CHAPTER 2: DETERMINANTS

2.1 Determinants by Cofactor Expansion

1. $M_{11} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 29$ $\qquad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 29$

$M_{12} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 21$ $\qquad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -21$

$M_{13} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 27$ $\qquad C_{13} = (-1)^{1+3} M_{13} = M_{13} = 27$

$M_{21} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -11$ $\qquad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 11$

$M_{22} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 13$ $\qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 13$

$M_{23} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = -5$ $\qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 5$

$M_{31} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = -19$ $\qquad C_{31} = (-1)^{3+1} M_{31} = M_{31} = -19$

$M_{32} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = -19$ $\qquad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 19$

$M_{33} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 19$ $\qquad C_{33} = (-1)^{3+3} M_{33} = M_{33} = 19$

2.

$$M_{11} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & 4 \end{vmatrix} = 6 \qquad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 6$$

$$M_{12} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 0 & 4 \end{vmatrix} = 12 \qquad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -12$$

$$M_{13} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix} = 3 \qquad C_{13} = (-1)^{1+3} M_{13} = M_{13} = 3$$

$$M_{21} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \qquad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -2$$

$$M_{22} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4 \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 4$$

$$M_{23} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -1$$

$$M_{31} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{31} = (-1)^{3+1} M_{31} = M_{31} = 0$$

$$M_{32} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 0$$

$$M_{33} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0 \qquad C_{33} = (-1)^{3+3} M_{33} = M_{33} = 0$$

3. **(a)**

$$M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

cofactor expansion along the first row

$$= 0 - 0 + 3(0) = 0$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13} = 0$$

(b) $M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 4(-12) + 1(-48) + 6(0) = -96$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 96$

(c) $M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = -4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} - 14\begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -4(-16) + 0 - 14(8) = -48$

$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -48$

(d) $M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} - 14\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -1(-16) + 0 - 14(-4) = 72$

$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -72$

4. **(a)** $M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(-3) + 1(-21) + 1(-3) = -30$

$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 30$

(b) $M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(2) - 3(-3) - 1(0) = 13$

$C_{44} = (-1)^{4+4} M_{44} = M_{44} = 13$

(c) $M_{41} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 3(-3) + 1(6) + 1(2) = -1$

$C_{41} = (-1)^{4+1} M_{41} = -M_{41} = 1$

(d) $M_{24} = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(0) - 3(0) - 1(0) = 0$

$C_{24} = (-1)^{2+4} M_{24} = M_{24} = 0$

5. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = (3)(4) - (5)(-2) = 12 + 10 = 22 \neq 0$. Inverse: $\frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & \frac{-5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$

6. $\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = (4)(2) - (1)(8) = 0$; The matrix is not invertible.

7. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (7)(-7) = 10 + 49 = 59 \neq 0$. Inverse: $\frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} \frac{-2}{59} & \frac{-7}{59} \\ \frac{7}{59} & \frac{-5}{59} \end{bmatrix}$

8. $\begin{vmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{vmatrix} = (\sqrt{2})(\sqrt{3}) - (\sqrt{6})(4) = \sqrt{6} - 4\sqrt{6} = -3\sqrt{6} \neq 0$. Inverse: $\frac{1}{-3\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ -4 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{3} \\ \frac{4}{3\sqrt{6}} & \frac{-1}{3\sqrt{3}} \end{bmatrix}$

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = \begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2) - 5(-3) = a^2 - 5a + 6 + 15 = a^2 - 5a + 21$

10. $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix} = \left|\begin{matrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{matrix}\right|\begin{matrix} -2 & 7 \\ 5 & 1 \\ 3 & 8 \end{matrix} = [-8 - 42 + 240] - [18 + 32 + 140] = 0$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \left|\begin{matrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{matrix}\right|\begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = [-20 - 7 + 72] - [20 + 84 + 6] = -65$

12. $$\begin{vmatrix}-1&1&2\\3&0&-5\\1&7&2\end{vmatrix} = \left|\begin{array}{ccc|cc}-1&1&2&-1&1\\3&0&-5&3&0\\1&7&2&1&7\end{array}\right| = \left[0-5+42\right]-\left[0+35+6\right]=-4$$

13. $$\begin{vmatrix}3&0&0\\2&-1&5\\1&9&-4\end{vmatrix} = \left|\begin{array}{ccc|cc}3&0&0&3&0\\2&-1&5&2&-1\\1&9&-4&1&9\end{array}\right| = \left[12+0+0\right]-\left[0+135+0\right]=-123$$

14. $$\begin{vmatrix}c&-4&3\\2&1&c^2\\4&c-1&2\end{vmatrix} = \left|\begin{array}{ccc|cc}c&-4&3&c&-4\\2&1&c^2&2&1\\4&c-1&2&4&c-1\end{array}\right| = \left[2c-16c^2+6(c-1)\right]-\left[12+(c-1)c^3-16\right]$$

$$=2c-16c^2+6c-6-12-c^4+c^3+16=-c^4+c^3-16c^2+8c-2$$

15. $$\det(A)=\begin{vmatrix}\lambda-2&1\\-5&\lambda+4\end{vmatrix}=(\lambda-2)(\lambda+4)-(1)(-5)=\lambda^2+2\lambda-3=(\lambda+3)(\lambda-1)$$

The determinant is zero if $\lambda=-3$ or $\lambda=1$.

16. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A)=\begin{vmatrix}\lambda-4&0&0\\0&\lambda&2\\0&3&\lambda-1\end{vmatrix} =(\lambda-4)\begin{vmatrix}\lambda&2\\3&\lambda-1\end{vmatrix}-0+0$$

$$=(\lambda-4)\left[\lambda(\lambda-1)-6\right]=(\lambda-4)\left[\lambda^2-\lambda-6\right]=(\lambda-4)(\lambda-3)(\lambda+2)$$

The determinant is zero if $\lambda=-2$, $\lambda=3$, or $\lambda=4$.

17. $$\det(A)=\begin{vmatrix}\lambda-1&0\\2&\lambda+1\end{vmatrix}=(\lambda-1)(\lambda+1)$$

The determinant is zero if $\lambda=1$ or $\lambda=-1$.

18. Calculate the determinant by a cofactor expansion along the third row:

$$\det(A)=\begin{vmatrix}\lambda-4&4&0\\-1&\lambda&0\\0&0&\lambda-5\end{vmatrix} =0-0+(\lambda-5)\begin{vmatrix}\lambda-4&4\\-1&\lambda\end{vmatrix}$$

$$=(\lambda-5)\left[(\lambda-4)\lambda+4\right]=(\lambda-5)\left[\lambda^2-4\lambda+4\right]=(\lambda-5)(\lambda-2)^2$$

The determinant is zero if $\lambda = 2$ or $\lambda = 5$.

19. **(a)** $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 + 0 = 3(-41) = -123$

(b) $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} = 3(-41) - 2(0) + 1(0) = -123$

(c) $-2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} = -2(0) - 1(-12) - 5(27) = -123$

(d) $-0 + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} = -1(-12) - 9(15) = -123$

(e) $1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = 1(0) - 9(15) - 4(-3) = -123$

(f) $0 - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -5(27) - 4(-3) = -123$

20. **(a)** $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} = (-1)(35) - 1(11) + 2(21) = -4$

(b) $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} = (-1)(35) - 3(-12) + 1(-5) = -4$

(c) $-3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 0 - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} = -3(-12) + 0 + 5(-8) = -4$

(d) $-1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 0 - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} = -1(11) + 0 - 7(-1) = -4$

(e) $1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 1(-5) - 7(-1) + 2(-3) = -4$

(f) $2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 2(21) + 5(-8) + 2(-3) = -4$

21. Calculate the determinant by a cofactor expansion along the second column:

$$-0 + 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 = 5(-8) = -40$$

22. Calculate the determinant by a cofactor expansion along the second row:

$$-1\begin{vmatrix} 3 & 1 \\ -3 & 5 \end{vmatrix} + 0 - (-4)\begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} = -1(18) + 0 + 4(-12) = -66$$

23. Calculate the determinant by a cofactor expansion along the first column:

$$1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} = 1(0) - 1(0) + 1(0) = 0$$

24. Calculate the determinant by a cofactor expansion along the second column:

$$\begin{aligned} -(k-1)\begin{vmatrix} 2 & 4 \\ 5 & k \end{vmatrix} + (k-3)\begin{vmatrix} k+1 & 7 \\ 5 & k \end{vmatrix} - (k+1)\begin{vmatrix} k+1 & 7 \\ 2 & 4 \end{vmatrix} \\ = -(k-1)(2k-20) + (k-3)((k+1)k-35) - (k+1)(4(k+1)-14) \\ = k^3 - 8k^2 - 10k + 95 \end{aligned}$$

25. Calculate the determinant by a cofactor expansion along the third column:

$$\det(A) = 0 - 0 + (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix}$$

Calculate the determinants in the third and fourth terms by a cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} = 3(24) - 3(8) + 5(16) = 128$$

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 3(2) - 3(8) + 5(-6) = -48$$

Therefore $\det(A) = 0 - 0 - 3(128) - 3(-48) = -240$.

26. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = 4\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} - 0 + 0 - 1\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} + 0$$

Calculate each of the two determinants by a cofactor expansion along its first row:

$$\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 & 3 \\ 6 & 2 & 3 \\ 4 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 0 = 3(0) - 3(0) - 1(0) - 0 = 0$$

$$\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & 3 \\ 9 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 9 & 4 & 3 \\ 2 & 2 & 3 \end{vmatrix} - 0 = 3(0) - 3(-6) + 3(-6) - 0 = 0$$

Therefore $\det(A) = 4(0) - 0 + 0 - 1(0) = 0$.

27. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (1)(-1)(1) = -1$.

28. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (2)(2)(2) = 8$.

29. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (0)(2)(3)(8) = 0$.

30. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(2)(3)(4) = 24$.

31. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(1)(2)(3) = 6$.

32. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (-3)(2)(-1)(3) = 18$.

33. **(a)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta)(\sin\theta) - (\cos\theta)(-\cos\theta) = \sin^2\theta + \cos^2\theta = 1$

(b) Calculate the determinant by a cofactor expansion along the third column:

$0 - 0 + 1\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0 - 0 + (1)(1) = 1$ (we used the result of part (a))

35. The minor M_{11} in both determinants is $\begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} = 1$. Expanding both determinants along the first row yields $d_1 + \lambda = d_2$.

37. If $n = 1$ then the determinant is 1.

If $n = 2$ then $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.

If $n = 3$ then a cofactor expansion will involves minors $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$. Therefore the determinant is 0.

By induction, we can show that the determinant will be 0 for all $n > 3$ as well.

