 and ±8.423 204 are roots accurate to six decimals is similar.



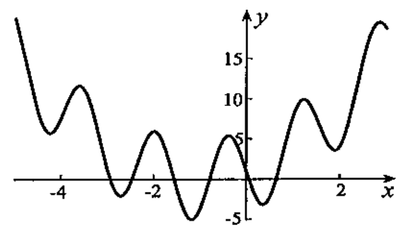

**10.** The plot shows two roots. My electronic device gives ±0.795 323 9 for the roots of $f(x) = \sec x - 2/(1 + x^4) = 0$. To verify that 0.795 324 is accurate to six decimals, we calculate $f(0.795\,323\,5) = -1.4 \times 10^{-6}$ and $f(0.795\,324\,5) = 2.1 \times 10^{-6}$.



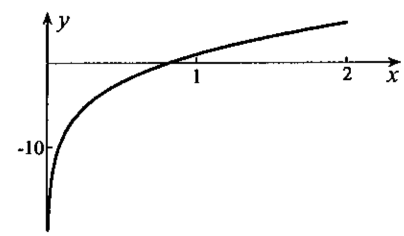
**11.** The plot shows two roots. My electronic device gives $-1.506\,052\,7$ and $-0.795\,823\,2$ for the roots of $f(x) = (x+1)^2 - \sin 4x = 0$. To verify that $-1.506\,053$ is accurate to six decimals, we calculate $f(-1.506\,052\,5) = -1.0 \times 10^{-6}$ and $f(-1.506\,053\,5) = 3.8 \times 10^{-6}$. Verification that $-0.795\,823$ is accurate to six decimals is similar.



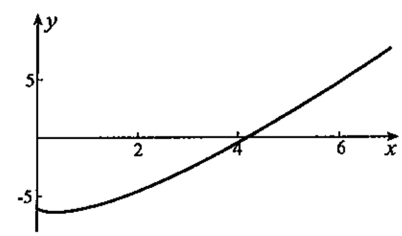
**12.** The plot shows six roots. My electronic device gives $-2.931\,137\,1$, $-2.467\,517\,5$, $-1.555\,365\,0$, $-0.787\,652\,8$, $0.056\,257\,6$ and $0.642\,850\,7$ for the roots of $f(x) = (x+1)^2 - 5\sin 4x = 0$. To verify that $-2.931\,137$ is accurate to six decimals, we calculate $f(-2.931\,136\,5) = -1.1 \times 10^{-5}$ and $f(-2.931\,137\,5) = 6.7 \times 10^{-6}$. Verification that $-2.467\,518$, $-1.555\,365$, $-0.787\,653$, $0.056\,258$, and $0.642\,851$ are accurate to six decimals is similar.



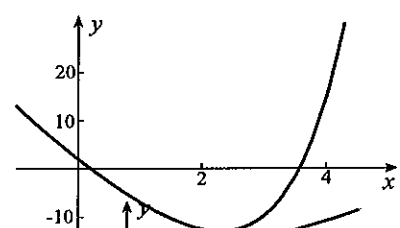
**13.** The plot shows one root. My electronic device gives $0.815\,553\,4$ for the root of $f(x) = x + 4\ln x = 0$. To verify that $0.815\,553$ is accurate to six decimals, we calculate $f(0.815\,552\,5) = -5.4 \times 10^{-6}$ and $f(0.815\,553\,5) = 4.8 \times 10^{-7}$.



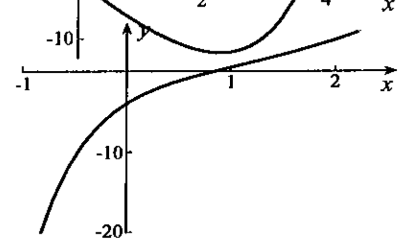
**14.** The plot shows one root. My electronic device gives $4.188\,760\,1$ for the root of $f(x) = x\ln x - 6 = 0$. To verify that $4.188\,760$ is accurate to six decimals, we calculate $f(4.188\,759\,5) = -1.5 \times 10^{-6}$ and $f(4.188\,760\,5) = 9.3 \times 10^{-7}$.



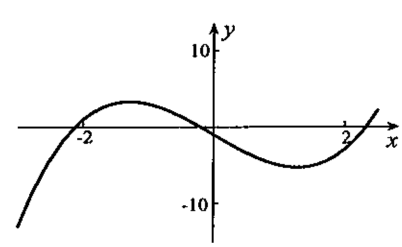
**15.** The plot shows two roots. My electronic device gives $0.204\,183\,6$ and $3.576\,065\,3$ for the roots of $f(x) = e^x + e^{-x} - 10x = 0$. To verify that $0.204\,184$ is accurate to six decimals, we calculate $f(0.204\,183\,5) = 9.5 \times 10^{-7}$ and $f(0.204\,184\,5) = -8.6 \times 10^{-6}$. The other root to six decimals is $3.576\,065$.



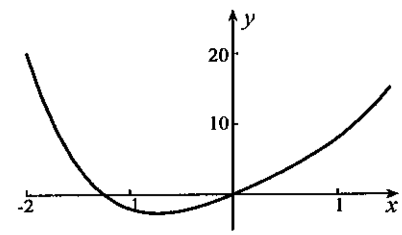
**16.** The plot shows one root. My electronic device gives $0.852\,605\,5$ for the root of $f(x) = x^2 - 4e^{-2x} = 0$. To verify that $0.852\,606$ is accurate to six decimals, we calculate $f(0.852\,605\,5) = -6.4 \times 10^{-9}$ and $f(0.852\,606\,5) = 3.2 \times 10^{-6}$.



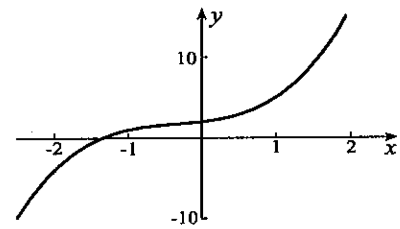
**17.** The plot shows three roots. My electronic device gives $-2.128\,4$, $-0.201\,6$, and $2.330\,1$ for the roots of $f(x) = x^3 - 5x - 1 = 0$. To verify that $2.330$ has error no greater than $10^{-3}$, we calculate $f(2.329) = -1.2 \times 10^{-2}$ and $f(2.331) = 1.1 \times 10^{-2}$. Verification that $-2.128$ and $-0.202$ have the same accuracy is similar.



