Chapter 4 The Exponential and Natural Logarithm Functions

4.1 Exponential Functions

1. $4^x = \left(2^2\right)^x = 2^{2x}$

$\left(\sqrt{3}\right)^x = \left(3^{1/2}\right)^x = 3^{(1/2)x}$

$\left(\frac{1}{9}\right)^x = \left(9^{-1}\right)^x = \left(\left(3^2\right)^{-1}\right)^x = 3^{-2x}$

2. $27^x = \left(3^3\right)^x = 3^{3x}$

$\left(\sqrt[3]{2}\right)^x = \left(2^{1/3}\right)^x = 2^{(1/3)x}$

$\left(\frac{1}{8}\right)^x = \left(8^{-1}\right)^x = \left(\left(2^3\right)^{-1}\right)^x = 2^{-3x}$

3. $8^{2x/3} = \left(2^3\right)^{2x/3} = 2^{3(2x/3)} = 2^{2x}$

$9^{3x/2} = \left(3^2\right)^{3x/2} = 3^{2(3x/2)} = 3^{3x}$

$16^{-3x/4} = \left(2^4\right)^{-3x/4} = 2^{4(-3x/4)} = 2^{-3x}$

4. $9^{-x/2} = \left(3^2\right)^{-x/2} = 3^{2(-x/2)} = 3^{-x}$

$8^{4x/3} = \left(2^3\right)^{4x/3} = 2^{3(4x/3)} = 2^{4x}$

$27^{-2x/3} = \left(3^3\right)^{-2x/3} = 3^{3(-2x/3)} = 3^{-2x}$

5. $\left(\frac{1}{4}\right)^{2x} = \left(4^{-1}\right)^{2x} = \left(\left(2^2\right)^{-1}\right)^{2x} = 2^{-4x}$

$\left(\frac{1}{8}\right)^{-3x} = \left(8^{-1}\right)^{-3x} = \left(\left(2^3\right)^{-1}\right)^{-3x} = 2^{9x}$

$\left(\frac{1}{81}\right)^{x/2} = \left(81^{-1}\right)^{x/2} = \left(\left(3^4\right)^{-1}\right)^{x/2} = 3^{-2x}$

6. $\left(\frac{1}{9}\right)^{2x} = \left(9^{-1}\right)^{2x} = \left(\left(3^2\right)^{-1}\right)^{2x} = 3^{-4x}$

$\left(\frac{1}{27}\right)^{x/3} = \left(27^{-1}\right)^{x/3} = \left(\left(3^3\right)^{-1}\right)^{x/3}$
$= 3^{-3(x/3)} = 3^{-x}$

$\left(\frac{1}{16}\right)^{-x/2} = \left(16^{-1}\right)^{-x/2} = \left(\left(2^4\right)^{-1}\right)^{-x/2}$
$= 2^{-4(-x/2)} = 2^{2x}$

7. $6^x \cdot 3^{-x} = (2\cdot 3)^x \cdot 3^{-x} = 2^x \cdot (3^x \cdot 3^{-x})$
$= 2^x \cdot (3^{x-x}) = 2^x$

$\frac{15^x}{5^x} = \frac{(3\cdot 5)^x}{5^x} = \frac{3^x \cdot 5^x}{5^x} = 3^x$

$\frac{12^x}{2^{2x}} = \frac{(3\cdot 4)^x}{2^{2x}} = \frac{3^x \cdot 4^x}{2^{2x}} = \frac{3^x \cdot (2^2)^x}{2^{2x}} = 3^x$

8. $7^{-x} \cdot 14^x = 7^{-x} \cdot (7\cdot 2)^x = 7^{-x} \cdot 7^x \cdot 2^x$
$= 7^{-x+x} \cdot 2^x = 2^x$

$\frac{2^x}{6^x} = \frac{2^x}{(2\cdot 3)^x} = \frac{2^x}{2^x \cdot 3^x} = \frac{1}{3^x} = 3^{-x}$

$\frac{3^{2x}}{18^x} = \frac{3^{2x}}{9^x \cdot 2^x} = \frac{3^{2x}}{3^{2x} \cdot 2^x} = \frac{1}{2^x} = 2^{-x}$

9. $\frac{3^{4x}}{3^{2x}} = 3^{4x} \cdot 3^{-2x} = 3^{4x-2x} = 3^{2x}$

$\frac{2^{5x+1}}{2\cdot 2^{-x}} = \frac{2^{5x+1}}{2^{1-x}} = 2^{5x+1} \cdot 2^{-(1-x)}$
$= 2^{5x+1-(1-x)} = 2^{6x}$

$\frac{9^{-x}}{27^{-x/3}} = \frac{\left(3^2\right)^{-x}}{\left(3^3\right)^{-x/3}} = \frac{3^{-2x}}{3^{-x}} = 3^{-2x} \cdot 3^x$
$= 3^{-2x+x} = 3^{-x}$

10. $\frac{2^x}{6^x} = \frac{2^x}{(2\cdot 3)^x} = \frac{2^x}{2^x \cdot 3^x} = \frac{1}{3^x} = 3^{-x}$

$\frac{3^{-5x}}{3^{-2x}} = 3^{-5x} \cdot 3^{2x} = 3^{-5x+2x} = 3^{-3x}$

$\frac{16^x}{8^{-x}} = \frac{(2^4)^x}{(2^3)^{-x}} = 2^{4x} \cdot 2^{3x} = 2^{4x+3x} = 2^{7x}$

11. $2^{3x} \cdot 2^{-5x/2} = 2^{3x-(5x/2)} = 2^{(6x/2)-(5x/2)}$
$= 2^{x/2}$

$3^{2x} \cdot \left(\frac{1}{3}\right)^{2x/3} = 3^{2x} \cdot \left(3^{-1}\right)^{2x/3} = 3^{2x} \cdot 3^{-2x/3}$
$= 3^{(6x/3)-(2x/3)} = 3^{4x/3} = 3^{(4/3)x}$

12. $2^{5x/4} \cdot \left(\frac{1}{2}\right)^x = 2^{5x/4} \cdot \left(2^{-1}\right)^x = 2^{5x/4} \cdot 2^{-4x/4}$
$= 2^{(5x/4)-(4x/4)} = 2^{x/4}$

$3^{-2x} \cdot 3^{5x/2} = 3^{-4x/2} \cdot 3^{5x/2} = 3^{(-4x/2)+(5x/2)}$
$= 3^{x/2}$

13. $\left(2^{-3x}\cdot 2^{-2x}\right)^{2/5}=\left(2^{-3x-2x}\right)^{2/5}$
$=\left(2^{-5x}\right)^{2/5}=2^{-2x}$

$\left(9^{1/2}\cdot 9^{4}\right)^{x/9}=\left(\left(3^2\right)^{1/2}\cdot\left(3^2\right)^4\right)^{x/9}$
$=\left(3^1\cdot 3^8\right)^{x/9}=\left(3^{1+8}\right)^{x/9}$
$=\left(3^9\right)^{x/9}=3^x$

14. $\left(3^{-x}\cdot 3^{x/5}\right)^5=3^{-5x}\cdot 3^{(x/5)5}=3^{-5x}\cdot 3^x$
$=3^{-5x+x}=3^{-4x}$

$\left(16^{1/4}\cdot 16^{-3/4}\right)^{3x}=\left(\left(2^4\right)^{1/4}\cdot\left(2^4\right)^{-3/4}\right)^{3x}$
$=\left(2^1\cdot 2^{-3}\right)^{3x}=\left(2^{1-3}\right)^{3x}$
$=\left(2^{-2}\right)^{3x}=2^{-6x}$

15. $f(x)=3^{-2x}=\left(3^{-2}\right)^x=\left(\frac{1}{9}\right)^x\Rightarrow b=\frac{1}{9}$

16. $8^{-x/3}=\left(8^{-1/3}\right)^x=\left(\frac{1}{2}\right)^x\Rightarrow b=\frac{1}{2}$

17. $5^{2x}=5^2\Rightarrow 2x=2\Rightarrow x=1$

18. $10^{-x}=10^2\Rightarrow -x=2\Rightarrow x=-2$

19. $(2.5)^{2x+1}=(2.5)^5\Rightarrow 2x+1=5\Rightarrow x=\frac{5-1}{2}=2$

20. $(3.2)^{x-3}=(3.2)^5\Rightarrow x-3=5\Rightarrow x=8$

21. $10^{1-x}=100\Rightarrow 10^{1-x}=10^2\Rightarrow 1-x=2\Rightarrow$
$x=-1$

22. $2^{4-x}=8\Rightarrow 2^{4-x}=2^3\Rightarrow 4-x=3\Rightarrow x=1$

23. $3(2.7)^{5x}=8.1\Rightarrow 2.7^{5x}=2.7\Rightarrow 5x=1\Rightarrow$
$x=\frac{1}{5}$

24. $4(2.7)^{2x-1}=10.8\Rightarrow 2.7^{2x-1}=2.7\Rightarrow$
$2x-1=1\Rightarrow x=\frac{1+1}{2}=1$

25. $\left(2^{x+1}\cdot 2^{-3}\right)^2=2\Rightarrow\left(2^{x+1-3}\right)^2=2\Rightarrow$
$\left(2^{x-2}\right)^2=2\Rightarrow 2^{2x-4}=2\Rightarrow 2x-4=1\Rightarrow$
$x=\frac{4+1}{2}=\frac{5}{2}$

26. $\left(3^{2x}\cdot 3^2\right)^4=3\Rightarrow\left(3^{2x+2}\right)^4=3\Rightarrow 3^{8x+8}=3\Rightarrow$
$8x+8=1\Rightarrow x=\frac{1-8}{8}=-\frac{7}{8}$

27. $2^{3x}=4\cdot 2^{5x}\Rightarrow 2^{3x}=2^2\cdot 2^{5x}\Rightarrow$
$2^{3x}=2^{2+5x}\Rightarrow 3x=2+5x\Rightarrow 2x=-2\Rightarrow$
$x=-1$

28. $3^{5x}\cdot 3^x-3=0\Rightarrow 3^{5x+x}=3\Rightarrow 5x+x=1\Rightarrow$
$6x=1\Rightarrow x=\frac{1}{6}$

29. $(1+x)2^{-x}-5\cdot 2^{-x}=0\Rightarrow 2^{-x}(1+x-5)=0\Rightarrow$
$2^{-x}(x-4)=0$

Since $2^{-x}\neq 0$ for every x, then $x=4$ is the only solution.