43. Calculate the determinant by a cofactor expansion along the first column:

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix}$$

$$= (x_2x_3^2 - x_3x_2^2) - (x_1x_3^2 - x_3x_1^2) + (x_1x_2^2 - x_2x_1^2) = \left[x_3^2(x_2 - x_1) - x_3(x_2^2 - x_1^2)\right] + x_1x_2^2 - x_2x_1^2.$$

Factor out $(x_2 - x_1)$ to get $(x_2 - x_1)\left[x_3^2 - x_2x_3 - x_1x_3 + x_1x_2\right] = (x_2 - x_1)\left[x_3^2 - (x_2 + x_1)x_3 + x_1x_2\right]$.

Since $x_3^2 - (x_2 + x_1)x_3 + x_1x_2 = (x_3 - x_1)(x_3 - x_2)$, the determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

True-False Exercises

(a) False. The determinant is $ad - bc$.

(b) False. E.g., $\det(I_2) = \det(I_3) = 1$.

(c) True. If $i + j$ is even then $(-1)^{i+j} = 1$ therefore $C_{ij} = (-1)^{i+j} M_{ij} = M_{ij}$.

(d) True. Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Then $C_{12} = (-1)^{1+2}\begin{vmatrix} b & e \\ c & f \end{vmatrix} = -(bf - ec)$ and $C_{21} = (-1)^{2+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix} = -(bf - ce)$ therefore $C_{12} = C_{21}$. In the same way, one can show $C_{13} = C_{31}$ and $C_{23} = C_{32}$.

(e) True. This follows from Theorem 2.1.1.

(f) True. In formulas (7) and (8), each cofactor C_{ij} is zero.

(g) False. The determinant of a lower triangular matrix is the *product* of the entries along the main diagonal.

(h) False. E.g. $\det(2I_2) = 4 \neq 2 = 2\det(I_2)$.

(i) False. E.g., $\det(I_2 + I_2) = 4 \neq 2 = \det(I_2) + \det(I_2)$.

(j) True. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right) = \begin{vmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{vmatrix} = (a^2 + bc)(bc + d^2) - (ab + bd)(ac + cd)$

$= a^2bc + a^2d^2 + b^2c^2 + bcd^2 - a^2bc - abcd - abcd - bcd^2 = a^2d^2 + b^2c^2 - 2abcd$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 = (ad - bc)^2 = a^2d^2 - 2adbc + b^2c^2$ therefore $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right) = \left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2$.

2.2 Evaluating Determinants by Row Reduction

1. $\det(A) = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = (-2)(4) - (3)(1) = -11$; $\det(A^T) = \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = (-2)(4) - (1)(3) = -11$

2. $\det(A) = \begin{vmatrix} -6 & 1 \\ 2 & -2 \end{vmatrix} = (-6)(-2) - (1)(2) = 10$; $\det(A^T) = \begin{vmatrix} -6 & 2 \\ 1 & -2 \end{vmatrix} = (-6)(-2) - (2)(1) = 10$

3. $\det(A)=\begin{vmatrix}2 & -1 & 3\\ 1 & 2 & 4\\ 5 & -3 & 6\end{vmatrix}=[24-20-9]-[30-24-6]=-5$;

$\det(A^T)=\begin{vmatrix}2 & 1 & 5\\ -1 & 2 & -3\\ 3 & 4 & 6\end{vmatrix}=[24-9-20]-[30-24-6]=-5$ (we used the arrow technique)

4. $\det(A)=\begin{vmatrix}4 & 2 & -1\\ 0 & 2 & -3\\ -1 & 1 & 5\end{vmatrix}=[40+6-0]-[2-12+0]=56$;

$\det(A^T)=\begin{vmatrix}4 & 0 & -1\\ 2 & 2 & 1\\ -1 & -3 & 5\end{vmatrix}=[40-0+6]-[2-12+0]=56$ (we used the arrow technique)

5. The third row of I_4 was multiplied by -5. By Theorem 2.2.4, the determinant equals -5.

6. -5 times the first row of I_3 was added to the third row. By Theorem 2.2.4, the determinant equals 1.

7. The second and the third rows of I_4 were interchanged. By Theorem 2.2.4, the determinant equals -1.

8. The second row of I_4 was multiplied by $-\frac{1}{3}$. By Theorem 2.2.4, the determinant equals $-\frac{1}{3}$.

9. $\begin{vmatrix}3 & -6 & 9\\ -2 & 7 & -2\\ 0 & 1 & 5\end{vmatrix}=3\begin{vmatrix}1 & -2 & 3\\ -2 & 7 & -2\\ 0 & 1 & 5\end{vmatrix}$ ← A common factor of 3 from the first row was taken through the determinant sign.

$=3\begin{vmatrix}1 & -2 & 3\\ 0 & 3 & 4\\ 0 & 1 & 5\end{vmatrix}$ ← 2 times the first row was added to the second row.

$=3(-1)\begin{vmatrix}1 & -2 & 3\\ 0 & 1 & 5\\ 0 & 3 & 4\end{vmatrix}$ ← The second and third rows were interchanged.

$=(3)(-1)\begin{vmatrix}1 & -2 & 3\\ 0 & 1 & 5\\ 0 & 0 & -11\end{vmatrix}$ ← -3 times the second row was added to the third row.

$=(3)(-1)(-11)\begin{vmatrix}1 & -2 & 3\\ 0 & 1 & 5\\ 0 & 0 & 1\end{vmatrix}$ ← A common factor of -11 from the last row was taken through the determinant sign.

$=(3)(-1)(-11)(1)=33$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 3\left[1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - 0 + 0\right] = 3\left[(1)(11)\right] = 33\,.$$

10.

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ 0 & 5 & -1 \end{vmatrix}$$

2 times the first row was added to the third row.

$$= 3(-1)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{vmatrix}$$

The second and third rows were interchanged.

$$= (3)(-1)(5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & -2 \end{vmatrix}$$

A common factor of 5 from the second row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -2 from the last row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)(1) = 30$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ 0 & 5 & -1 \end{vmatrix} = 3\left[1\begin{vmatrix} 0 & -2 \\ 5 & -1 \end{vmatrix} - 0 + 0\right] = 3\left[(1)(10)\right] = 30\,.$$

11.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

The first and second rows were interchanged.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$ ← -2 times the first row was added to the second row.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$ ← -2 times the second row was added to the third row.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$ ← -1 times the second row was added to the fourth row.

$$=(-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$ ← A common factor of -1 from the third row was taken through the determinant sign.

$$=(-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$ ← -1 times the third row was added to the fourth row.

$$=(-1)(-1)(6)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ ← A common factor of 6 from the third row was taken through the determinant sign.

$$=(-1)(-1)(6)(1)=6$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}=(-1)(1)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix}=(-1)(1)(1)\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$=(-1)(1)(1)(-6)=6$.

12. $$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix}=\begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 5 & -2 & 2 \end{vmatrix}$$ ← 2 times the first row was added to the second row.

$$= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix}$$ ← -5 times the first row was added to the third row.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 13 & 2 \end{vmatrix}$$ ← A common factor of -2 from the second row was taken through the determinant sign.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{17}{2} \end{vmatrix}$$ ← -13 times the second row was added to the third row.

$$= (-2)\left(\tfrac{17}{2}\right)\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$$ ← A common factor of $\frac{17}{2}$ from the last row was taken through the determinant sign.

$$= (-2)\left(\tfrac{17}{2}\right)(1) = -17$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = (1)\begin{vmatrix} -2 & 1 \\ 13 & 2 \end{vmatrix} = (1)(-17) = -17.$$

13.

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$ ← 2 times the first row was added to the second row.

$$= (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$ ← A common factor of -1 from the second row was taken through the determinant sign.

$$=(-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

-2 times the third row was added to the fourth row.

$$=(-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

-1 times the fourth row was added to the fifth row.

$$=(-1)(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

A common factor of 2 from the fifth row was taken through the determinant sign.

$$=(-1)(2)(1)=-2$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the third step above:

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)(1)\begin{vmatrix} 1 & -2 & -6 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$=(-1)(1)(1)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)(2) = -2 .$$

14.

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix}$$

-5 times the first row was added to the second row.

$$=\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix}$$

The first row was added to the third row.

$$=\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & -3 & -1\\ 0 & 12 & 0 & -1\end{vmatrix}$$

-2 times the first row was added to the fourth row.

$$=\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & -3 & -1\\ 0 & 0 & 108 & 23\end{vmatrix}$$

-12 times the second row was added to the fourth row.

$$=-3\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 108 & 23\end{vmatrix}$$

A common factor of -3 from the third row was taken through the determinant sign.

$$=-3\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & -13\end{vmatrix}$$

-108 times the third row was added to the fourth row.

$$=(-3)(-13)\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & 1\end{vmatrix}$$

A common factor of -13 from the third row was taken through the determinant sign.

$$=(-3)(-13)(1)=39$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix}1 & -2 & 3 & 1\\ 5 & -9 & 6 & 3\\ -1 & 2 & -6 & -2\\ 2 & 8 & 6 & 1\end{vmatrix}=\begin{vmatrix}1 & -2 & 3 & 1\\ 0 & 1 & -9 & -2\\ 0 & 0 & -3 & -1\\ 0 & 0 & 108 & 23\end{vmatrix}=(1)\begin{vmatrix}1 & -9 & -2\\ 0 & -3 & -1\\ 0 & 108 & 23\end{vmatrix}=(1)(1)\begin{vmatrix}-3 & -1\\ 108 & 23\end{vmatrix}$$

$=(1)(1)(1)(39)=39$.

15.

$$\begin{vmatrix}d & e & f\\ g & h & i\\ a & b & c\end{vmatrix}=(-1)\begin{vmatrix}a & b & c\\ g & h & i\\ d & e & f\end{vmatrix}$$

The first and third rows were interchanged.

$$=(-1)(-1)\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}$$

The second and third rows were interchanged.

$$=(-1)(-1)(-6)=-6$$

16. The first and the third rows were interchanged, therefore $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-6) = 6.$

17. $$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$ ← A common factor of 3 from the first row was taken through the determinant sign.

$$=3(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix}$$ ← A common factor of -1 from the second row was taken through the determinant sign.

$$=3(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of 4 from the third row was taken through the determinant sign.

$$=3(-1)(4)(-6)=72$$

18. $$\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{vmatrix}$$ ← The second row was added to the first row.

$$=-1\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of -1 from the second row was taken through the determinant sign.

$$=(-1)(-6)=6$$

19. $$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← -1 times the third row was added to the first row.

$$=-6$$

20. $$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$ ← -3 times the first row was added to the last row.