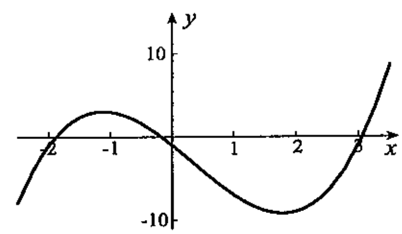
**18.** The plot shows two roots, one of which is $x = 0$. My electronic device gives $-1.248\,30$ for the other root of $f(x) = x^4 - x^3 + 2x^2 + 6x = 0$. To verify that $-1.248\,3$ has error no greater than $10^{-4}$, we calculate $f(-1.248\,2) = -1.1 \times 10^{-3}$ and $f(-1.248\,4) = 1.2 \times 10^{-3}$.



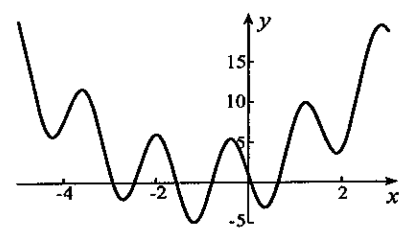

**19.** We rewrite the equation in the form $f(x) = x^3 + x^2 + x + 2 = 0$. The plot shows one root. My electronic device gives $-1.353\,210$ for the root. To verify that $-1.353\,21$ has error no greater than $10^{-5}$, we calculate $f(-1.353\,20) = 3.8 \times 10^{-5}$ and $f(-1.353\,22) = -3.8 \times 10^{-5}$.



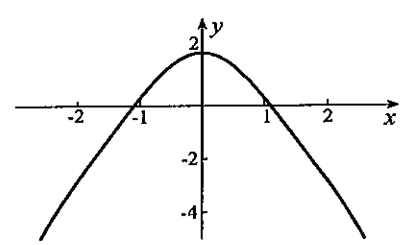

**20.** We rewrite the equation in the form $f(x) = x^3 - x^2 - 6x - 1 = 0$. The plot shows three roots. My electronic device gives $-1.892\,0$, $-0.172\,5$, and $3.064\,4$ for the roots. To verify that $-1.892$ has error no greater than $10^{-3}$, we calculate $f(-1.893) = -8.9 \times 10^{-3}$ and $f(-1.891) = 8.1 \times 10^{-3}$. Verification that $-0.172$ and $3.064$ also have the required accuracy is similar.



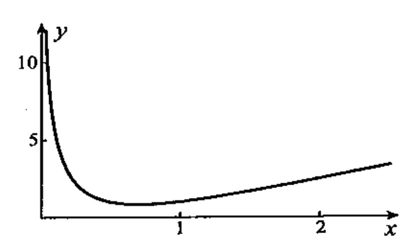

**21.** The plot shows six roots. My electronic device gives $-2.931\,1$, $-2.467\,5$, $-1.555\,4$, $-0.787\,7$, $0.056\,3$ and $0.642\,9$ for the roots of $f(x) = (x+1)^2 - 5\sin 4x = 0$. To verify that $-2.931$ has error no greater than $10^{-3}$, we calculate $f(-2.932) = 1.5 \times 10^{-2}$ and $f(-2.930) = -2.0 \times 10^{-2}$. Verification that $-2.468$, $-1.555$, $-0.788$, $0.056$, and $0.643$ have error no greater than $10^{-3}$ is similar.



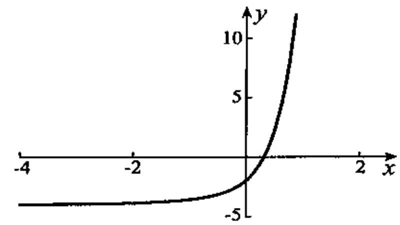

**22.** The plot shows two roots. My electronic device gives $\pm 1.098\,59$ for the roots of $f(x) = \cos^2 x - x^2 + 1 = 0$. To verify that $1.098\,6$ has error no greater than $10^{-4}$, we calculate $f(1.098\,5) = 2.6 \times 10^{-4}$ and $f(1.098\,7) = -3.4 \times 10^{-4}$.




**23.** The plot shows no solutions.



**24.** The plot shows one root. My electronic device gives $0.321\,21$ for the root of $f(x) = e^{3x} + e^x - 4 = 0$. To verify that $0.321\,2$ has error no greater than $10^{-4}$, we calculate $f(0.321\,1) = -1.0 \times 10^{-3}$ and $f(0.321\,3) = 8.2 \times 10^{-4}$.




**25.** To find $x$-coordinates of the points of intersection, we set $x^3 = x + 5$. My electronic device gives $x = 1.904\,161$ as the only solution of $f(x) = x^3 - x - 5 = 0$. The values $f(1.904\,160\,5) = -3.5 \times 10^{-6}$ and $f(1.904\,161\,5) = 6.3 \times 10^{-6}$ confirm six-decimal accuracy. Both equations $y = x^3$ and $y = x + 5$ give the same four decimals $y = 6.9042$. The point of intersection is therefore $(1.9042, 6.9042)$.

**26.** To find $x$-coordinates of the points of intersection, we set $(x+1)^2 = x^3 - 4x$. My electronic device gives $x = -1.891\,954$, $-0.172480$, and $3.064\,435$ as solutions of $f(x) = x^3 - 4x - (x+1)^2 = 0$. The values $f(-1.891\,954\,5) = -5.1 \times 10^{-7}$ and $f(-1.891\,953\,5) = 8.0 \times 10^{-6}$ confirm six-decimal accuracy of the first. Both equations $y = x^3 - 4x$ and $y = (x+1)^2$ give the same four decimals $y = 0.7956$. A point of intersection is therefore $(-1.8920, 0.7956)$. Similar procedures lead to the other points of intersection, $(-0.1725, 0.6848)$ and $(3.0644, 16.1596)$.

**27.** To find $x$-coordinates of the points of intersection, we set $x^4 - 20 = x^3 - 2x^2$. My electronic device gives $x = -1.726\,688$, and $2.130\,189$ as solutions of $f(x) = x^4 - x^3 + 2x^2 - 20 = 0$. The values $f(2.130\,188\,5) = -7.8 \times 10^{-6}$ and $f(2.130\,189\,5) = 2.6 \times 10^{-5}$ confirm six-decimal accuracy of the second. Both equations $y = x^4 - 20$ and $y = x^3 - 2x^2$ give the same four decimals $y = 0.5908$. A point of intersection is therefore $(2.1302, 0.5908)$. A similar procedure leads to the other point of intersection $(-1.7267, -11.1109)$.