30. $(2-3x)5^x+4\cdot 5^x=0\Rightarrow 5^x(2-3x+4)=0\Rightarrow$
$5^x(6-3x)=0$

Since $5^x\neq 0$ for every x, $x=2$ is the only solution.

31. $2^x-\frac{8}{2^{2x}}=0\Rightarrow 2^x-\frac{2^3}{2^{2x}}=0\Rightarrow$
$2^x-2^{3-2x}=0\Rightarrow 2^x=2^{3-2x}\Rightarrow$
$x=3-2x\Rightarrow x=1$

32. $2^x-\frac{1}{2^x}=0\Rightarrow 2^x-2^{-x}=0\Rightarrow 2^x=2^{-x}\Rightarrow$
$x=-x\Rightarrow x=0$

33. $2^{2x}-6\cdot 2^x+8=0\Rightarrow\left(2^x\right)^2-6\cdot 2^x+8=0$

Let $X=2^x\Rightarrow X^2-6X+8=0\Rightarrow$
$(X-2)(X-4)=0\Rightarrow X=2, X=4\Rightarrow$
$2^x=2$ or $2^x=4\Rightarrow x=1$ or $x=2$

34. $2^{2x+2}-17\cdot 2^x+4=0\Rightarrow$
$(2^x)^2 2^2-17\cdot 2^x+4=0$

Let $X=2^x\Rightarrow 4X^2-17X+4=0$
$X=\frac{17\pm\sqrt{(-17)^2-4(4)(4)}}{8}\Rightarrow X=\frac{1}{4}, X=4\Rightarrow$
$2^x=\frac{1}{4}$ or $2^x=4\Rightarrow x=-2$ or $x=2$

35. $3^{2x}-12\cdot 3^x+27=0\Rightarrow(3^x)^2-12\cdot 3^x+27=0$

Let $X=3^x\Rightarrow X^2-12X+27=0\Rightarrow$
$(X-3)(X-9)=0\Rightarrow X=3, X=9\Rightarrow$
$3^x=3$ or $3^x=9\Rightarrow x=1$ or $x=2$

36. $2^{2x} - 4 \cdot 2^x - 32 = 0 \Rightarrow (2^x)^2 - 4 \cdot 2^x - 32 = 0$

Let $X = 2^x \Rightarrow X^2 - 4X - 32 = 0 \Rightarrow$
$(X+4)(X-8) = 0 \Rightarrow X = -4, X = 8 \Rightarrow$
$2^x = -4$ (no solution) or $2^x = 8 \Rightarrow x = 3$

37. $2^{3+h} = 2^3 \cdot 2^h$ The missing factor is 2^h.

38. $5^{2+h} = 5^2 \cdot 5^h = 25 \cdot 5^h$

The missing factor is 5^h.

39. $2^{x+h} - 2^x = 2^x \cdot 2^h - 2^x = 2^x(2^h - 1)$

The missing factor is $2^h - 1$.

40. $5^{x+h} + 5^x = 5^x \cdot 5^h + 5^x = 5^x(5^h + 1)$

The missing factor is $5^h + 1$.

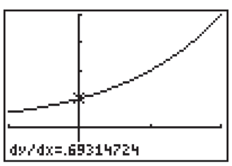
41. $$\begin{aligned} 3^{x/2} + 3^{-x/2} &= 3^{x-(x/2)} + 3^{-x/2} \\ &= 3^x \cdot 3^{-x/2} + 3^{-x/2} \\ &= 3^{-x/2}(3^x + 1) \end{aligned}$$

The missing factor is $3^x + 1$.

42. $5^{7x/2} - 5^{x/2} = 5^{x/2} \cdot 5^{6x/2} - 5^{x/2} \Rightarrow$
$5^{x/2}\left(5^{6x/2} - 1\right) = \sqrt{5^x}\left(5^{3x} - 1\right)$

The missing factor is $5^{3x} - 1$.

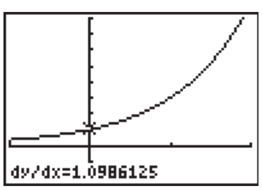
43.



$[-1, 2]$ by $[-1, 4]$

0.6931

44.



$[-1, 2]$ by $[-2, 8]$

1.0986

45. By trial and error, $b = 2.7$.

4.2 The Exponential Function e^x

1. If $h = .1$, then $\dfrac{3^h - 1}{h} \approx 1.16$.

If $h = .01$, then $\dfrac{3^h - 1}{h} \approx 1.10$.

If $h = .001$, then $\dfrac{3^h - 1}{h} \approx 1.10$.

Therefore, $\dfrac{d}{dx}(3^x)\Big|_{x=0} = \lim\limits_{x \to 0} \dfrac{3^h - 1}{h} \approx 1.1$.

2. If $h = .1$, then
$\dfrac{(2.7)^h - 1}{h} \approx \dfrac{1.10443 - 1}{.1} = 1.0443.$
If $h = .01$, then
$\dfrac{(2.7)^h - 1}{h} \approx \dfrac{1.00998 - 1}{.01} = .998.$
If $h = .001$, then
$\dfrac{(2.7)^h - 1}{h} \approx \dfrac{1.00099 - 1}{.001} = .99.$
Therefore,
$\dfrac{d}{dx}(2.7)^x \Big|_{x=0} = \lim\limits_{x \to 0} \dfrac{(2.7)^h - 1}{h} \approx .99.$

For exercises 3 and 4, we have
$m = \dfrac{d}{dx}\left(2^x\right)\Big|_{x=0} \approx .693$ from formula (2) in the text.

3. a. $\dfrac{d}{dx}(2^x)\Big|_{x=1} = m \cdot 2^1 = 2m$
$\approx 2(.693) = 1.386$

b. $\dfrac{d}{dx}(2^x)\Big|_{x=-2} = m \cdot 2^{-2} = \dfrac{1}{4}m$
$\approx \dfrac{1}{4}(0.693) = 0.1733$

4. a. $\dfrac{d}{dx}\left(2^x\right)\Big|_{x=1/2} = m \cdot 2^{1/2}$
$= \sqrt{2}m \approx \sqrt{2}(.693)$
$\approx .98005$

b. $\dfrac{d}{dx}(2^x)\Big|_{x=2} = m \cdot 2^2 = 4m$
$\approx 4(0.693) = 2.772$

5. a. $\dfrac{d}{dx}(e^x)\Big|_{x=1} = e^1 \approx 2.71828$

b. $\dfrac{d}{dx}(e^x)\Big|_{x=-1} = e^{-1} = \dfrac{1}{e} \approx .367879$

6. a. $\left.\frac{d}{dx}(e^x)\right|_{x=e} = e^e = 15.15426$

b. $\left.\frac{d}{dx}(e^x)\right|_{x=1/e} = e^{1/e} = 1.44467$

7. $\left(e^2\right)^x = e^{2x}$

$\left(\frac{1}{e}\right)^x = \left(e^{-1}\right)^x = e^{-x}$

8. $\left(e^3\right)^{x/5} = e^{3x/5} = e^{(3/5)x}$

$\left(\frac{1}{e^2}\right)^x = \left(e^{-2}\right)^x = e^{-2x}$

9. $\left(\frac{1}{e^3}\right)^{2x} = \left(e^{-3}\right)^{2x} = e^{-6x}$

$e^{1-x} \cdot e^{3x-1} = e^{2x}$

10. $\left(\frac{e^5}{e^3}\right)^x = \left(e^2\right)^x = e^{2x}$

$e^{4x+2} \cdot e^{x-2} = e^{5x}$

11. $\left(e^{4x} \cdot e^{6x}\right)^{3/5} = \left(e^{10x}\right)^{3/5} = e^{6x}$

$\frac{1}{e^{-2x}} = \left(e^{-2x}\right)^{-1} = e^{2x}$

12. $\sqrt{e^{-x} \cdot e^{7x}} = \sqrt{e^{6x}} = \left(e^{6x}\right)^{1/2} = e^{3x}$

$\frac{e^{-3x}}{e^{-4x}} = e^x$

13. $e^{5x} = e^{20} \Rightarrow 5x = 20 \Rightarrow x = 4$

14. $e^{1-x} = e^2 \Rightarrow 1 - x = 2 \Rightarrow x = -1$

15. $e^{x^2-2x} = e^8 \Rightarrow x^2 - 2x = 8 \Rightarrow$
$x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0 \Rightarrow$
$x = 4$ or $x = -2$

16. $e^{-x} = 1 \Rightarrow e^{-x} = e^0 \Rightarrow -x = 0 \Rightarrow x = 0$

17. $e^x(x^2-1) = 0 \Rightarrow e^x = 0$ (no solution) or
$x^2 - 1 = 0 \Rightarrow x = \pm 1$

18. $4e^x(x^2+1) = 0 \Rightarrow 4e^x = 0$ (no solution) or
$x^2 + 1 = 0$ (no solution)

19. The tangent passes through the point $\left(-1, e^{-1}\right) = \left(-1, \frac{1}{e}\right)$, or $(-1, 0.37)$. The slope of the tangent is given by $\left.\frac{dy}{dx}\right|_{x=-1} = \left.e^x\right|_{x=-1} = \frac{1}{e} \approx 0.37$. Thus, the equation of the tangent is $y - \frac{1}{e} = \frac{1}{e}(x+1)$ or $y = 0.37x + 0.74$.