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of 2 from the second row was taken through the determinant sign.

$$= (2)(-6) = -12$$

21. $$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$ ← A common factor of -3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← 4 times the second row was added to the last row.

$$= (-3)(-6) = 18$$

22. The third row is proportional to the first row, therefore by Theorem 2.2.5 $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix} = 0.$

(This can also be shown by adding -2 times the first row to the third, then performing a cofactor expansion of the resulting determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix}$ along the third row.)

23. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{vmatrix}$$ ← $-a$ times the first row was added to the second row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$ ← $-a^2$ times the first row was added to the third row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix}$$ ← $-(b+a)$ times the second row was added to the third row.

$$= (1)(b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

24. **(a)** Interchanging the first row and the third row and applying Theorem 2.1.2 yields

$$\det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)\det\begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{13} \end{bmatrix} = -a_{13}a_{22}a_{31}$$

(b) We interchange the first and the fourth row, as well as the second and the third row. Then we use Theorem 2.1.2 to obtain

$$\det\begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = (-1)(-1)\det\begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{14} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$$

Generally for any $n\times n$ matrix A such that $a_{ij}=0$ if $i+j\le n$ we have

$\det(A) = (-1)^n a_{1n}a_{2,n-1}\cdots a_{n1}$.

25.

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix} \quad \longleftarrow \quad \text{-1 times the first column was added to the third column.}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \longleftarrow \quad \text{-1 times the second column was added to the third column.}$$

26.

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ (1-t^2)b_1 & (1-t^2)b_2 & (1-t^2)b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \longleftarrow \quad \text{$-t$ times the first row was added to the second row.}$$

$$= (1-t^2)\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \longleftarrow \quad \text{A common factor of $1-t^2$ from the second row was taken through the determinant sign.}$$

$$= (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \longleftarrow \quad \text{$-t$ times the second row was added to the first row.}$$

27.
$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1 & -2b_1 & c_1 \\ a_2+b_2 & -2b_2 & c_2 \\ a_3+b_3 & -2b_3 & c_3 \end{vmatrix}$$
← -1 times the first column was added to the second column.

$$= -2\begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$$
← A common factor of -2 from the second column was taken through the determinant sign.

$$= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
← -1 times the second column was added to the first column.

28.
$$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix}$$
← $-t$ times the first column was added to the second column.

$$= \begin{vmatrix} a_1 & b_1 & c_1+rb_1 \\ a_2 & b_2 & c_2+rb_2 \\ a_3 & b_3 & c_3+rb_3 \end{vmatrix}$$
← $-s$ times the first column was added to the third column.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
← $-r$ times the second column was added to the third column.

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
← The matrix was transposed. (Theorem 2.2.2)

29. The second column vector is a scalar multiple of the fourth. By Theorem 2.2.5, the determinant is 0.

30. Adding the second, third, fourth, and fifth rows to the first results in the first row made up of zeros.

31. $$\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ -1 & 3 & 2 \end{vmatrix}\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & -4 \end{vmatrix} = \left(0-0+2\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right)\left(0-0+(-4)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}\right) = (2)(-12) = -24$$

32. $\det(M)=\begin{vmatrix}1&2&0\\0&1&2\\0&0&1\end{vmatrix}\begin{vmatrix}1&2\\0&1\end{vmatrix}=((1)(1)(1))((1)(1))=1$

33. In order to reverse the order of rows in 2×2 and 3×3 matrix, the first and the last rows can be interchanged, so $\det(B)=-\det(A)$.

For 4×4 and 5×5 matrices, two such interchanges are needed: the first and last rows can be swapped, then the second and the penultimate one can follow.

Thus, $\det(B)=(-1)(-1)\det(A)=\det(A)$ in this case.

Generally, to rows in an $n\times n$ matrix can be reversed by

- interchanging row 1 with row n,
- interchanging row 2 with row $n-1$,
- $\vdots$
- interchanging row $n/2$ with row $n-n/2$

where x is the greatest integer less than or equal to x (also known as the "floor" of x).

We conclude that $\det(B)=(-1)^{n/2}\det(A)$.

34.

$$\begin{vmatrix}a&b&b&b\\b&a&b&b\\b&b&a&b\\b&b&b&a\end{vmatrix}=\begin{vmatrix}a&b&b&b\\b-a&a-b&0&0\\b-a&0&a-b&0\\b-a&0&0&a-b\end{vmatrix}$$

-1 times the first row was added to each of the remaining rows.

$$=\begin{vmatrix}a+b&b&b&b\\b-a&a-b&0&0\\b-a&0&a-b&0\\0&0&0&a-b\end{vmatrix}$$

The last column was added to the first column.

$$=\begin{vmatrix}a+2b&b&b&b\\b-a&a-b&0&0\\0&0&a-b&0\\0&0&0&a-b\end{vmatrix}$$

The third column was added to the first column.

$$=\begin{vmatrix}a+3b&b&b&b\\0&a-b&0&0\\0&0&a-b&0\\0&0&0&a-b\end{vmatrix}$$

The second column was added to the first column.

$$=(a+3b)(a-b)^3$$

True-False Exercises

(a) True. $\det(B)=(-1)(-1)\det(A)=\det(A)$.

(b) True. $\det(B)=(4)(\tfrac{3}{4})\det(A)=3\det(A)$.

(c) False. $\det(B)=\det(A)$.

(d) False. $\det(B)=n(n-1)\cdots3\cdot2\cdot1\cdot\det(A)=(n!)\det(A)$.

(e) True. This follows from Theorem 2.2.5.

(f) True. Let B be obtained from A by adding the second row to the fourth row, so $\det(A)=\det(B)$. Since the fourth row and the sixth row of B are identical, by Theorem 2.2.5 $\det(B)=0$.

2.3 Properties of Determinants; Cramer's Rule

1. $\det(2A)=\begin{vmatrix}-2 & 4\\ 6 & 8\end{vmatrix}=(-2)(8)-(4)(6)=-40$

$(2)^2\det(A)=4\begin{vmatrix}-1 & 2\\ 3 & 4\end{vmatrix}=4\big((-1)(4)-(2)(3)\big)=(4)(-10)=-40$

2. $\det(-4A)=\begin{vmatrix}-8 & -8\\ -20 & 8\end{vmatrix}=(-8)(8)-(-8)(-20)=-224$

$(-4)^2\det(A)=16\begin{vmatrix}2 & 2\\ 5 & -2\end{vmatrix}=16\big((2)(-2)-(2)(5)\big)=(16)(-14)=-224$

3. We are using the arrow technique to evaluate both determinants.

$$\det(-2A)=\begin{vmatrix}-4 & 2 & -6\\ -6 & -4 & -2\\ -2 & -8 & -10\end{vmatrix}=(-160+8-288)-(-48-64+120)=-448$$

$$(-2)^3\det(A)=-8\begin{vmatrix}2 & -1 & 3\\ 3 & 2 & 1\\ 1 & 4 & 5\end{vmatrix}=(-8)\big((20-1+36)-(6+8-15)\big)=(-8)(56)=-448$$

4. We are using the cofactor expansion along the first column to evaluate both determinants.

$$\det(3A)=\begin{vmatrix}3 & 3 & 3\\ 0 & 6 & 9\\ 0 & 3 & -6\end{vmatrix}=3\begin{vmatrix}6 & 9\\ 3 & -6\end{vmatrix}=3\big((6)(-6)-(9)(3)\big)=(3)(-63)=-189$$

$$3^3\det(A)=27\begin{vmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 0 & 1 & -2\end{vmatrix}=(27)(1)\begin{vmatrix}2 & 3\\ 1 & -2\end{vmatrix}=27\big((2)(-2)-(3)(1)\big)=(27)(-7)=-189$$

5. We are using the arrow technique to evaluate the determinants in this problem.

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = (18-170+0)-(80+0-62) = -170;$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix} = (-22-120+510)-(660-20-102) = -170;$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = (45+0+0)-(75+0+0) = -30;$$

$\det(A) = (16+0+0)-(0+0+6) = 10$;

$\det(B) = (1-10+0)-(15+0-7) = -17$;

$\det(A+B) \neq \det(A)+\det(B)$

6. We are using the arrow technique to evaluate the determinants in this problem.

$$\det(AB) = \begin{vmatrix} 6 & 15 & 26 \\ 2 & -4 & -3 \\ -2 & 10 & 12 \end{vmatrix} = (-288+90+520)-(208-180+360) = -66;$$

$$\det(BA) = \begin{vmatrix} 5 & 8 & -3 \\ -6 & 14 & 7 \\ 5 & -2 & -5 \end{vmatrix} = (-350+280-36)-(-210-70+240) = -66;$$

$$\det(A+B) = \begin{vmatrix} 1 & 7 & -2 \\ 2 & 1 & 2 \\ -2 & 5 & 1 \end{vmatrix} = (1-28-20)-(4+10+14) = -75;$$

$\det(A) = (0+16+4)-(0+2+16) = 2$;

$\det(B) = (-2+0-12)-(0+18+1) = -33$;

$\det(A+B) \neq \det(A)+\det(B)$;

7. $\det(A) = (-6+0-20)-(-10+0-15) = -1 \neq 0$ therefore A is invertible by Theorem 2.3.3

8. $\det(A) = (-24+0+0)-(-18+0+0) = -6 \neq 0$ therefore A is invertible by Theorem 2.3.3

9. $\det(A) = (2)(1)(2) = 4 \neq 0$ therefore A is invertible by Theorem 2.3.3

10. $\det(A) = 0$ (second column contains only zeros) therefore A is not invertible by Theorem 2.3.3

11. $\det(A) = (24-24-16)-(24-16-24) = 0$ therefore A is not invertible by Theorem 2.3.3

12. $\det(A) = (1+0-81)-(8+36+0) = -124 \neq 0$ therefore A is invertible by Theorem 2.3.3

13. $\det(A)=(2)(1)(6)=12\neq 0$ therefore A is invertible by Theorem 2.3.3

14. $\det(A)=0$ (third column contains only zeros) therefore A is not invertible by Theorem 2.3.3

15. $\det(A)=(k-3)(k-2)-(-2)(-2)=k^2-5k+2=\left(k-\frac{5-\sqrt{17}}{2}\right)\left(k-\frac{5+\sqrt{17}}{2}\right)$. By Theorem 2.3.3, A is invertible if $k\neq\frac{5-\sqrt{17}}{2}$ and $k\neq\frac{5+\sqrt{17}}{2}$.

16. $\det(A)=k^2-4=(k-2)(k+2)$. By Theorem 2.3.3, A is invertible if $k\neq 2$ and $k\neq -2$.

17. $\det(A)=(2+12k+36)-(4k+18+12)=8+8k=8(1+k)$.
By Theorem 2.3.3, A is invertible if $k\neq -1$.

