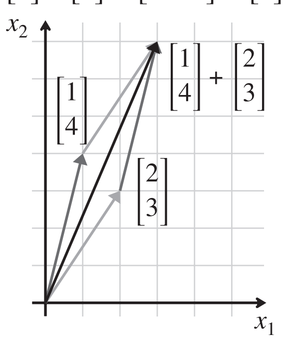
Chapter 1 Solutions

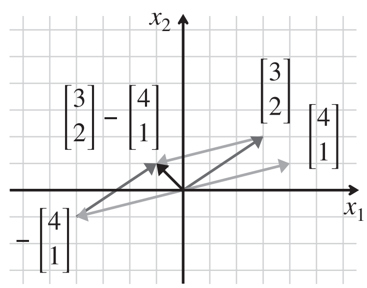
Section 1.1

A Practice Problems

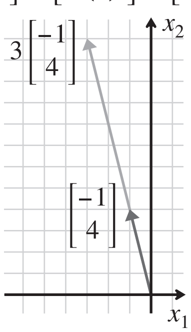
A1 $\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1+2\\4+3\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}$



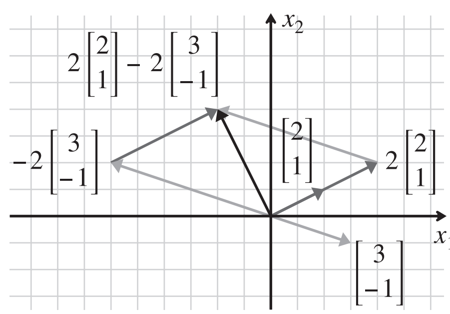
A2 $\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}3-4\\2-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$



A3 $3\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}3(-1)\\3(4)\end{bmatrix}=\begin{bmatrix}-3\\12\end{bmatrix}$



A4 $2\begin{bmatrix}2\\1\end{bmatrix}-2\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}6\\-2\end{bmatrix}=\begin{bmatrix}-2\\4\end{bmatrix}$



A5 $\begin{bmatrix}4\\-2\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix}=\begin{bmatrix}4+(-1)\\-2+3\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$

A6 $\begin{bmatrix}-3\\-4\end{bmatrix}-\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}-3-(-2)\\-4-5\end{bmatrix}=\begin{bmatrix}-1\\-9\end{bmatrix}$

A7 $-2\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}(-2)3\\(-2)(-2)\end{bmatrix}=\begin{bmatrix}-6\\4\end{bmatrix}$

A8 $\frac{1}{2}\begin{bmatrix}2\\6\end{bmatrix}+\frac{1}{3}\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}4/3\\1\end{bmatrix}=\begin{bmatrix}7/3\\4\end{bmatrix}$

A9 $\frac{2}{3}\begin{bmatrix}3\\1\end{bmatrix} - 2\begin{bmatrix}1/4\\1/3\end{bmatrix} = \begin{bmatrix}2\\2/3\end{bmatrix} - \begin{bmatrix}1/2\\2/3\end{bmatrix} = \begin{bmatrix}3/2\\0\end{bmatrix}$

A10 $\sqrt{2}\begin{bmatrix}\sqrt{2}\\\sqrt{3}\end{bmatrix} + 3\begin{bmatrix}1\\\sqrt{6}\end{bmatrix} = \begin{bmatrix}2\\\sqrt{6}\end{bmatrix} + \begin{bmatrix}3\\3\sqrt{6}\end{bmatrix} = \begin{bmatrix}5\\4\sqrt{6}\end{bmatrix}$

A11 $\begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}5\\1\\-2\end{bmatrix} = \begin{bmatrix}2-5\\3-1\\4-(-2)\end{bmatrix} = \begin{bmatrix}-3\\2\\6\end{bmatrix}$

A12 $\begin{bmatrix}2\\1\\-6\end{bmatrix} + \begin{bmatrix}-3\\1\\-4\end{bmatrix} = \begin{bmatrix}2+(-3)\\1+1\\-6+(-4)\end{bmatrix} = \begin{bmatrix}-1\\2\\-10\end{bmatrix}$

A13 $-6\begin{bmatrix}4\\-5\\-6\end{bmatrix} = \begin{bmatrix}(-6)4\\(-6)(-5)\\(-6)(-6)\end{bmatrix} = \begin{bmatrix}-24\\30\\36\end{bmatrix}$

A14 $-2\begin{bmatrix}-5\\1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\-1\end{bmatrix} = \begin{bmatrix}10\\-2\\-2\end{bmatrix} + \begin{bmatrix}-3\\0\\-3\end{bmatrix} = \begin{bmatrix}7\\-2\\-5\end{bmatrix}$

A15 $2\begin{bmatrix}2/3\\-1/3\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}7/3\\-4/3\\13/3\end{bmatrix}$

A16 $\sqrt{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \pi\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\\sqrt{2}\\\sqrt{2}\end{bmatrix} + \begin{bmatrix}-\pi\\0\\\pi\end{bmatrix} = \begin{bmatrix}\sqrt{2}-\pi\\\sqrt{2}\\\sqrt{2}+\pi\end{bmatrix}$

A17 (a) $2\vec{v} - 3\vec{w} = \begin{bmatrix}2\\4\\-4\end{bmatrix} - \begin{bmatrix}6\\-3\\9\end{bmatrix} = \begin{bmatrix}-4\\7\\-13\end{bmatrix}$

(b) $-3(\vec{v} + 2\vec{w}) + 5\vec{v} = -3\left(\begin{bmatrix}1\\2\\-2\end{bmatrix} + \begin{bmatrix}4\\-2\\6\end{bmatrix}\right) + \begin{bmatrix}5\\10\\-10\end{bmatrix} = -3\begin{bmatrix}5\\0\\4\end{bmatrix} + \begin{bmatrix}5\\10\\-10\end{bmatrix} = \begin{bmatrix}-15\\0\\-12\end{bmatrix} + \begin{bmatrix}5\\10\\-10\end{bmatrix} = \begin{bmatrix}-10\\10\\-22\end{bmatrix}$

(c) We have $\vec{w} - 2\vec{u} = 3\vec{v}$, so $2\vec{u} = \vec{w} - 3\vec{v}$ or $\vec{u} = \frac{1}{2}(\vec{w} - 3\vec{v})$. This gives

$$\vec{u} = \frac{1}{2}\left(\begin{bmatrix}2\\-1\\3\end{bmatrix} - \begin{bmatrix}3\\6\\-6\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}-1\\-7\\9\end{bmatrix} = \begin{bmatrix}-1/2\\-7/2\\9/2\end{bmatrix}$$

(d) We have $\vec{u} - 3\vec{v} = 2\vec{u}$, so $\vec{u} = -3\vec{v} = \begin{bmatrix}-3\\-6\\6\end{bmatrix}$.

A18 (a) $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \begin{bmatrix}3/2\\1/2\\1/2\end{bmatrix} + \begin{bmatrix}5/2\\-1/2\\-1\end{bmatrix} = \begin{bmatrix}4\\0\\-1/2\end{bmatrix}$

(b) $2(\vec{v} + \vec{w}) - (2\vec{v} - 3\vec{w}) = 2\begin{bmatrix}8\\0\\-1\end{bmatrix} - \left(\begin{bmatrix}6\\2\\2\end{bmatrix} - \begin{bmatrix}15\\-3\\-6\end{bmatrix}\right) = \begin{bmatrix}16\\0\\-2\end{bmatrix} - \begin{bmatrix}-9\\5\\8\end{bmatrix} = \begin{bmatrix}25\\-5\\-10\end{bmatrix}$

(c) We have $\vec{w} - \vec{u} = 2\vec{v}$, so $\vec{u} = \vec{w} - 2\vec{v}$. This gives $\vec{u} = \begin{bmatrix}5\\-1\\-2\end{bmatrix} - \begin{bmatrix}6\\2\\2\end{bmatrix} = \begin{bmatrix}-1\\-3\\-4\end{bmatrix}$.

(d) We have $\frac{1}{2}\vec{u} + \frac{1}{3}\vec{v} = \vec{w}$, so $\frac{1}{2}\vec{u} = \vec{w} - \frac{1}{3}\vec{v}$, or $\vec{u} = 2\vec{w} - \frac{2}{3}\vec{v} = \begin{bmatrix}10\\-2\\-4\end{bmatrix} - \begin{bmatrix}2\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}8\\-8/3\\-14/3\end{bmatrix}$.

A19

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \qquad \vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix} \qquad \vec{QR} = \vec{OR} - \vec{OQ} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{SR} = \vec{OR} - \vec{OS} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$$

Thus,

$$\vec{PQ} + \vec{QR} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix} = \vec{PS} + \vec{SR}$$

A20 The equation of the line is $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

A21 The equation of the line is $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -4 \\ -6 \end{bmatrix}, t \in \mathbb{R}$

A22 The equation of the line is $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ -11 \end{bmatrix}, t \in \mathbb{R}$

A23 The equation of the line is $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

For Problems A24 - A28, alternative correct answers are possible.

A24 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \quad t \in \mathbb{R}$$

A25 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + t\begin{bmatrix} -6 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$$

A26 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

A27 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

A28 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -1 \\ 1 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}, \quad t \in \mathbb{R}$$

A29 The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Hence, the parametric equation of the line is $\begin{cases} x_1 &= -1 + 3t \\ x_2 &= 2 - 5t, \end{cases} \; t \in \mathbb{R}$.

A scalar equation is $x_2 = 2 + \frac{-5}{3}(x_1 - (-1)) = -\frac{5}{3}x_1 + \frac{1}{3}$.

A30 The direction vector is $\vec{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence, the parametric equation of the line is $\begin{cases} x_1 &= 1 + t \\ x_2 &= 1 + t, \end{cases} \; t \in \mathbb{R}$.

A scalar equation is $x_2 = 1 + (x_1 - 1) = x_1$.

A31 The direction vector is $\vec{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Hence, the parametric equation of the line is $\begin{cases} x_1 &= 1 + 2t \\ x_2 &= 0 + 0t, \end{cases} \; t \in \mathbb{R}$.

A scalar equation is $x_2 = 0 + 0(x_1 - 1) = 0$.

A32 The direction vector is $\vec{d} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Hence, the parametric equation of the line is $\begin{cases} x_1 &= 1 - 2t \\ x_2 &= 3 + 2t, \end{cases} \quad t \in \mathbb{R}$.

A scalar equation is $x_2 = 3 + (-1)(x_1 - 1) = -x_1 + 4$.

A33 (a) Let P, Q, and R be three points in $\mathbb{R}^n$, with corresponding vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$. If P, Q, and R are collinear, then the directed line segments $\vec{PQ}$ and $\vec{PR}$ should define the same line. That is, the direction vector of one should be a non-zero scalar multiple of the direction vector of the other. Therefore, $\vec{PQ} = t\vec{PR}$, for some $t \in \mathbb{R}$.

(b) We have $\vec{PQ} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} = -2\vec{PQ}$, so they are collinear.

(c) We have $\vec{ST} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{SU} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$. Therefore, the points S, T, and U are not collinear because $\vec{SU} \neq t\vec{ST}$ for any real number t.

A34 For V2:

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{y} + \vec{x}$$

For V8:

$$(s + t)\vec{x} = (s + t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (s + t)x_1 \\ (s + t)x_2 \end{bmatrix} = \begin{bmatrix} sx_1 + tx_1 \\ sx_2 + tx_2 \end{bmatrix} = \begin{bmatrix} sx_1 \\ sx_2 \end{bmatrix} + \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = s\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\vec{x} + t\vec{x}$$

A35 We get that $\vec{F}_1 = \begin{bmatrix} 450 \\ 0 \end{bmatrix}$ and $\vec{F}_2 = \begin{bmatrix} 25 \\ 25\sqrt{3} \end{bmatrix}$. Thus, the net force is $\vec{F} = \begin{bmatrix} 475 \\ 25\sqrt{3} \end{bmatrix}$.

B Homework Problems

B1 $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ **B2** $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$ **B3** $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ **B4** $\begin{bmatrix} 5 \\ 16 \end{bmatrix}$

B5 $\begin{bmatrix} 5 \\ 15 \end{bmatrix}$ **B6** $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **B7** $\begin{bmatrix} 15 \\ -10 \end{bmatrix}$ **B8** $\begin{bmatrix} 3/4 \\ 19/4 \end{bmatrix}$

B9 $\begin{bmatrix} 2 \\ \sqrt{2} - \sqrt{18} \end{bmatrix}$ **B10** $\begin{bmatrix} 0 \\ -3 \\ -9 \end{bmatrix}$ **B11** $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ **B12** $\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$

B13 $\begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix}$ **B14** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **B15** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **B16** $\begin{bmatrix} 3 + \sqrt{2} \\ 0 \\ -1 \end{bmatrix}$

B17 (a) $\begin{bmatrix} 2 \\ 16 \\ 11 \end{bmatrix}$ (b) $\begin{bmatrix} 10 \\ -22 \\ -13 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 10 \\ 7 \end{bmatrix}$ (d) $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$

B18 (a) $\begin{bmatrix} 11 \\ 25 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 3/2 \\ 11/4 \end{bmatrix}$ (c) $\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ (d) $\vec{u} = \begin{bmatrix} 5/3 \\ 11/3 \\ 5/3 \end{bmatrix}$

B19 $\vec{PQ} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, $\vec{PR} = \begin{bmatrix} -4 \\ 0 \\ -3 \end{bmatrix}$, $\vec{PS} = \begin{bmatrix} 5 \\ -6 \\ 2 \end{bmatrix}$, $\vec{QR} = \begin{bmatrix} -7 \\ -1 \\ -2 \end{bmatrix}$, $\vec{SR} = \begin{bmatrix} -9 \\ 6 \\ -5 \end{bmatrix}$

B20 $\vec{PQ} = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, $\vec{PR} = \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$, $\vec{PS} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$, $\vec{QR} = \begin{bmatrix} -6 \\ -6 \\ -1 \end{bmatrix}$, $\vec{SR} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

B21 $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

B22 $\vec{x} = t\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

B23 $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

B24 $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, t \in \mathbb{R}$

B25 $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

B26 $\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

B27 $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$

B28 $\vec{x} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad t \in \mathbb{R}$

B29 $\vec{x} = t\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$

B30 $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$

B31 $\vec{x} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$

B32 $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1/2 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ -5/3 \\ -1/2 \end{bmatrix}, \quad t \in \mathbb{R}$

B33 $\begin{cases} x_1 = 2 + t \\ x_2 = 5 - 2t, \end{cases} t \in \mathbb{R}$; $x_2 = 5 - 2(x_1 - 2)$.

B34 $\begin{cases} x_1 = 3 + 3t \\ x_2 = -1 + 2t, \end{cases} t \in \mathbb{R}$; $x_2 = -1 + \frac{2}{3}(x_1 - 3)$.

B35 $\begin{cases} x_1 = t \\ x_2 = 3 - 8t, \end{cases} t \in \mathbb{R}$; $x_2 = 3 - 8x_1$.

B36 $\begin{cases} x_1 = -3 + 7t \\ x_2 = 1, \end{cases} t \in \mathbb{R}$; $x_2 = 1$.

B37 $\begin{cases} x_1 = 2 - 2t \\ x_2 = -3t, \end{cases} t \in \mathbb{R}$; $x_2 = \frac{3}{2}(x_1 - 2)$.

B38 $\begin{cases} x_1 = 5 + t \\ x_2 = -2 + 5t, \end{cases} t \in \mathbb{R}$; $x_2 = -2 + 5(x_1 - 5)$.

B39 collinear

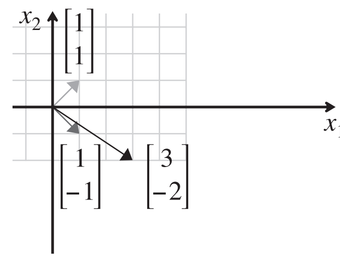
B40 not collinear

B41 collinear

C Conceptual Problems

C1 (a) We need to find t_1 and t_2 such that

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$$



That is, we need to solve the two equations in two unknowns $t_1 + t_2 = 3$ and $t_1 - t_2 = -2$. Using substitution and/or elimination we find that $t_1 = \frac{1}{2}$ and $t_2 = \frac{5}{2}$.

(b) We use the same approach as in part (a). We need to find t_1 and t_2 such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$$

Solving $t_1 + t_2 = x_1$ and $t_1 - t_2 = x_2$ by substitution and/or elimination gives $t_1 = \frac{1}{2}(x_1 + x_2)$ and $t_2 = \frac{1}{2}(x_1 - x_2)$.

(c) We have $x_1 = \sqrt{2}$ and $x_2 = \pi$, so we get $t_1 = \frac{1}{2}(\sqrt{2} + \pi)$ and $t_2 = \frac{1}{2}(\sqrt{2} - \pi)$.

C2 (a) $\vec{PQ} + \vec{QR} + \vec{RP}$ can be described informally as "start at P and move to Q, then move from Q to R, then from R to P; the net result is a zero change in position."

(b) We have $\vec{PQ} = \vec{q} - \vec{p}$, $\vec{QR} = \vec{r} - \vec{q}$, and $\vec{RP} = \vec{p} - \vec{r}$. Thus,

$$\vec{PQ} + \vec{QR} + \vec{RP} = \vec{q} - \vec{p} + \vec{r} - \vec{q} + \vec{p} - \vec{r} = \vec{0}$$

C3 Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then

$$s(t\vec{x}) = s\begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix} = \begin{bmatrix} s(tx_1) \\ s(tx_2) \\ s(tx_3) \end{bmatrix} = \begin{bmatrix} (st)x_1 \\ (st)x_2 \\ (st)x_3 \end{bmatrix} = (st)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (st)\vec{x}$$

C4 Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then,

$$s(\vec{x} + \vec{y}) = s\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} s(x_1 + y_1) \\ s(x_2 + y_2) \\ s(x_3 + y_3) \end{bmatrix} = \begin{bmatrix} sx_1 + sy_1 \\ sx_2 + sy_2 \\ sx_3 + sy_3 \end{bmatrix} = s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + s\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = s\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + s\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = s\vec{x} + s\vec{y}$$

C5 Assume that $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$, is a line in $\mathbb{R}^2$ passing through the origin. Then, there exists a real number t_1 such that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{p} + t_1\vec{d}$. Hence, $\vec{p} = -t_1\vec{d}$ and so $\vec{p}$ is a scalar multiple of $\vec{d}$. On the other hand, assume that $\vec{p}$ is a scalar multiple of $\vec{d}$. Then, there exists a real number t_1 such that $\vec{p} = t_1\vec{d}$. Hence, if we take $t = -t_1$, we get that the line with vector equation $\vec{x} = \vec{p} + t\vec{d}$ passes through the point $\vec{p} + (-t_1)\vec{d} = t_1\vec{d} - t_1\vec{d} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as required.