20. The tangent line is parallel to $y = x$, so the slope of the tangent line is 1.
$\frac{d}{dx}e^x = 1 \Rightarrow e^x = 1 \Rightarrow x = 0$
The tangent passes through the point (0, 1).

21. $\frac{d}{dx}e^x = e^x$ and $\frac{d^2}{dx^2}e^x = e^x$

e^x is always increasing and $e^x > 0$, so by the first derivative test, there are no relative extreme points. Because $\frac{d^2}{dx^2}e^x = e^x > 0$ for all values of x, then e^x is concave up.

22.

h	e^h	$\frac{e^h - 1}{h}$
0.01	1.01005	1.005
0.001	1.0010005	1.0005
0.0001	1.000100005	1.00005

The slope of e^x at $x = 0$ is 1.

23. The slope of the graph of e^x at (a, b) is $e^a = b$.

24. From exercise 23, we know that the slope of the tangent line at $\left(a, e^a\right)$ is e^a. Thus, the slope-point form of the equation of the tangent is $y - e^a = e^a(x - a)$.

25. $\frac{d}{dx}\left(3e^x - 7x\right) = \frac{d}{dx}\left(3e^x\right) - \frac{d}{dx}(7x) = 3e^x - 7$

26. $\frac{d}{dx}\left(\frac{2x + 4 - 5e^x}{4}\right)$
$= \frac{d}{dx}\left(\frac{2x}{4}\right) + \frac{d}{dx}\left(\frac{4}{4}\right) - \frac{d}{dx}\left(\frac{5e^x}{4}\right)$
$= \frac{1}{2} + 0 - \frac{5}{4}e^x = \frac{1}{2} - \frac{5}{4}e^x$

27. $$\frac{d}{dx}\left(xe^x\right) = x\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}x = xe^x + e^x = (x+1)e^x$$

28. $$\frac{d}{dx}\left(x^2+x+1\right)e^x = \left(x^2+x+1\right)\frac{d}{dx}e^x + e^x\frac{d}{dx}\left(x^2+x+1\right) = e^x\left(x^2+x+1\right) + e^x(2x+1) = e^x\left(x^2+3x+2\right)$$

29. $$\begin{aligned}\frac{d}{dx}\left[\left(8e^x\right)\left(1+2e^x\right)^2\right] &= \left(8e^x\right)\frac{d}{dx}\left[\left(1+2e^x\right)^2\right] + \left(1+2e^x\right)^2\frac{d}{dx}\left(8e^x\right)\\ &= \left(8e^x\right)(2)\left(1+2e^x\right)\left(2e^x\right) + \left(1+2e^x\right)^2\left(8e^x\right)\\ &= 32e^{2x}\left(1+2e^x\right) + 8e^x\left(1+2e^x\right)^2\\ &= 32e^{2x} + 64e^{3x} + 8e^x\left(1+4e^x+4e^{2x}\right)\\ &= 8e^x + 64e^{2x} + 96e^{3x}\\ &= 8e^x\left(1+8e^x+12e^{2x}\right)\\ &= 8e^x\left(1+6e^x\right)\left(1+2e^x\right)\end{aligned}$$

30. $$\begin{aligned}\frac{d}{dx}\left[\left(1+e^x\right)\left(1-e^x\right)\right] &= \left(1+e^x\right)\frac{d}{dx}\left(1-e^x\right) + \left(1-e^x\right)\frac{d}{dx}\left(1+e^x\right)\\ &= \left(1+e^x\right)\left(-e^x\right) + \left(1-e^x\right)\left(e^x\right)\\ &= -e^x - e^{2x} + e^x - e^{2x} = -2e^{2x}\end{aligned}$$

31. $$\frac{d}{dx}\left(\frac{e^x}{x+1}\right) = \frac{(x+1)\left(e^x\right) - e^x(1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$$

32. $$\frac{d}{dx}\left(\frac{x+1}{e^x}\right) = \frac{e^x(1) - (x+1)e^x}{\left(e^x\right)^2} = -\frac{x}{e^x}$$

33. $$\frac{d}{dx}\left(\frac{e^x-1}{e^x+1}\right) = \frac{\left(e^x+1\right)\left(e^x\right) - \left(e^x-1\right)\left(e^x\right)}{\left(e^x+1\right)^2} = \frac{e^{2x}+e^x-e^{2x}+e^x}{\left(e^x+1\right)^2} = \frac{2e^x}{\left(e^x+1\right)^2}$$

34. $$\frac{d}{dx}\sqrt{e^x+1} = \frac{d}{dx}\left(e^x+1\right)^{1/2} = \frac{1}{2}\left(e^x\right)\left(e^x+1\right)^{-1/2} = \frac{e^x}{2\sqrt{e^x+1}}$$

35. $y' = 1 - e^x$; $y'' = -e^x$

$1 - e^x = 0 \Rightarrow 1 = e^x \Rightarrow x = 0$

$-e^0 = -1 < 0$, so there is a maximum point at $\left(0,\ 0-e^0\right) = (0,\ -1)$.

36. $y = \left(x^2e^x\right)$

$y' = 2xe^x + x^2e^x = e^x\left(2x+x^2\right)$

$e^x\left(2x+x^2\right) = 0 \Rightarrow x(2+x) = 0 \Rightarrow$

$x = 0$ or $x = -2$

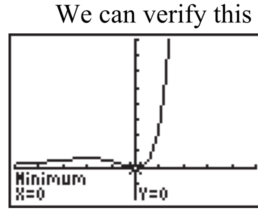
Thus, the extreme points are located at (0, 0) and $\left(-2,\ \frac{4}{e^2}\right)$.

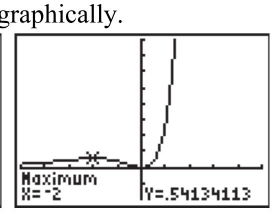
$$y'' = \frac{d}{dx}\left(e^x\left(2x+x^2\right)\right) = (2+2x)e^x + \left(2x+x^2\right)e^x = e^x\left(x^2+4x+2\right)$$

Evaluating y'' for $x = 0$ gives $e^0\left(0^2+4\cdot 0+2\right) = 2 > 0$, so (0, 0) is a minimum point.

Evaluating y'' for $x = -2$ gives $e^{-2}\left((-2)^2+4(-2)+2\right) = -\frac{2}{e^2} < 0$, so $\left(-2,\ \frac{4}{e^2}\right) = (-2,\ .54134113)$ is a maximum point.

We can verify this graphically.





37. $y = (1+x^2)e^x$

$y' = (1+x^2)e^x + e^x(2x) = e^x(x^2+2x+1)$

$y' = 0 \Rightarrow e^x(x^2+2x+1) = 0 \Rightarrow$

$x^2+2x+1 = 0 \Rightarrow (x+1)(x+1) = 0 \Rightarrow x = -1$

The tangent line is horizontal at $\left(-1, 2e^{-1}\right)$ or $\left(-1, \frac{2}{e}\right)$.

38. $y = e^x \Rightarrow y' = e^x \Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{x=a} = e^a$

$y = e^{-x} \Rightarrow y' = -e^{-x} \Rightarrow m_2 = \left.\frac{dy}{dx}\right|_{x=a} = -e^{-a}$

$m_1 \cdot m_2 = e^a(-e^{-a}) = -1$

Therefore, the tangent lines are perpendicular.

39. $\frac{dy}{dx} = xe^x + e^x$

$\left.\frac{dy}{dx}\right|_{x=0} = e^0 + 0 \cdot e^0 = 1$

The slope of the tangent line is 1.

40. $\frac{dy}{dx} = xe^x + e^x$

$\left.\frac{dy}{dx}\right|_{x=1} = e^1 + 1 \cdot e^1 = 2e$

The slope of the tangent line is $2e$.