18. $\det(A)=(1+0+0)-(0+2k+2k)=1-4k$. By Theorem 2.3.3, A is invertible if $k\neq\frac{1}{4}$.

19. $\det(A)=(-6+0-20)-(-10+0-15)=-1\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}-1 & 0\\ 4 & 3\end{vmatrix}=-3\quad C_{12}=-\begin{vmatrix}-1 & 0\\ 2 & 3\end{vmatrix}=3\quad C_{13}=\begin{vmatrix}-1 & -1\\ 2 & 4\end{vmatrix}=-2$$
$$C_{21}=-\begin{vmatrix}5 & 5\\ 4 & 3\end{vmatrix}=5\quad C_{22}=\begin{vmatrix}2 & 5\\ 2 & 3\end{vmatrix}=-4\quad C_{23}=-\begin{vmatrix}2 & 5\\ 2 & 4\end{vmatrix}=2$$
$$C_{31}=\begin{vmatrix}5 & 5\\ -1 & 0\end{vmatrix}=5\quad C_{32}=-\begin{vmatrix}2 & 5\\ -1 & 0\end{vmatrix}=-5\quad C_{33}=\begin{vmatrix}2 & 5\\ -1 & -1\end{vmatrix}=3$$

The matrix of cofactors is $\begin{bmatrix}-3 & 3 & -2\\ 5 & -4 & 2\\ 5 & -5 & 3\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}-3 & 5 & 5\\ 3 & -4 & -5\\ -2 & 2 & 3\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-1}\begin{bmatrix}-3 & 5 & 5\\ 3 & -4 & -5\\ -2 & 2 & 3\end{bmatrix}=\begin{bmatrix}3 & -5 & -5\\ -3 & 4 & 5\\ 2 & -2 & -3\end{bmatrix}$.

20. $\det(A)=(-24+0+0)-(-18+0+0)=-6\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}3 & 2\\ 0 & -4\end{vmatrix}=-12\quad C_{12}=-\begin{vmatrix}0 & 2\\ -2 & -4\end{vmatrix}=-4\quad C_{13}=\begin{vmatrix}0 & 3\\ -2 & 0\end{vmatrix}=6$$
$$C_{21}=-\begin{vmatrix}0 & 3\\ 0 & -4\end{vmatrix}=0\quad C_{22}=\begin{vmatrix}2 & 3\\ -2 & -4\end{vmatrix}=-2\quad C_{23}=-\begin{vmatrix}2 & 0\\ -2 & 0\end{vmatrix}=0$$
$$C_{31}=\begin{vmatrix}0 & 3\\ 3 & 2\end{vmatrix}=-9\quad C_{32}=-\begin{vmatrix}2 & 3\\ 0 & 2\end{vmatrix}=-4\quad C_{33}=\begin{vmatrix}2 & 0\\ 0 & 3\end{vmatrix}=6$$

The matrix of cofactors is $\begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$.

21. $\det(A) = (2)(1)(2) = 4 \neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{12} = -\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} = 0 \qquad C_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix} = 6 \qquad C_{22} = \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} = 4 \qquad C_{23} = -\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} = 6 \qquad C_{33} = \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2$$

The matrix of cofactors is $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 4 & 6 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{4}\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

22. $\det(A) = (2)(1)(6) = 12$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} = 6 \qquad C_{12} = -\begin{vmatrix} 8 & 0 \\ -5 & 6 \end{vmatrix} = -48 \qquad C_{13} = \begin{vmatrix} 8 & 1 \\ -5 & 3 \end{vmatrix} = 29$$

$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{22} = \begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix} = 12 \qquad C_{23} = -\begin{vmatrix} 2 & 0 \\ -5 & 3 \end{vmatrix} = -6$$

$$C_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 \qquad C_{32} = -\begin{vmatrix} 2 & 0 \\ 8 & 0 \end{vmatrix} = 0 \qquad C_{33} = \begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix} = 2$$

The matrix of cofactors is $\begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$.

23. $\begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ ← -2 times the first row was added to the second row; -1 times the first row was added to the third and fourth rows.

$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 8 \end{vmatrix}$ ← The third row and the fourth row were interchanged.

$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ ← -7 times the third row was added to the fourth row

$= -(1)(-1)(1)(1) = 1$

The determinant of A is nonzero therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = (80+54+12)-(48+90+12) = -4$$

$$C_{12} = -\begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -\left[(32+18+4)-(16+36+4)\right] = 2$$

$$C_{13} = \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = (12+45+6)-(6+54+10) = -7$$

$$C_{14} = -\begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(12+40+6)-(6+48+10)\right] = 6$$

$$C_{21} = -\begin{vmatrix} 3 & 1 & 1 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -\left[(48+27+6)-(24+54+6)\right] = 3$$

$$C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = (16+9+2)-(8+18+2) = -1$$

$$C_{23} = -\begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(6+27+3)-(3+27+6)\right] = 0$$

$$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = (6+24+3)-(3+24+6) = 0$$

$$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = (12+6+10)-(6+12+10) = 0$$

$$C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -\left[(4+2+4)-(2+4+4)\right] = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = (10+6+6)-(5+6+12) = -1$$

$$C_{34} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(10+6+6)-(5+6+12)\right] = 1$$

$$C_{41} = -\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -\left[(54+6+40)-(6+48+45)\right] = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = (18+2+16)-(2+16+18) = 0$$

$$C_{43} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -\left[(45+6+6)-(5+6+54)\right] = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = (40+6+6)-(5+6+48) = -7$$

The matrix of cofactors is $\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

24. $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$; $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$

25. $\det(A)=\begin{vmatrix}4&5&0\\11&1&2\\1&5&2\end{vmatrix}=(8+10+0)-(0+40+110)=-132$,

$\det(A_1)=\begin{vmatrix}2&5&0\\3&1&2\\1&5&2\end{vmatrix}=(4+10+0)-(0+20+30)=-36$,

$\det(A_2)=\begin{vmatrix}4&2&0\\11&3&2\\1&1&2\end{vmatrix}=(24+4+0)-(0+8+44)=-24$,

$\det(A_3)=\begin{vmatrix}4&5&2\\11&1&3\\1&5&1\end{vmatrix}=(4+15+110)-(2+60+55)=12$;

$x=\frac{\det(A_1)}{\det(A)}=\frac{-36}{-132}=\frac{3}{11}$, $y=\frac{\det(A_2)}{\det(A)}=\frac{-24}{-132}=\frac{2}{11}$, $z=\frac{\det(A_3)}{\det(A)}=\frac{12}{-132}=-\frac{1}{11}$.

26. $\det(A)=\begin{vmatrix}1&-4&1\\4&-1&2\\2&2&-3\end{vmatrix}=(3-16+8)-(-2+4+48)=-55$,

$\det(A_1)=\begin{vmatrix}6&-4&1\\-1&-1&2\\-20&2&-3\end{vmatrix}=(18+160-2)-(20+24-12)=144$,

$\det(A_2)=\begin{vmatrix}1&6&1\\4&-1&2\\2&-20&-3\end{vmatrix}=(3+24-80)-(-2-40-72)=61$,

$\det(A_3)=\begin{vmatrix}1&-4&6\\4&-1&-1\\2&2&-20\end{vmatrix}=(20+8+48)-(-12-2+320)=-230$;

$x=\frac{\det(A_1)}{\det(A)}=\frac{144}{-55}=-\frac{144}{55}$, $y=\frac{\det(A_2)}{\det(A)}=\frac{61}{-55}=-\frac{61}{55}$, $z=\frac{\det(A_3)}{\det(A)}=\frac{-230}{-55}=\frac{46}{11}$.

27. $\det(A)=\begin{vmatrix}1&-3&1\\2&-1&0\\4&0&-3\end{vmatrix}=(3+0+0)-(-4+0+18)=-11$,

$\det(A_1)=\begin{vmatrix}4&-3&1\\-2&-1&0\\0&0&-3\end{vmatrix}=-3\begin{vmatrix}4&-3\\-2&-1\end{vmatrix}=(-3)(-4-6)=30$,

$$\det(A_2) = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{vmatrix} = (6+0+0)-(-8+0-24) = 38,$$

$$\det(A_3) = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{vmatrix} = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = (4)(6+4) = 40;$$

$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{30}{-11} = -\frac{30}{11}$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{38}{-11} = -\frac{38}{11}$, $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{40}{-11} = -\frac{40}{11}$.

28. $$\det(A) = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 & 9 \\ -1 & 1 & 1 \\ 1 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 & 7 \\ -1 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= -\left[(12-14+9)-(-54-1-28)\right] + 4\left[(-24+7-9)-(27+2+28)\right]$$
$$+2\left[(-8-1+18)-(9-4-4)\right] - \left[(2-3+14)-(7-12+1)\right]$$

$$= -90-332+16-17 = -423$$

$$\det(A_1) = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$= -32\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 14 & 7 & 9 \\ 11 & 3 & 1 \\ -4 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 14 & -1 & 9 \\ 11 & 1 & 1 \\ -4 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 14 & -1 & 7 \\ 11 & 1 & 3 \\ -4 & -2 & 1 \end{vmatrix}$$

$$= -32\left[(12-14+9)-(-54-1-28)\right] + 4\left[(-168-28+99)-(-108+14-308)\right]$$
$$+2\left[(-56+4-198)-(-36-28+44)\right] - \left[(14+12-154)-(-28-84-11)\right]$$
$$= -2880+1220-460+5 = -2115$$

$$\det(A_2) = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ 1 & -4 & 1 & -4 \end{vmatrix}$$

$$= -1\begin{vmatrix} 14 & 7 & 9 \\ 11 & 3 & 1 \\ -4 & 1 & -4 \end{vmatrix} + 32\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & 14 & 9 \\ -1 & 11 & 1 \\ 1 & -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 14 & 7 \\ -1 & 11 & 3 \\ 1 & -4 & 1 \end{vmatrix}$$

$$= -\left[(-168-28+99)-(-108+14-308)\right] + 32\left[(-24+7-9)-(27+2+28)\right]$$

$$+2\left[(-88+14+36)-(99-8+56)\right]-\left[(22+42+28)-(77-24-14)\right]$$
$$=-305-2656-370-53=-3384$$

$$\det(A_3)=\begin{vmatrix}-1 & -4 & -32 & 1\\ 2 & -1 & 14 & 9\\ -1 & 1 & 11 & 1\\ 1 & -2 & -4 & -4\end{vmatrix}$$