C6 If the plane passes through the origin, then there exists $s, t \in \mathbb{R}$ such that

$$\vec{0} = \vec{p} + s\vec{u} + t\vec{v}$$

Hence,

$$\vec{p} = -s\vec{u} - t\vec{v}$$

and so $\vec{p}$ is a linear combination of $\vec{u}$ and $\vec{v}$.
On the other hand, if $\vec{p} = a\vec{u} + b\vec{v}$, then taking $s = -a$ and $t = -b$ gives

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v} = \vec{p} - a\vec{u} - b\vec{v} = \vec{0}$$

and hence the plane passes through the origin.

C7 A vector equation for the line segment from O to R is $\vec{x} = s\vec{OR}$, $0 \leq s \leq 1$. Similarly, a vector equation for the line segment from P to Q is $\vec{x} = \vec{p} + t\vec{PQ}$, $0 \leq t \leq 1$. The two lines intersect when

$$s\vec{OR} = \vec{p} + t\vec{PQ}$$

Since O, P, Q, R form a parallelogram, we know that $\vec{r} = \vec{p} + \vec{q}$. Hence, we get

$$\begin{aligned} s(\vec{r} - \vec{0}) &= \vec{p} + t(\vec{q} - \vec{p}) \\ s(\vec{p} + \vec{q}) &= \vec{p} + t\vec{q} - t\vec{p} \\ (s + t - 1)\vec{p} &= (-s + t)\vec{q} \end{aligned}$$

$\vec{p}$ and $\vec{q}$ cannot be scalar multiples of each other, as otherwise we would not have a parallelogram. Thus, for this equation to hold, we must have $s + t - 1 = 0$ and $-s + t = 0$. Solving, we find that $s = t = \frac{1}{2}$ as required.

C8 The line segment from A to B is $\vec{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t\begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}$, $0 \leq t \leq 1$. Thus, the point 1/3 of the way from A to B is

$$\vec{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}a_1 + \frac{1}{3}b_1 \\ \frac{2}{3}a_2 + \frac{1}{3}b_2 \end{bmatrix}$$

Hence, the coordinates are $\left(\frac{2}{3}a_1 + \frac{1}{3}b_1, \frac{2}{3}a_2 + \frac{1}{3}b_2\right)$.

C9 (a) Parametric equations for the plane are $\begin{cases} x_1 = 2 + s + t \\ x_2 = 1 + 2s + t \\ x_3 = 3s + 2t \end{cases}$ $s, t \in \mathbb{R}$.

(b) Subtracting the second equation from the first equation gives $x_1 - x_2 = 1 - s$, so $s = 1 - x_1 + x_2$. Then, the second equation gives

$$t = x_2 - 1 - 2s = x_2 - 1 - 2(1 - x_1 + x_2) = -3 + 2x_1 - x_2$$

The third equation now gives

$$x_3 = 3(1 - x_1 + x_2) + 2(-3 + 2x_1 - x_2) = -3 + x_1 + x_2$$

Hence, a scalar equation for the plane is $x_1 + x_2 - x_3 = 3$.

C10 (a) We solve $ax_1 + bt = c$ for x_1 to get $x_1 = \frac{c}{a} - \frac{b}{a}t$. Thus, parametric equations for the line are

$$\begin{cases} x_1 = \frac{c}{a} - \frac{b}{a}t \\ x_2 = t \end{cases} \quad t \in \mathbb{R}$$

(b) We have

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{c}{a} - \frac{b}{a}t \\ t \end{bmatrix} \\ &= \begin{bmatrix} c/a \\ 0 \end{bmatrix} + t\begin{bmatrix} -b/a \\ 1 \end{bmatrix}, t \in \mathbb{R} \end{aligned}$$

(c) From our work in (b), a vector equation for the line is

$$\vec{x} = \begin{bmatrix} 5/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(d) Parametric equations would be

$$\begin{cases} x_1 = 3 \\ x_2 = t \end{cases} \quad t \in \mathbb{R}$$

Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

C11 If $P(p_1, p_2)$ is on the line, then there exists $t_1 \in \mathbb{R}$ such that

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = t_1\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} td_1 \\ td_2 \end{bmatrix}$$

Thus, $p_1 = td_1$ and $p_2 = td_2$. If $d_1 = 0$, then $p_1 = 0$ and hence we have $p_1d_2 = 0 = p_2d_1$. If $d_1 \neq 0$, then $t = \frac{p_1}{d_1}$ and hence

$$p_2 = \frac{p_1}{d_1}d_2 \Rightarrow p_2d_1 = p_1d_2$$

On the other hand, assume $p_1d_2 = p_2d_1$. If $d_1 = 0$, then $p_1 = 0$ (if $d_2 = 0$, then L would not be a line). Hence, taking $t_2 = \frac{p_2}{d_2}$ gives

$$t_2\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t_2d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

If $d_1 \neq 0$, then we take $t_3 = \frac{p_1}{d_1}$ to get

$$t_3\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} t_3d_1 \\ t_3d_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ \frac{p_1}{d_1}d_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

C12 Let the two lines be $\vec{x} = \vec{a} + s\vec{b}$, $s \in \mathbb{R}$, and $\vec{x} = \vec{c} + t\vec{d}$, $t \in \mathbb{R}$. Since the lines are not parallel, we have $\vec{d} \neq k\vec{b}$ for any k. To determine whether there is a point of intersection, we try to solve $\vec{a} + s\vec{b} = \vec{c} + t\vec{d}$ for s and t. The components of this vector equation are

$$\begin{aligned} b_1 s - d_1 t &= c_1 - a_1 \\ b_2 s - d_2 t &= c_2 - a_2 \end{aligned}$$

Multiply the first equation by d_2 and the second equation by d_1 and subtract the second from the first to get

$$(b_1 d_2 - b_2 d_1)s = d_2(c_1 - a_1) - d_1(c_2 - a_2)$$

Now $b_1 d_2 - b_2 d_1 \neq 0$ since $\vec{d} \neq k\vec{b}$ for any k. Thus, we can solve this equation for s and then solve for t. Thus, there is a point of intersection.

Section 1.2

A Practice Problems

A1 Consider $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 3c_1 - c_2 \end{bmatrix}$. This gives

$$\begin{aligned} 3 &= c_1 + c_2 \\ 1 &= 3c_1 - c_2 \end{aligned}$$

Solving we find that $c_1 = 1$ and $c_2 = 2$. Thus, $\vec{x} \in \operatorname{Span} \mathcal{B}$.

A2 Consider $\begin{bmatrix} 8 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 \\ c_1 \end{bmatrix}$. Taking $c_1 = -4$ satisfies the equation. Thus, $\vec{x} \in \operatorname{Span} \mathcal{B}$.

A3 Consider $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c_1 \\ c_1 \end{bmatrix}$. For the first component, we require that $c_1 = -3$, but this does satisfy the second component. Thus, $\vec{x} \notin \operatorname{Span} \mathcal{B}$.

A4 Consider $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ -c_1 + 2c_2 \end{bmatrix}$. This gives

$$\begin{aligned} 2 &= 2c_1 + c_2 \\ 5 &= -c_1 + 2c_2 \end{aligned}$$

Solving we find that $c_1 = -1/5$ and $c_2 = 12/5$. Thus, $\vec{x} \in \operatorname{Span} \mathcal{B}$.

A5 Consider $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_3 \\ c_1 + c_2 \\ c_2 + c_3 \end{bmatrix}$. This gives

$$\begin{aligned} 1 &= c_1 + c_3 \\ 2 &= c_1 + c_2 \\ -1 &= c_2 + c_3 \end{aligned}$$

Solving we find that $c_1 = 2$, $c_2 = 0$, and $c_3 = -1$. Thus, $\vec{x} \in \operatorname{Span} \mathcal{B}$.

A6 Consider $\begin{bmatrix}0\\1\\3\end{bmatrix} = c_1\begin{bmatrix}1\\2\\2\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\0\\4\end{bmatrix} = \begin{bmatrix}c_1 + c_3\\2c_1 - c_2\\2c_1 + c_2 + 4c_3\end{bmatrix}$. This gives

$$\begin{aligned}0 &= c_1 + c_3\\1 &= 2c_1 - c_2\\3 &= 2c_1 + c_2 + 4c_3\end{aligned}$$

Adding the second and third equations gives $4 = 4c_1 + 4c_3$. Thus, $c_1 + c_3 = 1$ which contradicts the first equation. Hence, $\vec{x} \notin \operatorname{Span}\mathcal{B}$.

A7 Consider

$$\begin{bmatrix}0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}1\\3\end{bmatrix} + c_3\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}c_1 + c_2 + c_3\\2c_1 + 3c_2 + 4c_3\end{bmatrix}$$

This gives

$$\begin{aligned}c_1 + c_2 + c_3 &= 0\\2c_1 + 3c_2 + 4c_3 &= 0\end{aligned}$$

Subtracting two times the first equation from the second equation gives $c_2 + 2c_3 = 0$. Thus, if we take $c_3 = 1$, we get $c_2 = -2$ and hence $c_1 = 1$. Therefore, by definition, the set is linearly dependent.

A8 Consider

$$\begin{bmatrix}0\\0\end{bmatrix} = c_1\begin{bmatrix}3\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3c_1 - c_2\\c_1 + 3c_2\end{bmatrix}$$

This gives

$$\begin{aligned}3c_1 - c_2 &= 0\\c_1 + 3c_2 &= 0\end{aligned}$$

Solving we find that the only solution is $c_1 = c_2 = 0$, so the set is linearly independent.

A9 Consider

$$\begin{bmatrix}0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1\end{bmatrix}$$

This gives

$$\begin{aligned}c_1 + c_2 &= 0\\c_1 &= 0\end{aligned}$$

Solving we find that the only solution is $c_1 = c_2 = 0$, so the set is linearly independent.

A10 Observe that $2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}-4\\-6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, so the set is linearly dependent.

A11 Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}c_1\\2c_1\\c_1\end{bmatrix}$$

This gives $c_1 = 0$, so the set is linearly independent.

A12 Observe that

$$0\begin{bmatrix}1\\-3\\-2\end{bmatrix}+0\begin{bmatrix}4\\6\\1\end{bmatrix}+1\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

so the set is linearly dependent.

A13 Observe that

$$0\begin{bmatrix}1\\1\\0\end{bmatrix}+2\begin{bmatrix}1\\2\\-1\end{bmatrix}+1\begin{bmatrix}-2\\-4\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

so the set is linearly dependent.

A14 Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix}=c_1\begin{bmatrix}1\\-2\\1\end{bmatrix}+c_2\begin{bmatrix}2\\3\\4\end{bmatrix}+c_3\begin{bmatrix}0\\-1\\-2\end{bmatrix}=\begin{bmatrix}c_1+2c_2\\-2c_1+3c_2-c_3\\c_1+4c_2-2c_3\end{bmatrix}$$

This gives

$$\begin{aligned}c_1+2c_2&=0\\-2c_1+3c_2-c_3&=0\\c_1+4c_2-2c_3&=0\end{aligned}$$

Subtracting the first equation from the third equation gives $2c_2-2c_3=0$. Hence, $c_2=c_3$. The second equation then gives $0=-2c_1+3c_2-c_2=-2c_1+2c_2$. Thus, $c_1=c_2$. Therefore, the first equation gives $c_1=c_2=0$ and hence $c_3=c_2=0$. So, the set is linearly independent.

A15 Since the spanning set cannot be reduced, it is a line with vector equation $\vec{x}=s\begin{bmatrix}1\\0\end{bmatrix}$, $s\in\mathbb{R}$.

A16 Since $\begin{bmatrix}2\\-2\end{bmatrix}=-2\begin{bmatrix}-1\\1\end{bmatrix}$, we have $\operatorname{Span}\left\{\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}2\\-2\end{bmatrix}\right\}=\operatorname{Span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$. Since the spanning set cannot be reduced, it is a line with vector equation $\vec{x}=s\begin{bmatrix}-1\\1\end{bmatrix}$, $s\in\mathbb{R}$.

A17 Since $\begin{bmatrix}-2\\6\\-2\end{bmatrix}=-2\begin{bmatrix}1\\-3\\1\end{bmatrix}$, we have $\operatorname{Span}\left\{\begin{bmatrix}1\\-3\\1\end{bmatrix},\begin{bmatrix}-2\\6\\-2\end{bmatrix}\right\}=\operatorname{Span}\left\{\begin{bmatrix}1\\-3\\1\end{bmatrix}\right\}$. Since the spanning set cannot be reduced, it is a line with vector equation $\vec{x}=s\begin{bmatrix}1\\-3\\1\end{bmatrix}$, $s\in\mathbb{R}$.

A18 This is just two points in $\mathbb{R}^3$. A vector equation would be $\vec{x}=\begin{bmatrix}1\\-3\\1\end{bmatrix}$ or $\vec{x}=\begin{bmatrix}-2\\6\\-2\end{bmatrix}$.

A19 Since neither vector is a scalar multiple of the other, the set cannot be reduced. Thus, it is a plane with vector equation $\vec{x}=s\begin{bmatrix}1\\0\\-2\end{bmatrix}+t\begin{bmatrix}2\\1\\-1\end{bmatrix}$, $s,t\in\mathbb{R}$.

A20 It is just the origin with vector equation $\vec{x} = \vec{0}$.

A21 $\mathcal{B}$ does not form a basis for $\mathbb{R}^2$ since it does not span $\mathbb{R}^2$. For example, the vector $\begin{bmatrix}1\\0\end{bmatrix}$ is not in Span $\mathcal{B}$.

A22 We will prove $\mathcal{B}$ is a basis. Consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2c_1 + c_2\\3c_1\end{bmatrix}$$

This gives

$$\begin{aligned}2c_1 + c_2 &= x_1\\3c_1 &= x_2\end{aligned}$$

Solving, we get $c_1 = \frac{1}{3}x_2$ and $c_2 = x_1 - \frac{2}{3}x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^2$. Moreover, taking $x_1 = x_2 = 0$ gives the unique solution $c_1 = c_2 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^2$.

A23 Since $0\begin{bmatrix}2\\1\end{bmatrix} + 1\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, $\mathcal{B}$ is linearly dependent and hence is not a basis.

A24 $\mathcal{B}$ does not form a basis for $\mathbb{R}^2$ since the vector $\begin{bmatrix}1\\0\end{bmatrix}$ is not in Span $\mathcal{B}$.

A25 We will prove $\mathcal{B}$ is a basis. Consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-c_1 + c_2\\c_1 + 3c_2\end{bmatrix}$$

This gives

$$\begin{aligned}-c_1 + c_2 &= x_1\\c_1 + 3c_2 &= x_2\end{aligned}$$

Solving, we get $c_1 = -\frac{3}{4}x_1 + \frac{1}{4}x_2$ and $c_2 = \frac{1}{4}x_1 + \frac{1}{4}x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^2$. Moreover, taking $x_1 = x_2 = 0$ gives the unique solution $c_1 = c_2 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^2$.

A26 Since $1\begin{bmatrix}-1\\1\end{bmatrix} + 1\begin{bmatrix}1\\3\end{bmatrix} - 2\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, $\mathcal{B}$ is linearly dependent and hence is not a basis.

A27 Since $0\begin{bmatrix}1\\2\\1\end{bmatrix} + 1\begin{bmatrix}0\\0\\0\end{bmatrix} + 0\begin{bmatrix}1\\4\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\mathcal{B}$ is linearly dependent and hence is not a basis.

A28 Consider

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = c_1\begin{bmatrix}-1\\2\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}-c_1 + c_2\\2c_1 + c_2\\-c_1 + 2c_2\end{bmatrix}$$

This gives

$$\begin{aligned}-c_1 + c_2 &= x_1\\2c_1 + c_2 &= x_2\\-c_1 + 2c_2 &= x_3\end{aligned}$$

Subtracting the first equation from the second equation gives $3c_1 = x_2 - x_1$. Subtracting 2 times the first equation from the third gives $c_1 = x_3 - 2x_1$. Hence, for $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to be in the span, we must have $\frac{1}{3}(x_2 - x_1) = x_3 - 2x_1$. Since, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ does not satisfy this condition, it is not in Span $\mathcal{B}$. Therefore, $\mathcal{B}$ does not span $\mathbb{R}^3$ and hence is not a basis for $\mathbb{R}^3$.

A29 We will prove it is a basis. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_3 \\ c_1 + 2c_3 \end{bmatrix}$$

This gives

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_3 &= x_2 \\ c_1 + 2c_3 &= x_3 \end{aligned}$$

Solving we get $c_3 = x_2$, $c_1 = -2x_2 + x_3$, and $c_2 = x_1 + 2x_2 - x_3$. Hence, $\mathcal{B}$ spans $\mathbb{R}^3$. Moreover, taking $x_1 = x_2 = x_3 = 0$ gives the unique solution $c_1 = c_2 = c_3 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^3$.

A30 We will prove it is a basis. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_1 + c_2 \end{bmatrix}$$

This gives

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_2 + c_3 &= x_2 \\ c_1 + c_2 &= x_3 \end{aligned}$$

Solving we get $c_3 = x_1 - x_3$, $c_2 = -x_1 + x_2 + x_3$, and $c_1 = x_1 - x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^3$. Moreover, taking $x_1 = x_2 = x_3 = 0$ gives the unique solution $c_1 = c_2 = c_3 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^3$.

A31 (a) Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$$

This gives

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_2 &= x_2 \end{aligned}$$

Solving, we get $c_2 = x_2$ and $c_1 = x_1 - x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^2$. Moreover, taking $x_1 = x_2 = 0$ gives the unique solution $c_1 = c_2 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^2$.

(b) Taking $x_1 = 1$ and $x_2 = 0$ we find that the coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1$ and $c_2 = 0$.

Taking $x_1 = 0$ and $x_2 = 1$ we find that the coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -1$ and $c_2 = 1$.

Taking $x_1 = 1$ and $x_2 = 3$ we find that the coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = -2$ and $c_2 = 3$.

A32 (a) Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$$

This gives

$$\begin{aligned} c_1 + c_2 &= x_1 \\ c_1 - c_2 &= x_2 \end{aligned}$$

Solving, we get $c_1 = \frac{1}{2}x_1 + \frac{1}{2}x_2$ and $c_2 = \frac{1}{2}x_1 - \frac{1}{2}x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^2$. Moreover, taking $x_1 = x_2 = 0$ gives the unique solution $c_1 = c_2 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^2$.

(b) Taking $x_1 = 1$ and $x_2 = 0$ we find that the coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1/2$ and $c_2 = 1/2$.

Taking $x_1 = 0$ and $x_2 = 1$ we find that the coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = 1/2$ and $c_2 = -1/2$.

Taking $x_1 = 1$ and $x_2 = 3$ we find that the coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 2$ and $c_2 = -1$.

A33 (a) Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \\ 2c_1 - c_2 \end{bmatrix}$$

This gives

$$\begin{aligned} c_1 - c_2 &= x_1 \\ 2c_1 - c_2 &= x_2 \end{aligned}$$

Solving, we get $c_1 = -x_1 + x_2$ and $c_2 = -2x_1 + x_2$. Hence, $\mathcal{B}$ spans $\mathbb{R}^2$. Moreover, taking $x_1 = x_2 = 0$ gives the unique solution $c_1 = c_2 = 0$, so $\mathcal{B}$ is also linearly independent, and hence is a basis for $\mathbb{R}^2$.

