CHAPTER 1
FUNDAMENTAL CONCEPTS

1.1 We use the first three steps of Eq. 1.11

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Adding the above, we get

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Adding and subtracting $\nu\frac{\sigma_x}{E}$ from the first equation,

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Similar expressions can be obtained for ε_y, and ε_z.

From the relationship for γ_{yz} and Eq. 1.12,

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \qquad \text{etc.}$$

Above relations can be written in the form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where **D** is the material property matrix defined in Eq. 1.15. ■

1.2 Plane strain condition implies that

$$\varepsilon_z = 0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

which gives

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

We have, $\sigma_x = 20000$ psi $\quad \sigma_y = -10000$ psi $\quad E = 30\times10^6$ psi $\quad \nu = 0.3$.
On substituting the values,

$$\sigma_z = 3000 \text{ psi} \qquad ■$$

1.3 Displacement field

$$u = 10^{-4}\left(-x^2 + 2y^2 + 6xy\right)$$
$$v = 10^{-4}\left(3x + 6y - y^2\right)$$
$$\frac{\partial u}{\partial x} = 10^{-4}(-2x + 6y) \quad \frac{\partial u}{\partial y} = 10^{-4}(4y + 6x)$$
$$\frac{\partial v}{\partial x} = 3 \times 10^{-4} \qquad \frac{\partial v}{\partial y} = 10^{-4}(6 + 2y)$$
$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

■

1.4 On inspection, we note that the displacements u and v are given by

$$u = 0.1\,y + 4$$
$$v = 0$$

It is then easy to see that

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.1$$

■

1.5 The displacement field is given as

$$u = 1 + 3x + 4x^3 + 6xy^2$$
$$v = xy - 7x^2$$

(a) The strains are then given by

$$\varepsilon_x = \frac{\partial u}{\partial x} = 3 + 12x^2 + 6y^2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = x$$

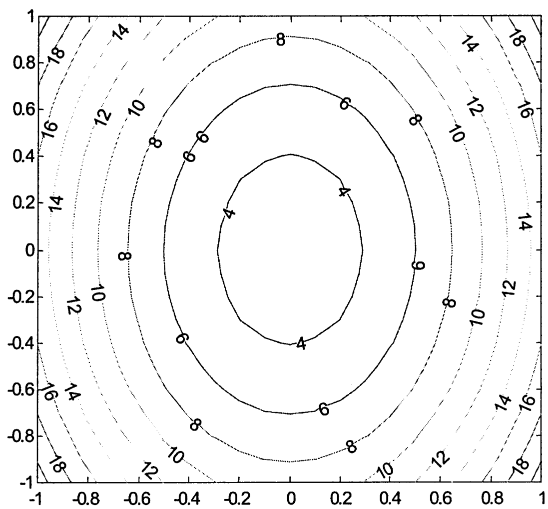
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 12xy + y - 14x$$

(b) In order to draw the contours of the strain field using MATLAB, we need to create a script file, which may be edited as a text file and save with ".m" extension. The file for plotting ε_x is given below

file "prob1p5b.m"

```
[X,Y] = meshgrid(-1:.1:1,-1:.1:1);
Z = 3.+12.*X.^2+6.*Y.^2;
[C,h] = contour(X,Y,Z);
clabel(C,h);
```

On running the program, the contour map is shown as follows:



Contours of ε_x

Contours of ε_y and γ_{xy} are obtained by changing Z in the script file. The numbers on the contours show the function values.

(c) The maximum value of ε_x is at any of the corners of the square region. The maximum value is 21.

■

1.6

$$\sigma_x = 40\text{ MPa} \quad \sigma_y = 20\text{ MPa} \quad \sigma_z = 30\text{ MPa}$$
$$\tau_{yz} = -30\text{ MPa} \quad \tau_{xz} = 15\text{ MPa} \quad \tau_{xy} = 10\text{ MPa}$$
$$\mathbf{n} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}}$$

From Eq.1.8 we get

$$\begin{aligned} T_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ &= 35.607\text{ MPa} \\ T_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ &= -6.213\text{ MPa} \\ T_z &= \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z \\ &= 13.713\text{ MPa} \\ \sigma_n &= T_x n_x + T_y n_y + T_z n_z \\ &= 24.393\text{ MPa} \end{aligned}$$

■

1.7 From the derivation made in P1.1, we have

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right]$$

which can be written in the form

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu\varepsilon_v\right]$$

and

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

Lame's constants λ and μ are defined in the expressions

$$\sigma_x = \lambda\varepsilon_v + 2\mu\varepsilon_x$$
$$\tau_{yz} = \mu\gamma_{yz}$$