$$=-1\begin{vmatrix}-1 & 14 & 9\\ 1 & 11 & 1\\ -2 & -4 & -4\end{vmatrix}+4\begin{vmatrix}2 & 14 & 9\\ -1 & 11 & 1\\ 1 & -4 & -4\end{vmatrix}-32\begin{vmatrix}2 & -1 & 9\\ -1 & 1 & 1\\ 1 & -2 & -4\end{vmatrix}-1\begin{vmatrix}2 & -1 & 14\\ -1 & 1 & 11\\ 1 & -2 & -4\end{vmatrix}$$

$$=-\left[(44-28-36)-(-198+4-56)\right]+4\left[(-88+14+36)-(99-8+56)\right]$$
$$-32\left[(-8-1+18)-(9-4-4)\right]-\left[(-8-11+28)-(14-44-4)\right]$$
$$=-230-740-256-43=-1269$$

$$\det(A_4)=\begin{vmatrix}-1 & -4 & 2 & -32\\ 2 & -1 & 7 & 14\\ -1 & 1 & 3 & 11\\ 1 & -2 & 1 & -4\end{vmatrix}$$

$$=-1\begin{vmatrix}-1 & 7 & 14\\ 1 & 3 & 11\\ -2 & 1 & -4\end{vmatrix}+4\begin{vmatrix}2 & 7 & 14\\ -1 & 3 & 11\\ 1 & 1 & -4\end{vmatrix}+2\begin{vmatrix}2 & -1 & 14\\ -1 & 1 & 11\\ 1 & -2 & -4\end{vmatrix}+32\begin{vmatrix}2 & -1 & 7\\ -1 & 1 & 3\\ 1 & -2 & 1\end{vmatrix}$$

$$=-\left[(12-154+14)-(-84-11-28)\right]+4\left[(-24+77-14)-(42+22+28)\right]$$
$$+2\left[(-8-11+28)-(14-44-4)\right]+32\left[(2-3+14)-(7-12+1)\right]$$
$$=5-212+86+544=423$$

$x_1=\frac{\det(A_1)}{\det(A)}=\frac{-2115}{-423}=5$, $\qquad x_2=\frac{\det(A_2)}{\det(A)}=\frac{-3384}{-423}=8$,

$x_3=\frac{\det(A_3)}{\det(A)}=\frac{-1269}{-423}=3$, $\qquad x_4=\frac{\det(A_4)}{\det(A)}=\frac{423}{-423}=-1$

29. $\det(A)=0$ therefore Cramer's rule does not apply.

30. $\det(A)=\cos^2\theta+\sin^2\theta=1$ is nonzero for all values of θ, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11}=\cos\theta & C_{12}=\sin\theta & C_{13}=0\\ C_{21}=-\sin\theta & C_{22}=\cos\theta & C_{23}=0\\ C_{31}=0 & C_{32}=0 & C_{33}=\cos^2\theta+\sin^2\theta=1\end{array}$$

The matrix of cofactors is

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

and the adjoint matrix is

$$\operatorname{adj}(A)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

From Theorem 2.3.6, we have

$$A^{-1}=\tfrac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{1}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

31. $\det(A)=\begin{vmatrix}4 & 1 & 1 & 1\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & 8\\ 1 & 1 & 1 & 2\end{vmatrix}=-424$; $\det(A_2)=\begin{vmatrix}4 & 6 & 1 & 1\\ 3 & 1 & -1 & 1\\ 7 & -3 & -5 & 8\\ 1 & 3 & 1 & 2\end{vmatrix}=0$; $y=\frac{\det(A_2)}{\det(A)}=\frac{0}{-424}=0$

32. **(a)** $A=\begin{bmatrix}4 & 1 & 1 & 1\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & 8\\ 1 & 1 & 1 & 2\end{bmatrix}$,

$$A_1=\begin{bmatrix}6 & 1 & 1 & 1\\ 1 & 7 & -1 & 1\\ -3 & 3 & -5 & 8\\ 3 & 1 & 1 & 2\end{bmatrix},\ A_2=\begin{bmatrix}4 & 6 & 1 & 1\\ 3 & 1 & -1 & 1\\ 7 & -3 & -5 & 8\\ 1 & 3 & 1 & 2\end{bmatrix},\ A_3=\begin{bmatrix}4 & 1 & 6 & 1\\ 3 & 7 & 1 & 1\\ 7 & 3 & -3 & 8\\ 1 & 1 & 3 & 2\end{bmatrix},\ A_4=\begin{bmatrix}4 & 1 & 1 & 6\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & -3\\ 1 & 1 & 1 & 3\end{bmatrix};$$

$x=\frac{\det(A_1)}{\det(A)}=\frac{-424}{-424}=1$, $y=\frac{\det(A_2)}{\det(A)}=\frac{0}{-424}=0$, $z=\frac{\det(A_3)}{\det(A)}=\frac{-848}{-424}=2$, $w=\frac{\det(A_4)}{\det(A)}=\frac{0}{-424}=0$

(b) The augmented matrix of the system $\begin{bmatrix}4 & 1 & 1 & 1 & 6\\ 3 & 7 & -1 & 1 & 1\\ 7 & 3 & -5 & 8 & -3\\ 1 & 1 & 1 & 2 & 3\end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$ therefore the system has only one solution: $x=1$, $y=0$, $z=2$, and $w=0$.

(c) The method in part (b) requires fewer computations.

33. **(a)** $\det(3A)=3^3\det(A)=(27)(-7)=-189$ (using Formula (1))

(b) $\det(A^{-1})=\frac{1}{\det(A)}=\frac{1}{-7}=-\frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det\left(2A^{-1}\right)=2^3\det\left(A^{-1}\right)=\frac{8}{\det(A)}=\frac{8}{-7}=-\frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det\left(\left(2A\right)^{-1}\right)=\frac{1}{\det(2A)}=\frac{1}{2^3\det(A)}=\frac{1}{(8)(-7)}=-\frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

(e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}=-\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}=-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=-(-7)=7$ (in the first step we interchanged the last two columns applying Theorem 2.2.3(b); in the second step we transposed the matrix applying Theorem 2.2.2)

34. **(a)** $\det(-A)=\det((-1)A)=(-1)^4\det(A)=\det(A)=-2$ (using Formula (1))

(b) $\det\left(A^{-1}\right)=\frac{1}{\det(A)}=\frac{1}{-2}=-\frac{1}{2}$ (using Theorem 2.3.5)

(c) $\det\left(2A^T\right)=2^4\det\left(A^T\right)=16\det(A)=-32$ (using Formula (1) and Theorem 2.2.2)

(d) $\det\left(A^3\right)=\det(AAA)=\det(A)\det(A)\det(A)=(-2)^3=-8$ (using Theorem 2.3.4)

35. **(a)** $\det(3A)=3^3\det(A)=(27)(7)=189$ (using Formula (1))

(b) $\det\left(A^{-1}\right)=\frac{1}{\det(A)}=\frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det\left(2A^{-1}\right)=2^3\det\left(A^{-1}\right)=\frac{8}{\det(A)}=\frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det\left(\left(2A\right)^{-1}\right)=\frac{1}{\det(2A)}=\frac{1}{2^3\det(A)}=\frac{1}{(8)(7)}=\frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

True-False Exercises

(a) False. By Formula (1), $\det(2A)=2^3\det(A)=8\det(A)$.

(b) False. E.g. $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have $\det(A)=\det(B)=0$ but $\det(A+B)=1\neq 2\det(A)$.

(c) True. By Theorems 2.3.4 and 2.3.5,
$\det\left(A^{-1}BA\right)=\det\left(A^{-1}\right)\det(B)\det(A)=\frac{1}{\det(A)}\det(B)\det(A)=\det(B)$.

(d) False. A square matrix A is invertible if and only if $\det(A)\neq 0$.

(e) True. This follows from Definition 1.

(f) True. This is Formula (8).

(g) True. If $\det(A)\neq 0$ then by Theorem 2.3.8 $A\mathbf{x}=0$ must have only the trivial solution, which contradicts our assumption. Consequently, $\det(A)=0$.

(h) True. If the reduced row echelon form of A is I_n then by Theorem 2.3.8 $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b}$, which contradicts our assumption. Consequently, the reduced row echelon form of A cannot be I_n.

(i) True. Since the reduced row echelon form of E is I then by Theorem 2.3.8 $E\mathbf{x}=0$ must have only the trivial solution.

(j) True. If A is invertible, so is A^{-1}. By Theorem 2.3.8, each system has only the trivial solution.

(k) True. From Theorem 2.3.6, $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A)=\det(A)A^{-1}$. Consequently,
$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A)=\left(\frac{1}{\det(A)}A\right)\left(\det(A)\,A^{-1}\right)=\frac{\det(A)}{\det(A)}\left(AA^{-1}\right)=I_n$ so $\left(\operatorname{adj}(A)\right)^{-1}=\frac{1}{\det(A)}A$.

(l) False. If the kth row of A contains only zeros then all cofactors C_{jk} where $j\neq i$ are zero (since each of them involves a determinant of a matrix with a zero row). This means the matrix of cofactors contains at least one zero row, therefore $\operatorname{adj}(A)$ has a *column* of zeros.

Chapter 2 Supplementary Exercises

1. **(a)** Cofactor expansion along the first row: $\begin{vmatrix}-4 & 2\\ 3 & 3\end{vmatrix}=(-4)(3)-(2)(3)=-12-6=-18$

(b) $\begin{vmatrix}-4 & 2\\ 3 & 3\end{vmatrix}=-\begin{vmatrix}3 & 3\\ -4 & 2\end{vmatrix}$ ← The first and second rows were interchanged.

$=-(3)\begin{vmatrix}1 & 1\\ -4 & 2\end{vmatrix}$ ← A common factor of 3 from the first row was taken through the determinant sign.

$=-(3)\begin{vmatrix}1 & 1\\ 0 & 6\end{vmatrix}$ ← 4 times the first row was added to the second row

$=-(3)(1)(6)=-18$ ← Use Theorem 2.1.2.

2. **(a)** Cofactor expansion along the first row: $\begin{vmatrix}7 & -1\\ -2 & -6\end{vmatrix}=(7)(-6)-(-1)(-2)=-42-2=-44$

(b) $\begin{vmatrix}7 & -1\\ -2 & -6\end{vmatrix}=-\begin{vmatrix}-2 & -6\\ 7 & -1\end{vmatrix}$ ← The first and second rows were interchanged.

$=-(-2)\begin{vmatrix}1 & 3\\ 7 & -1\end{vmatrix}$ ← A common factor of -2 from the first row was taken through the determinant sign.