(b) Taking $x_1 = 1$ and $x_2 = 0$ we find that the coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = -1$ and $c_2 = -2$.

Taking $x_1 = 0$ and $x_2 = 1$ we find that the coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = 1$ and $c_2 = 1$.

Taking $x_1 = 1$ and $x_2 = 3$ we find that the coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 2$ and $c_2 = 1$.

A34 Assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. For a contradiction, assume without loss of generality that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$. Then $\vec{v}_1 = t\vec{v}_2$ and hence $\vec{v}_1 - t\vec{v}_2 = \vec{0}$. This contradicts the fact that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent since the coefficient of $\vec{v}_1$ is non-zero.

On the other hand, assume that $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent. Then there exists $c_1, c_2 \in \mathbb{R}$ not both zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Without loss of generality assume that $c_1 \neq 0$. Then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$ and hence $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

A35 To prove this, we will prove that both sets are a subset of the other.

Let $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Then there exists $c_1, c_2 \in \mathbb{R}$ such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Since $t \neq 0$ we get

$$\vec{x} = c_1\vec{v}_1 + \frac{c_2}{t}(t\vec{v}_2)$$

so $\vec{x} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$. Thus, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$.

If $\vec{y} \in \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$, then there exists $d_1, d_2 \in \mathbb{R}$ such that

$$\vec{y} = d_1\vec{v}_1 + d_2(t\vec{v}_2) = d_1\vec{v}_1 + (d_2 t)\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

Hence, we also have $\operatorname{Span}\{\vec{v}_1, t\vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Therefore, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, t\vec{v}_2\}$.

B Homework Problems

B1 $\begin{bmatrix}3\\2\end{bmatrix} = \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$

B2 $\vec{x} \notin \operatorname{Span}\mathcal{B}$

B3 $\begin{bmatrix}2\\-2\end{bmatrix} = -\frac{2}{3}\begin{bmatrix}-3\\3\end{bmatrix}$

B4 $\begin{bmatrix}1\\0\end{bmatrix} = \frac{2}{5}\begin{bmatrix}2\\-1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}$

B5 $\begin{bmatrix}3\\1\\4\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\0\\1\end{bmatrix}$

B6 $\vec{x} \notin \operatorname{Span}\mathcal{B}$

B7 $\frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

B8 $0\begin{bmatrix}8\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

B9 Linearly independent

B10 $-2\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}4\\-10\end{bmatrix}$

B11 Linearly independent

B12 $0\begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

B13 $2\begin{bmatrix}2\\-1\\1\end{bmatrix} + 0\begin{bmatrix}1\\5\\3\end{bmatrix} = \begin{bmatrix}4\\-2\\2\end{bmatrix}$

B14 $\frac{1}{2}\begin{bmatrix}4\\2\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\6\\3\end{bmatrix} = \begin{bmatrix}3\\4\\2\end{bmatrix}$

B15 A line. $\vec{x} = s\begin{bmatrix}1\\0\end{bmatrix}, s \in \mathbb{R}$

B16 All of $\mathbb{R}^2$. $\vec{x} = s\begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}2\\3\end{bmatrix}, s, t \in \mathbb{R}$

B17 A line. $\vec{x} = s\begin{bmatrix}1\\2\end{bmatrix}, s \in \mathbb{R}$

B18 The origin. $\vec{x} = \vec{0}$.

B19 A line. $\vec{x} = s\begin{bmatrix}3\\1\\1\end{bmatrix}, s \in \mathbb{R}$

B20 A line. $\vec{x} = s\begin{bmatrix}2\\1\\3\end{bmatrix}, s \in \mathbb{R}$

B21 A basis

B22 Not a basis

B23 A basis

B24 A basis

B25 Not a basis

B26 A basis

B27 Not A basis

B28 A basis

B29 (a) Show $\mathcal{B}$ is a linearly independent spanning set.

(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1$, $c_2 = 1$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = 0$, $c_2 = 1$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 2$, $c_2 = 3$.

B30 (a) Show $\mathcal{B}$ is a linearly independent spanning set.

(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 3/5$, $c_2 = -1/5$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -1/5$, $c_2 = 2/5$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 0$, $c_2 = 1$.

B31 (a) Show $\mathcal{B}$ is a linearly independent spanning set.

(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1/2$, $c_2 = 0$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -1/6$, $c_2 = 1/3$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = 0$, $c_2 = 1$.

B32 (a) Show $\mathcal{B}$ is a linearly independent spanning set.

(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = 1/5$, $c_2 = 2/5$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -2/5$, $c_2 = 1/5$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = -1$, $c_2 = 1$.

B33 (a) Show $\mathcal{B}$ is a linearly independent spanning set.

(b) The coordinates of $\vec{e}_1$ with respect to $\mathcal{B}$ are $c_1 = -5/13$, $c_2 = -1/13$.
The coordinates of $\vec{e}_2$ with respect to $\mathcal{B}$ are $c_1 = -3/13$, $c_2 = 2/13$.
The coordinates of $\vec{x}$ with respect to $\mathcal{B}$ are $c_1 = -14/13$, $c_2 = 5/13$.

Section 1.3

A Practice Problems

A1 $\left\|\begin{bmatrix}2\\-5\end{bmatrix}\right\| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$

A2 $\left\|\begin{bmatrix}2/\sqrt{29}\\-5/\sqrt{29}\end{bmatrix}\right\| = \sqrt{(2/\sqrt{29})^2 + (-5/\sqrt{29})^2} = \sqrt{4/29 + 25/29} = 1$

A3 $\left\|\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

A4 $\left\|\begin{bmatrix}2\\3\\-2\end{bmatrix}\right\| = \sqrt{2^2+3^2+(-2)^2} = \sqrt{17}$

A5 $\left\|\begin{bmatrix}1\\1/5\\-3\end{bmatrix}\right\| = \sqrt{1^2+(1/5)^2+(-3)^2} = \sqrt{251}/5$

A6 $\left\|\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}\right\| = \sqrt{(1/\sqrt{3})^2+(1/\sqrt{3})^2+(-1/\sqrt{3})^2} = 1$

A7 The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix}-4\\1\end{bmatrix}-\begin{bmatrix}2\\3\end{bmatrix}\right\| = \left\|\begin{bmatrix}-6\\-2\end{bmatrix}\right\| = \sqrt{(-6)^2+(-2)^2} = 2\sqrt{10}$.

A8 The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix}-3\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\| = \left\|\begin{bmatrix}-4\\0\\3\end{bmatrix}\right\| = \sqrt{(-4)^2+0^2+3^2} = 5$.

A9 The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix}-3\\5\\1\end{bmatrix}-\begin{bmatrix}4\\-6\\1\end{bmatrix}\right\| = \left\|\begin{bmatrix}-7\\11\\0\end{bmatrix}\right\| = \sqrt{(-7)^2+11^2+0^2} = \sqrt{170}$.

A10 The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix}4\\6\\-2\end{bmatrix}-\begin{bmatrix}2\\1\\1\end{bmatrix}\right\| = \left\|\begin{bmatrix}2\\5\\-3\end{bmatrix}\right\| = \sqrt{2^2+5^2+(-3)^2} = \sqrt{38}$.

A11 $\begin{bmatrix}1\\3\\2\end{bmatrix}\cdot\begin{bmatrix}2\\-2\\2\end{bmatrix} = 1(2)+3(-2)+2(2) = 0$. Hence these vectors are orthogonal.

A12 $\begin{bmatrix}-3\\1\\7\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\1\end{bmatrix} = (-3)(2)+1(-1)+7(1) = 0$. Hence these vectors are orthogonal.

A13 $\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\2\end{bmatrix} = 2(-1)+1(4)+1(2) = 4 \neq 0$. Therefore, these vectors are not orthogonal.

A14 $\begin{bmatrix}4\\1\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\4\\3\end{bmatrix} = 4(-1)+1(4)+0(3) = 0$. Hence these vectors are orthogonal.

A15 $\begin{bmatrix}0\\0\\0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0(x_1)+0(x_2)+0(x_3) = 0$. Hence these vectors are orthogonal.

A16 $\begin{bmatrix}1/3\\2/3\\-1/3\end{bmatrix}\cdot\begin{bmatrix}3/2\\0\\-3/2\end{bmatrix} = \frac{1}{3}\left(\frac{3}{2}\right)+\frac{2}{3}(0)+\left(-\frac{1}{3}\right)\left(-\frac{3}{2}\right) = 1$. Therefore, these vectors are not orthogonal.

A17 The vectors are orthogonal when $0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ k \end{bmatrix} = 3(2) + (-1)k = 6 - k$.
Thus, the vectors are orthogonal only when $k = 6$.

A18 The vectors are orthogonal when $0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} k \\ k^2 \end{bmatrix} = 3(k) + (-1)(k^2) = 3k - k^2 = k(3 - k)$.
Thus, the vectors are orthogonal only when $k = 0$ or $k = 3$.

A19 The vectors are orthogonal when $0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -k \\ k \end{bmatrix} = 1(3) + 2(-k) + 3(k) = 3 + k$.
Thus, the vectors are orthogonal only when $k = -3$.

A20 The vectors are orthogonal when $0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k \\ k \\ -k \end{bmatrix} = 1(k) + 2(k) + 3(-k) = 0$.
Therefore, the vectors are always orthogonal.

A21 The scalar equation of the plane is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 + 1 \\ x_2 - 2 \\ x_3 + 3 \end{bmatrix} \\ &= 2(x_1 + 1) + 4(x_2 - 2) + (-1)(x_3 + 3) \\ &= 2x_1 + 2 + 4x_2 - 8 - x_3 - 3 \\ 9 &= 2x_1 + 4x_2 - x_3 \end{aligned}$$

A22 The scalar equation of the plane is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 2 \\ x_2 - 5 \\ x_3 - 4 \end{bmatrix} \\ &= 3(x_1 - 2) + 0(x_2 - 5) + 5(x_3 - 4) \\ &= 3x_1 - 6 + 5x_3 - 20 \\ 26 &= 3x_1 + 5x_3 \end{aligned}$$

A23 The scalar equation of the plane is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 - 1 \end{bmatrix} \\ &= 3(x_1 - 1) + (-4)(x_2 + 1) + 1(x_3 - 1) \\ &= 3x_1 - 3 - 4x_2 - 4 + x_3 - 1 \\ 8 &= 3x_1 - 4x_2 + x_3 \end{aligned}$$

A24 $\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (-5)(5) - 1(2) \\ 2(-2) - 1(5) \\ 1(1) - (-5)(-2) \end{bmatrix} = \begin{bmatrix} -27 \\ -9 \\ -9 \end{bmatrix}$

A25 $\begin{bmatrix}2\\-3\\-5\end{bmatrix}\times\begin{bmatrix}4\\-2\\7\end{bmatrix}=\begin{bmatrix}(-3)(7)-(-5)(-2)\\(-5)(4)-2(7)\\2(-2)-(-3)(4)\end{bmatrix}=\begin{bmatrix}-31\\-34\\8\end{bmatrix}$

A26 $\begin{bmatrix}-1\\0\\-1\end{bmatrix}\times\begin{bmatrix}0\\4\\5\end{bmatrix}=\begin{bmatrix}0(5)-(-1)(4)\\(-1)(0)-(-1)(5)\\(-1)(4)-0(0)\end{bmatrix}=\begin{bmatrix}4\\5\\-4\end{bmatrix}$

A27 $\begin{bmatrix}1\\2\\0\end{bmatrix}\times\begin{bmatrix}-1\\-3\\0\end{bmatrix}=\begin{bmatrix}2(0)-0(-3)\\0(-1)-1(0)\\1(-3)-2(-1)\end{bmatrix}=\begin{bmatrix}0\\0\\-1\end{bmatrix}$

A28 $\begin{bmatrix}4\\-2\\6\end{bmatrix}\times\begin{bmatrix}-2\\1\\-3\end{bmatrix}=\begin{bmatrix}(-2)(-3)-6(1)\\6(-2)-4(-3)\\4(1)-(-2)(-2)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

A29 $\begin{bmatrix}3\\1\\3\end{bmatrix}\times\begin{bmatrix}3\\1\\3\end{bmatrix}=\begin{bmatrix}1(3)-3(1)\\3(3)-3(3)\\3(1)-1(3)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

A30 (a) $\vec{u}\times\vec{u}=\begin{bmatrix}4(2)-2(4)\\2(-1)-(-1)(2)\\(-1)(4)-4(-1)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(b) We have

$$\vec{u}\times\vec{v}=\begin{bmatrix}4(-1)-2(1)\\2(3)-(-1)(-1)\\(-1)(1)-4(3)\end{bmatrix}=\begin{bmatrix}-6\\5\\-13\end{bmatrix}$$
$$-\vec{v}\times\vec{u}=-\begin{bmatrix}1(2)-(-1)(4)\\(-1)(-1)-3(2)\\3(4)-1(-1)\end{bmatrix}=-\begin{bmatrix}6\\-5\\13\end{bmatrix}=\vec{u}\times\vec{v}$$

(c) We have

$$\vec{u}\times 3\vec{w}=\begin{bmatrix}-1\\4\\2\end{bmatrix}\times\begin{bmatrix}6\\-9\\-3\end{bmatrix}=\begin{bmatrix}4(-3)-2(-9)\\2(6)-(-1)(-3)\\(-1)(-9)-4(6)\end{bmatrix}=\begin{bmatrix}6\\9\\-15\end{bmatrix}$$
$$3(\vec{u}\times\vec{w})=3\begin{bmatrix}4(-1)-2(-3)\\2(2)-(-1)(-1)\\(-1)(-3)-4(2)\end{bmatrix}=3\begin{bmatrix}2\\3\\-5\end{bmatrix}=\begin{bmatrix}6\\9\\-15\end{bmatrix}$$

(d) We have

$$\vec{u} \times (\vec{v} + \vec{w}) = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4(-2) - 2(-2) \\ 2(5) - (-1)(-2) \\ (-1)(-2) - 4(5) \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -18 \end{bmatrix}$$

$$\vec{u} \times \vec{v} + \vec{u} \times \vec{w} = \begin{bmatrix} -6 \\ 5 \\ -13 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -18 \end{bmatrix}$$

(e) We have

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1(-1) - (-1)(-3) \\ (-1)(2) - 3(-1) \\ 3(-3) - 1(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ -11 \end{bmatrix} = -14$$

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 5 \\ -13 \end{bmatrix} = -14$$

(f) From part (e) we have $\vec{u} \cdot (\vec{v} \times \vec{w}) = -14$. Then

$$\vec{v} \cdot (\vec{u} \times \vec{w}) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} = 14 = -\vec{u} \cdot (\vec{v} \times \vec{w})$$

A31 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -10 \end{bmatrix}$. Thus, a scalar equation for the plane is

$$x_1 - 4x_2 - 10x_3 = 1(1) - 4(4) - 10(7) = -85$$

A32 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$. Thus, a scalar equation for the plane is

$$2x_1 - 2x_2 + 3x_3 = 2(2) - 2(3) + 3(-1) = -5$$

A33 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 6 \end{bmatrix}$. Thus, a scalar equation for the plane is

$$-5x_1 - 2x_2 + 6x_3 = -5(1) - 2(-1) + 6(3) = 15$$

A34 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -17 \\ -1 \\ 10 \end{bmatrix}$. Thus, a scalar equation for the plane is

$$-17x_1 - x_2 + 10x_3 = -17(0) - (0) + 10(0) = 0$$

For Problems A35 - A40, alternate answers are possible.

A35 We can rewrite the equation as $x_3 = -2x_1 + 3x_2$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}$$

A36 We can rewrite the equation as $x_2 = 5 - 4x_1 + 2x_3$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 5 - 4x_1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad x_1, x_3 \in \mathbb{R}$$

A37 We can rewrite the equation as $x_1 = 1 - 2x_2 - 2x_3$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - 2x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

A38 We can rewrite the equation as $x_1 = \frac{7}{3} - \frac{5}{3}x_2 + \frac{4}{3}x_3$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} - \frac{5}{3}x_2 + \frac{4}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

A39 We can rewrite the equation as $x_2 = 2x_1 + 3x_3$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + 3x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}$$

A40 We can rewrite the equation as $x_2 = 3 - 2x_1 - 3x_3$. Thus, a vector equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3 - 2x_1 - 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad x_1, x_3 \in \mathbb{R}$$

A41 We have that the vectors $\vec{PQ} = \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 39 \\ 12 \\ 10 \end{bmatrix}$. Then, since $P(2, 1, 5)$ is a point on the plane we get a scalar equation of the plane is

$$39x_1 + 12x_2 + 10x_3 = 39(2) + 12(1) + 10(5) = 140$$

A42 We have that the vectors $\vec{PQ} = \begin{bmatrix} -5 \\ -1 \\ -2 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -2 \\ 3 \\ -5 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} -5 \\ -1 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -21 \\ -17 \end{bmatrix}$. Then, since $P(3, 1, 4)$ is a point on the plane we get a scalar equation of the plane is

$$11x_1 - 21x_2 - 17x_3 = 11(3) - 21(1) - 17(4) = -56$$

A43 We have that the vectors $\vec{PQ} = \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix} \times \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \\ -19 \end{bmatrix}$. Then, since $P(-1, 4, 2)$ is a point on the plane we get a scalar equation of the plane is

$$-12x_1 + 3x_2 - 19x_3 = -12(-1) + 3(4) - 19(2) = -14$$

A44 We have that the vectors $\vec{PQ} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$. Then, since $R(0, 0, 0)$ is a point on the plane we get a scalar equation of the plane is $-2x_2 = 0$ or $x_2 = 0$.