41. $\frac{dy}{dx} = \frac{(1+2e^x)e^x - e^x(2e^x)}{(1+2e^x)^2}$

$= \frac{e^x}{(1+2e^x)^2}$

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{e^0}{(1+2e^0)^2} = \frac{1}{9}$

$(x_1, y_1) = \left(0, \frac{1}{3}\right), m = \frac{1}{9}$

$y - \frac{1}{3} = \frac{1}{9}(x-0) \Rightarrow y = \frac{1}{9}x + \frac{1}{3}$

42. $\frac{dy}{dx} = \frac{(x+e^x)e^x - e^x(1+e^x)}{(x+e^x)^2}$

$= \frac{xe^x - e^x}{(x+e^x)^2}$

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{0 \cdot e^0 - e^0}{(0+e^0)^2} = -1$

$(x_1, y_1) = (0, 1), m = -1$

$y - 1 = -1(x-0) \Rightarrow y = -x + 1$

43. $f(x) = e^x(1+x)^2$

$f'(x) = e^x(2)(1+x) + e^x(1+x)^2$

$= e^x(1+x)(3+x)$

$= e^x(x^2+4x+3)$

$f''(x) = e^x(2x+4) + (x^2+4x+3)(e^x)$

$= e^x(x^2+6x+7)$

44. $f(x) = \frac{e^x}{x}$

$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

$f''(x) = \frac{x^2(xe^x + e^x - e^x) - (xe^x - e^x)(2x)}{x^4}$

$= \frac{e^x(x^2 - 2x + 2)}{x^3}$

45. **a.** $\frac{d}{dx}\left(5e^x\right) = 5\left(e^x\right) + e^x(0) = 5e^x$

b. $\frac{d}{dx}\left(e^x\right)^{10} = 10\left(e^x\right)^9 e^x = 10e^{10x}$

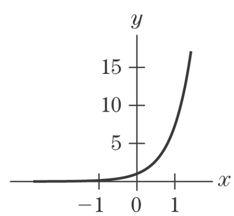
c. $\frac{d}{dx}\left(e^{2+x}\right) = \frac{d}{dx}\left(e^2 \cdot e^x\right)$

$= e^2(e^x) + e^x(0) = e^{2+x}$

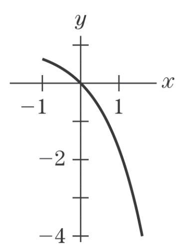
46. **a.** $\frac{d}{dx}\left(e^{4x}\right) = \frac{d}{dx}\left[\left(e^x\right)^4\right] = 4(e^x)^3 \cdot e^x = 4e^{4x}$

b. $\frac{d}{dx}\left(e^{kx}\right) = \frac{d}{dx}\left(e^x\right)^k = k\left(e^x\right)^{k-1} e^x = ke^{kx}$

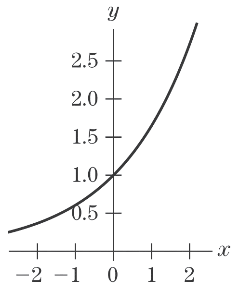
47.



48.



49.



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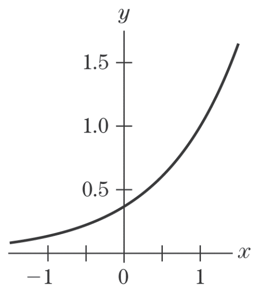
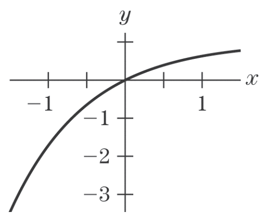
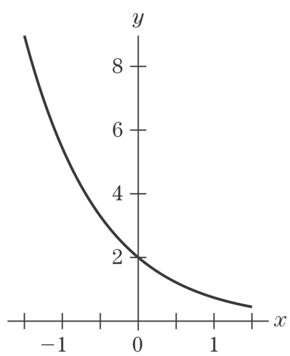


51.



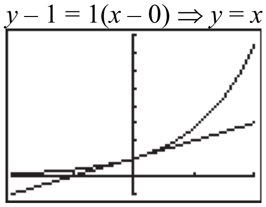
52.



53. $\dfrac{dy}{dx} = e^x = 1$ at $x = 0$.

$y = 1$ at $x = 0$

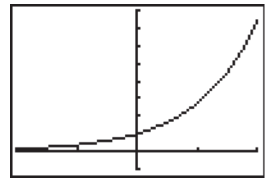
$y - 1 = 1(x - 0) \Rightarrow y = x + 1$



[−2, 2] by [−1, 8]

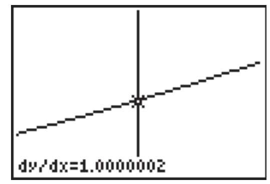
The graph confirms the answer.

54. **a.**



[−2, 2] by [−1, 8]

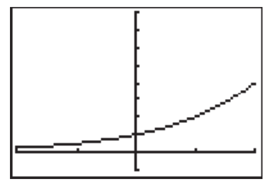
b.



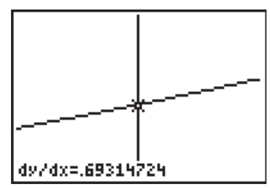
[−0.125, 0.125] by [0.7510, 1.315]

Estimate the slope to be 1.

c.



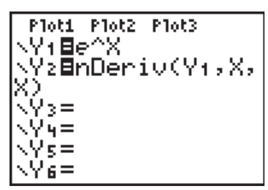
[−2, 2] by [−1, 8]

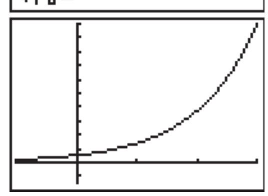


[−0.125, 0.125] by [0.7510, 1.315]

Estimate the slope to be .7.

55.





[−1, 3] by [−3, 20]

56. $x = 0.1: \dfrac{10^x - 1}{x} \approx 2.5893$

$x = 0.01: \dfrac{10^x - 1}{x} \approx 2.3293$

$x = 0.001: \dfrac{10^x - 1}{x} \approx 2.3052$

$x = 0.0001: \dfrac{10^x - 1}{x} \approx 2.3029$

$x = 0.00001: \dfrac{10^x - 1}{x} \approx 2.3026$

$x = 0.000001: \dfrac{10^x - 1}{x} \approx 2.3026$

$\dfrac{d}{dx}(10^x)\Big|_{x=0} \approx 2.3026$

$\dfrac{d}{dx}(10^x) = m \cdot 10^x$ where $m = \dfrac{d}{dx}(10^x)\Big|_{x=0}$

4.3 Differentiation of Exponential Functions

1. $\dfrac{d}{dx}\left(e^{2x+3}\right) = e^{2x+3}\dfrac{d}{dx}(2x+3)$
$= e^{2x+3}(2) = 2e^{2x+3}$

2. $\dfrac{d}{dx}\left(e^{-3x-2}\right) = e^{-3x-2}\dfrac{d}{dx}(-3x-2) = -3e^{-3x-2}$

3. $\dfrac{d}{dx}\left(e^{4x^2-x}\right) = e^{4x^2-x}\dfrac{d}{dx}\left(4x^2-x\right)$
$= (8x-1)e^{4x^2-x}$

4. $\dfrac{d}{dx}\left(e^{(1+x)^3}\right) = e^{(1+x)^3}\dfrac{d}{dx}\left((1+x)^3\right)$
$= e^{(1+x)^3}\left(3(1+x)^2\right)(1)$
$= 3(1+x)^2 e^{(1+x)^3}$

5. $\dfrac{d}{dx}\left(e^{e^x}\right) = e^{e^x}\dfrac{d}{dx}\left(e^x\right) = e^{e^x}e^x$

6. $\dfrac{d}{dx}\left(e^{1/x}\right) = e^{1/x}\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = e^{1/x}\left(-\dfrac{1}{x^2}\right) = -\dfrac{e^{1/x}}{x^2}$

7. $\dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = e^{\sqrt{x}}\left(\dfrac{1}{2}x^{-1/2}\right)$
$= \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

8. $\dfrac{d}{dx}\left(e^{\sqrt{x^2+1}}\right)$
$= e^{\sqrt{x^2+1}}\dfrac{d}{dx}\left(\sqrt{x^2+1}\right)$
$= e^{\sqrt{x^2+1}}\left(\dfrac{1}{2}\left(x^2+1\right)^{-1/2}\right)\dfrac{d}{dx}\left(x^2+1\right)$
$= e^{\sqrt{x^2+1}}\left(\dfrac{1}{2}\left(x^2+1\right)^{-1/2}\right)(2x)$
$= e^{\sqrt{x^2+1}}\left(\dfrac{x}{\sqrt{x^2+1}}\right)$

9. $\dfrac{d}{dx}\left(-7e^{x/7}\right) = -7\dfrac{d}{dx}\left(e^{x/7}\right) = -7e^{x/7}\dfrac{d}{dx}\left(\dfrac{x}{7}\right)$
$= -7e^{x/7}\left(\dfrac{1}{7}\right) = -e^{x/7}$

10. $\dfrac{d}{dx}10e^{(-x-2)/5} = 10\dfrac{d}{dx}e^{(-x-2)/5}$
$= 10e^{(-x-2)/5}\dfrac{d}{dx}\left(\dfrac{-x-2}{5}\right)$
$= 10e^{(-x-2)/5}\left(-\dfrac{1}{5}\right)$
$= -2e^{(-x-2)/5}$

11. $\dfrac{d}{dt}\left(4e^{0.05t} - 23e^{0.01t}\right)$
$= \dfrac{d}{dt}\left(4e^{0.05t}\right) - \dfrac{d}{dt}\left(23e^{0.01t}\right)$
$= 4\dfrac{d}{dt}\left(e^{0.05t}\right) - 23\dfrac{d}{dt}\left(e^{0.01t}\right)$
$= 4e^{0.05t}\dfrac{d}{dt}(0.05t) - 23e^{0.01t}\dfrac{d}{dt}(0.01t)$
$= 4e^{0.05t}(0.05) - 23e^{0.01t}(0.01)$
$= 0.2e^{0.05t} - 0.23e^{0.01t}$