$=-(-2)\begin{vmatrix}1 & 3\\ 0 & -22\end{vmatrix}$ ← -7 times the first row was added to the second row

$$= -(-2)(1)(-22)$$ ← Use Theorem 2.1.2.

$$= -44$$

3. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = -0 + 2\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 5 \\ -3 & 1 \end{vmatrix}$$

$$= 0 + 2\left[(-1)(1) - (2)(-3)\right] - (-1)\left[(-1)(1) - (5)(-3)\right]$$

$$= 0 + (2)(5) - (-1)(14) = 0 + 10 + 14 = 24$$

(b)

$$\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}$$ ← A common factor of -1 from the first row was taken through the determinant sign.

$$= (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ 0 & -14 & -5 \end{vmatrix}$$ ← 3 times the first row was added to the third row.

$$= (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & -12 \end{vmatrix}$$ ← 7 times the second row was added to the third.

$$= (-1)(1)(2)(-12) = 24$$ ← Use Theorem 2.1.2.

4. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} -5 & -6 \\ -8 & -9 \end{vmatrix} - (-2)\begin{vmatrix} -4 & -6 \\ -7 & -9 \end{vmatrix} + (-3)\begin{vmatrix} -4 & -5 \\ -7 & -8 \end{vmatrix}$$

$$= (-1)\left[(-5)(-9) - (-6)(-8)\right] - (-2)\left[(-4)(-9) - (-6)(-7)\right]$$
$$+ (-3)\left[(-4)(-8) - (-5)(-7)\right]$$

$$= (-1)(-3) - (-2)(-6) + (-3)(-3) = 3 - 12 + 9 = 0$$

(b)

$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix}$$ ← A common factor of -1 from the first row was taken through the determinant sign.

$$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{vmatrix}$$ ← 4 times the first row was added to the second row and 7 times the first row was added to the third row

$$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{vmatrix}$$ ← -2 times the second row was added to the third row

$$= (-1)(0) = 0$$ ← Use Theorem 2.2.1.

5. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (3)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}$$

$$= (3)\left[(1)(2)-(1)(4)\right] - 0 + (-1)\left[(1)(4)-(1)(0)\right]$$

$$= (3)(-2) - 0 + (-1)(4) = -6 + 0 - 4 = -10$$

(b) $$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 2 \end{vmatrix}$$ ← The first and second rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{vmatrix}$$ ← -3 times the first row was added to the second.

$$= (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$ ← The second row was added to the third row

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & -4 \end{vmatrix}$$ ← The second and third rows were interchanged.

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{vmatrix}$$ ← 3 times the second row was added to the third.

$$= (-1)(-1)(1)(1)(-10) = -10$$ ← Use Theorem 2.1.2.

6. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + 0 - 2\begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= -3\left[(1)(2)-(4)(-2)\right] - 2\left[(-5)(-2)-(1)(1)\right] = (-3)(10) - 2(9) = -30 - 18 = -48$$

(b) $$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 2 \\ -5 & 1 & 4 \end{vmatrix}$$ ← The first and third rows were interchanged.

$$= -\begin{vmatrix} 1 & -2 & 2 \\ 0 & 6 & -4 \\ 0 & -9 & 14 \end{vmatrix}$$ ← -3 times the first row was added to the second row and 5 times the first row was added to the third row

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & -9 & 14 \end{vmatrix}$$ ← A common factor of 6 from the second row was taken through the determinant sign.

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 8 \end{vmatrix}$$ ← 9 times the second row was added to the third row.

$$= -6(1)(1)(8) = -48$$ ← Use Theorem 2.1.2.

7. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant. In each of the 3x3 determinants, we expand along the second row:

$$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix}$$

$$= 3\left(-0 + (-1)\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}\right) - 6\left(-1\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 4 \\ -9 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ -9 & -2 \end{vmatrix}\right)$$

$$+0 - 1\left(-1\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} + 0 - (-1)\begin{vmatrix} -2 & 3 \\ -9 & 2 \end{vmatrix}\right)$$

$$= 3\big(0 - 1(-2) - 1(-8)\big) - 6\big(-1(10) - 1(32) - 1(13)\big) + 0 - 1\big(-1(-8) + 0 + 1(23)\big)$$

$$= 3(10) - 6(-55) + 0 - 1(31)$$

$$= 329$$

(b) $$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & -1 & 1 \\ -2 & 3 & 1 & 4 \\ 3 & 6 & 0 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix}$$ ← The first and third rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 6 & 3 & -2 \\ 0 & 2 & -11 & 11 \end{vmatrix}$$

2 times the first row was added to the second, -3 times the first row was added to the third and 9 times the first row was added to the fourth.

$$= (-1)\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 0 & 5 & -14 \\ 0 & 0 & -\frac{31}{3} & 7 \end{vmatrix}$$

-2 times the second row was added to the third and $-\frac{2}{3}$ times the second row was added to the fourth.

$$= (-1)\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & -1 & 6 \\ 0 & 0 & 5 & -14 \\ 0 & 0 & 0 & -\frac{329}{15} \end{vmatrix}$$

$\frac{31}{15}$ times the third row was added to the fourth.

$$= (-1)(1)(3)(5)\left(-\tfrac{329}{15}\right) = 329$$

Use Theorem 2.1.2.

8. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant, as well as in each of the 3x3 determinants:

$$\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \\ -3 & -2 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 4 & 2 & 1 \\ 1 & 3 & 4 \\ -4 & -2 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 4 & 3 & 1 \\ 1 & 2 & 4 \\ -4 & -3 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ -4 & -3 & -2 \end{vmatrix}$$

$$= -1\left(3\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix}\right) + 2\left(4\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix}\right)$$

$$-3\left(4\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right) + 4\left(4\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right)$$

$$\begin{aligned}&=-\big((3)(5)-(2)(10)+5\big)+(2)(4)(5)-2\big((4)(5)-(2)(15)+10\big)\\&-3\big((4)(10)-(3)(15)+5\big)+4\big((4)(5)-(3)(10)+(2)(5)\big)\\&=0+0+0+0\\&=0\end{aligned}$$

(b) $\begin{vmatrix}-1&-2&-3&-4\\4&3&2&1\\1&2&3&4\\-4&-3&-2&-1\end{vmatrix}=\begin{vmatrix}-1&-2&-3&-4\\4&3&2&1\\0&0&0&0\\-4&-3&-2&-1\end{vmatrix}$ ← The first row was added to the third row.

$=0$ ← Use Theorem 2.2.1.

9. $\begin{vmatrix}-1&5&2\\0&2&-1\\-3&1&1\end{vmatrix}=\left|\begin{array}{ccc|cc}-1&5&2&-1&5\\0&2&-1&0&2\\-3&1&1&-3&1\end{array}\right|=\big[-2+15+0\big]-\big[-12+1+0\big]=24$

$\begin{vmatrix}-1&-2&-3\\-4&-5&-6\\-7&-8&-9\end{vmatrix}=\left|\begin{array}{ccc|cc}-1&-2&-3&-1&-2\\-4&-5&-6&-4&-5\\-7&-8&-9&-7&-8\end{array}\right|=\big[-45-84-96\big]-\big[-105-48-72\big]=0$

$\begin{vmatrix}3&0&-1\\1&1&1\\0&4&2\end{vmatrix}=\left|\begin{array}{ccc|cc}3&0&-1&3&0\\1&1&1&1&1\\0&4&2&0&4\end{array}\right|=\big[6+0-4\big]-\big[0+12+0\big]=-10$

$\begin{vmatrix}-5&1&4\\3&0&2\\1&-2&2\end{vmatrix}=\left|\begin{array}{ccc|cc}-5&1&4&-5&1\\3&0&2&3&0\\1&-2&2&1&-2\end{array}\right|=\big[0+2-24\big]-\big[0+20+6\big]=-48$

10. **(a)** e.g. $\begin{vmatrix}4&0&3&6\\8&0&5&0\\7&3&7&10\\13&0&10&0\end{vmatrix}=-3\begin{vmatrix}4&3&6\\8&5&0\\13&10&0\end{vmatrix}=-18\begin{vmatrix}8&5\\13&10\end{vmatrix}=(-18)(15)=-270$ was easy to calculate by cofactor expansions (first, we expanded along the second column, then along the third column), but would be more difficult to calculate using elementary row operations.

(b) e.g., $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$ of Exercise 8 was easy to calculate using elementary row operations, but more difficult using cofactor expansion.

11. In Exercise 1: $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -18 \neq 0$ therefore the matrix is invertible.

In Exercise 2: $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44 \neq 0$ therefore the matrix is invertible.

In Exercise 3: $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = 24 \neq 0$ therefore the matrix is invertible.

In Exercise 4: $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = 0$ therefore the matrix is not invertible.

12. In Exercise 5: $\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = -10 \neq 0$ therefore the matrix is invertible.

In Exercise 6: $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48 \neq 0$ therefore the matrix is invertible.

In Exercise 7: $\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 329 \neq 0$ therefore the matrix is invertible.

In Exercise 8: $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = 0$ therefore the matrix is not invertible.

13. $\begin{vmatrix} 5 & b-3 \\ b-2 & -3 \end{vmatrix} = (5)(-3) - (b-3)(b-2) = -15 - b^2 + 2b + 3b - 6 = -b^2 + 5b - 21$

14. $\begin{vmatrix} 3 & -4 & a \\ a^2 & 1 & 2 \\ 2 & a-1 & 4 \end{vmatrix} = \begin{vmatrix} 4a^2+3 & 0 & 8+a \\ a^2 & 1 & 2 \\ -a^3+a^2+2 & 0 & -2a+6 \end{vmatrix}$ ← 4 times the second row was added to the first row and $1-a$ times the second row was added to the last row.

$$= -0 + 1\begin{vmatrix} 4a^2+3 & 8+a \\ -a^3+a^2+2 & -2a+6 \end{vmatrix} - 0$$ Cofactor expansion along the second column.

$$= \left(4a^2+3\right)\left(-2a+6\right) - \left(8+a\right)\left(-a^3+a^2+2\right)$$

$$= a^4 - a^3 + 16a^2 - 8a + 2$$

15. $$\begin{vmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$$ The first row and the fifth row were interchanged.

$$= (-1)(-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$$ The second row and the fourth row were interchanged.

$$= (-1)(-1)(5)(2)(-1)(-4)(-3) = -120$$

16. $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) - (-1)(3) = -x^2 + x + 3$;

Adding -2 times the first row to the second row, then performing cofactor expansion along the second row yields

$$\begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 0 & x & 0 \\ 1 & 3 & x-5 \end{vmatrix} = x\begin{vmatrix} 1 & -3 \\ 1 & x-5 \end{vmatrix} = x(x-5+3) = x^2 - 2x$$

Solve the equation

$$-x^2 + x + 3 = x^2 - 2x$$
$$-2x^2 + 3x + 3 = 0$$

From quadratic formula $x = \frac{-3+\sqrt{9+24}}{-4} = \frac{3-\sqrt{33}}{4}$ or $x = \frac{-3-\sqrt{9+24}}{-4} = \frac{3+\sqrt{33}}{4}$.