A45 We have that the vectors $\vec{PQ} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$. Then, since $P(0, 2, 1)$ is a point on the plane we get a scalar equation of the plane is

$$3x_1 + 3x_2 + 6x_3 = 3(0) + 3(2) + 6(1) = 12 \text{ or } x_1 + x_2 + 2x_3 = 4$$

A46 We have that the vectors $\vec{PQ} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \\ -5 \end{bmatrix}$. Then, since $R(1, 0, 1)$ is a point on the plane we get a scalar equation of the plane is

$$14x_1 - 4x_2 - 5x_3 = 14(1) - 4(0) - 5(1) = 9$$

A47 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$. Then, since $P(1, -3, -1)$ is a point on the plane we get a scalar equation of the plane is

$$2x_1 - 3x_2 + 5x_3 = 2(1) - 3(-3) + 5(-1) = 6$$

A48 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then, since $P(0, -2, -4)$ is a point on the plane we get a scalar equation of the plane is

$$x_2 = -2$$

A49 A normal vector for the plane is $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Then, since $P(1, 2, 1)$ is a point on the plane we get a scalar equation of the plane is

$$x_1 - x_2 + 3x_3 = 1(1) - 1(2) + 3(1) = 2$$

A50 The line of intersection must lie in both planes and hence it must be orthogonal to both normal vectors. Hence, a direction vector of the line is

$$\vec{d} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -11 \end{bmatrix}$$

To find a point on the line we set $x_3 = 0$ in the equations of both planes to get $x_1 + 3x_2 = 5$ and $2x_1 - 5x_2 = 7$. Solving the two equations in two unknowns gives the solution $x_1 = \frac{46}{11}$ and $x_2 = \frac{3}{11}$. Thus, an equation of the line is

$$\vec{x} = \begin{bmatrix} 46/11 \\ 3/11 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ -11 \end{bmatrix}, \quad t \in \mathbb{R}$$

A51 A direction vector of the line is

$$\vec{d} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

To find a point on the line we set $x_3 = 0$ to get $2x_1 = 7$ and $x_2 = 4$. Thus, an equation of the line is

$$\vec{x} = \begin{bmatrix} 7/2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$$

A52 A direction vector of the line is

$$\vec{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}$$

To find a point on the line we set $x_3 = 0$ in the equations of both planes to get $x_1 - 2x_2 = 1$ and $3x_1 + 4x_2 = 5$. Solving the two equations in two unknowns gives the solution $x_1 = \frac{7}{5}$ and $x_2 = \frac{1}{5}$. Thus, an equation of the line is

$$\vec{x} = \begin{bmatrix} 7/5 \\ 1/5 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}, \quad t \in \mathbb{R}$$

A53 A direction vector of the line is

$$\vec{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}$$

Clearly $(0, 0, 0)$ is on both lines. Hence, an equation of the line is

$$\vec{x} = t\begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix}, \quad t \in \mathbb{R}$$

A54 The area of the parallelogram is

$$\left\|\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}\right\| = \sqrt{35}$$

A55 The area of the parallelogram is

$$\left\|\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\| = \sqrt{11}$$

A56 As specified in the hint, we write the vectors as $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Hence, the area of the parallelogram is

$$\left\|\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 \\ 0 \\ -13 \end{bmatrix}\right\| = 13$$

A57 $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. Therefore, $\vec{u}$ lies in the plane through the origin that contains $\vec{v}$ and $\vec{w}$. We can also see this by observing that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ has volume zero; this can happen only if the three vectors lie in a common plane.

A58 We have

$$\begin{aligned}(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) &= \vec{u} \times (\vec{u} + \vec{v}) - \vec{v} \times (\vec{u} + \vec{v}) \\ &= \vec{u} \times \vec{u} + \vec{u} \times \vec{v} - \vec{v} \times \vec{u} - \vec{v} \times \vec{v} \\ &= \vec{0} + \vec{u} \times \vec{v} + \vec{u} \times \vec{v} - \vec{0} \\ &= 2(\vec{u} \times \vec{v})\end{aligned}$$

as required.

B Homework Problems

B1 $\sqrt{17}$ **B2** $\sqrt{13}$ **B3** 0
B4 1 **B5** 1 **B6** $\sqrt{3/2}$

B7 $\sqrt{26}$ **B8** $\sqrt{17}$ **B9** $\sqrt{41}$
B10 $\sqrt{11}$ **B11** $\sqrt{24}$ **B12** $\sqrt{14}$
B13 $\sqrt{57}$

B14 Not orthogonal **B15** Not orthogonal **B16** Orthogonal
B17 Not orthogonal **B18** Not orthogonal **B19** Orthogonal

B20 $k = 0$ **B21** $k = 0, -3$
B22 $k = 2/7$ **B23** $k = 0, 5$
B24 $x_1 - x_2 + 5x_3 = 4$ **B25** $3x_1 + 3x_2 - 4x_3 = 17$
B26 $-2x_2 - x_3 = -5$ **B27** $x_1 + 3x_2 + x_3 = 11$
B28 $5x_1 - 6x_2 + 3x_3 = 0$

B29 $\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ **B30** $\begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}$

B31 $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **B32** $\begin{bmatrix} 1 \\ -11 \\ -16 \end{bmatrix}$

B33 $\begin{bmatrix} -1 \\ 11 \\ 16 \end{bmatrix}$ **B34** $\begin{bmatrix} -4 \\ 20 \\ -12 \end{bmatrix}$

B35 (a) $\vec{u} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\vec{u} \times \vec{v} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = -\vec{v} \times \vec{u}$

(c) $\vec{u} \times 2\vec{w} = \begin{bmatrix} -6 \\ -8 \\ 2 \end{bmatrix} = 2(\vec{u} \times \vec{w})$ (d) $\vec{u} \times (\vec{v} + \vec{w}) = \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix} = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(e) $\vec{u} \cdot (\vec{v} \times \vec{w}) = -3 = \vec{w} \cdot (\vec{u} \times \vec{v})$ (f) $\vec{u} \cdot (\vec{v} \times \vec{w}) = -3 = -\vec{v} \cdot (\vec{u} \times \vec{w})$

B36 $-2x_1 - 4x_2 + 5x_3 = -15$ **B37** $x_1 - 7x_2 - 5x_3 = -35$
B38 $x_1 - x_2 - x_3 = 1$ **B39** $x_1 + 11x_2 + 14x_3 = 0$
B40 $x_1 + x_3 = 0$ **B41** $5x_1 + 2x_2 - 3x_3 = 0$

B42 $\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$ **B43** $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, x_1, x_3 \in \mathbb{R}$

B44 $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$ **B45** $\vec{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$

B46 $\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, x_1, x_2 \in \mathbb{R}$

B47 $\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}, x_2, x_3 \in \mathbb{R}$

B48 $x_1 + 11x_2 + 2x_3 = 43$

B49 $8x_1 - x_2 + 2x_3 = 25$

B50 $x_1 + 2x_2 + 2x_3 = 6$

B51 $7x_1 + x_2 - 14x_3 = -6$

B52 $-2x_1 - 6x_2 + x_3 = -31$

B53 $-19x_1 + 22x_2 - 21x_3 = -6$

B54 $4x_1 + x_2 + 2x_3 = 6$

B55 $-x_1 + 2x_2 - 3x_3 = -23$

B56 $2x_1 + 3x_3 = 12$

B57 $-x_1 - 5x_2 + 3x_3 = -6$

B58 $2x_1 + 3x_2 - 4x_3 = 0$

B59 $4x_1 + 2x_2 + 2x_3 = 0$

B60 $\vec{x} = \begin{bmatrix} 11/7 \\ -2/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}, t \in \mathbb{R}$

B61 $\vec{x} = \begin{bmatrix} 1/2 \\ 3/4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}, t \in \mathbb{R}$

B62 $\vec{x} = \begin{bmatrix} 9/7 \\ 1/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, t \in \mathbb{R}$

B63 $\vec{x} = \begin{bmatrix} 7/4 \\ 1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 2 \\ -16 \end{bmatrix}, t \in \mathbb{R}$

B64 $\vec{x} = \begin{bmatrix} 4/3 \\ -10/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 11 \\ -3 \end{bmatrix}, t \in \mathbb{R}$

B65 $\sqrt{75}$

B66 $\sqrt{65}$

B67 $\sqrt{120}$

B68 19

B69 2

B70 6

C Conceptual Problems

C1 (a) First, we know that $\vec{d} \neq \vec{0}$ as otherwise the vector equation would not be a line. Intuitively, if there is no point of intersection, the line is parallel to the plane. Hence, the direction vector of the line must be orthogonal to the normal to the plane. Therefore, we will have that $\vec{d} \cdot \vec{n} = 0$. Since the point P cannot be on the plane, it cannot satisfy the equation of the plane, so $\vec{p} \cdot \vec{n} \neq k$.

(b) Substitute $\vec{x} = \vec{p} + t\vec{d}$ into the equation of the plane to see whether for some t, $\vec{x}$ satisfies the equation of the plane.

$$\vec{n} \cdot (\vec{p} + t\vec{d}) = k$$

Isolate the term in t: $t(\vec{n} \cdot \vec{d}) = k - \vec{n} \cdot \vec{p}$.

There is one solution for t (and thus, one point of intersection of the line and the plane) exactly when $\vec{n} \cdot \vec{d} \neq 0$. If $\vec{n} \cdot \vec{d} = 0$, there is no solution for t unless we also have $\vec{n} \cdot \vec{p} = k$. In this case the equation is satisfied for all t and the line lies in the plane. Thus, to have no point of intersection, it is necessary and sufficient that $\vec{n} \cdot \vec{d} = 0$ and $\vec{n} \cdot \vec{p} \neq k$.

C2 (a) We have $\vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + x_3^2 \geq 0$.

(b) If $\vec{x} \cdot \vec{x} = 0$, then $x_1^2 + x_2^2 + x_3^2 = 0$ which implies $x_1 = x_2 = x_3 = 0$ as required. On the other hand $\vec{0} \cdot \vec{0} = 0^2 + 0^2 + 0^2 = 0$.

(c) We have $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3 = y_1x_1 + y_2x_2 + y_3x_3 = \vec{y} \cdot \vec{x}$.

(d) We have

$$\begin{aligned}\vec{x}\cdot(s\vec{y}+t\vec{z}) &= x_1(sy_1+tz_1)+x_2(sy_2+tz_2)+x_3(sy_3+tz_3)\\ &= s[x_1y_1+x_2y_2+x_3y_3]+t[x_1z_1+x_2z_2+x_3z_3]\\ &= s(\vec{x}\cdot\vec{y})+t(\vec{x}\cdot\vec{z})\end{aligned}$$

C3 (a) $\begin{bmatrix}x_1\\x_2\end{bmatrix}\cdot\begin{bmatrix}0\\0\end{bmatrix}=x_1(0)+x_2(0)=0$

(b) $\vec{x}\cdot\vec{0}=\vec{x}\cdot 0(\vec{y})=0(\vec{x}\cdot\vec{y})=0$

C4 Let $\vec{x}$ be any point that is equidistant from P and Q. Then $\vec{x}$ satisfies $\|\vec{x}-\vec{p}\|=\|\vec{x}-\vec{q}\|$, or equivalently, $\|\vec{x}-\vec{p}\|^2=\|\vec{x}-\vec{q}\|^2$. Hence,

$$\begin{aligned}(\vec{x}-\vec{p})\cdot(\vec{x}-\vec{p}) &= (\vec{x}-\vec{q})\cdot(\vec{x}-\vec{q})\\ \vec{x}\cdot\vec{x}-\vec{x}\cdot\vec{p}-\vec{p}\cdot\vec{x}+\vec{p}\cdot\vec{p} &= \vec{x}\cdot\vec{x}-\vec{x}\cdot\vec{q}-\vec{q}\cdot\vec{x}+\vec{q}\cdot\vec{q}\\ -2\vec{p}\cdot\vec{x}+2\vec{q}\cdot\vec{x} &= \vec{q}\cdot\vec{q}-\vec{p}\cdot\vec{p}\\ 2(\vec{q}-\vec{p})\cdot\vec{x} &= \|\vec{q}\|^2-\|\vec{p}\|^2\end{aligned}$$

This is the equation of a plane with normal vector $2(\vec{q}-\vec{p})$.

C5 (a) A point $\vec{x}$ on the plane must satisfy $\left\|\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right\|=\left\|\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right\|$. Square both sides and simplify.

$$\begin{aligned}\left(\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right)\cdot\left(\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right) &= \left(\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right)\cdot\left(\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right)\\ \vec{x}\cdot\vec{x}-2\begin{bmatrix}2\\2\\5\end{bmatrix}\cdot\vec{x}+33 &= \vec{x}\cdot\vec{x}-2\begin{bmatrix}-3\\4\\1\end{bmatrix}\cdot\vec{x}+26\\ 2\left(\begin{bmatrix}-3\\4\\1\end{bmatrix}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right)\cdot\vec{x} &= 26-33\\ 5x_1-2x_2+4x_3 &= 7/2\end{aligned}$$

(b) A point equidistant from the points is $\frac{1}{2}\left(\begin{bmatrix}2\\2\\5\end{bmatrix}+\begin{bmatrix}-3\\4\\1\end{bmatrix}\right)=\begin{bmatrix}-1/2\\3\\3\end{bmatrix}$. The vector joining the two points, $\vec{n}=\begin{bmatrix}2\\2\\5\end{bmatrix}-\begin{bmatrix}-3\\4\\1\end{bmatrix}=\begin{bmatrix}5\\-2\\4\end{bmatrix}$ must be orthogonal to the plane. Thus, the equation of the plane is

$$\vec{n}\cdot(\vec{x}-\vec{p})=\begin{bmatrix}5\\-2\\4\end{bmatrix}\cdot\begin{bmatrix}x_1+\frac{1}{2}\\x_2-3\\x_3-3\end{bmatrix}$$

which gives

$$5x_1-2x_2+4x_3=7/2$$

C6 (a) The statement is false. If $\vec{x} = \vec{0}$, $\vec{y} = \vec{e}_1$ and $\vec{z} = \vec{e}_2$, then $\vec{x} \cdot \vec{y} = 0 = \vec{x} \cdot \vec{z}$ but $\vec{y} \neq \vec{z}$.

(b) No, it does not. If $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, then $\vec{x} \cdot \vec{y} = 0 = \vec{x} \cdot \vec{z}$ but $\vec{y} \neq \vec{z}$.

C7 If X is a point on the line through P and Q, then for some $t \in \mathbb{R}$, $\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$ Hence,

$$\begin{aligned} \vec{x} \times (\vec{q} - \vec{p}) &= (\vec{p} + t(\vec{q} - \vec{p})) \times (\vec{q} - \vec{p}) \\ &= \vec{p} \times \vec{q} - \vec{p} \times \vec{p} + t(\vec{q} - \vec{p}) \times (\vec{q} - \vec{p}) = \vec{p} \times \vec{q} \end{aligned}$$

C8 (a) Let $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$. We have $\vec{n} \cdot \vec{e}_1 = \|\vec{n}\| \|\vec{e}_1\| \cos\alpha$. But, $\|\vec{n}\| = 1$ and $\|\vec{e}_1\| = 1$, so $\vec{n} \cdot \vec{e}_1 = \cos\alpha$. But, $\vec{n} \cdot \vec{e}_1 = n_1$, so $n_1 = \cos\alpha$. Similarly, $n_2 = \cos\beta$ and $n_3 = \cos\gamma$, so $\vec{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$.

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \|\vec{n}\|^2 = 1$, because $\vec{n}$ is a unit vector.

(c) In $\mathbb{R}^2$, the unit vector is $\vec{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \end{bmatrix}$, where α is the angle between $\vec{n}$ and $\vec{e}_1$ and β is the angle between $\vec{n}$ and $\vec{e}_2$. But in the plane $\alpha + \beta = \frac{\pi}{2}$, so $\cos\beta = \cos(\pi/2 - \alpha) = \sin\alpha$. Now let $\theta = \alpha$, and we have

$$1 = \|\vec{n}\|^2 = \cos^2\alpha + \cos^2\beta = \cos^2\theta + \sin^2\theta$$

C9 The statement is false. For any non-zero vector $\vec{u}$ and any vector $\vec{v} \in \mathbb{R}^3$, let $\vec{w} = \vec{v} + t\vec{u}$ for any $t \in \mathbb{R}$, $t \neq 0$. Then

$$\vec{u} \times \vec{w} = \vec{u} \times (\vec{v} + t\vec{u}) = \vec{u} \times \vec{v}$$

but $\vec{v} \neq \vec{w}$.

C10 If $\vec{v} \times \vec{w} = \vec{0}$, then $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{0}$ which clearly satisfies the equation $\vec{x} = s\vec{v} + t\vec{w}$. Assume $\vec{n} = \vec{v} \times \vec{w} \neq \vec{0}$. Then $\vec{n}$ is orthogonal to both $\vec{v}$ and $\vec{w}$ and hence it is a normal vector of the plane through the origin containing $\vec{v}$ and $\vec{w}$. Then, $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \vec{n}$ is orthogonal to $\vec{n}$ so it lies in the plane through the origin with normal vector $\vec{n}$. That is, it is in the plane containing $\vec{v}$ and $\vec{w}$. Hence, there exists $s, t \in \mathbb{R}$ such that $\vec{u} \times (\vec{v} \times \vec{w}) = s\vec{v} + t\vec{w}$.

C11 (a) We have $\vec{e}_1 \times (\vec{e}_2 \times \vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (\vec{e}_1 \times \vec{e}_2) \times \vec{e}_3$.

(b) Take $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then $\vec{e}_1 \times (\vec{e}_2 \times \vec{w}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ while $(\vec{e}_1 \times \vec{e}_2) \times \vec{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

C12 Consider $\vec{0} = c_1\vec{x} + c_2\vec{y}$. Taking the dot product of both sides with $\vec{x}$ gives

$$\begin{aligned} \vec{0} \cdot \vec{x} &= (c_1\vec{x} + c_2\vec{y}) \cdot \vec{x} \\ 0 &= c_1(\vec{x} \cdot \vec{x}) + c_2(\vec{x} \cdot \vec{y}) \\ 0 &= c_1\|\vec{x}\|^2 + 0 \end{aligned}$$

But, $\|\vec{x}\| \neq 0$ since $\vec{x} \neq \vec{0}$. Thus, $c_1 = 0$. Similarly, taking the dot product of both sides with respect to $\vec{y}$ gives $c_2 = 0$. Thus, $\{\vec{x}, \vec{y}\}$ is linearly independent.

C13 Consider $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Taking the dot product of both sides with $\vec{v}_1$ gives

$$\begin{aligned}\vec{x} \cdot \vec{v}_1 &= (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot \vec{v}_1 \\ \vec{x} \cdot \vec{v}_1 &= c_1\|\vec{v}_1\|^2 + 0\end{aligned}$$

Since $\vec{v}_1 \neq \vec{0}$ (as otherwise $\mathcal{B}$ would be linearly dependent), we get that $c_1 = \frac{\vec{x}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}$ as required. The proof for c_2 is the same.

Section 1.4

A Practice Problems

A1 $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

A2 $\begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 10 \\ -5 \end{bmatrix}$

A3 $\begin{bmatrix} 3 \\ -4 \\ -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \\ 5 \\ 3 \end{bmatrix}$

A4 $2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -3 \end{bmatrix}$

A5 The set is a subspace of $\mathbb{R}^2$ by Theorem 1.4.2.

A6 Since the condition of the set contains the square of a variable in it, we suspect that it is not a subspace. To prove it is not a subspace we just need to find one example where the set is not closed under linear combinations. Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Observe that $\vec{x}$ and $\vec{y}$ are in the set since $x_1^2 - x_2^2 = 1^2 - 1^2 = 0 = x_3$ and $y_1^2 - y_2^2 = 2^2 - 1^2 = 3 = y_3$, but $\vec{x} + \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ is not in the set since $3^2 - 2^2 = 5 \neq 3$.

A7 Since the condition of the set only contains linear variables, we suspect that this is a subspace. To prove it is a subspace we need to show that it satisfies the definition of a subspace.
Call the set S. First, observe that S is a subset of $\mathbb{R}^3$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in S. Then they must satisfy the condition of S, so $x_1 = x_3$ and $y_1 = y_3$. We now need to show that $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \end{bmatrix}$ satisfies the conditions of the set. In particular, we need to show that the first entry of $s\vec{x} + t\vec{y}$ equals its third entry. Since $x_1 = x_3$ and $y_1 = y_3$ we get $sx_1 + ty_1 = sx_3 + ty_3$ as required. Thus, S is a subspace of $\mathbb{R}^3$.