**28.** To find $x$-coordinates of the points of intersection, we set $x/(x+1) = x^2 + 2$. My electronic device gives $x = -1.353\,210$ as the only solution of $f(x) = x^3 + x^2 + x + 2 = 0$. The values $f(-1.353\,210\,5) = -2.0 \times 10^{-6}$ and $f(-1.353\,209\,5) = 1.8 \times 10^{-6}$ confirm six-decimal accuracy. Both equations $y = x/(x+1)$ and $y = x^2 + 2$ give the same four decimals $y = 3.8312$. The point of intersection is therefore $(-1.3532, 3.8312)$.

**29.** My electronic device gives $x = 3.926\,602$ as the smallest positive solution of $f(x) = \tan x - (e^x - e^{-x})/(e^x + e^{-x}) = 0$. The values $f(3.926\,601\,5) = -1.6 \times 10^{-6}$ and $f(3.926\,602\,5) = 3.8 \times 10^{-7}$ confirm six-decimal accuracy. When divided by $20\pi$, the smallest frequency is 0.0625. A similar procedure for the second frequency gives 0.1125.

**30.** (a) My electronic device gives $t = 3.833$ as the solution of $f(t) = 1181(1 - e^{-t/10}) - 98.1t = 0$. Since $y(3.825) = 0.14$ and $y(3.835) = -0.03$, it follows that to 2 decimals $t = 3.83$ s.

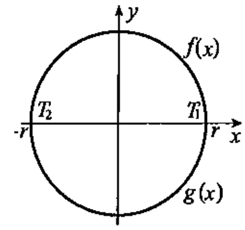
(b) When we set $0 = y = 20t - 4.905t^2$, the positive solution is 4.08 s.

**31.** My electronic device gives $z = 0.012\,957$ as the solution of $f(z) = 2Pz - e^{Lz} + e^{-Lz} = 0$ when $P = 80$ and $L = 70$. Since $f(0.012\,956\,5) = 2.3 \times 10^{-5}$ and $f(0.0129\,957\,5) = -1.9 \times 10^{-5}$, we can say that the solution is $z = 0.012\,957$ to six decimals. This gives $T = \rho g/(2z) = 189.3$.

**32.** To simplify calculations, we set $z = c/\lambda$. Then, $z$ must satisfy the equation $f(z) = (5 - z)e^z - 5 = 0$. My electronic device gives $z = 4.965\,114\,232$. With this approximation for $z$, we obtain $\lambda = c/z = 0.000\,028\,974$. For a seven decimal answer, we use $g(\lambda) = (5\lambda - c)e^{c/\lambda} - 5\lambda$ to calculate $g(0.000\,028\,95) = -1.7 \times 10^{-5}$ and $g(0.000\,029\,05) = 5.1 \times 10^{-5}$. Thus, to 7 decimals, $\lambda = 0.000\,029\,0$.

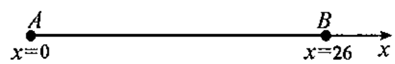
**33.** Consider the function $g(x) = f(x) - x$. Since the range of $f(x)$ is $a \le x \le b$, it follows that $f(a) \ge a$ and $f(b) \le b$. Consequently, $g(a) = f(a) - a \ge 0$ and $g(b) = f(b) - b \le 0$. If $f(a) = a$, then $x = a$ is a solution of $f(x) = x$. If $f(b) = b$, then $x = b$ is a solution. When $f(a) \ne a$ and $f(b) \ne b$, the Zero Intermediate Value Theorem implies that there is at least one solution of $g(x) = 0$ in the interval $a < x < b$. This gives a solution of $f(x) = x$.

**34.** Suppose we let $T_1$ and $T_2$ be the temperatures at the points $(r, 0)$ and $(-r, 0)$. Then, $F(r) = f(r) - g(-r) = T_1 - T_2$ and $F(-r) = f(-r) - g(r) = T_2 - T_1$. If $T_1 = T_2$, then temperatures are the same at the points $(r, 0)$ and $(-r, 0)$. Otherwise, values of $F(x)$ have opposite signs at $x = r$ and $x = -r$.




This implies that there is a value of $x$ between $-r$ and $r$ at which $F(x) = 0$. At this value, $f(x) = g(-x)$, and these give equal temperatures at points opposite each other on the equator.

**35.** Let $f_1(t)$ be the position of the runner on Saturday at any time $t$ on the course taking time $t = 0$ at 7:00 a.m. Choose $x = 0$ at $A$ and $x = 26$ at $B$, so that $f_1(0) = 0$ and $f_1(T_1) = 26$, where $T_1$ is the time to finish the marathon on Saturday. Similarly, let $f_2(t)$ be the position of the runner on Sunday with $f_2(0) = 26$ and $f_2(T_2) = 0$, where $T_2$ is her finish time on Sunday. Consider the function $f(t) = f_1(t) - f_2(t)$. If $T$ is the smaller of $T_1$ and $T_2$, then, $f(0) = f_1(0) - f_2(0) = -26$ and $f(T) = f_1(T) - f_2(T) > 0$. Consequently, there is a value of $t$ between $t = 0$ and $t = T$ at which $f(t) = 0$, and at this time $f_1(t) = f_2(t)$; that is, the runner is at this position at the same times on the two days.




**36.** (a) Suppose $f(x)$ is continuous on an interval $I$. Let $c$ and $d$, where $d > c$, be any two points in the range of $f(x)$ and $e$ be any number between $c$ and $d$. There exist values $a$ and $b$ in $I$ such that $d = f(b)$ and $c = f(a)$. If we define a function $F(x) = f(x) - e$, then

$$F(a)F(b) = [f(a) - e][f(b) - e] = (c - e)(d - e) < 0.$$

Since $F(x)$ is continuous, the zero intermediate value theorem guarantees the existence of a number $z$ between $a$ and $b$ for which $0 = F(z) = f(z) - e$. This means that $e$ is in the range of $f(x)$. Hence, the range of $f(x)$ is an interval.

(b) Not necessarily. For example, $f(x) = 1$ maps every interval on the $x$-axis onto a single point.

## REVIEW EXERCISES

**1.** Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 4$. We find that $x = 2$ is a solution. We factor $x - 2$ from the cubic,

$$x^3 - x^2 - 4 = (x - 2)(x^2 + x + 2).$$

Since the discriminant of the quadratic is negative, the only real solution is $x = 2$.

**2.** Possible rational solutions are $\pm 1$, $\pm 3$, $\pm 9$, $\pm 27$, $\pm 1/2$, $\pm 3/2$, $\pm 9/2$, $\pm 27/2$. We find that $x = 3$ is a solution. We factor $x - 3$ from the cubic,

$$2x^3 - 9x^2 + 27 = (x - 3)(2x^2 - 3x - 9) = (x - 3)(2x + 3)(x - 3) = (x - 3)^2(2x + 3).$$

Solutions are $x = 3$ with multiplicity 2 and $x = -3/2$.