12. $\dfrac{d}{dt}\left(2e^{t/2} - 0.4e^{0.001t}\right)$
$= \dfrac{d}{dt}\left(2e^{t/2}\right) - \dfrac{d}{dt}\left(0.4e^{0.001t}\right)$
$= 2\dfrac{d}{dt}\left(e^{t/2}\right) - 0.4\dfrac{d}{dt}\left(e^{0.001t}\right)$
$= 2e^{t/2}\dfrac{d}{dt}\left(\dfrac{t}{2}\right) - 0.4e^{0.001t}\dfrac{d}{dt}(0.001t)$
$= 2e^{t/2}\left(\dfrac{1}{2}\right) - 0.4e^{0.001t}(0.001)$
$= e^{t/2} - 0.0004e^{0.001t}$

13. $f(t)=\left(t^2+2e^t\right)e^{t-1}$

To differentiate, use the product rule.

$$\begin{aligned}
&\frac{d}{dt}\left[\left(t^2+2e^t\right)e^{t-1}\right]\\
&=e^{t-1}\frac{d}{dt}\left(t^2+2e^t\right)+\left(t^2+2e^t\right)\frac{d}{dt}\left(e^{t-1}\right)\\
&=e^{t-1}\left(2t+2e^t\right)+\left(t^2+2e^t\right)\left(e^{t-1}\right)\\
&=e^{t-1}\left(2t+2e^t+t^2+2e^t\right)\\
&=e^{t-1}\left(t^2+2t+4e^t\right)
\end{aligned}$$

14. $f(t)=\left(t^3-3t\right)e^{1+t}$

To differentiate, use the product rule.

$$\begin{aligned}
&\frac{d}{dt}\left[\left(t^3-3t\right)e^{1+t}\right]\\
&=e^{1+t}\frac{d}{dt}\left(t^3-3t\right)+\left(t^3-3t\right)\frac{d}{dt}\left(e^{1+t}\right)\\
&=e^{1+t}\left(3t^2-3\right)+\left(t^3-3t\right)\left(e^{1+t}\right)\\
&=e^{1+t}\left(t^3+3t^2-3t-3\right)
\end{aligned}$$

15. $f(x)=\left(x+\frac{1}{x}\right)e^{2x}$

To differentiate, use the product rule.

$$\begin{aligned}
&\frac{d}{dx}\left[\left(x+\frac{1}{x}\right)e^{2x}\right]\\
&=e^{2x}\frac{d}{dx}\left(x+\frac{1}{x}\right)+\left(x+\frac{1}{x}\right)\frac{d}{dx}\left(e^{2x}\right)\\
&=e^{2x}\left(1-\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)\left(2e^{2x}\right)\\
&=e^{2x}\left[1-\frac{1}{x^2}+2\left(x+\frac{1}{x}\right)\right]\\
&=e^{2x}\left(1+2x+\frac{2}{x}-\frac{1}{x^2}\right)
\end{aligned}$$

16. $$\begin{aligned}\frac{d}{dx}\left(e^{e^{e^x}}\right)&=e^{e^{e^x}}\frac{d}{dx}\left(e^{e^x}\right)=e^{e^{e^x}}e^{e^x}\frac{d}{dx}\left(e^x\right)\\&=e^{e^{e^x}}e^{e^x}e^x\end{aligned}$$

17. $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

To differentiate, use the quotient rule.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)\\
&=\frac{\left(e^x-e^{-x}\right)\frac{d}{dx}\left(e^x+e^{-x}\right)-\left(e^x+e^{-x}\right)\frac{d}{dx}\left(e^x-e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}\\
&=\frac{\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)-\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}\\
&=\frac{\left(e^{2x}-2e^xe^{-x}+e^{-2x}\right)-\left(e^{2x}+2e^xe^{-x}+e^{-2x}\right)}{\left(e^x-e^{-x}\right)^2}\\
&=\frac{-2-2}{\left(e^x-e^{-x}\right)^2}=-\frac{4}{\left(e^x-e^{-x}\right)^2}
\end{aligned}$$

18. $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

To differentiate, use the quotient rule.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\\
&=\frac{\left(e^x+e^{-x}\right)\frac{d}{dx}\left(e^x-e^{-x}\right)-\left(e^x-e^{-x}\right)\frac{d}{dx}\left(e^x+e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}\\
&=\frac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}\\
&=\frac{\left(e^{2x}+2e^xe^{-x}+e^{-2x}\right)-\left(e^{2x}-2e^xe^{-x}+e^{-2x}\right)}{\left(e^x+e^{-x}\right)^2}\\
&=\frac{2-(-2)}{\left(e^x+e^{-x}\right)^2}=\frac{4}{\left(e^x+e^{-x}\right)^2}
\end{aligned}$$

19. $$\begin{aligned}\frac{d}{dx}\sqrt{e^x+1}&=\frac{d}{dx}\left(e^x+1\right)^{1/2}\\
&=\frac{1}{2}\left(e^x+1\right)^{-1/2}\frac{d}{dx}\left(e^x+1\right)\\
&=\frac{1}{2}\left(e^x+1\right)^{-1/2}\left(e^x\right)\\
&=\frac{e^x}{2\sqrt{e^x+1}}\end{aligned}$$

20. $\dfrac{d}{dx}\left(e^{ex}\right)=e^{ex}\dfrac{d}{dx}(ex)=e^{ex}e=e^{ex+1}$

21. $f(x) = \left(e^{3x}\right)^5 = e^{15x}$

$$\frac{d}{dx}\left(e^{3x}\right)^5 = \frac{d}{dx}\left(e^{15x}\right) = e^{15x}\frac{d}{dx}(15x) = 15e^{15x}$$

22. $f(x) = e^x e^{2x} e^{3x} = e^{6x}$

$$\frac{d}{dx}\left(e^x e^{2x} e^{3x}\right) = \frac{d}{dx}e^{6x} = e^{6x}\frac{d}{dx}(6x) = 6e^{6x}$$

23. $f(x) = \dfrac{1}{\sqrt{e^x}} = \left(e^x\right)^{-1/2} = e^{-x/2}$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{e^x}}\right) = \frac{d}{dx}\left(e^{-x/2}\right) = e^{-x/2}\frac{d}{dx}\left(-\frac{x}{2}\right) = -\frac{e^{-x/2}}{2} = -\frac{1}{2\sqrt{e^x}}$$

24. $f(t) = e^{3t}\left(e^{2t} - e^{4t}\right) = e^{5t} - e^{7t}$

$$\frac{d}{dt}\left[e^{3t}\left(e^{2t} - e^{4t}\right)\right] = \frac{d}{dt}\left(e^{5t} - e^{7t}\right) = \frac{d}{dt}e^{5t} - \frac{d}{dt}e^{7t} = 5e^{5t} - 7e^{7t}$$

25. $f(x) = \dfrac{e^x + 5e^{2x}}{e^x} = 1 + 5e^x$

$$\frac{d}{dx}\frac{e^x + 5e^{2x}}{e^x} = \frac{d}{dx}\left(1 + 5e^x\right) = 5e^x$$

26. $f(x) = \sqrt{e^{3x}} = \left(e^{3x}\right)^{1/2} = e^{3x/2}$

$$\frac{d}{dx}\sqrt{e^{3x}} = \frac{d}{dx}e^{3x/2} = e^{3x/2}\frac{d}{dx}\frac{3x}{2} = \frac{3e^{3x/2}}{2}$$

27. $\dfrac{d}{dx}\left[(1+x)e^{-3x}\right] = (1+x)e^{-3x}(-3) + e^{-3x}(1) = (-3-3x)e^{-3x} + e^{-3x} = (-2-3x)e^{-3x}$

$(-2-3x)e^{-3x} = 0 \Rightarrow -2-3x = 0 \Rightarrow x = -\dfrac{2}{3}$

$$\frac{d}{dx}\left[(-2-3x)e^{-3x}\right] = (-2-3x)e^{-3x}(-3) + e^{-3x}(-3) = (6+9x)e^{-3x} - 3e^{-3x} = (3+9x)e^{-3x} = 3(1+3x)e^{-3x}$$

At $x = -\dfrac{2}{3}$, $3(1+3x)e^{-3x} < 0$, so there is a maximum at $x = -\dfrac{2}{3}$.

28. $\dfrac{d}{dx}\left[(1-x)e^{2x}\right] = (1-x)e^{2x}(2) + e^{2x}(-1) = (2-2x)e^{2x} - e^{2x} = (1-2x)e^{2x}$

$(1-2x)e^{2x} = 0 \Rightarrow 1-2x = 0 \Rightarrow x = \dfrac{1}{2}$

$$\frac{d}{dx}\left[(1-2x)e^{2x}\right] = (1-2x)e^{2x}(2) + e^{2x}(-2) = (2-4x)e^{2x} - 2e^{2x} = -4xe^{2x}$$

At $x = \dfrac{1}{2}$, $-4xe^{2x} < 0$, so there is a maximum at $x = \dfrac{1}{2}$.