A8 Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set S. First, observe that S is a subset of $\mathbb{R}^2$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ in S. Then they must satisfy the condition of S, so $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$. Then $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \end{bmatrix}$ satisfies the conditions of the set since $(sx_1 + ty_1) + (sx_2 + ty_2) = s(x_1 + x_2) + t(y_1 + y_2) = s(0) + t(0) = 0$. Thus, S is a subspace of $\mathbb{R}^2$.

A9 The condition of the set involves multiplication of entries, so we suspect that it is not a subspace. Observe that if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\vec{x}$ is in the set since $x_1x_2 = 1(1) = 1 = x_3$, but $2\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is not in the set since $2(2) = 4 \neq 2$. Therefore, the set is not a subspace.

A10 At first glance this might not seem like a subspace since we are adding the vector $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. However, the key observation to make is that $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, the set can be written as $S = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ and hence is a subspace by Theorem 1.4.2.

A11 Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set S. By definition S is a subset of $\mathbb{R}^4$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ in S, then $x_1 + x_2 + x_3 + x_4 = 0$ and $y_1 + y_2 + y_3 + y_4 = 0$. We have $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \\ sx_4 + ty_4 \end{bmatrix}$ satisfies the conditions of the set since $(sx_1+ty_1)+(sx_2+ty_2)+(sx_3+ty_3)+(sx_4+ty_4) = s(x_1+x_2+x_3+x_4)+t(y_1+y_2+y_3+y_4) = s(0)+t(0) = 0$. Thus, S is a subspace of $\mathbb{R}^4$.

A12 The set clearly does not contain the zero vector and hence cannot be a subspace.

A13 The conditions of the set only contain linear variables, but we notice that the first equation $x_1+2x_3 = 5$ excludes $x_1 = x_3 = 0$. Hence the zero vector is not in the set so it is not a subspace.

A14 The conditions of the set involve a multiplication of variables, so we suspect that it is not a subspace. We take $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then, $\vec{x}$ is in the set since $x_1 = 1 = 1(1) = x_3x_4$ and $x_2 - x_4 = 1 - 1 = 0$. But, $2\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ is not in the set since $2 \neq 2(2)$.

A15 Since the conditions of the set only contains linear variables, we suspect that this is a subspace. Call the set S. By definition S is a subset of $\mathbb{R}^4$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ in S, then $2x_1 = 3x_4$, $x_2 - 5x_3 = 0$, $2y_1 = 3y_4$, and $y_2 - 5y_3 = 0$. We have $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \\ sx_4 + ty_4 \end{bmatrix}$ satisfies the conditions of the set since $2(sx_1 + ty_1) = 2sx_1 + 2ty_1 = 3sx_4 + t3y_4 = 3(sx_4 + ty_4)$ and $(sx_2 + ty_2) - 5(sx_3 - ty_3) = s(x_2 - 5x_3) + t(y_2 - 5y_3) = s(0) + t(0) = 0$. Thus, S is a subspace of $\mathbb{R}^4$.

A16 Since $x_3 = 2$ the zero vector cannot be in the set, so it is not a subspace.

For Problems A17 - A20, alternative correct answers are possible.

A17 $1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A18 $0\begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A19 $1\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A20 It is difficult to determine a linear combination by inspection, so we set up a system of equations. Consider

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\\-2\\3\end{bmatrix} + c_2\begin{bmatrix}1\\2\\1\\3\end{bmatrix} + c_3\begin{bmatrix}1\\-1\\-8\\3\end{bmatrix} = \begin{bmatrix}c_1 + c_2 + c_3\\c_1 + 2c_2 - c_3\\-2c_1 + c_2 - 8c_3\\3c_1 + 3c_2 + 3c_3\end{bmatrix}$$

This gives us the system of equations

$$\begin{aligned}c_1 + c_2 + c_3 &= 0\\c_1 + 2c_2 - c_3 &= 0\\-2c_1 + c_2 - 8c_3 &= 0\\3c_1 + 3c_2 + 3c_3 &= 0\end{aligned}$$

Adding the first equation and the second equation gives $2c_1 + 3c_2 = 0$. Subtracting the first equation from the second equation gives $c_2 - 2c_3 = 0$. Thus, if we take $c_3 = 1$, we get $c_2 = 2$ and hence $c_1 = -3$. Indeed, we find that

$$(-3)\begin{bmatrix}1\\1\\-2\\3\end{bmatrix} + 2\begin{bmatrix}1\\2\\1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\\-8\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

A21 Observe that $0\begin{bmatrix}1\\1\\0\\3\end{bmatrix} + 2\begin{bmatrix}1\\2\\-1\\1\end{bmatrix} + \begin{bmatrix}-2\\-4\\2\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, so the set is linearly dependent.

A22 Observe that $-2\begin{bmatrix}1\\1\\2\\1\end{bmatrix} + \begin{bmatrix}2\\2\\4\\2\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, so the set is linearly dependent.

A23 Consider $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_1 + c_2\\c_2\\c_1 + c_2\end{bmatrix}$. Comparing entries gives $c_1 = c_2 = 0$, so the set is linearly independent.

A24 Consider $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}3\\2\\1\\2\end{bmatrix} + c_2\begin{bmatrix}4\\4\\-5\\0\end{bmatrix} + c_3\begin{bmatrix}3\\3\\-2\\1\end{bmatrix} = \begin{bmatrix}3c_1+4c_2+3c_3\\2c_1+4c_2+3c_3\\c_1-5c_2-2c_3\\2c_1+c_3\end{bmatrix}$. This gives

$$\begin{aligned}3c_1+4c_2+3c_3&=0\\2c_1+4c_2+3c_3&=0\\c_1-5c_2-2c_3&=0\\2c_1+c_3&=0\end{aligned}$$

Subtracting the second equation from the first gives $c_1 = 0$. Then, the third equation gives $c_3 = 0$ and any of the other equations gives $c_2 = 0$. Thus, the set is linearly independent.

A25 By the definition of P, every $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in P$ satisfies $2x_1 + x_2 + x_3 = 0$. Solving this for x_2 gives $x_2 = -2x_1 - x_3$. Consider

$$\begin{bmatrix}x_1\\-2x_1-x_3\\x_3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\-2\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\-2c_1-c_2\end{bmatrix}$$

Solving we find that $c_1 = x_1$, $c_2 = -2x_1 - x_3$. Observe that $-2c_1 - c_2 = -2x_1 - (-2x_1 - x_3) = x_3$ so the third equation is also satisfied. Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\0\\-2\end{bmatrix} + c_2\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\-2c_1-c_2\end{bmatrix}$$

Comparing entries we get that $c_1 = c_2 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

NOTE: We could have solved the equation for the plane P for x_3 instead.

A26 By the definition of P, every $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in P$ satisfies $3x_1 + x_2 - 2x_3 = 0$. Solving this for x_2 gives $x_2 = -3x_1 + 2x_3$. Consider

$$\begin{bmatrix}x_1\\-3x_1+2x_3\\x_3\end{bmatrix} = c_1\begin{bmatrix}1\\-3\\0\end{bmatrix} + c_2\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}c_1\\-3c_1+2c_2\\c_2\end{bmatrix}$$

Solving we find that $c_1 = x_1$, $c_2 = x_3$ (observe that $-3c_1 + 2c_2 = -3x_1 + 2x_3$ so the second equation is also satisfied). Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\-3\\0\end{bmatrix} + c_2\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}c_1\\-3c_1+2c_2\\c_2\end{bmatrix}$$

Comparing entries we get that $c_1 = c_2 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

A27 By the definition of P, every $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in P$ satisfies $3x_1 + x_2 - 2x_3 = 0$. Solving this for x_2 gives $x_2 = -3x_1 + 2x_3$. Consider

$$\begin{bmatrix} x_1 \\ -3x_1 + 2x_3 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{bmatrix}$$

Solving we find that $c_1 = x_1$, $c_2 = -3x_1 + 2x_3$ (observe that $\frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{3}{2}x_1 + \frac{1}{2}(-3x_1 + 2x_3) = x_3$ so the third equation is also satisfied). Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 \end{bmatrix}$$

Comparing entries we get that $c_1 = c_2 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

A28 By the definition of P, every $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in P$ satisfies $x_1 + x_2 + x_3 - x_4 = 0$. Solving this for x_4 gives $x_4 = x_1 + x_2 + x_3$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

Solving we find that $c_1 = x_1$, $c_2 = x_2$, $c_3 = x_3$ (observe that $c_1 + c_2 + c_3 = x_1 + x_2 + x_3 = x_4$ so the fourth equation is also satisfied). Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

Comparing entries we get that $c_1 = c_2 = c_3 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

For Problems A29 - A32, alternative correct answers are possible.

A29 We observe that neither vector is a scalar multiple of the other. Hence, this is a linearly independent set of two vectors in $\mathbb{R}^4$. Hence, it is a plane in $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

A30 The set $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ is a subset of the standard basis for $\mathbb{R}^4$ and hence is a linearly independent set of three vectors in $\mathbb{R}^4$. Hence, the span of this set is a hyperplane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

A31 Observe that the second and third vectors are just scalar multiples of the first vector. Hence, by Theorem 1.4.3, we can write

$$\operatorname{Span}\left\{\begin{bmatrix}3\\1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}6\\2\\-2\\0\end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix}3\\1\\-1\\0\end{bmatrix}\right\}$$

Therefore, it is a line in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}3\\1\\-1\\0\end{bmatrix}\right\}$.

A32 Observe that the third vector is the sum of the first two vectors. Hence, by Theorem 1.4.3 we can write

$$\operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$$

Since $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$ is linearly independent, we get that it spans a plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$.

A33 If $\vec{x} = \vec{p} + t\vec{d}$ is a subspace of $\mathbb{R}^n$, then it contains the zero vector. Hence, there exists t_1 such that $\vec{0} = \vec{p} + t_1\vec{d}$. Thus, $\vec{p} = -t_1\vec{d}$ and so $\vec{p}$ is a scalar multiple of $\vec{d}$. On the other hand, if $\vec{p}$ is a scalar multiple of $\vec{d}$, say $\vec{p} = t_1\vec{d}$, then we have $\vec{x} = \vec{p} + t\vec{d} = t_1\vec{d} + t\vec{d} = (t_1 + t)\vec{d}$. Hence, the set is $\operatorname{Span}\{\vec{d}\}$ and thus is a subspace.

A34 Assume there is a non-empty subset $\mathcal{B}_1 = \{\vec{v}_1, \dots, \vec{v}_\ell\}$ of $\mathcal{B}$ that is linearly dependent. Then there exists c_i not all zero such that

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell = c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell + 0\vec{v}_{\ell+1} + \cdots + 0\vec{v}_n$$

which contradicts the fact that $\mathcal{B}$ is linearly independent. Hence, $\mathcal{B}_1$ must be linearly independent.

A35 (a) Assume that $\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$.

Since $\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ our assumption implies that $\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$. Consequently, there exists $b_1, \dots, b_{k-1} \in \mathbb{R}$ such that

$$\vec{v}_k = b_1\vec{v}_1 + \cdots + b_{k-1}\vec{v}_{k-1}$$

Therefore, $\vec{v}_k$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_{k-1}$ as required.

(b) If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_{k-1}$, then, by definition of linear combination, there exist $c_1, \ldots, c_{k-1} \in \mathbb{R}$ such that

$$c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = \vec{v}_k \tag{1}$$

To prove that $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ we will show that the sets are subsets of each other.

By definition of span, for any $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ there exist $d_1, \ldots, d_k \in \mathbb{R}$ such that

$$\vec{x} = d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_k\vec{v}_k$$

Using equation (1) to substitute in for $\vec{v}_k$ gives

$$\vec{x} = d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1} + d_k(c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1})$$

Rearranging using properties from Theorem 1.1.1 gives

$$\vec{x} = (d_1 + d_kc_1)\vec{v}_1 + \cdots + (d_{k-1} + d_kc_{k-1})\vec{v}_{k-1}$$

Thus, by definition, $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ and hence

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$$

Now, if $\vec{y} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$, then there exists $a_1, \ldots, a_{k-1} \in \mathbb{R}$ such that

$$\begin{aligned}\vec{y} &= a_1\vec{v}_1 + \cdots + a_{k-1}\vec{v}_{k-1}\\ &= a_1\vec{v}_1 + \cdots + a_{k-1}\vec{v}_{k-1} + 0\vec{v}_k\end{aligned}$$

Thus, $\vec{y} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. Hence, we also have $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\} \subseteq \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ and so

$$\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$$

A36 The linear combination represent how much material is required to produce 100 thingamajiggers and 250 whatchamacallits.

B Homework Problems

B1 $\begin{bmatrix} 9 \\ 4 \\ 16 \\ 23 \end{bmatrix}$ **B2** $\begin{bmatrix} 7 \\ 6 \\ 5 \\ -3 \\ -1 \end{bmatrix}$ **B3** $\begin{bmatrix} 4 \\ -5 \\ -5 \\ 3 \\ 0 \end{bmatrix}$ **B4** $\begin{bmatrix} -5 \\ 2 \\ 12 \\ 5 \\ 20 \end{bmatrix}$

B5 It is a subspace of $\mathbb{R}^2$. **B6** It is not a subspace of $\mathbb{R}^2$. **B7** It is not a subspace of $\mathbb{R}^2$.
B8 It is a subspace of $\mathbb{R}^3$. **B9** It is not a subspace of $\mathbb{R}^3$. **B10** It is a subspace of $\mathbb{R}^3$.
B11 It is not a subspace of $\mathbb{R}^3$. **B12** It is a subspace of $\mathbb{R}^3$. **B13** It is not a subspace of $\mathbb{R}^3$.

B14 It is a subspace of $\mathbb{R}^3$.

B15 It is a subspace of $\mathbb{R}^4$.

B16 It is a subspace of $\mathbb{R}^4$.

B17 It is not a subspace of $\mathbb{R}^4$.

B18 It is a subspace of $\mathbb{R}^4$.

B19 $\begin{bmatrix}1\\2\\5\\1\end{bmatrix} + \begin{bmatrix}2\\-1\\1\\1\end{bmatrix} - \begin{bmatrix}3\\1\\6\\2\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

B20 $3\begin{bmatrix}4\\1\\-2\\1\end{bmatrix} - 3\begin{bmatrix}3\\1\\1\\2\end{bmatrix} - \begin{bmatrix}3\\0\\-9\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

B21 $0\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \begin{bmatrix}2\\2\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \begin{bmatrix}-2\\-2\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

B22 $0\begin{bmatrix}1\\1\\2\\5\end{bmatrix} + 0\begin{bmatrix}3\\1\\5\\2\end{bmatrix} + 1\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

B23 Linearly independent

B24 Linearly independent

B25 $\begin{bmatrix}2\\1\\1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\\0\\4\end{bmatrix} = \begin{bmatrix}3\\0\\1\\7\end{bmatrix}$

B26 $\frac{1}{2}\begin{bmatrix}-2\\5\\1\\4\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\3\\1\\2\end{bmatrix} = \begin{bmatrix}0\\4\\1\\3\end{bmatrix}$

B27 Show $\mathcal{B}$ spans P and is linearly independent.

B28 Show $\mathcal{B}$ spans P and is linearly independent.

B29 Show $\mathcal{B}$ spans P and is linearly independent.

B30 A plane. A basis is $\left\{\begin{bmatrix}1\\3\\1\\2\end{bmatrix}, \begin{bmatrix}4\\1\\3\\-2\end{bmatrix}\right\}$.

B31 A plane. A basis is $\left\{\begin{bmatrix}1\\2\\-2\\1\end{bmatrix}, \begin{bmatrix}3\\2\\4\\-1\end{bmatrix}\right\}$.

B32 A hyperplane. A basis is $\left\{\begin{bmatrix}1\\2\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix}\right\}$

B33 A line. A basis is $\left\{\begin{bmatrix}-1\\-2\\1\\-3\end{bmatrix}\right\}$.

B34 A hyperplane. A basis is $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\\0\end{bmatrix}\right\}$.

B35 A line. A basis is $\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}\right\}$.

C Conceptual Problems

C1 Let $\vec{x} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ and let $s, t \in \mathbb{R}$. Then

$$(s+t)\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}(s+t)x_1\\\vdots\\(s+t)x_n\end{bmatrix} = \begin{bmatrix}sx_1 + tx_1\\\vdots\\sx_n + tx_n\end{bmatrix} = \begin{bmatrix}sx_1\\\vdots\\sx_n\end{bmatrix} + \begin{bmatrix}tx_1\\\vdots\\tx_n\end{bmatrix} = s\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} + t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$$

C2 Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, and let $t \in \mathbb{R}$.

$$t(\vec{x} + \vec{y}) = t\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} t(x_1 + y_1) \\ \vdots \\ t(x_n + y_n) \end{bmatrix} = \begin{bmatrix} tx_1 + ty_1 \\ \vdots \\ tx_n + ty_n \end{bmatrix}$$
$$= \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} + \begin{bmatrix} ty_1 \\ \vdots \\ ty_n \end{bmatrix} = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + t\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = t\vec{x} + t\vec{y}$$

C3 By the definition of spanning, every $\vec{x} \in \operatorname{Span}\mathcal{B}$ can be written as a linear combination of the vectors in $\mathcal{B}$. Now, assume that we have $\vec{x} = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$ and $\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$. Then, we have

$$\begin{aligned} s_1\vec{v}_1 + \cdots + s_k\vec{v}_k &= t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \\ (s_1\vec{v}_1 + \cdots + s_k\vec{v}_k) - (t_1\vec{v}_1 + \cdots + t_k\vec{v}_k) &= \vec{0} \\ (s_1 - t_1)\vec{v}_1 + \cdots + (s_k - t_k)\vec{v}_k &= \vec{0} \end{aligned}$$

Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, this implies that $s_i - t_i = 0$ for $1 \leq i \leq k$. That is, $s_i = t_i$. Therefore, there is a unique linear combination of the vectors in $\mathcal{B}$ which equals $\vec{x}$.

C4 If $\vec{v}_i = \vec{0}$, then we have that

$$0\vec{v}_1 + \cdots + 0\vec{v}_{i-1} + 1\vec{v}_i + 0\vec{v}_{i+1} + \cdots + 0\vec{v}_k = \vec{0}$$

Hence, by definition, $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

C5 (a) By definition $U \cap V$ is a subset of $\mathbb{R}^n$, and $\vec{0} \in U$ and $\vec{0} \in V$ since they are both subspaces. Thus, $\vec{0} \in U \cap V$. Let $\vec{x}, \vec{y} \in U \cap V$. Then $\vec{x}, \vec{y} \in U$ and $\vec{x}, \vec{y} \in V$. Since U is a subspace, we have that $s\vec{x} + t\vec{y} \in U$ for all $s, t \in \mathbb{R}$. Similarly, V is a subspace, so $s\vec{x} + t\vec{y} \in V$ for all $s, t \in \mathbb{R}$. Hence, $s\vec{x} + t\vec{y} \in U \cap V$. Thus, $U \cap V$ is a subspace of $\mathbb{R}^n$.