17. It was shown in the solution of Exercise 1 that $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -18$. The determinant is nonzero, therefore by Theorem 2.3.3, the matrix $A = \begin{bmatrix} -4 & 2 \\ 3 & 3 \end{bmatrix}$ is invertible.

The cofactors are:

$$\begin{aligned} C_{11} &= 3 & C_{12} &= -3 \\ C_{21} &= -2 & C_{22} &= -4 \end{aligned}$$

The matrix of cofactors is $\begin{bmatrix} 3 & -3 \\ -2 & -4 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-18} \begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{9} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}$.

18. It was shown in the solution of Exercise 2 that $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44$. The determinant is nonzero, therefore by Theorem 2.3.3, the matrix $A = \begin{bmatrix} 7 & -1 \\ -2 & -6 \end{bmatrix}$ is invertible.

The cofactors are:

$$\begin{aligned} C_{11} &= -6 & C_{12} &= 2 \\ C_{21} &= 1 & C_{22} &= 7 \end{aligned}$$

The matrix of cofactors is $\begin{bmatrix} -6 & 2 \\ 1 & 7 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-44} \begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & -\frac{1}{44} \\ -\frac{1}{22} & -\frac{7}{44} \end{bmatrix}$.

19. It was shown in the solution of Exercise 3 that $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = 24$. The determinant is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}2 & -1\\1 & 1\end{vmatrix}=3 \quad C_{12}=-\begin{vmatrix}0 & -1\\-3 & 1\end{vmatrix}=3 \quad C_{13}=\begin{vmatrix}0 & 2\\-3 & 1\end{vmatrix}=6$$
$$C_{21}=-\begin{vmatrix}5 & 2\\1 & 1\end{vmatrix}=-3 \quad C_{22}=\begin{vmatrix}-1 & 2\\-3 & 1\end{vmatrix}=5 \quad C_{23}=-\begin{vmatrix}-1 & 5\\-3 & 1\end{vmatrix}=-14$$
$$C_{31}=\begin{vmatrix}5 & 2\\2 & -1\end{vmatrix}=-9 \quad C_{32}=-\begin{vmatrix}-1 & 2\\0 & -1\end{vmatrix}=-1 \quad C_{33}=\begin{vmatrix}-1 & 5\\0 & 2\end{vmatrix}=-2$$

The matrix of cofactors is $\begin{bmatrix}3 & 3 & 6\\-3 & 5 & -14\\-9 & -1 & -2\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}3 & -3 & -9\\3 & 5 & -1\\6 & -14 & -2\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{24}\begin{bmatrix}3 & -3 & -9\\3 & 5 & -1\\6 & -14 & -2\end{bmatrix}=\begin{bmatrix}\frac{1}{8} & -\frac{1}{8} & -\frac{3}{8}\\\frac{1}{8} & \frac{5}{24} & -\frac{1}{24}\\\frac{1}{4} & -\frac{7}{12} & -\frac{1}{12}\end{bmatrix}$.

20. It was shown in the solution of Exercise 4 that $\begin{vmatrix}-1 & -2 & -3\\-4 & -5 & -6\\-7 & -8 & -9\end{vmatrix}=0$ therefore by Theorem 2.3.3, the matrix is not invertible.

21. It was shown in the solution of Exercise 5 that $\begin{vmatrix}3 & 0 & -1\\1 & 1 & 1\\0 & 4 & 2\end{vmatrix}=-10$. The determinant is nonzero, therefore by Theorem 2.3.3, $A=\begin{bmatrix}3 & 0 & -1\\1 & 1 & 1\\0 & 4 & 2\end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}1 & 1\\4 & 2\end{vmatrix}=-2 \quad C_{12}=-\begin{vmatrix}1 & 1\\0 & 2\end{vmatrix}=-2 \quad C_{13}=\begin{vmatrix}1 & 1\\0 & 4\end{vmatrix}=4$$
$$C_{21}=-\begin{vmatrix}0 & -1\\4 & 2\end{vmatrix}=-4 \quad C_{22}=\begin{vmatrix}3 & -1\\0 & 2\end{vmatrix}=6 \quad C_{23}=-\begin{vmatrix}3 & 0\\0 & 4\end{vmatrix}=-12$$
$$C_{31}=\begin{vmatrix}0 & -1\\1 & 1\end{vmatrix}=1 \quad C_{32}=-\begin{vmatrix}3 & -1\\1 & 1\end{vmatrix}=-4 \quad C_{33}=\begin{vmatrix}3 & 0\\1 & 1\end{vmatrix}=3$$

The matrix of cofactors is $\begin{bmatrix}-2 & -2 & 4\\-4 & 6 & -12\\1 & -4 & 3\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}-2 & -4 & 1\\-2 & 6 & -4\\4 & -12 & 3\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-10}\begin{bmatrix}-2 & -4 & 1\\-2 & 6 & -4\\4 & -12 & 3\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & \frac{2}{5} & -\frac{1}{10}\\\frac{1}{5} & -\frac{3}{5} & \frac{2}{5}\\-\frac{2}{5} & \frac{6}{5} & -\frac{3}{10}\end{bmatrix}$.

22. It was shown in the solution of Exercise 6 that $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48$. The determinant is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 0 & 2 \\ -2 & 2 \end{vmatrix} = 4 \qquad C_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = -4 \qquad C_{13} = \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} = -6$$

$$C_{21} = -\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} = -10 \quad C_{22} = \begin{vmatrix} -5 & 4 \\ 1 & 2 \end{vmatrix} = -14 \quad C_{23} = -\begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix} = -9$$

$$C_{31} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} -5 & 4 \\ 3 & 2 \end{vmatrix} = 22 \quad C_{33} = \begin{vmatrix} -5 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

The matrix of cofactors is $\begin{bmatrix} 4 & -4 & -6 \\ -10 & -14 & -9 \\ 2 & 22 & -3 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-48}\begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{5}{24} & -\frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & -\frac{11}{24} \\ \frac{1}{8} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}$.

23. It was shown in the solution of Exercise 7 that $\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 329$. The determinant of A is nonzero therefore by Theorem 2.3.3, $A = \begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = (-6+2+0) - (-8-6+0) = 10$$

$$C_{12} = -\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = -\left[(4-9-8) - (36+4+2)\right] = 55$$

$$C_{13} = \begin{vmatrix} -2 & 3 & 4 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = (0-27+8) - (0-4+6) = -21$$

$$C_{14} = -\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = -\left[(0+27+2)-(0+4-6)\right] = -31$$

$$C_{21} = -\begin{vmatrix} 6 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = -\left[(-12+0+0)-(-2-12+0)\right] = -2$$

$$C_{22} = \begin{vmatrix} 3 & 0 & 1 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = (-6+0-2)-(9-6+0) = -11$$

$$C_{23} = -\begin{vmatrix} 3 & 6 & 1 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = -\left[(0-54+2)-(0+6+12)\right] = 70$$

$$C_{24} = \begin{vmatrix} 3 & 6 & 0 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = (0+54+0)-(0-6-12) = 72$$

$$C_{31} = \begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 2 & -2 & 2 \end{vmatrix} = (12+0-6)-(2-48+0) = 52$$

$$C_{32} = -\begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 4 \\ -9 & -2 & 2 \end{vmatrix} = -\left[(6+0+4)-(-9-24+0)\right] = -43$$

$$C_{33} = \begin{vmatrix} 3 & 6 & 1 \\ -2 & 3 & 4 \\ -9 & 2 & 2 \end{vmatrix} = (18-216-4)-(-27+24-24) = -175$$

$$C_{34} = -\begin{vmatrix} 3 & 6 & 0 \\ -2 & 3 & 1 \\ -9 & 2 & -2 \end{vmatrix} = -\left[(-18-54+0)-(0+6+24)\right] = 102$$

$$C_{41} = -\begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & -1 & 1 \end{vmatrix} = -\left[(6+0-3)-(0-24+0)\right] = -27$$

$$C_{42} = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = (3+0+2)-(1-12+0) = 16$$

$$C_{43} = -\begin{vmatrix} 3 & 6 & 1 \\ -2 & 3 & 4 \\ 1 & 0 & 1 \end{vmatrix} = -\left[(9+24+0)-(3+0-12)\right] = -42$$

$$C_{44}=\begin{vmatrix}3&6&0\\-2&3&1\\1&0&-1\end{vmatrix}=(-9+6+0)-(0+0+12)=-15$$

The matrix of cofactors is $\begin{bmatrix}10&55&-21&-31\\-2&-11&70&72\\52&-43&-175&102\\-27&16&-42&-15\end{bmatrix}$ and $\operatorname{adj}(A)=\begin{bmatrix}10&-2&52&-27\\55&-11&-43&16\\-21&70&-175&-42\\-31&72&102&-15\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{329}\begin{bmatrix}10&-2&52&-27\\55&-11&-43&16\\-21&70&-175&-42\\-31&72&102&-15\end{bmatrix}=\begin{bmatrix}\frac{10}{329}&-\frac{2}{329}&\frac{52}{329}&-\frac{27}{329}\\\frac{55}{329}&-\frac{11}{329}&-\frac{43}{329}&\frac{16}{329}\\-\frac{3}{47}&\frac{10}{47}&-\frac{25}{47}&-\frac{6}{47}\\-\frac{31}{329}&\frac{72}{329}&\frac{102}{329}&-\frac{15}{329}\end{bmatrix}$.

24. It was shown in the solution of Exercise 8 that $\begin{vmatrix}-1&-2&-3&-4\\4&3&2&1\\1&2&3&4\\-4&-3&-2&-1\end{vmatrix}=0$ therefore by Theorem 2.3.3, the matrix is not invertible.