**3.** Possible rational solutions are $\pm 1$, $\pm 5$, $\pm 1/2$, $\pm 5/2$. We find that $x = 1$ is a solution. We factor $x - 1$ from the quartic,

$$2x^4 - x^3 - 9x^2 + 13x - 5 = (x - 1)(2x^3 + x^2 - 8x + 5).$$

Possible rational zeros of the cubic are the same. We find that $x = 1$ is a zero, and factor $x - 1$ from the cubic,

$$2x^4 - x^3 - 9x^2 + 13x - 5 = (x - 1)^2(2x^2 + 3x - 5) = (x - 1)^2(x - 1)(2x + 5) = (x - 1)^3(2x + 5).$$

Solutions are $x = 1$ with multiplicity 3 and $x = -5/2$.

**4.** The list of possible rational solutions here is formidable. Perhaps we can be a little ingenious. The fact that $36x^4$ and 225 are perfect squares suggests investigating whether the poynomial is the square of a quadratic expression. A little experimentation reveals that

$$36x^4 + 12x^3 - 179x^2 - 30x + 225 = (6x^2 + x - 15)^2 = (3x+5)^2(2x-3)^2.$$

Solutions are therefore $x = -5/3$ and $x = 3/2$ each of multiplicity 2.

**5.** The distance between the points is $\sqrt{(4+1)^2 + (2-3)^2} = \sqrt{26}$. The midpoint of the line segment is $\big((4-1)/2, (3+2)/2\big) = (3/2, 5/2)$.

**6.** The distance between the points is $\sqrt{(2+3)^2 + (1+4)^2} = 5\sqrt{2}$. The midpoint of the line segment is $\big((2-3)/2, (1-4)/2\big) = (-1/2, -3/2)$.

**7.** Since the slope of the line is 1/2, its equation is $y - 3 = (1/2)(x-2)$ or $2y = x + 4$.

**8.** Since the slope of the line joining $(-2, 1)$ and the origin is $-1/2$, and the midpoint of the line segment joining $(1, 3)$ and $(-1, 5)$ is $(0, 4)$, the equation of the required line is $y - 4 = 2(x - 0)$, or, $y = 2x + 4$.

**9.** If we set $4y - 11 = \sqrt{y^2 + 9}$, we see that the solution is $y = 4$. The point of intersection is $(5, 4)$. Since the slope of the required line is $-4$, its equation is $y - 4 = -4(x - 5)$ or $4x + y = 24$.

**10.** If we substitute $y = x^2$ into the second equation, $5x = 6 - x^4$, or $x^4 + 5x - 6 = 0$. Possible rational solutions are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 6$. We find that $x = 1$ is a solution and factor $x - 1$ from the quartic

$$x^4 + 5x - 6 = (x-1)(x^3 + x^2 + x + 6) = 0.$$

We now see that $x = -2$ is a zero of the cubic so that

$$x^4 + 5x - 6 = (x-1)(x+2)(x^2 - x + 3) = 0.$$

The curves therefore intersect at the points $(-2, 4)$ and $(1, 1)$. The equation of the line joining these points is $y - 1 = -(x - 1) \Longrightarrow x + y = 2$.

**11.** This function is defined for all reals.

**12.** For $x^2 - 5$ to be nonnegative, $|x| \geq \sqrt{5}$.

**13.** Since $x^2 + 3x + 2 = (x+1)(x+2)$, the function is defined for all $x$ except $x = -1$ and $x = -2$.

**14.** Since $x^3 + 2x^2 + x = x(x+1)^2$, we cannot set $x = 0$ or $x = -1$.

**15.** This function is defined for all reals.

**16.** $x \geq 0$

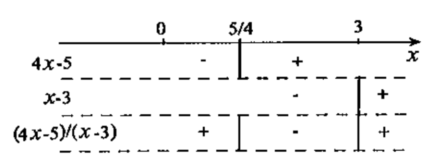
**17.** Since $x^2 + 4x - 6 = 0$ when $x = \big(-4 \pm \sqrt{16+24}\big)/2 = -2 \pm \sqrt{10}$, and is negative between these values, we must restrict $x$ to the intervals $x \leq -2 - \sqrt{10}$ and $x \geq -2 + \sqrt{10}$. These can be combined into $|x + 2| \geq \sqrt{10}$.

**18.** Since $2x^2 + 4x - 5 = 0$ when $x = \big(-4 \pm \sqrt{16+40}\big)/4 = -1 \pm \sqrt{14}/2$, and is negative between these values, we must restrict $x$ to the intervals $x < -1 - \sqrt{14}/2$ and $x > -1 + \sqrt{14}/2$. These can be combined into $|x + 1| > \sqrt{14}/2$.

**19.** If we express the function in the form

$$f(x) = \sqrt{\frac{4x-5}{x-3}},$$

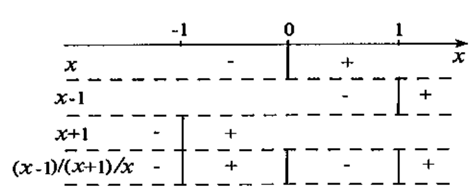
the sign diagram to the right indicates that the function is defined for $x \leq 5/4$ and $x > 3$.




**20.** We require

$$0 \leq x - \frac{1}{x} = \frac{x^2 - 1}{x} = \frac{(x-1)(x+1)}{x}.$$

The sign diagram indicates that this occurs for $-1 \leq x < 0$, and $x \geq 1$.

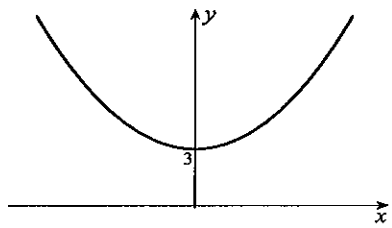



**21.** Straight line **22.** Parabola **23.** None of these **24.** Ellipse **25.** Hyperbola **26.** None of these **27.** Circle **28.** Parabola **29.** Ellipse **30.** Circle **31.** Hyperbola **32.** Hyperbola

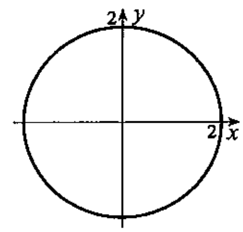
**33.** With formula 1.16, the distance is $\left|\dfrac{2(1)-(-3)+3}{\sqrt{2^2+(-1)^2}}\right| = \dfrac{8}{\sqrt{5}}$.