29. $\dfrac{d}{dx}\left(\dfrac{3-4x}{e^{2x}}\right) = \dfrac{d}{dx}\left[(3-4x)e^{-2x}\right] = (3-4x)e^{-2x}(-2) + e^{-2x}(-4) = (-6+8x)e^{-2x} - 4e^{-2x} = (8x-10)e^{-2x}$

$(8x-10)e^{-2x} = 0 \Rightarrow 8x-10 = 0 \Rightarrow x = \dfrac{5}{4}$

$$\frac{d}{dx}\left[(8x-10)e^{-2x}\right] = (8x-10)e^{-2x}(-2) + e^{-2x}(8) = (-16x+20)e^{-2x} + 8e^{-2x} = (-16x+28)e^{-2x}$$

At $x = \dfrac{5}{4}$, $(-16x+28)e^{-2x} > 0$, so there is a minimum at $x = \dfrac{5}{4}$.

30. $\dfrac{d}{dx}\left(\dfrac{4x-1}{e^{x/2}}\right) = \dfrac{d}{dx}\left[(4x-1)e^{-x/2}\right] = (4x-1)e^{-x/2}\left(-\dfrac{1}{2}\right) + e^{-x/2}(4) = \left(-2x+\dfrac{1}{2}\right)e^{-x/2} + 4e^{-x/2} = \left(\dfrac{9}{2} - 2x\right)e^{-x/2}$

$\left(\dfrac{9}{2} - 2x\right)e^{-x/2} = 0 \Rightarrow \dfrac{9}{2} - 2x = 0 \Rightarrow x = \dfrac{9}{4}$

(*continued on next page*)

(*continued*)

$$\frac{d}{dx}\left[\left(\frac{9}{2}-2x\right)e^{-x/2}\right]$$
$$=\left(\frac{9}{2}-2x\right)e^{-x/2}\left(-\frac{1}{2}\right)+e^{-x/2}(-2)$$
$$=\left(-\frac{9}{4}+x\right)e^{-x/2}-2e^{-x/2}$$
$$=\left(x-\frac{17}{4}\right)e^{-x/2}$$

At $x=\frac{9}{4}$, $\left(x-\frac{17}{4}\right)e^{-x/2}<0$, so there is a maximum at $x=\frac{9}{4}$.

31. $$\frac{d}{dx}\left[(5x-2)e^{1-2x}\right]$$
$$=(5x-2)e^{1-2x}(-2)+e^{1-2x}(5)$$
$$=(-10x+4)e^{1-2x}+5e^{1-2x}$$
$$=(9-10x)e^{1-2x}$$

$$(9-10x)e^{1-2x}=0\Rightarrow 9-10x=0\Rightarrow x=\frac{9}{10}$$

$$\frac{d}{dx}\left[(9-10x)e^{1-2x}\right]$$
$$=(9-10x)e^{1-2x}(-2)+e^{1-2x}(-10)$$
$$=(-18+20x)e^{1-2x}-10e^{1-2x}$$
$$=(20x-28)e^{1-2x}$$

At $x=\frac{9}{10}$, $(20x-28)e^{1-2x}<0$, so there is a maximum at $x=\frac{9}{10}$.

32. $$\frac{d}{dx}\left[(2x-5)e^{3x-1}\right]$$
$$=(2x-5)e^{3x-1}(3)+e^{3x-1}(2)$$
$$=(6x-15)e^{3x-1}+2e^{3x-1}$$
$$=(6x-13)e^{3x-1}$$

$$(6x-13)e^{3x-1}=0\Rightarrow 6x-13=0\Rightarrow x=\frac{13}{6}$$

$$\frac{d}{dx}\left[(6x-13)e^{3x-1}\right]$$
$$=(6x-13)e^{3x-1}(3)+e^{3x-1}(6)$$
$$=(18x-39)e^{3x-1}+6e^{3x-1}$$
$$=(18x-33)e^{3x-1}$$

At $x=\frac{13}{6}$, $(18x-33)e^{3x-1}>0$, so there is a minimum at $x=\frac{13}{6}$.

33. a. $f(t)=3e^{0.06t}+2e^{0.02t}$

$f(0)=3e^{0.06(0)}+2e^{0.02(0)}=5$

The initial dollar amount invested is \$5000.

b. $f(5)=3e^{0.06(5)}+2e^{0.02(5)}\approx 6.25992$

After five years, the value of the portfolio is \$6259.92.

c. $$f'(t)=3e^{0.06t}(0.06)+2e^{0.02t}(0.02)$$
$$=0.18e^{0.06t}+0.04e^{0.02t}$$
$$f'(5)=0.18e^{0.06(5)}+0.04e^{0.02(5)}$$
$$\approx 0.28718$$

After five years, the investment is appreciating at about \$287.18 per year.

34. $v'(t)=2000e^{-.35t}(-.35)=-700e^{-.35t}$

$v'(3)=-700e^{-1.05}\approx -244.96$

After 3 years, the computer's value is falling at a rate of about \$244.96 per year.

35. a. $f(t)=31.87e^{0.096t}$

If $t=0$ represents 1997, then $t=18$ represents 2015.

$$f(18)=31.87e^{0.096(18)}$$
$$\approx \$179.408 \text{ million}$$

b. $$f'(t)=31.87e^{0.096t}(0.096)$$
$$=3.05952e^{0.096t}$$
$$f'(18)=3.05952e^{0.096(18)}\approx 17.223$$

The painting is appreciating at about \$17.223 million per year in 2015.

c. $$f(23)=31.87e^{0.096(23)}$$
$$\approx \$289.937 \text{ million}$$
$$f'(23)=3.05952e^{0.096(23)}\approx 27.834$$

The painting is appreciating at about \$27.834 million per year in 2020.

36. $v'(t)=100,000e^{t/5}\left(\frac{1}{5}\right)=20,000e^{t/5}$

$v'(5)=20,000e\approx 54,366$

The painting will be appreciating at approximately \$54,366 per year.

37. a. $f(8)\approx 45$ m/sec

b. $f'(0)=10$ m/sec^2

c. $f(t) = 30$ when $t \approx 4$ sec

d. $f'(t) = 5$ when $t \approx 4$ sec

38. $v(9) = 65\left(1 - e^{-0.16 \cdot 9}\right) \approx 49.6$

$v'(t) = 65(-e^{-.16t})(-.16) = 10.4e^{-.16t}$

$v'(9) = 10.4e^{-1.44} \approx 2.46$

When $t = 9$ seconds, the speed is approximately 49.6 m/sec and the acceleration is approximately 2.46 m/sec^2.

39. $f'(t) = (-1)\left(0.05 + e^{-0.4t}\right)^{-2}\left(e^{-0.4t}\right)(-0.4)$

$$= \frac{0.4e^{-0.4t}}{\left(0.05 + e^{-0.4t}\right)^2}$$

$$f'(7) = \frac{.4e^{-2.8}}{(.05 + e^{-2.8})^2} \approx 2$$

The rate of growth is about 2 inches per week.

40. a. $f'(10) \approx 2$ cm/week

b. $f(t) = 10$ when $t \approx 5$ weeks.

c. $f'(t) = 2$ when $t \approx 3$ weeks and $t \approx 10$ weeks.

d. The maximum rate of growth appears to be approximately 4 cm/week.

41. $y = e^{-2e^{-0.01x}}$

$$\frac{dy}{dx} = e^{-2e^{-0.01x}}\left[-2e^{-0.01x}(-.01)\right]$$

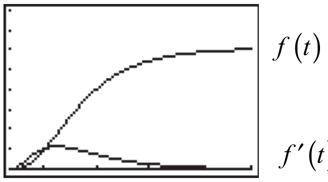
$$= .02e^{-2e^{-0.01x}}e^{-0.01x}$$

42. $y = e^{-(1/10)e^{-x/2}}$

$$\frac{dy}{dx} = e^{-(1/10)e^{-x/2}}\left(-\frac{1}{10}\right)\left(e^{-x/2}\right)\left(-\frac{1}{2}\right)$$

$$= \frac{1}{20}e^{-(1/10)e^{-x/2}}e^{-x/2}$$

43. a.



[1, 15] by [0, 8]

The tumor's volume appears to stabilize around 6 ml.

b. $f(5) \approx 3.2$ ml

c. $f(t) = 5$ when $t \approx 7.7$ weeks

d. $f'(5) \approx 0.97$ ml/week

e. $f'(t)$ is maximum when $t \approx 3.74$ weeks

f. The maximum growth rate appears to be approximately 1.13 ml/week.

44. a. $f(t) = 11$ when $t \approx 8$ weeks.

b. $f'(t) = 1$ when $t \approx 3.1$ weeks and $t \approx 11.9$ weeks.

c. $f'(t)$ is maximum when $t \approx 7.5$ weeks and the maximum rate of growth appears to be 2 in./week.