(b) Consider the subspaces $U = \left\{\begin{bmatrix} x_1 \\ 0 \end{bmatrix} \mid x_1 \in \mathbb{R}\right\}$ and $V = \left\{\begin{bmatrix} 0 \\ x_2 \end{bmatrix} \mid x_2 \in \mathbb{R}\right\}$ of $\mathbb{R}^2$. Then $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$, but $\vec{x} + \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in U and not in V, so it is not in $U \cup V$. Thus, $U \cup V$ is not a subspace.

(c) Since U and V are subspaces of $\mathbb{R}^n$, $\vec{u}, \vec{v} \in \mathbb{R}^n$ for any $\vec{u} \in U$ and $\vec{v} \in V$, so $\vec{u} + \vec{v} \in \mathbb{R}^n$ since $\mathbb{R}^n$ is closed under addition. Hence, $U + V$ is a subset of $\mathbb{R}^n$. Also, since U and V are subspace of $\mathbb{R}^n$, we have $\vec{0} \in U$ and $\vec{0} \in V$, thus $\vec{0} = \vec{0} + \vec{0} \in U + V$. Pick any vectors $\vec{x}, \vec{y} \in U + V$. Then, there exists vectors $\vec{u}_1, \vec{u}_2 \in U$ and $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{x} = \vec{u}_1 + \vec{v}_1$ and $\vec{y} = \vec{u}_2 + \vec{v}_2$. We have $s\vec{x} + t\vec{y} = s(\vec{u}_1 + \vec{v}_1) + t(\vec{u}_2 + \vec{v}_2) = (s\vec{u}_1 + t\vec{u}_2) + (s\vec{v}_1 + t\vec{v}_2)$ with $s\vec{u}_1 + t\vec{u}_2 \in U$ and $s\vec{v}_1 + t\vec{v}_2 \in V$ since U and V are both subspaces. Hence, $s\vec{x} + t\vec{y} \in U + V$ for all $s, t \in \mathbb{R}$. Therefore, $U + V$ is a subspace of $\mathbb{R}^n$.

C6 There are many possible solutions.

(a) Pick $\vec{p} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

(b) Pick $\vec{p} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$.

(c) Pick $\vec{p} = \begin{bmatrix}1\\3\\1\\1\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.

(d) Pick $\vec{p} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}3\\0\\0\\0\end{bmatrix}$.

C7 If $\vec{x} \in \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$, then

$$\vec{x} = c_1\vec{v}_1 + c_2(s\vec{v}_1 + t\vec{v}_2) = (c_1 + sc_2)\vec{v}_1 + c_2t\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

Hence, $\operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.
Since $t \neq 0$ we get that $\vec{v}_2 = \frac{-s}{t}\vec{v}_1 + \frac{1}{t}(s\vec{v}_1 + t\vec{v}_2)$. Hence, if $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, then

$$\begin{aligned}\vec{v} &= b_1\vec{v}_1 + b_2\vec{v}_2 = b_1\vec{v}_1 + b_2\left(\frac{-s}{t}\vec{v}_1 + \frac{1}{t}(s\vec{v}_1 + t\vec{v}_2)\right)\\ &= \left(b_1 - \frac{b_2 s}{t}\right)\vec{v}_1 + \frac{b_2}{t}(s\vec{v}_1 + t\vec{v}_2) \in \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}\end{aligned}$$

Thus, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$. Hence $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$.

C8 A subspace S of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that has the additional properties that S is non-empty and that $s\vec{x} + t\vec{y} \in S$ for all $\vec{x}, \vec{y} \in S$ and $s, t \in \mathbb{R}$. That is, every subspace of $\mathbb{R}^n$ must be a subset of $\mathbb{R}^n$, but not every subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

C9 TRUE. We can rearrange the equation to get $-t\vec{v}_1 + \vec{v}_2 = \vec{0}$ with at least one non-zero coefficient (the coefficient of $\vec{v}_2$). Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent by definition.

C10 FALSE. If $\vec{v}_2 = \vec{0}$ and $\vec{v}_1$ is any non-zero vector, then $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent by Problem C4.

C11 FALSE. If $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\0\\0\end{bmatrix}$. Then, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, but $\vec{v}_1$ cannot be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

C12 TRUE. If $\vec{v}_1 = s\vec{v}_2 + t\vec{v}_3$, then we have $\vec{v}_1 - s\vec{v}_2 - t\vec{v}_3 = \vec{0}$ with at least one non-zero coefficient (the coefficient of $\vec{v}_1$). Hence, by definition, the set is linearly dependent.

C13 FALSE. The set $\{\vec{0}\} = \text{Span}\{\vec{0}\}$ is a subspace by Theorem 1.4.2.

C14 TRUE. By Theorem 1.4.2.

Section 1.5

A Practice Problems

A1 $\begin{bmatrix} 5 \\ 3 \\ -6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 4 \\ 0 \end{bmatrix} = 5(3) + 3(2) + (-6)(4) + 1(0) = -3$

A2 $\begin{bmatrix} 1 \\ -2 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1/2 \\ 1/2 \\ -1 \end{bmatrix} = 1(2) + (-2)(1/2) + (-2)(1/2) + 4(-1) = -4$

A3 $\begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 1(2) + 4(-1) + (-1)(-1) + 1(1) = 0$

A4 $\left\| \begin{bmatrix} \sqrt{2} \\ 1 \\ -\sqrt{2} \\ -1 \end{bmatrix} \right\| = \sqrt{(\sqrt{2})^2 + 1^2 + (-\sqrt{2})^2 + (-1)^2} = \sqrt{6}$

A5 $\left\| \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\| = \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} = 1$

A6 $\left\| \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2 + (-1)^2 + 3^2} = \sqrt{15}$

A7 We have $\|\vec{x}\| = \sqrt{1^2 + 2^2 + 5^2} = \sqrt{30}$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

A8 We have $\|\vec{x}\| = \sqrt{3^2 + (-2)^2 + (-1)^2 + 1^2} = \sqrt{15}$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{15}}\begin{bmatrix} 3 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

A9 We have $\|\vec{x}\| = \sqrt{(-2)^2 + 1^2 + 0^2 + 1^2} = \sqrt{6}$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$$

A10 We have $\|\vec{x}\| = \sqrt{1^2 + 2^2 + 5^2 + (-3)^2} = \sqrt{39}$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{39}}\begin{bmatrix}1\\2\\5\\-3\end{bmatrix}$$

A11 We have $\|\vec{x}\| = \sqrt{(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2} = 1$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

A12 We have $\|\vec{x}\| = \sqrt{1^2 + 0^2 + 1^2 + 0^2 + 1^2} = \sqrt{3}$. Thus, a unit vector in the direction of $\vec{x}$ is

$$\hat{\vec{x}} = \frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}$$

A13 We have $\|\vec{x}\| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$, $\|\vec{y}\| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$, $\|\vec{x}+\vec{y}\| = \left\|\begin{bmatrix}6\\4\\6\end{bmatrix}\right\| = \sqrt{6^2 + 4^2 + 6^2} = 2\sqrt{22}$, and $|\vec{x}\cdot\vec{y}| = 4(2) + 3(1) + 1(5) = 16$. The triangle inequality is satisfied since

$$2\sqrt{22} \approx 9.38 \leq \sqrt{26} + \sqrt{30} \approx 10.58$$

The Cauchy-Schwarz inequality is also satisfied since $16 \leq \sqrt{26(30)} \approx 27.93$.

A14 We have $\|\vec{x}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$, $\|\vec{y}\| = \sqrt{(-3)^2 + 2^2 + 4^2} = \sqrt{29}$, $\|\vec{x} + \vec{y}\| = \left\|\begin{bmatrix}-2\\1\\6\end{bmatrix}\right\| = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41}$, and $|\vec{x}\cdot\vec{y}| = 1(-3) + (-1)(2) + 2(4) = 3$. The triangle inequality is satisfied since

$$\sqrt{41} \approx 6.40 \leq \sqrt{6} + \sqrt{29} \approx 7.83$$

The Cauchy-Schwarz inequality is satisfied since $3 \leq \sqrt{6(29)} \approx 13.19$.

A15 A scalar equation of the hyperplane is $3x_1 + x_2 + 4x_3 = 3(1) + 1(1) + 4(-1) = 0$.

A16 A scalar equation of the hyperplane is $x_2 + 3x_3 + 3x_4 = 0(2) + 1(-2) + 3(0) + 3(1) = 1$.

A17 A scalar equation of the hyperplane is $3x_1 - 2x_2 - 5x_3 + x_4 = 3(2) - 2(1) - 5(1) + 1(5) = 4$.

A18 A scalar equation of the hyperplane is $2x_1 - 4x_2 + x_3 - 3x_4 = 2(3) - 4(1) + 1(0) - 3(7) = -19$.

A19 A scalar equation of the hyperplane is $x_1 - 4x_2 + 5x_3 - 2x_4 = 1(0) - 4(0) + 5(0) - 2(0) = 0$.

A20 A scalar equation of the hyperplane is $x_2 + 2x_3 + x_4 + x_5 = 0(1) + 1(0) + 2(1) + 1(2) + 1(1) = 5$.

A21 $\vec{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **A22** $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ **A23** $\vec{n} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}$ **A24** $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}$ **A25** $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

A26 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-5}{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$
$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

A27 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{12/5}{1}\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix}$$
$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix} = \begin{bmatrix} -136/25 \\ 102/25 \end{bmatrix}$$

A28 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{5}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$
$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

A29 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-4/3}{1}\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 8/9 \\ -8/9 \end{bmatrix}$$
$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} -4/9 \\ 8/9 \\ -8/9 \end{bmatrix} = \begin{bmatrix} 40/9 \\ 1/9 \\ -19/9 \end{bmatrix}$$

A30 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{0}{6}\begin{bmatrix}1\\1\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1\\-1\\2\\-1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\\-1\end{bmatrix}$$

A31 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-1/2\\0\\0\\-1/2\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}2\\3\\2\\-3\end{bmatrix} - \begin{bmatrix}-1/2\\0\\0\\-1/2\end{bmatrix} = \begin{bmatrix}5/2\\3\\2\\-5/2\end{bmatrix}$$

A32 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{0}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3\\-3\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix}$$

A33 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{17}\begin{bmatrix}2\\3\\-2\end{bmatrix} = \begin{bmatrix}-2/17\\-3/17\\2/17\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}4\\-1\\3\end{bmatrix} - \begin{bmatrix}-2/17\\-3/17\\2/17\end{bmatrix} = \begin{bmatrix}70/17\\-14/17\\49/17\end{bmatrix}$$

A34 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-14}{6}\begin{bmatrix}-2\\1\\-1\end{bmatrix} = \begin{bmatrix}14/3\\-7/3\\7/3\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}5\\-1\\3\end{bmatrix} - \begin{bmatrix}14/3\\-7/3\\7/3\end{bmatrix} = \begin{bmatrix}1/3\\4/3\\2/3\end{bmatrix}$$

A35 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{9}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\-3\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}4\\1\\-2\end{bmatrix} - \begin{bmatrix}3/2\\3/2\\-3\end{bmatrix} = \begin{bmatrix}5/2\\-1/2\\1\end{bmatrix}$$

A36 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-5}{15}\begin{bmatrix}-1\\2\\1\\-3\end{bmatrix} = \begin{bmatrix}1/3\\-2/3\\-1/3\\1\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}2\\-1\\2\\1\end{bmatrix} - \begin{bmatrix}1/3\\-2/3\\-1/3\\1\end{bmatrix} = \begin{bmatrix}5/3\\-1/3\\7/3\\0\end{bmatrix}$$

A37 We have

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{6}\begin{bmatrix}2\\0\\1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\0\\-1/6\\-1/6\end{bmatrix}$$

$$\text{perp}_{\vec{v}}(\vec{u}) = \vec{u} - \text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1\\2\\-1\\2\end{bmatrix} - \begin{bmatrix}-1/3\\0\\-1/6\\-1/6\end{bmatrix} = \begin{bmatrix}-2/3\\2\\-5/6\\13/6\end{bmatrix}$$

A38 (a) A unit vector in the direction of $\vec{u}$ is

$$\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u} = \begin{bmatrix}2/7\\6/7\\3/7\end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{u}}(\vec{F}) = \frac{\vec{F}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{110}{49}\begin{bmatrix}2\\6\\3\end{bmatrix} = \begin{bmatrix}220/49\\660/49\\330/49\end{bmatrix}$$

(c) We get

$$\text{perp}_{\vec{u}}(\vec{F}) = \vec{F} - \text{proj}_{\vec{u}}(\vec{F}) = \begin{bmatrix}10\\18\\-6\end{bmatrix} - \begin{bmatrix}220/49\\660/49\\330/49\end{bmatrix} = \begin{bmatrix}270/49\\222/49\\-624/49\end{bmatrix}$$

A39 (a) A unit vector in the direction of $\vec{u}$ is

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{u}}(\vec{F}) = \frac{\vec{F}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{16}{14}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 24/7 \\ 8/7 \\ -16/7 \end{bmatrix}$$

(c) We get

$$\text{perp}_{\vec{u}}(\vec{F}) = \vec{F} - \text{proj}_{\vec{u}}(\vec{F}) = \begin{bmatrix} 3 \\ 11 \\ 2 \end{bmatrix} - \begin{bmatrix} 24/7 \\ 8/7 \\ -16/7 \end{bmatrix} = \begin{bmatrix} -3/7 \\ 69/7 \\ 30/7 \end{bmatrix}$$

A40 We first pick a point P on the line, say $P(1,4)$. Then the point R on the line that is closest to $Q(0,0)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ})$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ}) = \frac{\vec{PQ}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{-6}{8}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(5/2, 5/2)$. The distance from R to Q is

$$\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \left\|\begin{bmatrix} -1 \\ -4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -5/2 \\ -5/2 \end{bmatrix}\right\| = \frac{5}{\sqrt{2}}$$

A41 We first pick the point $P(3,7)$ on the line. Then the point R on the line that is closest to $Q(2,5)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ})$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ}) = \frac{\vec{PQ}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{7}{17}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix} = \begin{bmatrix} 58/17 \\ 91/17 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(58/17, 91/17)$. The distance from R to Q is

$$\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \left\|\begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -24/17 \\ -6/17 \end{bmatrix}\right\| = \frac{6}{\sqrt{17}}$$

A42 We first pick the point $P(2, 2, -1)$ on the line. Then the point R on the line that is closest to $Q(1, 0, 1)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ})$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ}) = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{5}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix} = \begin{bmatrix} 17/6 \\ 1/3 \\ -1/6 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(17/6, 1/3, -1/6)$. The distance from R to Q is

$$\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \left\|\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -11/6 \\ -1/3 \\ 7/6 \end{bmatrix}\right\| = \sqrt{\frac{29}{6}}$$

A43 We first pick the point $P(1, 1, -1)$ on the line. Then the point R on the line that is closest to $Q(2, 3, 2)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ})$ where $\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}}(\vec{PQ}) = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{12}{18}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 11/3 \\ -1/3 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(5/3, 11/3, -1/3)$. The distance from R to Q is

$$\|\text{perp}_{\vec{d}}(\vec{PQ})\| = \left\|\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 1/3 \\ -2/3 \\ 7/3 \end{bmatrix}\right\| = \sqrt{6}$$

A44 We first pick any point P on the plane (that is, any point $P(x_1, x_2, x_3)$ such that $3x_1 - x_2 + 4x_3 = 5$). We pick $P(0, -5, 0)$. Then the distance from Q to the plane is the length of the projection of $\vec{PQ} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$ onto a normal vector of the plane, say $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$. Thus, the distance is

$$\|\text{proj}_{\vec{n}}(\vec{PQ})\| = \left|\frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|}\right| = \frac{2}{\sqrt{26}}$$

A45 We pick the point $P(0,0,-1)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{13}{\sqrt{38}}$$

A46 We pick the point $P(0,0,-5)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{4}{\sqrt{5}}$$

A47 We pick the point $P(2,0,0)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \sqrt{6}$$

A48 We pick the point $P(2,2,1)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{3}{\sqrt{11}}$$

A49 We pick the point $P(0,5,0)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{13}{\sqrt{21}}$$

A50 We pick the point $P(6,0,0)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}}(\vec{PQ}) \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{5}{\sqrt{3}}$$

A51 Pick a point P on the hyperplane, say $P(0,0,0,0)$. Then the point R on the hyperplane that is closest to $Q(1,0,0,1)$ satisfies $\vec{OR} = \vec{OQ} + \text{proj}_{\vec{n}}(\vec{QP})$ where $\vec{n}$ is a normal vector of the hyperplane. We have $\vec{QP} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, so

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{-3}{7}\begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -6/7 \\ 3/7 \\ -3/7 \\ -3/7 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 3/7 \\ -3/7 \\ 4/7 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(1/7, 3/7, -3/7, 4/7)$.

A52 We pick the point $P(1,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}$ for the hyperplane. Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{14}\begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/14 \\ -2/14 \\ 3/14 \\ 0 \end{bmatrix} = \begin{bmatrix} 15/14 \\ 13/7 \\ 17/14 \\ 3 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(15/14, 13/7, 17/14, 3)$.

A53 We pick the point $P(0,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix}$ for the hyperplane. Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 4 \end{bmatrix} + \frac{-18}{27}\begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2/3 \\ -8/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14/3 \\ 1/3 \\ 10/3 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(0, 14/3, 1/3, 10/3)$.

A54 We pick the point $P(4,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ for the hyperplane. Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + \frac{-5}{7}\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -5/7 \\ -10/7 \\ -5/7 \\ 5/7 \end{bmatrix} = \begin{bmatrix} -12/7 \\ 11/7 \\ 9/7 \\ -9/7 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(-12/7, 11/7, 9/7, -9/7)$.

A55 The volume of the parallelepiped is

$$\left| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right| = 1$$

A56 The volume of the parallelepiped is

$$\left| \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 28 \\ 8 \\ -6 \end{bmatrix} \right| = 126$$

A57 The volume of the parallelepiped is

$$\left| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -15 \\ 6 \end{bmatrix} \right| = |-5| = 5$$

A58 The volume of the parallelepiped is

$$\left| \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -7 \\ 0 \end{bmatrix} \right| = |-35| = 35$$

A59 By Hooke's Law, we have that

$$\begin{aligned} 3.0 &= 1k \\ 6.5 &= 2k \\ 9.0 &= 3k \end{aligned}$$

Let $\vec{p} = \begin{bmatrix} 3.0 \\ 6.5 \\ 9.0 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We want to find the value of k that makes the vector $k\vec{d}$ closest to the point $P(3, 6.5, 9)$. We interpret $k\vec{d}$ as the line L with vector equation

$$\vec{x} = k \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad k \in \mathbb{R}$$

The vector on L that is closest to P is the projection of P onto L. Moreover, we know that the coefficient k of the projection is

$$k = \frac{\vec{p} \cdot \vec{d}}{\|\vec{d}\|^2} = \frac{43}{14} \approx 3.07$$

Thus, based on the data, the best approximation of k would be $k \approx 3.07$.