On inspection,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

μ is same as the shear modulus G. ■

1.8

$$\varepsilon = 1.2\times10^{-5}$$
$$\Delta T = 30^0 C$$
$$E = 200\,\text{GPa}$$
$$\alpha = 12\times10^{-6}\,/^0\,C$$
$$\varepsilon_0 = \alpha\Delta T = 3.6\times10^{-4}$$
$$\sigma = E(\varepsilon-\varepsilon_0) = -69.6\,\text{MPa}$$

■

1.9

$$\varepsilon_x = \frac{du}{dx} = 1 + 2x^2$$
$$\delta = \int_0^L \frac{du}{dx}dx = \left(x + \frac{2}{3}x^3\right)\Bigg|_0^L$$
$$= L\left(1+\frac{2}{3}L^2\right)$$

■

1.10 Following the steps of Example 1.1, we have

$$\begin{bmatrix}(80+40+50) & -80\\ -80 & 80\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}60\\ 50\end{Bmatrix}$$

Above matrix form is same as the set of equations:

$$170\,q_1 - 80\,q_2 = 60$$

$$-80\,q_1 + 80\,q_2 = 50$$

Solving for q_1 and q_2, we get

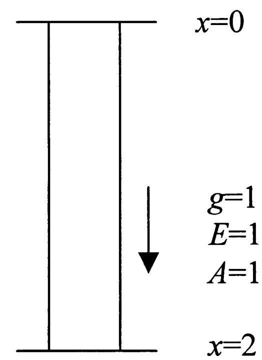
$$q_1 = 1.222\text{ mm}$$

$$q_2 = 1.847\text{ mm}$$

■

1.11

The potential energy Π is given by



$$\Pi = \frac{1}{2}\int_0^2 EA\left(\frac{du}{dx}\right)^2 dx - \int_0^2 ugA\,dx$$

Consider the polynomial from Example 1.2,

$$u = a_3\left(-2x + x^2\right)$$

$$\frac{du}{dx} = (-2+2x)a_3 = 2(-1+x)a_3$$

On substituting the above expressions and integrating, the first term of becomes

$$2a_3^{\,2}\left(\frac{2}{3}\right)$$

and the second term

$$\int_0^2 ugA\,dx = \int_0^2 u\,dx = a_3\left(-x^2 + \frac{x^3}{3}\right)\Big|_0^2$$

$$= -\frac{4}{3}a_3$$

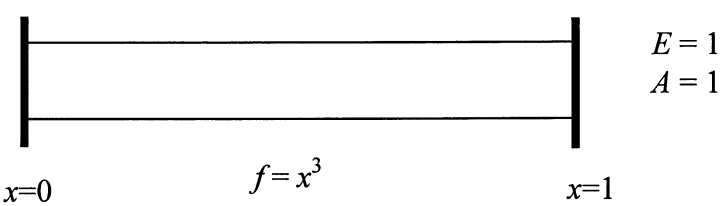
Thus

$$\Pi = \frac{4}{3}\left(a_3^{\,2} + a_3\right)$$

$$\frac{\partial \Pi}{\partial a_3} = 0 \Rightarrow \quad a_3 = -\frac{1}{2}$$

this gives $\quad u_{x=1} = -\frac{1}{2}(-2+1) = 0.5$ ■

1.12



We use the displacement field defined by $u = a_0 + a_1x + a_2x^2$.

$u = 0$ at $x = 0 \Rightarrow a_0 = 0$

$u = 0$ at $x = 1 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$

We then have $u = a_1 x(1-x)$, and $du/dx = a_1(1-x)$.

The potential energy is now written as

$$\Pi = \frac{1}{2}\int_0^1 \left(\frac{du}{dx}\right)^2 dx - \int_0^1 fu dx$$
$$= \frac{1}{2}\int_0^1 a_1^2 (1-2x)^2 dx - \int_0^1 x^3 a_1 x(1-x) dx$$
$$= \frac{1}{2}\int_0^1 a_1^2 (1-4x+4x^2) dx - \int_0^1 a_1 (x^4 - x^5) dx$$
$$= \frac{1}{2} a_1^2 \left(1 - \frac{4}{2} + \frac{4}{3}\right) - a_1 \left(\frac{1}{5} - \frac{1}{6}\right)$$
$$= \frac{a_1^2}{6} - \frac{a_1}{30}$$

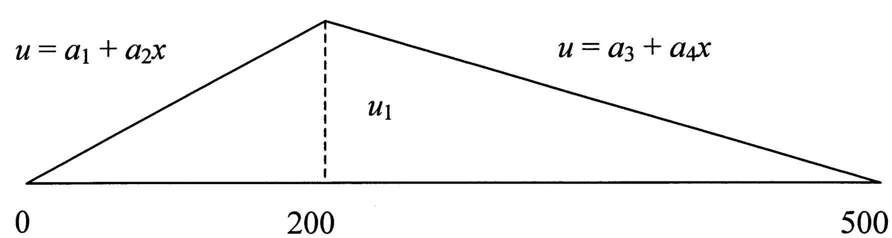
$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \Rightarrow \quad \frac{a_1}{3} - \frac{1}{30} = 0$$