4.4 The Natural Logarithm Function

1. $\ln(\sqrt{e}) = \ln e^{1/2} = \frac{1}{2}$

2. $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$

3. $e^x = 5 \Rightarrow \ln(e^x) = \ln(5) \Rightarrow x = \ln 5$

4. $e^{-x} = 3.2 \Rightarrow \ln(e^{-x}) = \ln(3.2) \Rightarrow$
$-x = \ln(3.2) \Rightarrow x = -\ln 3.2$

5. $\ln x = -1 \Rightarrow e^{\ln x} = e^{-1} \Rightarrow x = \frac{1}{e}$

6. $\ln x = 4.5 \Rightarrow e^{\ln x} = e^{4.5} \Rightarrow x = e^{4.5}$

7. $\ln e^{-3} = -3$

8. $e^{\ln 4.1} = 4.1$

9. $e^{e^{\ln 1}} = e^1 = e$

10. $\ln\left(e^{-2\ln e}\right) = -2\ln e = -2$

11. $\ln(\ln e) = \ln(1) = 0$

12. $e^{4\ln 1} = e^{4(0)} = 1$

13. $e^{2\ln x} = \left(e^{\ln x}\right)^2 = (x)^2 = x^2$

14. $e^{x\ln 2} = \left(e^{\ln 2}\right)^x = (2)^x = 2^x$

15. $e^{-2\ln 7} = \left(e^{\ln 7}\right)^{-2} = (7)^{-2} = \frac{1}{49}$

16. $\ln\left(e^{-2}e^4\right) = \ln\left(e^2\right) = 2$

17. $e^{\ln x+\ln 2}=e^{\ln x}e^{\ln 2}=(x)(2)=2x$

18. $e^{\ln 3-2\ln x}=e^{\ln 3}e^{-2\ln x}=\dfrac{e^{\ln 3}}{e^{2\ln x}}=\dfrac{e^{\ln 3}}{\left(e^{\ln x}\right)^2}=\dfrac{3}{x^2}$

19. $e^{2x}=5\Rightarrow \ln(e^{2x})=\ln 5\Rightarrow 2x=\ln 5\Rightarrow$
$x=\dfrac{1}{2}\ln 5$

20. $e^{1-3x}=4\Rightarrow \ln(e^{1-3x})=\ln 4\Rightarrow$
$1-3x=\ln 4\Rightarrow x=\dfrac{\ln 4-1}{-3}=\dfrac{1-\ln 4}{3}$

21. $\ln(4-x)=\dfrac{1}{2}\Rightarrow e^{\ln(4-x)}=e^{1/2}\Rightarrow$
$4-x=e^{1/2}\Rightarrow x=4-e^{1/2}$

22. $\ln 3x=2\Rightarrow e^{\ln 3x}=e^2\Rightarrow 3x=e^2\Rightarrow x=\dfrac{1}{3}e^2$

23. $\ln x^2=9\Rightarrow e^{\ln x^2}=e^9\Rightarrow x^2=e^9\Rightarrow x=\pm e^{9/2}$

24. $e^{x^2}=25\Rightarrow \ln(e^{x^2})=\ln 25\Rightarrow x^2=\ln 25\Rightarrow$
$x=\pm\sqrt{\ln 25}$

25. $6e^{-.00012x}=3\Rightarrow e^{-.00012x}=\dfrac{1}{2}\Rightarrow$
$\ln(e^{-.00012x})=\ln\left(\dfrac{1}{2}\right)\Rightarrow -.00012x=\ln\dfrac{1}{2}\Rightarrow$
$x=\dfrac{\ln\frac{1}{2}}{-.00012}=-\dfrac{\ln .5}{.00012}$

26. $4-\ln x=0\Rightarrow 4=\ln x\Rightarrow e^4=e^{\ln x}\Rightarrow x=e^4$

27. $\ln 3x=\ln 5\Rightarrow e^{\ln 3x}=e^{\ln 5}\Rightarrow 3x=5\Rightarrow x=\dfrac{5}{3}$

28. $\ln(x^2-5)=0\Rightarrow e^{\ln(x^2-5)}=e^0\Rightarrow$
$x^2-5=1\Rightarrow x=\pm\sqrt{6}$

29. $\ln(\ln 3x)=0\Rightarrow e^{\ln(\ln 3x)}=e^0\Rightarrow \ln 3x=1\Rightarrow$
$e^{\ln 3x}=e^1\Rightarrow 3x=e\Rightarrow x=\dfrac{e}{3}$

30. $2\ln x=7\Rightarrow \ln x=\dfrac{7}{2}\Rightarrow e^{\ln x}=e^{7/2}\Rightarrow x=e^{7/2}$

31. $2e^{x/3}-9=0\Rightarrow e^{x/3}=\dfrac{9}{2}\Rightarrow$
$\ln(e^{x/3})=\ln\left(\dfrac{9}{2}\right)\Rightarrow x=3\ln\left(\dfrac{9}{2}\right)$

32. $e^{\sqrt{x}}=\sqrt{e^x}\Rightarrow e^{\sqrt{x}}=\left(e^x\right)^{1/2}=e^{x/2}\Rightarrow$
$\sqrt{x}=\dfrac{1}{2}x\Rightarrow x=\dfrac{1}{4}x^2\Rightarrow 4x=x^2\Rightarrow$
$x^2-4x=0\Rightarrow x(x-4)=0\Rightarrow x=0$ or $x=4$

33. $5\ln 2x=8\Rightarrow \ln 2x=\dfrac{8}{5}\Rightarrow e^{\ln 2x}=e^{8/5}\Rightarrow$
$2x=e^{8/5}\Rightarrow x=\dfrac{1}{2}e^{8/5}$

34. $750e^{-.4x}=375\Rightarrow e^{-.4x}=\dfrac{1}{2}\Rightarrow$
$\ln(e^{-.4x})=\ln\left(\dfrac{1}{2}\right)\Rightarrow -.4x=\ln\left(\dfrac{1}{2}\right)\Rightarrow$
$x=-\dfrac{\ln\left(\frac{1}{2}\right)}{.4}$ or $x=-\dfrac{(-\ln 2)}{.4}=\dfrac{\ln 2}{.4}$

35. $\left(e^2\right)^x\cdot e^{\ln 1}=4\Rightarrow e^{2x}\cdot 1=4\Rightarrow$
$\ln(e^{2x})=\ln 4\Rightarrow 2x=\ln 4\Rightarrow x=\dfrac{1}{2}\ln 4$

36. $e^{5x}\cdot e^{\ln 5}=2\Rightarrow e^{5x}\cdot 5=2\Rightarrow$
$\ln(e^{5x})=\ln\left(\dfrac{2}{5}\right)\Rightarrow 5x=\ln\left(\dfrac{2}{5}\right)\Rightarrow x=\dfrac{1}{5}\ln\left(\dfrac{2}{5}\right)$

37. $4e^x\cdot e^{-2x}=6\Rightarrow e^x\cdot e^{-2x}=\dfrac{3}{2}\Rightarrow e^{x-2x}=\dfrac{3}{2}\Rightarrow$
$\ln(e^{-x})=\ln\left(\dfrac{3}{2}\right)\Rightarrow -x=\ln\left(\dfrac{3}{2}\right)\Rightarrow x=-\ln\dfrac{3}{2}$

38. $(e^x)^2\cdot e^{2-3x}=4\Rightarrow e^{2x+2-3x}=4\Rightarrow$
$\ln(e^{2-x})=\ln 4\Rightarrow 2-x=\ln 4\Rightarrow x=2-\ln 4$

39. $f(x)=-5x+e^x;\ f'(x)=-5+e^x;\ f''(x)=e^x$
$f'(x)=0\Rightarrow -5+e^x=0\Rightarrow \ln(e^x)=\ln 5\Rightarrow$
$x=\ln 5$. Thus $f'(\ln 5)=0$ and the y-coordinate is $y=f(\ln 5)=-5\ln 5+e^{\ln 5}$ $=-5\ln 5+5=5-5\ln 5.$ Using the second derivative test, $f''(\ln 5)=e^{\ln 5}=5>0,$ so (ln 5, 5 –5 ln5) is the minimum.

40. $f(x)=-5x+e^x;\ f'(x)=-5+e^x;\ f''(x)=e^x$
Because $e^x>0$ for all values of x, the graph is always concave up.

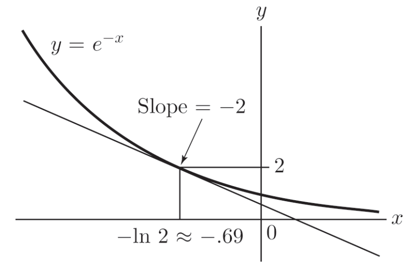
41. a. $f(x) = -5x + e^x$; $f'(x) = -5 + e^x$;
$f''(x) = e^x$
Solve $f'(x) = 3$ to find the x-coordinate of the point(s) on the graph where the tangent line has slope 3.
$-5 + e^x = 3 \Rightarrow e^x = 8 \Rightarrow x = \ln 8 \approx 2.08$

b. Solve $f'(x) = -7$ to find the x-coordinate of the point(s) on the graph where the tangent line has slope 3.
$-5 + e^x = -7 \Rightarrow e^x = -2$
This equation has no solution, so there are no points on the graph where the tangent has slope –7.

42. $f(x) = -1 + (x-1)^2 e^x$
$f'(x) = (x-1)^2 e^x + 2(x-1)e^x$
$= (x-1)(x+1)e^x$
$f''(x) = (x-1)(x+1)e^x + (x+1)e^x + (x-1)e^x$
$= (x^2 + 2x - 1)e^x$
$f'(x) = 0 \Rightarrow (x-1)(x+1)e^x = 0 \Rightarrow$
$(x-1)(x+1) = 0 \Rightarrow x = \pm 1$
When $x = 1$ the y-coordinate is
$f(1) = -1 + (1-1)^2 e^1 = -1.$ Using the second derivative test,
$f''(1) = \left(1^2 + 2(1) - 1\right)e^1 = 2e > 0,$ so (1, –1) is a relative minimum.
When $x = -1$, the y-coordinate is
$f(-1) = -1 + (-1-1)^2 e^{-1} = \dfrac{4}{e} - 1.$
$f''(-1) = \left((-1)^2 + 2(-1) - 1\right)e^{-1} = -\dfrac{2}{e} < 0,$ so
$\left(-1, \dfrac{4}{e} - 1\right)$ is a relative maximum.