**34.** With formula 1.16, the distance is $\left|\dfrac{(-2)+3(-5)-4}{\sqrt{1^2+3^2}}\right| = \dfrac{21}{\sqrt{10}}$.

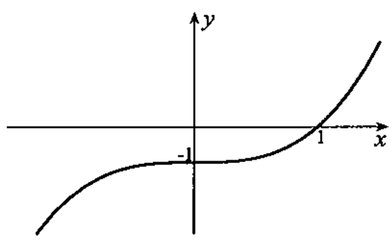
**35.** This is the parabola $y = 2x^2$ shifted upward 3 units.



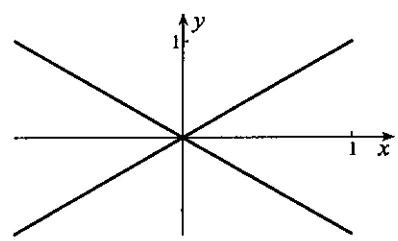

**36.** This is the circle $x^2 + y^2 = 4$ with centre $(0, 0)$ and radius 2.



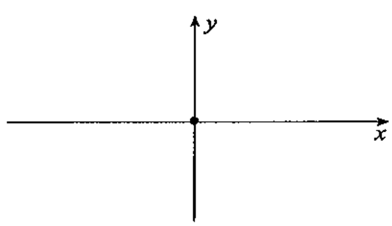

**37.** This is the cubic $y = x^3$ shifted downward 1 unit.



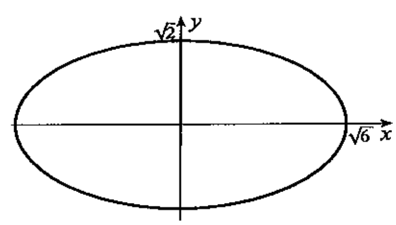

**38.** This equation describes two straight lines $y = \pm x$.



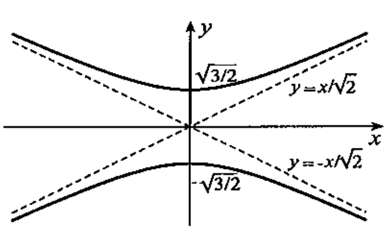

**39.** Only the point $(0, 0)$ satisfies this equation.



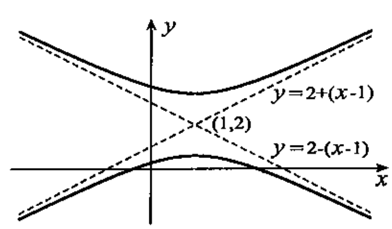

**40.** This is an ellipse with $x$-intercepts $\pm\sqrt{6}$ and $y$-intercepts $\pm\sqrt{2}$.



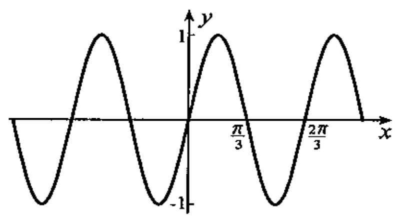

**41.** This is a hyperbola with $y$-intercepts $\pm\sqrt{3/2}$ and asymptotes $y = \pm x/\sqrt{2}$.



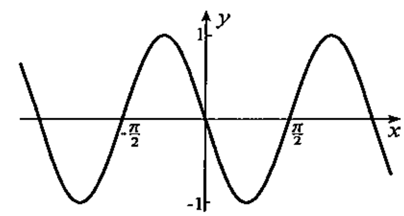

**42.** When we complete squares on $x$- and $y$-terms, $(y-2)^2 - (x-1)^2 = 2$. Asymptotes for this hyperbola are $y = 2 \pm (x-1)$ intersecting at $(1, 2)$.



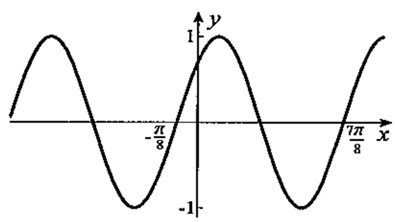
**43.** The period is $2\pi/3$.



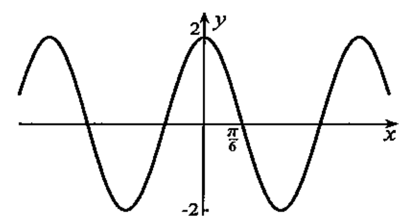

**44.** When we expand the cosine function,
$y = -\sin 2x$.



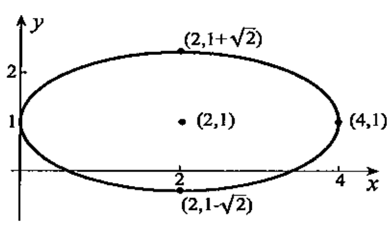

**45.** The period is $\pi$, and the phase shift is $\pi/8$.



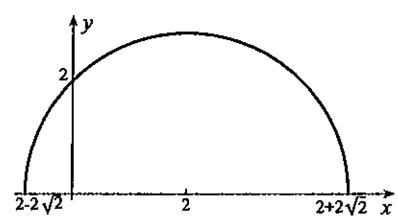

**46.** When we expand the sine function,
$y = 2\cos 3x$.



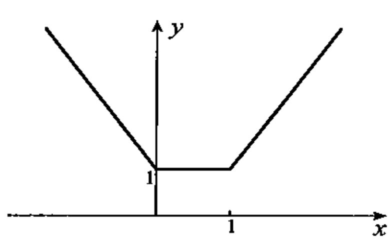

**47.** Completion of squares on $x$- and $y$-terms gives $(x-2)^2 + 2(y-1)^2 = 4$. This is an ellipse.



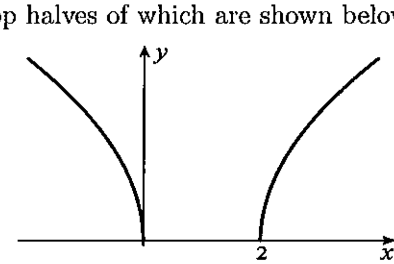

**48.** If we square the equation $y^2 = -x^2 + 4x + 4$, and then complete the square on the $x$-terms, $(x-2)^2 + y^2 = 8$. This is a circle with centre $(2,0)$ and radius $2\sqrt{2}$. The original equation describes the top half of the circle.