B Homework Problems

B1 -2

B2 23

B3 -3

B4 $\sqrt{18}$

B5 $\sqrt{27}$

B6 $\sqrt{10}/3$

B7 -12

B8 $\begin{bmatrix}-20\\-40\\0\\20\end{bmatrix}$

B9 $\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

B10 $\frac{1}{\sqrt{26}}\begin{bmatrix}5\\0\\0\\1\end{bmatrix}$

B11 $\frac{1}{\sqrt{14}}\begin{bmatrix}3\\2\\-1\\0\end{bmatrix}$

B12 $\frac{1}{\sqrt{18}}\begin{bmatrix}-2\\1\\-2\\3\end{bmatrix}$

B13 $\frac{1}{\sqrt{7/18}}\begin{bmatrix}1/3\\1/2\\1/6\\0\end{bmatrix}$

B14 $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$

B15 We have $\|\vec{x}\| = \sqrt{21}$, $\|\vec{y}\| = \sqrt{35}$, $|\vec{x}\cdot\vec{y}| = 25$, and $\|\vec{x}+\vec{y}\| = \sqrt{106}$. Indeed we have $25 \leq \sqrt{21}\sqrt{35}$ and $\sqrt{106} \leq \sqrt{21} + \sqrt{35}$.

B16 We have $\|\vec{x}\| = \sqrt{14}$, $\|\vec{y}\| = \sqrt{12}$, $|\vec{x}\cdot\vec{y}| = 4$, and $\|\vec{x}+\vec{y}\| = \sqrt{34}$. Indeed we have $4 \leq \sqrt{14}\sqrt{12}$ and $\sqrt{34} \leq \sqrt{14} + \sqrt{12}$.

B17 $2x_1 + 2x_2 + 6x_3 - x_4 = 19$

B18 $x_1 + 5x_2 + 9x_3 + 2x_4 = 46$

B19 $2x_1 + x_2 + 2x_3 + x_4 = 10$

B20 $x_2 + 2x_3 + x_4 = 3$

B21 $\begin{bmatrix}3\\1\end{bmatrix}$

B22 $\begin{bmatrix}1\\2\\7\end{bmatrix}$

B23 $\begin{bmatrix}3\\-5\\1\\-1\end{bmatrix}$

B24 $\begin{bmatrix}1\\0\\-3\\9\end{bmatrix}$

B25 $\begin{bmatrix}2\\0\\1\\0\\3\end{bmatrix}$

B26 $\begin{bmatrix}-2\\-1\\-2\\2\\-2\end{bmatrix}$

B27 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}3/2\\3/2\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1/2\\1/2\end{bmatrix}$

B28 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}4\\-6\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\0\end{bmatrix}$

B29 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}16/25\\12/25\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}9/25\\-12/25\end{bmatrix}$

B30 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-92/25\\69/25\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}42/25\\56/25\end{bmatrix}$

B31 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}9/2\\0\\9/2\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-5/2\\-4\\5/2\end{bmatrix}$

B32 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-4/3\\8/3\\8/3\end{bmatrix}$

B33 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1/3\\0\\1/3\\1/3\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}8/3\\3\\-13/3\\5/3\end{bmatrix}$

B34 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1\\3\\2\\1\end{bmatrix}$

B35 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}-1\\1\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}4\\4\end{bmatrix}$

B36 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}5\\0\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}0\\3\end{bmatrix}$

B37 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}1\\1\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}2\\-2\end{bmatrix}$

B38 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix}7/5\\21/5\end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix}18/5\\-6/5\end{bmatrix}$

B39 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

B40 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$

B41 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

B42 $\text{proj}_{\vec{v}}(\vec{u}) = \begin{bmatrix} 2/9 \\ 0 \\ 1/9 \\ 2/9 \end{bmatrix}$, $\text{perp}_{\vec{v}}(\vec{u}) = \begin{bmatrix} -11/9 \\ 2 \\ -10/9 \\ 16/9 \end{bmatrix}$

B43 (a) $\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (b) $\text{proj}_{\vec{u}}(\vec{F}) = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ (c) $\text{perp}_{\vec{u}}(\vec{F}) = \begin{bmatrix} -11/3 \\ 14/3 \\ 4/3 \end{bmatrix}$

B44 (a) $\frac{1}{\sqrt{19}}\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$ (b) $\text{proj}_{\vec{u}}(\vec{F}) = \begin{bmatrix} -2/19 \\ -6/19 \\ 6/19 \end{bmatrix}$ (c) $\text{perp}_{\vec{u}}(\vec{F}) = \begin{bmatrix} 78/19 \\ 63/19 \\ 89/19 \end{bmatrix}$

B45 $(16/5, -28/5)$, 1

B46 $(16/9, 13/9, 4/9)$, $\sqrt{50}/3$

B47 $(5/3, -1/3, -1/3)$, $\sqrt{14/3}$

B48 $(14/3, 4/3, -2/3)$, $\sqrt{11/3}$

B49 $26/\sqrt{38}$

B50 $7/\sqrt{21}$

B51 $4\sqrt{3}$

B52 $4/\sqrt{6}$

B53 $(32/17, 1, -2/17, -20/17)$

B54 $(10/9, 26/9, 1/18, 7/6)$

B55 $(3/4, 7/4, 11/4, 21/4)$

B56 $(2, 1/2, 1, -3/2)$

B57 2

B58 21

B59 40

B60 48

C Conceptual Problems

C1 (a) False. One possible counterexample is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -97 \end{bmatrix}$.

(b) Our counterexample in part (a) has $\vec{u} \neq \vec{0}$ so the result does not change.

C2 Since $\vec{x} = \vec{x} - \vec{y} + \vec{y}$,

$$\|\vec{x}\| = \|\vec{x} - \vec{y} + \vec{y}\| = \|(\vec{x} - \vec{y}) + \vec{y}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y}\|$$

So, $\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\|$. This is almost what we require, but the left-hand side might be negative. So, by a similar argument with $\vec{y}$, and using the fact that $\|\vec{y} - \vec{x}\| = \|\vec{x} - \vec{y}\|$, we obtain $\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{x} - \vec{y}\|$. From this equation and the previous one, we can conclude that

$$\big|\|\vec{x}\| - \|\vec{y}\|\big| \leq \|\vec{x} - \vec{y}\|$$

C3 We have

$$\begin{aligned} \|\vec{v}_1 + \vec{v}_2\|^2 &= (\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2) = \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 \\ &= \|\vec{v}_1\|^2 + 0 + 0 + \|\vec{v}_2\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 \end{aligned}$$

C4 By Theorem 1.5.2 (2) we have that $\left\|\frac{1}{\|\vec{x}\|}\vec{x}\right\| = \left|\frac{1}{\|\vec{x}\|}\right| \|\vec{x}\| = 1$.

C5 Consider $\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$. Taking the dot product of both sides with respect to $\vec{v}_i$ gives

$$0 = \vec{0}\cdot\vec{v}_i = (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k)\cdot\vec{v}_i = c_i(\vec{v}_i\cdot\vec{v}_i)$$

Since $\vec{v}_i \neq \vec{0}$, we have that $\vec{v}_i\cdot\vec{v}_i \neq 0$ by Theorem 1.5.2 (1). Hence, we have $c_i = 0$. Since this applies for all $1 \leq i \leq k$, we have that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

C6 By definition, $S^\perp$ is a subset of $\mathbb{R}^n$. Moreover, since $\vec{0}\cdot\vec{v} = 0$ for all $\vec{v} \in S$ we have that $\vec{0} \in S^\perp$. Let $\vec{w}_1, \vec{w}_2 \in S^\perp$. Then, $\vec{w}_1\cdot\vec{v} = 0$ and $\vec{w}_2\cdot\vec{v} = 0$ for all $\vec{v} \in S$. Hence, we have that

$$(s\vec{w}_1 + t\vec{w}_2)\cdot\vec{v} = s(\vec{w}_1\cdot\vec{v}) + t(\vec{w}_2\cdot\vec{v}) = s(0) + t(0) = 0$$

for all $\vec{v} \in S$ and $s, t \in \mathbb{R}$. Hence, $S^\perp$ is a subspace of $\mathbb{R}^n$.

C7 (a) We have

$$\begin{aligned} C(s\vec{x} + t\vec{y}) &= \text{proj}_{\vec{u}}(\text{proj}_{\vec{v}}(s\vec{x} + t\vec{y})) = \text{proj}_{\vec{u}}(s\,\text{proj}_{\vec{v}}(\vec{x}) + t\,\text{proj}_{\vec{v}}(\vec{y})) \\ &= s\,\text{proj}_{\vec{u}}(\text{proj}_{\vec{v}}(\vec{x})) + t\,\text{proj}_{\vec{u}}(\text{proj}_{\vec{v}}(\vec{y})) = sC(\vec{x}) + tC(\vec{y}) \end{aligned}$$

(b) If $C(\vec{x}) = \vec{0}$ for all $\vec{x}$, then certainly

$$\vec{0} = C(\vec{v}) = \text{proj}_{\vec{u}}(\text{proj}_{\vec{v}}(\vec{v})) = \text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$$

Hence, $\vec{v}\cdot\vec{u} = 0$, and the vectors $\vec{u}$ and $\vec{v}$ are orthogonal to each other.

C8

$$\text{proj}_{-\vec{u}}(\vec{x}) = \frac{\vec{x}\cdot(-\vec{u})}{\|-\vec{u}\|^2}(-\vec{u}) = \frac{-(\vec{x}\cdot\vec{u})}{\|\vec{u}\|^2}(-\vec{u}) = \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \text{proj}_{\vec{u}}(\vec{x})$$

Geometrically, $\text{proj}_{-\vec{u}}(\vec{x})$ is a vector along the line through the origin with direction vector $-\vec{u}$, and this line is the same as the line with direction vector $\vec{u}$. We have that $\text{proj}_{-\vec{u}}(\vec{x})$ is the point on this line that is closest to $\vec{x}$ and this is the same as $\text{proj}_{\vec{u}}(\vec{x})$.

C9 (a)

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y})\cdot(\vec{x} + \vec{y}) = \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y} \\ &= \|\vec{x}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \end{aligned}$$

Hence, $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x}\cdot\vec{y} = 0$.

(b) Following the hint, we subtract and add $\text{proj}_{\vec{d}}(\vec{p})$:

$$\begin{aligned} \|\vec{p} - \vec{q}\|^2 &= \|\vec{p} - \text{proj}_{\vec{d}}(\vec{p}) + \text{proj}_{\vec{d}}(\vec{p}) - \vec{q}\|^2 \\ &= \left\|\text{perp}_{\vec{d}}(\vec{p}) + \left(\frac{\vec{p}\cdot\vec{d}}{\|\vec{d}\|^2} - t\right)\vec{d}\right\|^2 \end{aligned}$$

Since, $\vec{d}\cdot\text{perp}_{\vec{d}}(\vec{p}) = 0$, we can apply the result of (a) to get

$$\|\vec{p} - \vec{q}\|^2 = \|\,\text{perp}_{\vec{d}}(\vec{p})\|^2 + \|\,\text{proj}_{\vec{d}}(\vec{p}) - \vec{q}\|^2$$

Since $\vec{p}$ and $\vec{d}$ are given, $\text{perp}_{\vec{d}}(\vec{p})$ is fixed, so to make this expression as small as possible choose $\vec{q} = \text{proj}_{\vec{d}}(\vec{p})$. Thus, the distance from the point $\vec{p}$ to a point on the line is minimized by the point $\vec{q} = \text{proj}_{\vec{d}}(\vec{p})$ on the line.

C10

$$\vec{OP} + \text{perp}_{\vec{n}}(\vec{PQ}) = \vec{OP} + \left(\vec{PQ} - \text{proj}_{\vec{n}}\left(\vec{PQ}\right)\right)$$
$$= \left(\vec{OP} + \vec{PQ}\right) + \text{proj}_{\vec{n}}\left(-\vec{PQ}\right) = \vec{OQ} + \text{proj}_{\vec{n}}\left(\vec{QP}\right)$$

C11 (a)

$$\text{perp}_{\vec{u}}(\vec{x}) = \vec{x} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} 2/3 \\ 11/3 \\ 13/3 \end{bmatrix}$$
$$\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) = \frac{\text{perp}_{\vec{u}}(\vec{x})\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b)

$$\begin{aligned}
\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) &= \left[\left(\vec{x} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right)\cdot\frac{\vec{u}}{\|\vec{u}\|^2}\right]\vec{u} \\
&= \left[\frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2} - \frac{(\vec{x}\cdot\vec{u})(\vec{u}\cdot\vec{u})}{\|\vec{u}\|^4}\right]\vec{u} \\
&= \left[\frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\right]\vec{u} \\
&= \vec{0}
\end{aligned}$$

(c) $\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}(\vec{x})) = \vec{0}$ since $\text{perp}_{\vec{u}}(\vec{x})$ is orthogonal to $\vec{u}$ and $\text{proj}_{\vec{u}}$ maps vectors orthogonal to $\vec{u}$ to the zero vector.

C12 (a) We have

$$\|\vec{e}_1\| = \sqrt{1^2 + 0^2 + 0^2} = 1$$
$$\|\vec{e}_2\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$
$$\|\vec{e}_3\| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Thus, each standard basis vector is a unit vector. We also have

$$\vec{e}_1\cdot\vec{e}_2 = 1(0) + 0(1) + 0(0) = 0$$
$$\vec{e}_1\cdot\vec{e}_3 = 1(0) + 0(0) + 0(1) = 0$$
$$\vec{e}_2\cdot\vec{e}_3 = 0(0) + 1(0) + 0(1) = 0$$

Hence, each vector is orthogonal to every other vector in the set. So, the set $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is orthonormal.

(b) If each vector is a unit vector, then they are all non-zero. Hence, the result follows from Problem C5.

Chapter 1 Quiz

E1 We have $\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

E2 A vector orthogonal to $\vec{x}$ and $\vec{y}$ is $\vec{x} \times \vec{y} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$. The length of $\vec{x} \times \vec{y}$ is $\sqrt{2^2 + (-1)^2 + 7^2} = \sqrt{54}$.

Thus, a unit vector that is orthogonal to both $\vec{x}$ and $\vec{y}$ is $\frac{1}{\sqrt{54}}\begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$.

E3 $\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} 4/5 \\ 4/15 \\ 8/15 \\ -4/15 \end{bmatrix}$.

$\text{perp}_{\vec{u}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) = \begin{bmatrix} 1/5 \\ -4/15 \\ 22/15 \\ 49/15 \end{bmatrix}$.

E4 Any direction vector of the line is a non-zero scalar multiple of the directed line segment between P and Q. Thus, we can take $\vec{d} = \vec{PQ} = \begin{bmatrix} 5-(-2) \\ -2-1 \\ 1-(-4) \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$. Thus, since $P(-2, 1, -4)$ is a point on the line we get that a vector equation of the line is

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

E5 Every vector in the plane satisfies $x_1 = 3 + 2x_3$. Hence, they satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3+2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

for $x_2, x_3 \in \mathbb{R}$. This is a vector equation for the plane.

E6 We have that the vectors $\vec{PQ} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ 14 \end{bmatrix}$. Then, since $P(1, -1, 0)$ is a point on the plane we get a scalar equation of the plane is

$$16x_1 - 2x_2 + 14x_3 = 16(1) - 2(-1) + 14(0) = 18$$

or $8x_1 - x_2 + 7x_3 = 9$.

E7 Observe that $\begin{bmatrix}2\\6\\4\end{bmatrix}+\begin{bmatrix}1\\3\\3\end{bmatrix}=\begin{bmatrix}3\\9\\7\end{bmatrix}$. Hence, by Theorem 1.4.3, we have that

$$\operatorname{Span}\left\{\begin{bmatrix}2\\6\\4\end{bmatrix},\begin{bmatrix}1\\3\\3\end{bmatrix},\begin{bmatrix}3\\9\\7\end{bmatrix}\right\}=\operatorname{Span}\left\{\begin{bmatrix}2\\6\\4\end{bmatrix},\begin{bmatrix}1\\3\\3\end{bmatrix}\right\}$$

Since $\mathcal{B}=\left\{\begin{bmatrix}2\\6\\4\end{bmatrix},\begin{bmatrix}1\\3\\3\end{bmatrix}\right\}$ cannot be reduced further (it is linearly independent), it is a basis for the spanned set which is a plane in $\mathbb{R}^3$.

E8 Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix}=c_1\begin{bmatrix}1\\2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\3\end{bmatrix}+c_3\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}c_1+c_2+2c_3\\2c_1-c_2\\c_1+3c_2+c_3\end{bmatrix}$$

This gives the system

$$\begin{aligned}c_1+c_2+2c_3&=0\\2c_1-c_2&=0\\c_1+3c_2+c_3&=0\end{aligned}$$

Adding the first and the second equation gives $3c_1+2c_3=0$. Hence, we have $c_1=-\frac{2}{3}c_3$. From the second equation we have $c_2=2c_1=-\frac{4}{3}c_3$. Thus, the third equation gives

$$0=-\frac{2}{3}c_3-4c_3+c_3=-\frac{11}{3}c_3$$

Thus, $c_3=0$ which implies that $c_1=c_2=0$. Therefore, the set is linearly independent.

E9 (a) To show that $\left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\right\}$ is a basis, we need to show that it spans $\mathbb{R}^2$ and that it is linearly independent.

Consider

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=t_1\begin{bmatrix}1\\2\end{bmatrix}+t_2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}t_1-t_2\\2t_1+2t_2\end{bmatrix}$$

This gives $x_1=t_1-t_2$ and $x_2=2t_1+2t_2$. Solving using substitution and elimination we get $t_1=\frac{1}{4}(2x_1+x_2)$ and $t_2=\frac{1}{4}(-2x_1+x_2)$. Hence, every vector $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ can be written as

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{1}{4}(2x_1+x_2)\begin{bmatrix}1\\2\end{bmatrix}+\frac{1}{4}(-2x_1+x_2)\begin{bmatrix}-1\\2\end{bmatrix}$$

So, it spans $\mathbb{R}^2$. Moreover, if $x_1=x_2=0$, then our calculations above show that $t_1=t_2=0$, so the set is also linearly independent. Therefore, it is a basis for $\mathbb{R}^2$.

(b) Taking $x_1 = 3$ and $x_2 = 5$ in our work above gives $t_1 = \frac{1}{4}(6+5) = \frac{11}{4}$ and $t_2 = \frac{1}{4}(-6+5) = -\frac{1}{4}$. So, these are the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$. Indeed we have

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{11}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(c) Since $\vec{y} = 2\vec{x}$, the coordinates of $\vec{y}$ with respect to the basis $\mathcal{B}$ are $t_1 = \frac{11}{2}$ and $t_2 = -\frac{1}{2}$. Indeed we have

$$\begin{bmatrix} 6 \\ 10 \end{bmatrix} = \frac{11}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

E10 Observe that $0 \neq 3 - 5(0)$ so $\vec{0} \notin S$, so S is not a subspace.