25. $A=\begin{bmatrix}\frac{3}{5}&-\frac{4}{5}\\\frac{4}{5}&\frac{3}{5}\end{bmatrix}$, $\det(A)=\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(-\frac{4}{5}\right)\left(\frac{4}{5}\right)=\frac{9}{25}+\frac{16}{25}=1$; $A_1=\begin{bmatrix}x&-\frac{4}{5}\\y&\frac{3}{5}\end{bmatrix}$, $A_2=\begin{bmatrix}\frac{3}{5}&x\\\frac{4}{5}&y\end{bmatrix}$;

$$x'=\frac{\det(A_1)}{\det(A)}=\frac{3}{5}x+\frac{4}{5}y,\quad y'=\frac{\det(A_2)}{\det(A)}=\frac{3}{5}y-\frac{4}{5}x$$

26. $A=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$, $A_1=\begin{bmatrix}x&-\sin\theta\\y&\cos\theta\end{bmatrix}$, $A_2=\begin{bmatrix}\cos\theta&x\\\sin\theta&y\end{bmatrix}$;

$$x'=\frac{\det(A_1)}{\det(A)}=\frac{x\cos\theta+y\sin\theta}{\cos^2\theta+\sin^2\theta}=x\cos\theta+y\sin\theta,\quad y'=\frac{\det(A_2)}{\det(A)}=\frac{y\cos\theta-x\sin\theta}{\cos^2\theta+\sin^2\theta}=y\cos\theta-x\sin\theta$$

27. The coefficient matrix of the given system is $A=\begin{bmatrix}1&1&\alpha\\1&1&\beta\\\alpha&\beta&1\end{bmatrix}$. Coefficient expansion along the first row yields

$$\det(A)=1\begin{vmatrix}1&\beta\\\beta&1\end{vmatrix}-1\begin{vmatrix}1&\beta\\\alpha&1\end{vmatrix}+\alpha\begin{vmatrix}1&1\\\alpha&\beta\end{vmatrix}$$

$$=1-\beta^2-(1-\alpha\beta)+\alpha(\beta-\alpha)=-\alpha^2+2\alpha\beta-\beta^2=-(\alpha-\beta)^2$$

By Theorem 2.3.8, the given system has a nontrivial solution if and only if $\det(A)=0$, i.e., $\alpha=\beta$.

28. According to the arrow technique (see Example 7 in Section 2.1), the determinant of a 3×3 matrix can be expressed as a sum of six terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

If each entry of A is either 0 or 1, then each of the terms must be either 0 or ± 1. The largest value 3 would result from the terms $1+1+1-0-0-0$, however, this is not possible since the first three terms all equal 1 would require that all nine matrix entries be equal 1, making the determinant 0.

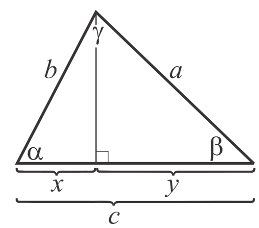
The largest value of the determinant that is actually attainable is 2, e.g., let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

29. **(a)** We will justify the third equality, $a\cos\beta + b\cos\alpha = c$ by considering three cases:

CASE I: $\alpha \le \frac{\pi}{2}$ and $\beta \le \frac{\pi}{2}$

Referring to the figure on the right side, we have $x = b\cos\alpha$ and $y = a\cos\beta$.

Since $x + y = c$ we obtain, $a\cos\beta + b\cos\alpha = c$.

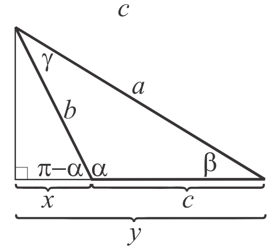


CASE II: $\alpha > \frac{\pi}{2}$ and $\beta < \frac{\pi}{2}$

Referring to the picture on the right side, we can write $x = b\cos(\pi - \alpha) = -b\cos\alpha$ and $y = a\cos\beta$

This time we can write $c = y - x = a\cos\beta - (-b\cos\alpha)$

therefore once again $a\cos\beta + b\cos\alpha = c$.



CASE III: $\beta > \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$ (similarly to case II, $c = b\cos\alpha - a\cos(\pi - \beta) = b\cos\alpha + a\cos\beta$)

The first two equations can be justified in the same manner.

Denoting $X = \cos\alpha$, $Y = \cos\beta$, and $Z = \cos\gamma$ we can rewrite the linear system as

$$\begin{array}{rcrcrcl} & & cY & + & bZ & = & a \\ cX & & & + & aZ & = & b \\ bX & + & aY & & & = & c \end{array}$$

We have $\det(A) = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = [0 + abc + abc] - [0+0+0] = 2abc$ and

$\det(A_1) = \begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = [0 + ac^2 + ab^2] - [0 + a^3 + 0] = a(b^2 + c^2 - a^2)$ therefore by Cramer's rule

$$\cos\alpha = X = \frac{\det(A_1)}{\det(A)} = \frac{a(b^2 + c^2 - a^2)}{2abc} = \frac{b^2 + c^2 - a^2}{2bc}.$$

(b) Using the results obtained in part (a) along with

$$\det(A_2)=\begin{vmatrix}0 & a & b\\ c & b & a\\ b & c & 0\end{vmatrix}=\left[0+a^2b+bc^2\right]-\left[b^3+0+0\right]=b\left(a^2+c^2-b^2\right) \text{ and}$$

$$\det(A_3)=\begin{vmatrix}0 & c & a\\ c & 0 & b\\ b & a & c\end{vmatrix}=\left[0+b^2c+a^2c\right]-\left[0+0+c^3\right]=c\left(a^2+b^2-c^2\right) \text{ therefore by Cramer's rule}$$

$$\cos\beta=Y=\frac{\det(A_2)}{\det(A)}=\frac{a^2+c^2-b^2}{2ac} \text{ and } \cos\gamma=Z=\frac{\det(A_3)}{\det(A)}=\frac{a^2+b^2-c^2}{2ab}.$$

31. From Theorem 2.3.6, $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A)=\det(A)A^{-1}$. Consequently,

$$\left(\tfrac{1}{\det(A)}A\right)\operatorname{adj}(A)=\left(\tfrac{1}{\det(A)}A\right)\left(\det(A)\ A^{-1}\right)=\tfrac{\det(A)}{\det(A)}\left(AA^{-1}\right)=I_n \text{ so } \left(\operatorname{adj}(A)\right)^{-1}=\tfrac{1}{\det(A)}A.$$

Using Theorem 2.3.5, we can also write $\operatorname{adj}\left(A^{-1}\right)=\det\left(A^{-1}\right)\left(A^{-1}\right)^{-1}=\frac{1}{\det(A)}A.$

33. The equality $A\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ means that the homogeneous system $A\mathbf{x}=\mathbf{0}$ has a nontrivial solution $\mathbf{x}=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$.

Consequently, it follows from Theorem 2.3.8 that $\det(A)=0$.

34. **(b)** $\frac{1}{2}\begin{vmatrix}3 & 3 & 1\\ 4 & 0 & 1\\ -2 & -1 & 1\end{vmatrix}=-\frac{19}{2}$ is the negative of the area of the triangle because it is being traced clockwise; (reversing the order of the points would change the orientation to counterclockwise, and thereby result in the positive area: $\frac{1}{2}\begin{vmatrix}-2 & -1 & 1\\ 4 & 0 & 1\\ 3 & 3 & 1\end{vmatrix}=\frac{19}{2}$.

37. In the special case that $n=3$, the augmented matrix for the system (13) of Section 1.10 is $\begin{bmatrix}1 & x_1 & x_1^2 & y_1\\ 1 & x_2 & x_2^2 & y_2\\ 1 & x_3 & x_3^2 & y_3\end{bmatrix}$.

We apply Cramer's Rule to the coefficient matrix $A=\begin{bmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2\\ 1 & x_3 & x_3^2\end{bmatrix}$.

$A_1=\begin{bmatrix}y_1 & x_1 & x_1^2\\ y_2 & x_2 & x_2^2\\ y_3 & x_3 & x_3^2\end{bmatrix}$, $A_2=\begin{bmatrix}1 & y_1 & x_1^2\\ 1 & y_2 & x_2^2\\ 1 & y_3 & x_3^2\end{bmatrix}$, and $A_3=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}$ so the coefficients of the desired interpolating polynomial $y=a_0+a_1x+a_2x^2$ are: $a_0=\frac{\det(A_1)}{\det(A)}$, $a_1=\frac{\det(A_2)}{\det(A)}$, and $a_2=\frac{\det(A_3)}{\det(A)}$. From the result of Exercise 43 of Section 2.1, $\det(A)=(x_2-x_1)(x_3-x_1)(x_3-x_2)$.

$$\det\left(\begin{bmatrix} y_1 & x_1 & x_1^2 \\ y_2 & x_2 & x_2^2 \\ y_3 & x_3 & x_3^2 \end{bmatrix}\right) = y_3x_1x_2\left(x_2 - x_1\right) - y_2x_1x_3\left(x_3 - x_1\right) + y_1x_2x_3\left(x_3 - x_2\right),$$

$$\det\left(\begin{bmatrix} 1 & y_1 & x_1^2 \\ 1 & y_2 & x_2^2 \\ 1 & y_3 & x_3^2 \end{bmatrix}\right) = -y_3\left(x_2^2 - x_1^2\right) + y_2\left(x_3^2 - x_1^2\right) - y_1\left(x_3^2 - x_2^2\right),$$

and $\det\left(\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\right) = y_3\left(x_2 - x_1\right) - y_2\left(x_3 - x_1\right) + y_1\left(x_3 - x_2\right).$

Therefore,

$$a_0 = \frac{y_3x_1x_2(x_2 - x_1) - y_2x_1x_3(x_3 - x_1) + y_1x_2x_3(x_3 - x_2)}{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)} = \frac{y_3x_1x_2}{(x_3 - x_1)(x_3 - x_2)} - \frac{y_2x_1x_3}{(x_2 - x_1)(x_3 - x_2)} + \frac{y_1x_2x_3}{(x_3 - x_1)(x_2 - x_1)},$$

$$a_1 = \frac{-y_3\left(x_2^2 - x_1^2\right) + y_2\left(x_3^2 - x_1^2\right) - y_1\left(x_3^2 - x_2^2\right)}{\left(x_2 - x_1\right)\left(x_3 - x_1\right)\left(x_3 - x_2\right)} = \frac{y_2\left(x_3 + x_1\right)}{\left(x_3 - x_2\right)\left(x_2 - x_1\right)} - \frac{y_3\left(x_2 + x_1\right)}{\left(x_3 - x_1\right)\left(x_3 - x_2\right)} - \frac{y_1\left(x_3 + x_2\right)}{\left(x_3 - x_1\right)\left(x_2 - x_1\right)}$$

and $a_2 = \dfrac{y_3\left(x_2 - x_1\right) - y_2\left(x_3 - x_1\right) + y_1\left(x_3 - x_2\right)}{\left(x_2 - x_1\right)\left(x_3 - x_1\right)\left(x_3 - x_2\right)} = \dfrac{y_3}{\left(x_3 - x_1\right)\left(x_3 - x_2\right)} - \dfrac{y_2}{\left(x_2 - x_1\right)\left(x_3 - x_2\right)} + \dfrac{y_1}{\left(x_3 - x_1\right)\left(x_2 - x_1\right)}.$

38. No. For instance, $T\left(1,0,0,1\right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ so that $T\left(1,0,0,1\right) + T\left(1,0,0,1\right) = 2$

but $T\left(\left(1,0,0,1\right) + \left(1,0,0,1\right)\right) = T\left(2,0,0,2\right) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ which shows that additivity fails.