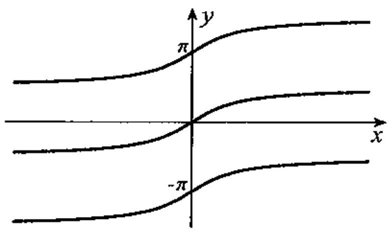

**49.** We add ordinates of $y = |x|$ and $y = |x-1|$.



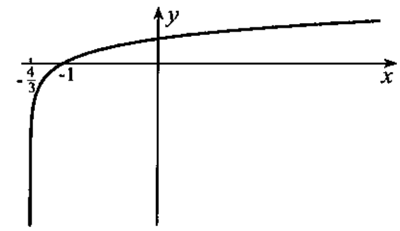

**50.** When we square the equation,
$y^2 = |x-1| - 1$, or, $y^2 + 1 = |x-1|$.
Thus, $x - 1 = \pm(y^2+1)$, or, $x = 1 \pm (y^2+1)$
$= -y^2,\ y^2 + 2$. This is two parabolas, the top halves of which are shown below.



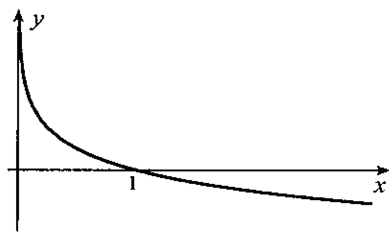
**51.** We interchange axes in Figure 1.90c.



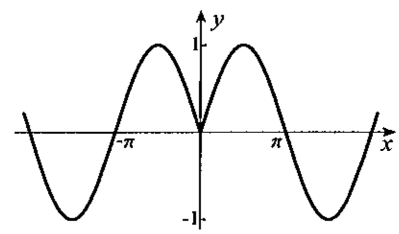

**52.** We must have $x > -4/3$.
The $x$-intercept is $-1$.



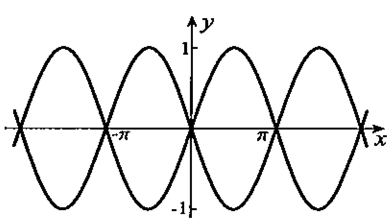

**53.** We interchange axes in Figure 1.115 and reflected in the $x$-axis.



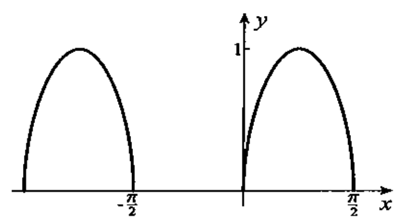

**54.** We graph this even function by reflecting that part of $y = \sin x$ to the right of the $y$-axis in the $y$-axis.



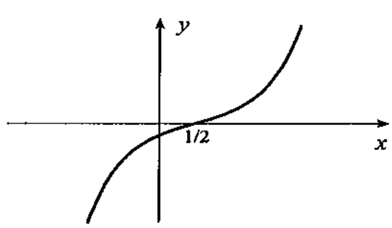

**55.** We draw $y = |\sin x|$ and then take its refection in the $x$-axis.



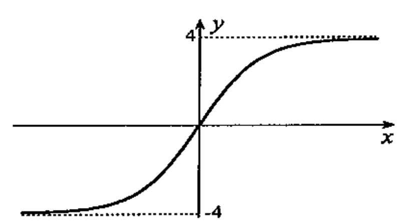

**56.** We take square roots of ordinates of $y = \sin 2x$.




**57.** The graph has the shape of $\sinh x$ shifted 1/2 unit to the right.




**58.** The graph has the shape of $\tanh x$ with asymptotes $y = \pm 4$.




**59.** With slopes $-1/2$ and 3, equation 1.60 gives $\theta = \text{Tan}^{-1}\left|\dfrac{3 + 1/2}{1 - 3/2}\right| = 1.43$ radians.

**60.** With slopes 1/4 and $-2/3$, equation 1.60 gives $\theta = \text{Tan}^{-1}\left|\dfrac{1/4 + 2/3}{1 - 1/6}\right| = 0.833$ radians.

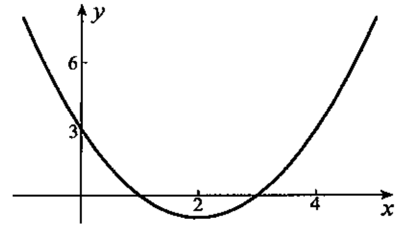
**61.** $f(x) = 1$

**62.** Since $(x + 1)(2 - x)$ is nonnegative only for $-1 \le x \le 2$, the function $\sqrt{(x + 1)(2 - x)}$ is only defined for these values of $x$.

**63.** $f(x) = 1/(x^2 - 1)$

**64.** Since $\sqrt{-x}$ is defined only for $x \le 0$, a function is $1 + \sqrt{-x}$.

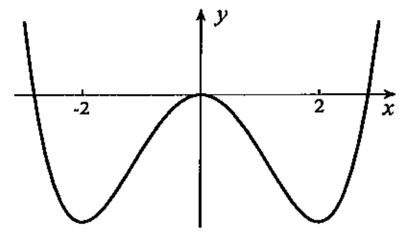
**65.** The graph shows that $f(x)$ has an inverse on the intervals $x \le 2$ and $x \ge 2$. If we set $y = x^2 - 4x + 3$ $= (x-2)^2 - 1$, then $(x-2)^2 = y + 1$, from which $x = 2 \pm \sqrt{y+1}$. The equation $x = 2 - \sqrt{y+1}$ describes the left half of the curve, and $x = 2 + \sqrt{y+1}$ describes the right half. For $x \le 2$ then, the inverse function is $f^{-1}(x) = 2 - \sqrt{x+1}$, and for $x \ge 2$, it is $f^{-1}(x) = 2 + \sqrt{x+1}$.




**66.** The graph shows that $f(x)$ has an inverse on the intervals $x \le -2$, $-2 \le x \le 0$, $0 \le x \le 2$, and $x \ge 2$. If we set $y = x^4 - 8x^2$, then $(x^2)^2 - 8(x^2) - y = 0$. The quadratic formula gives

$$x^2 = \frac{8 \pm \sqrt{64+4y}}{2} = 4 \pm \sqrt{16+y},$$

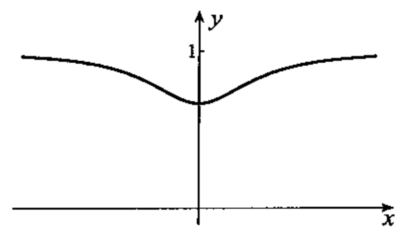
from which $x = \pm\sqrt{4 \pm \sqrt{16+y}}$. Since $x = -\sqrt{4+\sqrt{16+y}}$, $x = -\sqrt{4-\sqrt{16-y}}$, $x = \sqrt{4-\sqrt{16-y}}$, and $x = \sqrt{4+\sqrt{16+y}}$ describe parts of the graph on each of the above intervals, respectively, it follows that inverse functions on these intervals are:



