CHAPTER 2

2.1) ENTROPY AND TEMPERATURE

(a) $\sigma = \log g = \log C + (3N/2)\log U$.

$(\partial\sigma/\partial U)_N = 1/\tau = (3N/2)(1/U)$, hence $U = 3N\tau/2$.

(b) $(\partial^2\sigma/\partial U^2)N = (3N/2)(-1/U^2) < 0$.

2.2) PARAMAGNETISM

States with spin excess 2s have the energy $U = -2smB$, hence $s = -U/2mB$. Insertion into (40) yields (41). From (41) obtain $(\partial\sigma/\partial U)_{N,B} = 1/\tau = -U/m^2B^2N$. Solve for U: $U = \langle U\rangle = -m^2B^2N/\tau = -MB$. Hence $M/Nm = mB/\tau$.

2.3) QUANTUM HARMONIC OSCILLATOR

(a) From (1.55):

$$\sigma = \log g = \log(N+n-1)! - \log n! - \log(N-1)!$$

$$\cong \log(N+n)! - \log n! - \log N!$$

Replacing N-1 by N is equivalent to neglecting a term $-\log[(N+n)/N]$, which for large N is small compared to the factorial terms. With the Stirling approximation:

$$\sigma \cong (N+n)\log(N+n) - n\log n - N\log N$$

$$= (N+n)\log[(N+n)/N] - n\log(n/N)$$

$$= N[(1+n/N)\log(1+n/N) - (n/N)\log(n/N)].$$

(b) Set $n = U/\hbar\omega$ and $N\hbar\omega = U_0(N)$:

$$\sigma(U,N) = N[(1+U/U_0)\log(1+U/U_0) - (U/U_0)\log(U/U_0)].$$

With the abbreviation $x = U/U_0$:

$$1/\tau = (\partial\sigma/\partial U)_N = (1/U_0)(\partial\sigma/\partial x)_N = (N/U_0)[\log(1+x)-\log x]$$
$$= (1/\hbar\omega)\log(1+U_0/U).$$

Solving for U/N yields $U/N = \hbar\omega/[\exp(\hbar\omega/\tau) - 1]$.

Alternate method: For large n:

$$d(\log n!)/dn \cong \log n! - \log(n-1)! = \log n.$$

With this,

$$1/\tau = (1/\hbar\omega)(\partial\sigma/\partial n)_N = (1/\hbar\omega)[\log(N+n) - \log n]$$
$$= (1/\hbar\omega)\log(1+N/n),$$

which is the equivalent to the result above.

2.4) THE MEANING OF NEVER

(a) The probability a correct key will be struck is 1/44. The probability a sequence of 10^5 keys will be correct is

$$(1/44)^{100,000} = 10^{-164,345}.$$

(b) On a single typewriter the number of keys that can be struck in 10^{18}s at 10 keys/s is 10^{19}. The Hamlet sequence may start with any of these except the last 10^5-1. Thus there are $10^{19} - 10^5 + 1 \cong 10^{19}$ possible starts. The probability of a correct Hamlet sequence on one typewriter is

$$10^{19}\times10^{-164,345} = 10^{-164,326}.$$

For 10^{10} monkeys:

$$10^{10}\times10^{-164,326} = 10^{-164,316}.$$

Comment. One UCSB student suggested that the monkeys be permitted at least as many typographical errors as in the lecture notes from which the main text was prepared. It is not difficult to show that even 1000 errors would have increased the probability by less than a factor

$$(43\times 10^5)^{1000} \cong 10^{6633}$$

to a value that is still negligible.

2.5) ADDITIVITY OF ENTROPY FOR TWO SPIN SYSTEMS

(a) From (17), with $N_1 = N_2 = 10^{22}$, $\delta = 10^{11}$:

$$g_1 g_2 = (g_1 g_2)_{max} \exp(-4\delta^2/N_1) = (g_1 g_2)_{max} \times \exp(-4),$$

$$(g_1 g_2)/(g_1 g_2)_{max} \cong 0.0183.$$

Compare this with 10^{-174} for $\delta = 10^{12}$.

(b) From (1.35):

$$g(N,s) = (2/\pi N)^{\frac{1}{2}} \times 2^N \times \exp(-2s^2/N).$$

This may be applied both to the two individual systems ($N = N_1 = N_2$), and to the combined system ($N = N_1 + N_2$). For the two individual systems in their most probable configuration we have from (14), $\hat{s}_1 = \hat{s}_2$. Hence

$$(g_1 g_2)_{max} = [g(N_1,\hat{s}_1)]^2 = (2/\pi N_1) \times 2^{2N_1} \times \exp(-4\hat{s}_1{}^2/N_1) .$$

For the combined system, $s = 2\hat{s}_1$:

$$\sum_{s_1} g_1(N_1,s_1) g_2(N_x,s-s_1) = g(2N_1,2\hat{s}1)$$

$$= (1/\pi N_1)^{\frac{1}{2}} \times 2^{2N_1} \times \exp(-4\hat{s}_1{}^2/N_1) .$$

The two results differ by the factor

$$g(2N_1,2\hat{s}_1)/[g(N_1,\hat{s}_1)]^2 = (N_1/2\pi)^{1/2} \cong 4\times10^{10} \quad .$$

(c) The true entropy of the interacting combined system is

$$\sigma = \log g(2N_1,2\hat{s}_1) = 2N_1 \log 2 - 4\hat{s}_1{}^2/N_1 - \tfrac{1}{2}\log(\pi N_1)$$

$$\cong 1.3863\times10^{22} - 10^{18} - 25.9 \cong 1.3862\times10^{22} \quad .$$

The error made in the entropy by estimating the entropy as $\log(g_1g_2)_{max} = \sigma_1+\sigma_2$ is $\log(4\times10^{10}) \cong 24.4$. This is a fractional error of 1.76×10^{-21}.

2.6) <u>INTEGRATED DEVIATION</u>

For $N_1 = N_2$, (17) becomes

$$g_1g_2 = (g_1g_2)_{max} \exp(-4\delta^2/N_1) \quad .$$

$$\sum_{\delta>\delta_1} g_1g_2 \cong (g_1g_2)_{max} \int_{\delta_1}^{\infty} \exp(-4\delta^2/N_1)d\delta$$

$$= (g_1g_2)_{max} (N_1/4)^{1/2} \int_{x}^{\infty} \exp(-t^2)dt \quad ,$$

where $x = 2\delta/N^{1/2}$. We count both positive and negative δ with $|\delta| > \delta_1$. Then

$$P(|\delta|>\delta_1) = \int_{x}^{\infty} \exp(-t^2)dt/\int_{0}^{\infty} \exp(-t^2)dt$$

$$= (2/\pi^{1/2}) \operatorname{erfc}(x) \cong \frac{1}{2\delta_1}\left(\frac{N_1}{\pi}\right)^{1/2} \exp(-4\delta_1{}^2/N_1)$$

$$= 0.28 \times \exp(-400) = \exp(-403.6) = 10^{-175.3} \quad .$$