This yields, $a_1 = 0.1$

Displacemen $u = 0.1x(1-x)$

Stress $\sigma = E du/dx = 0.1(1-x)$ ■

1.13 Let u_1 be the displacement at $x = 200$ mm. Piecewise linear displacement that is continuous in the interval $0 \le x \le 500$ is represented as shown in the figure.



$0 \le x \le 200$

$u = 0$ at $x = 0 \Rightarrow a_1 = 0$

$u = u_1$ at $x = 200 \Rightarrow a_2 = u_1/200$

$\Rightarrow\ u = (u_1/200)x \qquad du/dx = u_1/200$

$200 \le x \le 500$

$u = 0$ at $x = 500 \ \Rightarrow\ a_3 + 500\,a_4 = 0$

$u = u_1$ at $x = 200 \ \Rightarrow\ a_3 + 200\,a_4 = u_1$

$\Rightarrow\ a_4 = -u_1/300 \qquad a_3 = (5/3)u_1$

$\Rightarrow\ u = (5/3)u_1 - (u_1/300)x \qquad du/dx = -\,u_1/200$

$$\Pi = \frac{1}{2}\int_0^{200} E_{al}A_1\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_{200}^{500} E_{st}A_2\left(\frac{du}{dx}\right)^2 dx - 10000u_1$$

$$\Pi = \frac{1}{2}E_{al}A_1\left(\frac{u_1}{200}\right)^2 200 + \frac{1}{2}E_{st}A_2\left(-\frac{u_1}{300}\right)^2 300 - 10000u_1$$

$$= \frac{1}{2}\left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1^{\,2} - 10000u_1$$

$$\frac{\partial\Pi}{\partial u_1} = 0 \ \Rightarrow\ \left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1 - 10000 = 0$$

Note that using the units MPa (N/mm^2) for modulus of elasticity and mm^2 for area and mm for length will result in displacement in mm, and stress in MPa.

Thus, $E_{al} = 70000$ MPa, $E_{st} = 200000$, and $A_1 = 900$ mm^2, $A_2 = 1200$ mm^2. On substituting these values into the above equation, we get

$$u_1 = 0.009 \text{ mm}$$

This is precisely the solution obtained from strength of materials approach ■

1.14

In the Galerkin method, we start from the equilibrium equation

$$\frac{d}{dx}EA\frac{du}{dx} + g = 0$$

Following the steps of Example 1.3, we get

$$\int_0^2 -EA\frac{du}{dx}\frac{d\phi}{dx}dx + \int_0^2 g\phi\,dx$$

Introducing

$$u = \left(2x - x^2\right)u_1, \text{ and}$$
$$\phi = \left(2x - x^2\right)\phi_1$$

where u_1 and ϕ_1 are the values of u and ϕ at $x = 1$ respectively,

$$\phi_1\left[-u_1\int_0^2(1-2x)^2\,dx+\int_0^2\left(2x-x^2\right)dx\right]=0$$

On integrating, we get

$$\phi_1\left(-\frac{8}{3}u_1+\frac{4}{3}\right)=0$$

This is to be satisfied for every ϕ_1, which gives the solution

$$u_1 = 0.5 \qquad \blacksquare$$

1.15 We use

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$
$$u = 0 \text{ at } x = 0$$
$$u = 0 \text{ at } x = 2$$

This implies that

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4_3 + 8a_4$$

and

$$u = a_3\left(x^2 - 2x\right) + a_4\left(x^3 - 4x\right)$$
$$\frac{du}{dx} = 2a_3(x-1) + a_4\left(3x^2 - 4\right)$$

a_3 and a_4 are considered as independent variables in

$$\Pi = \frac{1}{2}\int_0^2\left[2a_3(x-1)+a_4\left(3x^2-4\right)\right]^2 dx - 2\left(-a_3 - 3a_4\right)$$

on expanding and integrating the terms, we get

$$\Pi = 1.333a_3^{\,2} + 12.8a_4^{\,2} + 8a_3a_4 + 2a_3 + 6a_4$$

We differentiate with respect to the variables and equate to zero.

$$\frac{\partial\Pi}{\partial a_3} = 2.667a_3 + 8a_4 + 2 = 0$$
$$\frac{\partial\Pi}{\partial a_4} = 8a_3 + 25.6a_4 + 6 = 0$$