E11 If $d \neq 0$, then $a_1(0) + a_2(0) + a_3(0) = 0 \neq d$, so $\vec{0} \notin S$ and thus, S is not a subspace of $\mathbb{R}^3$.

On the other hand, assume $d = 0$. Observe that, by definition, S is a subset of $\mathbb{R}^3$ and that $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$ since taking $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ satisfies $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in S$. Then they must satisfy the condition of the set, so $a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $a_1y_1 + a_2y_2 + a_3y_3 = 0$.

To show that S is a subspace, we must show that $s\vec{x} + t\vec{y}$ satisfies the condition of S. We have $s\vec{x} + t\vec{y} = \begin{bmatrix} sx_1 + ty_1 \\ sx_2 + ty_2 \\ sx_3 + ty_3 \end{bmatrix}$ and

$$\begin{aligned} a_1(sx_1 + ty_1) + a_2(sx_2 + ty_2) + a_3(sx_3 + ty_3) &= s(a_1x_1 + a_2x_2 + a_3x_3) + t(a_1y_1 + a_2y_2 + a_3y_3) \\ &= s(0) + t(0) = 0 \end{aligned}$$

Therefore, S is a subspace of $\mathbb{R}^3$.

E12 By the definition of P, every $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in P$ satisfies $x_1 - 3x_2 + x_3 = 0$. Solving this for x_3 gives $x_3 = -x_1 + 3x_2$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ -x_1 + 3x_2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ -c_1 + 3c_2 \end{bmatrix}$$

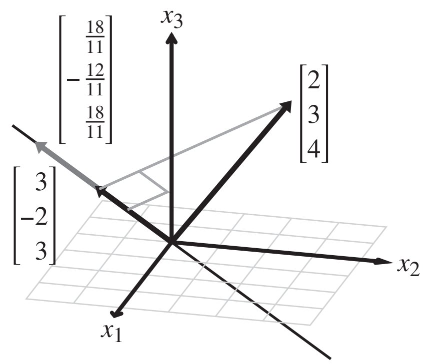
Solving we find that $c_1 = x_1$, $c_2 = x_2$ (observe that $-c_1 + 3c_2 = -x_1 + 3x_2$ so the third equation is also satisfied). Thus, $\mathcal{B}$ spans P. Now consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ -c_1 + 3c_2 \end{bmatrix}$$

Comparing entries we get that $c_1 = c_2 = 0$. Hence, $\mathcal{B}$ is also linearly independent.

Since $\mathcal{B}$ is linearly independent and spans P, it is a basis for P.

E13 Since the origin $O(0,0,0)$ is on the line, we get that the point Q on the line closest to P is given by $\vec{OQ} = \text{proj}_{\vec{d}}\left(\vec{OP}\right)$, where $\vec{d} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ is a direction vector of the line. Hence,



$$\vec{OQ} = \frac{\vec{OP} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \begin{bmatrix} 18/11 \\ -12/11 \\ 18/11 \end{bmatrix}$$

and the closest point is $Q(18/11, -12/11, 18/11)$.

E14 Let $Q(0,0,0,1)$ be a point in the hyperplane. We have that a normal vector to the plane is $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Then, the point R in the hyperplane closest to P satisfies $\vec{PR} = \text{proj}_{\vec{n}}\left(\vec{PQ}\right)$. Hence,

$$\vec{OR} = \vec{OP} + \text{proj}_{\vec{n}}\left(\vec{PQ}\right) = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \\ -1/2 \\ 3/2 \end{bmatrix}$$

Then the distance from the point to the line is the length of $\vec{PR}$.

$$\|\vec{PR}\| = \left\|\begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}\right\| = 1$$

E15 The volume of the parallelepiped determined by $\vec{u} + k\vec{v}$, $\vec{v}$, and $\vec{w}$ is

$$\begin{aligned} |(\vec{u} + k\vec{v}) \cdot (\vec{v} \times \vec{w})| &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(\vec{v} \cdot (\vec{v} \times \vec{w}))| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(0)| \end{aligned}$$

which equals the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

E16 FALSE. The points $P(0,0,0)$, $Q(0,0,1)$, and $R(0,0,2)$ lie in every plane of the form $t_1x_1 + t_2x_2 = 0$ with t_1 and t_2 not both zero.

E17 TRUE. This is the definition of a line reworded in terms of a spanning set.

E18 TRUE. By definition of the plane $\{\vec{v}_1, \vec{v}_2\}$ spans the plane. If $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent, then the set would not satisfy the definition of a plane, so $\{\vec{v}_1, \vec{v}_2\}$ must be linearly independent. Hence, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for the plane.

E19 FALSE. The dot product of the zero vector with itself is 0.

E20 FALSE. Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, $\text{proj}_{\vec{x}}\, \vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, while $\text{proj}_{\vec{y}}\, \vec{x} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$.

E21 FALSE. If $\vec{y} = \vec{0}$, then $\text{proj}_{\vec{x}}(\vec{y}) = \vec{0}$. Thus, $\{\text{proj}_{\vec{x}}(\vec{y}), \text{perp}_{\vec{x}}(\vec{y})\}$ contains the zero vector so it is linearly dependent.

E22 TRUE. We have

$$\|\vec{u} \times (\vec{v} + 3\vec{u})\| = \|\vec{u} \times \vec{v} + 3(\vec{u} \times \vec{u})\| = \|\vec{u} \times \vec{v} + \vec{0}\| = \|\vec{u} \times \vec{v}\|$$

so the parallelograms have the same area.

Chapter 1 Further Problems

F1 The statement is true. Rewrite the conditions in the form

$$\vec{u} \cdot (\vec{v} - \vec{w}) = 0, \qquad \vec{u} \times (\vec{v} - \vec{w}) = \vec{0}$$

The first condition says that $\vec{v} - \vec{w}$ is orthogonal to $\vec{u}$, so the angle θ between $\vec{u}$ and $\vec{v} - \vec{w}$ is $\frac{\pi}{2}$ radians. Thus, $\sin\theta = 1$, so the second condition tells us that

$$0 = \|\vec{u} \times (\vec{v} - \vec{w})\| = \|\vec{u}\|\|\vec{v} - \vec{w}\| \sin\theta = \|\vec{u}\|\|\vec{v} - \vec{w}\|$$

Since $\|\vec{u}\| \neq 0$, it follows that $\|\vec{v} - \vec{w}\| = 0$ and hence $\vec{v} = \vec{w}$.

F2 Since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors, $\vec{u} \times \vec{v}$ is a unit vector orthogonal to the plane containing $\vec{u}$ and $\vec{v}$. Then $\text{perp}_{\vec{u}\times\vec{v}}(\vec{x})$ is orthogonal to $\vec{u} \times \vec{v}$, so it lies in the plane containing $\vec{u}$ and $\vec{v}$. Therefore, for some $s, t \in \mathbb{R}$, $\text{perp}_{\vec{u}\times\vec{v}}(\vec{x}) = s\vec{u} + t\vec{v}$. Now since $\vec{u} \cdot \vec{u} = 1$, $\vec{u} \cdot \vec{v} = 0$, and $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$,

$$s = \vec{u} \cdot (s\vec{u} + t\vec{v}) = \vec{u} \cdot \text{perp}_{\vec{u}\times\vec{v}}(\vec{x}) = \vec{u} \cdot (\vec{x} - \text{proj}_{\vec{u}\times\vec{v}}(\vec{x})) = \vec{u} \cdot \vec{x} - 0$$

Similarly, $t = \vec{v} \cdot \vec{x}$. Hence,

$$\text{perp}_{\vec{u}\times\vec{v}}(\vec{x}) = (\vec{u} \cdot \vec{x})\vec{u} + (\vec{v} \cdot \vec{x})\vec{v} = \text{proj}_{\vec{u}}(\vec{x}) + \text{proj}_{\vec{v}}(\vec{x})$$

F3 (a) We can calculate that both sides of the equation are equal to

$$\begin{bmatrix} u_2v_1w_2 - u_2v_2w_1 + u_3v_1w_3 - u_3v_3w_1 \\ -u_1v_1w_2 + u_1v_2w_1 + u_3v_2w_3 - u_3v_3w_2 \\ -u_1v_1w_3 + u_1v_3w_1 - u_2v_2w_3 + u_2v_3w_2 \end{bmatrix}$$

(b) Using part (a), we get that

$$\begin{aligned} &\vec{u}\times(\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \\ &\quad ((\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}) + ((\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}) + ((\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}) = \vec{0} \end{aligned}$$

F4 If $a = b = 0$, then Span $\mathcal{B} = \{s\begin{bmatrix} c \\ d \end{bmatrix} \mid s \in \mathbb{R}\} \neq \mathbb{R}^2$. Thus, at least one of a or b is non-zero. Assume $a \neq 0$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_1 \begin{bmatrix} a \\ b \end{bmatrix} + t_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} t_1 a + t_2 c \\ t_1 b + t_2 d \end{bmatrix}$$

Since $a \neq 0$, we get $t_1 = \frac{x_1}{a} - t_2 \frac{c}{a}$. Hence,

$$x_2 = \frac{bx_1}{a} - t_2\left(\frac{bc}{a} - d\right)$$

If $\frac{bc}{a} - d = 0$, then $x_2 = \frac{bx_1}{a}$ and hence $\mathcal{B}$ could not span $\mathbb{R}^2$. Thus, $\frac{bc}{a} - d \neq 0$ which we rewrite as $ad - bc \neq 0$. Then, we get that

$$t_2 = \frac{1}{ad - bc}(-bx_1 + ax_2)$$
$$t_1 = \frac{1}{ad - bc}(dx_1 - cx_2)$$

This implies that if $ad - bc \neq 0$, then $\mathcal{B}$ spans $\mathbb{R}^2$ and is linearly independent.

F5 (a) Let $\vec{w} = \text{perp}_{\vec{v}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$. Then,

$$\vec{w} \cdot \vec{v}_1 = (\vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1) \cdot \vec{v}_1 = \vec{v}_2 \cdot \vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}(\vec{v}_1 \cdot \vec{v}_1) = 0$$

Hence, $\{\vec{v}_1, \vec{w}\}$ is an orthogonal set.

Observe that $\vec{w} \neq \vec{0}$ as otherwise we would have $\vec{v}_2 = \frac{\vec{v}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$ which would contradict $\{\vec{v}_1, \vec{v}_2\}$ being linearly independent.

Hence, by Problem C5 in Section 1.5, we have that $\{\vec{v}_1, \vec{w}\}$ is linearly independent.

Also, by Problem C7 in Section 1.4, we have that $P = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{v}_1, \vec{w}\}$.

Thus, $\{\vec{v}_1, \vec{w}\}$ is also a basis for P.

(b) Let $\vec{y} = \vec{v}_1 \times \vec{w}$. Then, we have that $\{\vec{v}_1, \vec{w}, \vec{y}\}$ is an orthogonal set. Moreover, we know $\vec{y} \neq \vec{0}$ since $\{\vec{v}_1, \vec{w}\}$ is linearly independent. Then, by Problem C5 in Section 1.5, we have that $\{\vec{v}_1, \vec{w}, \vec{y}\}$ is linearly independent.

Let $\vec{x} \in \mathbb{R}^3$. Our work with finding the nearest point $\vec{r}$ shows us that $\vec{r} = \vec{x} + \text{proj}_{\vec{y}}(\vec{x})$ where $\vec{r} \in \text{Span}\{\vec{v}_1, \vec{w}\}$. Let $\vec{r} = c_1\vec{v}_1 + c_2\vec{w}$. Then, we have that

$$c_1\vec{v}_1 + c_2\vec{w} = \vec{x} + \frac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}\vec{y}$$
$$\vec{x} = c_1\vec{v}_1 c_2\vec{w} - \frac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}\vec{y}$$

Thus, every $\vec{x} \in \mathbb{R}^3$ is a linear combination of $\vec{v}_1$, $\vec{w}$, and $\vec{y}$. Thus, $\{\vec{v}_1, \vec{w}, \vec{y}\}$ also spans $\mathbb{R}^3$ as required.

(c) Since $\{\vec{v}_1, \vec{w}, \vec{y}\}$ is a basis for $\mathbb{R}^3$, for any $\vec{x} \in \mathbb{R}^3$, there exists unique $d_1, d_2, d_3 \in \mathbb{R}$ such that

$$\vec{x} = d_1\vec{v}_1 + d_2\vec{w} + d_3\vec{y}$$

Taking the dot product of both sides with respect to $\vec{v}_1$ gives

$$\vec{x} \cdot \vec{v}_1 = d_1(\vec{v}_1 \cdot \vec{v}_1) + 0$$

Hence, $d_1 = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$. Similarly, we get $d_2 = \frac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2}$, and $d_3 = \frac{\vec{x} \cdot \vec{y}}{\|\vec{y}\|^2}$.

F6 (a) By definition, $\mathbb{U} \oplus \mathbb{W}$ is a subset of $\mathbb{R}^n$. Since $\mathbb{U}$ and $\mathbb{W}$ are subspaces we have $\vec{0} \in \mathbb{U}$ and $\vec{0} \in \mathbb{W}$. Thus, $\vec{0} = \vec{0} + \vec{0} \in \mathbb{U} \oplus \mathbb{W}$ so $\mathbb{U} \oplus \mathbb{W}$ is non-empty.

Let $\vec{x}, \vec{y} \in \mathbb{U} \oplus \mathbb{W}$ and $s, t \in \mathbb{R}$. Then, $\vec{x} = \vec{u}_1 + \vec{w}_1$ and $\vec{y} = \vec{u}_2 + \vec{w}_2$ where $\vec{u}_1, \vec{u}_2 \in \mathbb{U}$ and $\vec{w}_1, \vec{w}_2 \in \mathbb{W}$. Since $\mathbb{U}$ and $\mathbb{W}$ are subspaces we have that

$$s\vec{u}_1 + t\vec{u}_2 \in \mathbb{U} \text{ and } s\vec{w}_1 + t\vec{w}_2 \in \mathbb{W}$$

Thus,

$$s\vec{x} + t\vec{y} = s(\vec{u}_1 + \vec{w}_1) + t(\vec{u}_2 + \vec{w}_2) = s\vec{u}_1 + t\vec{u}_2 + s\vec{w}_1 + t\vec{w}_2 \in \mathbb{U} \oplus \mathbb{W}$$

Therefore, $\mathbb{U} \oplus \mathbb{W}$ is a subspace of $\mathbb{R}^n$.

(b) Let $\vec{x} \in \mathbb{U} \oplus \mathbb{W}$. Then, $\vec{x} = \vec{u} + \vec{w}$ for $\vec{u} \in \mathbb{U}$ and $\vec{w} \in \mathbb{W}$. Then we can write

$$\begin{aligned}\vec{u} &= a_1\vec{u}_1 + \cdots + a_k\vec{u}_k \\ \vec{w} &= b_1\vec{w}_1 + \cdots + b_\ell\vec{w}_\ell\end{aligned}$$

Thus,

$$\vec{x} = a_1\vec{u}_1 + \cdots + a_k\vec{u}_k + b_1\vec{w}_1 + \cdots + b_\ell\vec{w}_\ell$$

Hence, $\text{Span}\{\vec{u}_1, \dots, \vec{u}_k, \vec{w}_1, \dots, \vec{w}_\ell\} = \mathbb{U} \oplus \mathbb{W}$.

Consider

$$c_1\vec{u}_1 + \cdots + c_k\vec{u}_k + c_{k+1}\vec{w}_1 + \cdots + c_{k+\ell}\vec{w}_\ell = \vec{0}$$

This implies that

$$c_1\vec{u}_1 + \cdots + c_k\vec{u}_k = -c_{k+1}\vec{w}_1 - \cdots - c_{k+\ell}\vec{w}_\ell$$

The vector on the left is in $\mathbb{U}$ and the vector on the right is in $\mathbb{W}$. Hence, both vectors must be the zero vector. Therefore, $c_1 = \cdots = c_{k+\ell} = 0$ since $\{\vec{u}_1, \dots, \vec{u}_k\}$ and $\{\vec{w}_1, \dots, \vec{w}_\ell\}$ are both linearly independent.

F7 (a) We have

$$\begin{aligned}\|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2\end{aligned}$$

By subtraction,

$$\frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2 = \vec{u} \cdot \vec{v}$$

(b) By addition of the above expressions,

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

(c) The vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are the diagonal vectors of the parallelogram. Take the equation of part (a) and divide by $\|\vec{u}\|\|\vec{v}\|$ to obtain an expression for the cosine of the angle between $\vec{u}$ and $\vec{v}$, in terms of the lengths of $\vec{u}$, $\vec{v}$, and the diagonal vectors. The cosine is 0 if and only if the diagonals are of equal length. In this case, the parallelogram is a rectangle.

Part (b) says that the sum of the squares of the two diagonal lengths is the sum of the squares of the lengths of all four sides of the parallelogram. You can also see that this is true by using the cosine law and the fact that if the angle between $\vec{u}$ and $\vec{v}$ is θ, then the angle at the next vertex of the parallelogram is $\pi - \theta$.

F8 P, Q, and R are collinear if and only if for some scalar t, $\vec{PQ} = t\vec{PR}$. Thus, $\vec{q} - \vec{p} = t(\vec{r} - \vec{p})$, or $\vec{q} = (1-t)\vec{p} + t\vec{r}$. Then

$$\begin{aligned}(\vec{p}\times\vec{q}) + (\vec{q}\times\vec{r}) + (\vec{r}\times\vec{p}) &= \vec{p}\times((1-t)\vec{p}+t\vec{r}) + ((1-t)\vec{p}+t\vec{r})\times\vec{r} + \vec{r}\times\vec{p}\\ &= t\vec{p}\times\vec{r} + \vec{p}\times\vec{r} - t\vec{p}\times\vec{r} + \vec{r}\times\vec{p} = \vec{0}\end{aligned}$$

since $\vec{p}\times\vec{r} = -\vec{r}\times\vec{p}$.

F9 (a) Suppose that the skew lines are $\vec{x} = \vec{p} + s\vec{c}$ and $\vec{x} = \vec{q} + t\vec{d}$. Then the cross-product of the two direction vectors $\vec{n} = \vec{c}\times\vec{d}$ is perpendicular to both lines, so the plane through P with normal $\vec{n}$ contains the first line, and the plane through Q with normal $\vec{n}$ contains the second line. Since the two planes have the same normal vector, they are parallel planes.

(b) We find that $\vec{n} = \begin{bmatrix}2\\0\\1\end{bmatrix}\times\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}-1\\-5\\2\end{bmatrix}$. Thus, the equation of the plane passing through $P(1,4,2)$ is $-1x_1 - 5x_2 + 2x_3 = -17$. Hence, we find that the distance from the point $Q(2,-3,1)$ to this plane is $32/\sqrt{30}$ which is the distance between the skew lines.