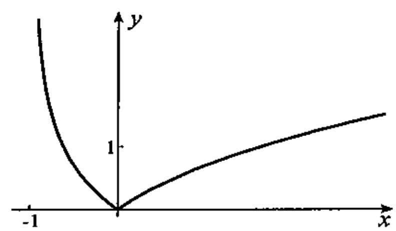

$$f^{-1}(x) = -\sqrt{4+\sqrt{16+x}} \text{ for } x \le -2; \quad f^{-1}(x) = -\sqrt{4-\sqrt{16-x}} \text{ for } -2 \le x \le 0;$$

$$f^{-1}(x) = \sqrt{4-\sqrt{16-x}} \text{ for } 0 \le x \le 2; \quad f^{-1}(x) = \sqrt{4+\sqrt{16+x}} \text{ for } x \ge 2.$$

**67.** The graph shows that $f(x)$ has an inverse on the intervals $x \le 0$ and $x \ge 0$. If we set $y = (x^2+2)/(x^2+3)$, then $(x^2+3)y = x^2+2$, from which $(y-1)x^2 = 2-3y$. Thus $x = \pm\sqrt{(2-3y)/(y-1)}$. The inverse function for $x \le 0$ is $f^{-1}(x) = -\sqrt{(2-3x)/(x-1)}$, and that for $x \ge 0$ is $f^{-1}(x) = \sqrt{(2-3x)/(x-1)}$.



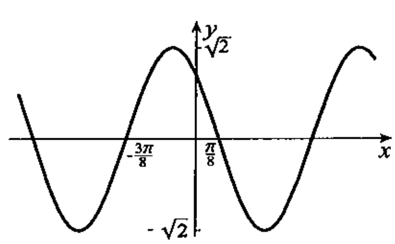

**68.** The graph shows that $f(x)$ has an inverse for $-1 < x \le 0$ and for $x \ge 0$. If we set $y = \sqrt{x^2/(x+1)}$, then $x^2 = (x+1)y^2$, from which $x^2 - y^2x - y^2 = 0$. The quadratic formula gives $x = (y^2 \pm \sqrt{y^4+4y^2})/2$. Since $x = (y^2 - \sqrt{y^4+4y^2})/2$ describes the graph for $-1 < x \le 0$, and $x = (y^2 + \sqrt{y^4+4y^2})/2$ describes the graph for $x \ge 0$, inverse functions are $f^{-1}(x) = (x^2 - \sqrt{x^4+4x^2})/2$ for $-1 < x \le 0$, and $f^{-1}(x) = (x^2 + \sqrt{x^4+4x^2})/2$ for $x \ge 0$.




**69.** If we set $f(x) = \cos 2x - \sin 2x = A\sin(2x+\phi)$, and expand the right side,

$\cos 2x - \sin 2x = A(\sin 2x \cos\phi + \cos 2x \sin\phi)$.

This will be true for all $x$ if we set $A\cos\phi = -1$ and $A\sin\phi = 1$. When these are squared and added, $2 = (-1)^2 + 1^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. If we choose $A = \sqrt{2}$, then $\sin\phi = 1/\sqrt{2}$ and $\cos\phi = -1/\sqrt{2}$. These are satisfied by $\phi = 3\pi/4$. The amplitude is $\sqrt{2}$, the period is $\pi$, and the phase shift is $-3\pi/8$. Angles for which $f(x) = 0$ are defined by $0 = \sqrt{2}\sin(2x + 3\pi/4) \Longrightarrow 2x + 3\pi/4 = n\pi$, where $n$ is an integer. These give $x = -3\pi/8 + n\pi/2$. The second smallest positive solution is $x = 5\pi/8$, which we could also have seen from the graph.




**70.** Since $2\sin 2x$ has period $\pi$, and $3\cos 3x$ has period $2\pi/3$, the function $f(x)$ has period $2\pi$.

**71.** This quadratic in $\cos x$ can be factored $0 = \cos^2 x + 5\cos x - 6 = (\cos x + 6)(\cos x - 1)$. Thus, either $\cos x = -6$ or $\cos x = 1$. The former is impossible, and solutions of the latter are $x = 2n\pi$, $n$ an integer.

**72.** One solution for $2x$ of $\sin 2x = 1/4$ is $2x = \text{Sin}^{-1}(1/4) = 0.253$. All solutions are given by

$$2x = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.253\right) + 2n\pi \qquad \Longrightarrow \qquad x = \frac{\pi}{4} \pm 0.659 + n\pi, \quad n \text{ an integer.}$$

**73.** We can rewrite the equation in the form $\sin(x+1) = \pm 1/\sqrt{3}$. One solution of $\sin(x+1) = 1/\sqrt{3}$ for $x+1$ is $x + 1 = \text{Sin}^{-1}(1/\sqrt{3}) = 0.6155$. All solutions are given by

$$x + 1 = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - 0.6155\right) + 2n\pi \qquad \Longrightarrow \qquad x = -1 + \frac{\pi}{2} \pm 0.955 + 2n\pi, \quad n \text{ an integer.}$$

From $\sin(x+1) = -1/\sqrt{3}$, we obtain the solutions

$$x = -1 - \frac{\pi}{2} \pm 0.955 + 2n\pi, \quad n \text{ an integer.}$$

**74.** Since $5 - 2\pi$ is in the principal value range for the inverse tangent function, we may take tangents of both sides of the equation,

$$3x + 2 = \tan(5 - 2\pi) \qquad \Longrightarrow \qquad x = -\frac{2}{3} + \frac{1}{3}\tan(5 - 2\pi) = -1.79.$$

**75.** If $\cos 2x = \sin x$, then $0 = \sin x - (1 - 2\sin^2 x) = 2\sin^2 x + \sin x - 1 = (2\sin x - 1)(\sin x + 1)$. Thus, either $\sin x = -1$, the solutions of which are $x = 2n\pi - \pi/2 = (4n-1)\pi/2$, where $n$ is an integer, or, $\sin x = 1/2$. Solutions of this equation are $x = \pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$. These can be represented more compactly in the form $x = 2n\pi + \pi/2 \pm \pi/3 = (4n+1)\pi/2 \pm \pi/3$.

**76.** If we write $\ln[\sin x(1 + \sin x)] = \ln(3/2)$, and exponentiate both sides to base $e$, we obtain $\sin x(1 + \sin x) = 3/2$, or, $2\sin^2 x + 2\sin x - 3 = 0$. Thus, $\sin x = \dfrac{-2 \pm \sqrt{4 + 24}}{4} = \dfrac{-1 \pm \sqrt{7}}{2}$. Since $\sin x$ must be nonnegative in the original equation, we take $\sin x = (\sqrt{7} - 1)/2$. Solutions of this equation are

$$x = \{\text{Sin}^{-1}[(\sqrt{7} - 1)/2] + 2n\pi, \quad \pi - \text{Sin}^{-1}[(\sqrt{7} - 1)/2] + 2n\pi\} = \{0.966 + 2n\pi, \quad 2.175 + 2n\pi\}.$$

These can be expressed in the form $x = \dfrac{\pi}{2} \pm \left(\dfrac{\pi}{2} - 0.966\right) + 2n\pi = \left(\dfrac{4n+1}{2}\right)\pi \pm 0.604$.

**77.** If $3\,\text{Sin}^{-1}(e^{x+2}) = 2$, then $e^{x+2} = \sin(2/3)$, from which $x = -2 + \ln[\sin(2/3)] = -2.48$.

**78.** If $3\sin(e^{x+2}) = 2$, then $\sin(e^{x+2}) = 2/3$. This implies that

$$e^{x+2} = \{\text{Sin}^{-1}(2/3) + 2n\pi, \quad \pi - \text{Sin}^{-1}(2/3) + 2n\pi\} = \{0.73 + 2n\pi, \quad \pi - 0.73 + 2n\pi\}.$$

These values can be represented more compactly as

$$e^{x+2} = \frac{\pi}{2} \pm 0.84 + 2n\pi = \left(\frac{4n+1}{2}\right)\pi \pm 0.84.$$

Because $e^{x+2}$ must be positive, $n$ must be a nonnegative integer. Finally then,

$$x = \ln\left[\left(\frac{4n+1}{2}\right)\pi \pm 0.84\right] - 2, \text{ where } n \geq 0.$$

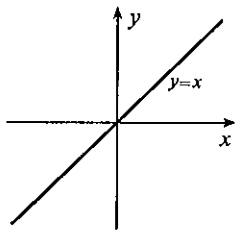
**79.** Since $\text{Tan}^{-1}(1/4) = 0.245$, it follows that $x\cosh 2 = 0.245 + n\pi$, were $n$ is an integer. Hence, $x = (0.245 + n\pi)/\cosh 2 = 0.065 + 0.27n\pi$.

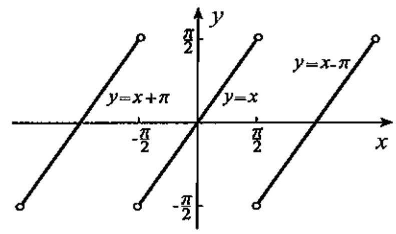
**80.** If $4 = \sinh x = (e^x - e^{-x})/2$, then multiplication by $e^x$ gives $e^{2x} - 8e^x - 1 = 0$. Thus,

$$e^x = \frac{8 \pm \sqrt{64 + 4}}{2} = 4 \pm \sqrt{17}.$$

Since $e^x$ must be positive, $e^x = 4 + \sqrt{17}$, and $x = \ln(4 + \sqrt{17}) = 2.09$.

**81.** Since $\tan x$ is the inverse of $\mathrm{Tan}^{-1}x$, for all $x$, it follows that $f(x) = x$. Its graph is shown to the left below.
(b) On the interval $-\pi/2 \le x \le \pi/2$, the function $\mathrm{Tan}^{-1}x$ is the inverse $\tan x$. For these values of $x$ then, $f(x) = x$. Since $\tan x$ is $\pi$-periodic, so also is $f(x)$, and the graph is shown to the right below.



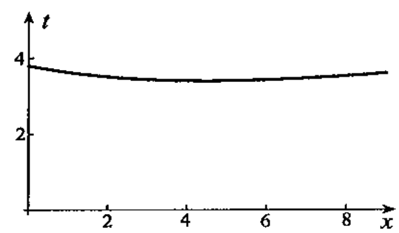

**82.** To solve the equation $x(t) = 0$, we divide by $\cos 4t$,

$$\tan 4t = -\frac{1}{10} \quad \Longrightarrow \quad 4t = \mathrm{Tan}^{-1}(-0.1) + n\pi = -0.09967 + n\pi \quad \Longrightarrow \quad t = \frac{1}{4}(-0.09967 + n\pi),$$

where $n$ is an integer. The smallest positive solution is 0.760 (when $n = 1$).

**83.** Travel time on water is distance on water $\sqrt{x^2+36}$ divided by rowing speed; travel time on land is distance on land $9 - x$ divided by walking speed. Total travel time is therefore

$$t = f(x) = \frac{\sqrt{x^2+36}}{3} + \frac{9-x}{5}.$$




**84.** My electronic device gives 1.526 0 for the only root of $f(x) = x^3 - 2x^2 + 4x - 5 = 0$. To verify that 1.526 is accurate to three decimals, we calculate $f(1.525\,5) = -2.2 \times 10^{-3}$ and $f(1.526\,5) = 2.6 \times 10^{-3}$.

**85.** My electronic device gives 1.409 6 and $-0.6367$ for roots of $f(x) = x^2 - 1 - \sin x = 0$. To verify that 1.410 is accurate to three decimals, we calculate $f(1.409\,5) = -3.3 \times 10^{-4}$ and $f(1.410\,5) = 2.3 \times 10^{-3}$. A similar calculation confirms $-0.637$ as the other root.

**86.** My electronic device gives $-11.618\,69$, $-0.873\,80$, and $0.492\,49$ for roots of $f(x) = x^3 + 12x^2 + 4x - 5 = 0$. To verify that 0.4925 has error less than $10^{-4}$, we calculate $f(0.4924) = -1.5 \times 10^{-3}$ and $f(0.4926) = 1.8 \times 10^{-3}$. A similar calculation confirms $-11.6187$ and $-0.8738$ as the other roots.

**87.** My electronic device gives $-4.938\,52$, $-3.697\,99$, $-0.041\,61$, and $2.842\,22$ for roots of $f(x) = x^2 - 1 - 24\sin x = 0$. To verify that 2.8422 has error less than $10^{-4}$, we calculate $f(2.8421) = -3.3 \times 10^{-3}$ and $f(2.8423) = 2.4 \times 10^{-3}$. A similar calculation confirms $-4.9385$, $-3.6980$, and $-0.0416$ as the other roots.

**88.** Let us take a coordinate system as shown, in which case $a^2 = b^2 + d^2$. Slopes of $AB$ and $OC$ are $d/(b-a)$ and $d/(b+a)$. The product of these is

$$\left(\frac{d}{b-a}\right)\left(\frac{d}{b+a}\right) = \frac{d^2}{b^2 - a^2} = -1.$$

Hence the diagonals are perpendicular.