43. a. $y = e^{-x}$
$\dfrac{dy}{dx} = -e^{-x} \Rightarrow -e^{-x} = -2 \Rightarrow e^{-x} = 2 \Rightarrow$
$\ln(e^{-x}) = \ln 2 \Rightarrow -x = \ln 2 \Rightarrow$
$x = -\ln 2 \Rightarrow x = \ln \dfrac{1}{2}$
$y = e^{-(-\ln 2)} = 2$
The tangent line has slope –2 at $\left(\ln \frac{1}{2}, 2\right)$.

b.



44. Solve $(x-1)^2 \ln(x+1) = 0$ for $x > -1$.
$(x-1)^2 = 0 \Rightarrow x = 1$
$\ln(x+1) = 0 \Rightarrow e^{\ln(x+1)} = e^0 \Rightarrow x + 1 = 1 \Rightarrow$
$x = 0$
x-intercepts: $(1, 0), (0, 0)$

45. $f(x) = e^{-x} + 3x$; $f'(x) = -e^{-x} + 3$;
$f''(x) = e^{-x}$
$f'(x) = 0 \Rightarrow -e^{-x} + 3 = 0 \Rightarrow \ln(e^{-x}) = \ln 3 \Rightarrow$
$-x = \ln 3$; $x = -\ln 3$. Thus $f'(-\ln 3) = 0$ and the y-coordinate is $f(-\ln 3) = e^{\ln 3} - 3\ln$
$= 3 - 3\ln 3$. Using the second derivative test,
$f''(-\ln 3) = e^{\ln 3} = 3 > 0$, so $(-\ln 3, 3 - 3\ln 3)$ is a relative minimum

46. $f(x) = 5x - 2e^x$; $f'(x) = 5 - 2e^x$; $f''(x) = -2e^x$
$f'(x) = 0 \Rightarrow 5 - 2e^x = 0 \Rightarrow 2e^x = 5; e^x = \dfrac{5}{2} \Rightarrow$
$\ln(e^x) = \ln\left(\dfrac{5}{2}\right) \Rightarrow x = \ln\left(\dfrac{5}{2}\right)$
$f\left(\ln \dfrac{5}{2}\right) = 5\ln\left(\dfrac{5}{2}\right) - 2e^{\ln(5/2)} = 5\ln\left(\dfrac{5}{2}\right) - 5$
$= 5\left(\ln \dfrac{5}{2} - 1\right)$
Using the second derivative test,
$f''\left(\ln \dfrac{5}{2}\right) = -2e^{\ln(5/2)} = -5 < 0$, so
$\left(\ln \dfrac{5}{2}, 5\left(\ln \dfrac{5}{2} - 1\right)\right)$ is a relative maximum.

47. $e^{0.05t} - 4e^{-0.06t} = 0 \Rightarrow e^{0.05t} = 4e^{-0.06t} \Rightarrow$
$\ln\left(e^{0.05t}\right) = \ln\left(4e^{-0.06t}\right) \Rightarrow$
$.05t = \ln 4 - .06t \Rightarrow .11t = \ln 4 \Rightarrow$
$t = \dfrac{\ln 4}{.11} \approx 12.6$

48. $4e^{0.01t} - 3e^{0.04t} = 0 \Rightarrow 4e^{0.01t} = 3e^{0.04t} \Rightarrow$
$\ln\left(4e^{0.01t}\right) = \ln\left(3e^{0.04t}\right) \Rightarrow$
$\ln 4 + .01t = \ln 3 + .04t$
$\ln 4 - \ln 3 = .03t \Rightarrow t = \dfrac{\ln 4 - \ln 3}{.03} \approx 9.59$

49. $f(t) = 5(e^{-.01t} - e^{-.51t})$, for $t \ge 0$.
$f'(t) = 5(-.01e^{-.01t} + .51e^{-.51t})$
$f'(t) = 0 \Rightarrow 5(-.01e^{-.01t} + .51e^{-.51t}) = 0 \Rightarrow$
$-e^{-.01t} + 51e^{-.51t} = 0 \Rightarrow 51e^{-.51t} = e^{-.01t} \Rightarrow$
$51e^{-.51t} \cdot e^{.51t} = e^{-.01t} \cdot e^{.51t} \Rightarrow 51(1) = e^{.5t} \Rightarrow$
$\ln 51 = \ln(e^{.5t}) \Rightarrow \ln 51 = .5t \Rightarrow t = 2\ln 51$.
Thus the maximum must occur at $t = 2 \ln 51 \approx 7.86$.

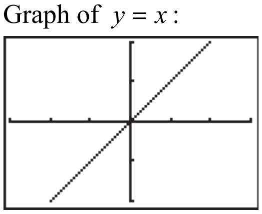
50. a. $v = K\ln\left(\dfrac{x}{x_0}\right) = 0$
$\ln\left(\dfrac{x}{x_0}\right) = 0 \Rightarrow e^{\ln(x/x_0)} = e^0 = 1 \Rightarrow$
$\dfrac{x}{x_0} = 1 \Rightarrow x = x_0 = .7$ cm

b. $v = K\ln\left(\dfrac{x}{x_0}\right) = 1200 \Rightarrow \ln\left(\dfrac{x}{x_0}\right) = \dfrac{1200}{K} \Rightarrow$
$e^{\ln(x/x_0)} = e^{1200/K} \Rightarrow \dfrac{x}{x_0} = e^{1200/K} \Rightarrow$
$x = .7e^{1200/300} = .7e^4 \approx 38.2$ cm

51. Using $b^x = e^{kx}$ where $k = \ln b$, we have $k = \ln 2$.

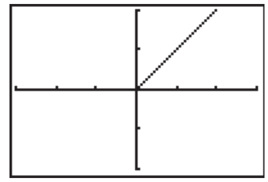
52. $2^{-x/5} = \left(2^{-1/5}\right)^x = \left(\dfrac{1}{2^{1/5}}\right)^x = \left(\dfrac{1}{\sqrt[5]{2}}\right)^x$, so
$\left(\dfrac{1}{\sqrt[5]{2}}\right)^x = e^{kx}$.
Using $b^x = e^{kx}$ where $k = \ln b$, we have
$k = \ln\left(\dfrac{1}{\sqrt[5]{2}}\right) = -\ln\sqrt[5]{2} = -\dfrac{1}{5}\ln 2$.

53. Graph of $y = x$:



$[-3, 3]$ by $[-2, 2]$

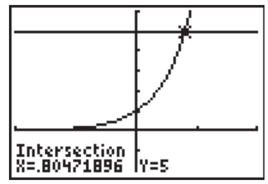
Graph of $y = e^{\ln x}$:



$[-3, 3]$ by $[-2, 2]$

The graph of $y = e^{\ln x}$ is the same as the graph of $y = x$ for $x > 0$.

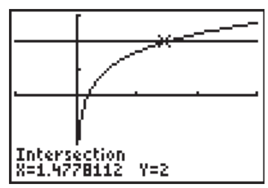
54.



$[-2, 2]$ by $[-2, 6]$

Using the graph, the x-coordinate of their point of intersection is approximately .8047.
$e^{2x} = 5 \Rightarrow \ln(e^{2x}) = \ln 5 \Rightarrow 2x = \ln 5 \Rightarrow$
$x = \dfrac{1}{2}\ln 5 \approx .8047$

55.



$[-1, 3]$ by $[-3, 3]$

Using the graph, the x-coordinate of their point of intersection is approximately 1.4778.
$\ln 5x = 2 \Rightarrow e^{\ln 5x} = e^2 \Rightarrow 5x = e^2 \Rightarrow$
$x = \dfrac{1}{5}e^2 \approx 1.4778$

4.5 The Derivative of ln *x*

1. $\dfrac{d}{dx}\left(3\ln x + \ln 2\right) = \dfrac{d}{dx}\left(3\ln x\right) + \dfrac{d}{dx}\ln 2 = \dfrac{3}{x}$

2. $\dfrac{d}{dx}\left(\dfrac{\ln x}{\ln 3}\right) = \dfrac{d}{dx}\left(\dfrac{1}{\ln 3}\ln x\right) = \dfrac{1}{\ln 3}\cdot\dfrac{1}{x} = \dfrac{1}{x\ln 3}$

3. $\frac{d}{dx}\left(\frac{x^2 \ln x}{2}\right) = \frac{1}{2}\frac{d}{dx}\left(x^2 \ln x\right)$

Use the product rule.

$$\frac{d}{dx}\left(\frac{x^2 \ln x}{2}\right) = \frac{1}{2}\frac{d}{dx}\left(x^2 \ln x\right) = \frac{1}{2}\left[\ln x \frac{d}{dx}x^2 + x^2 \frac{d}{dx}\ln x\right] = \frac{1}{2}\left(2x \ln x + \frac{x^2}{x}\right) = x \ln x + \frac{x}{2} = x\left(\ln x + \frac{1}{2}\right)$$

4. $$\frac{d}{dx}\left(\frac{3\ln x}{x}\right) = 3\frac{d}{dx}\left(\frac{\ln x}{x}\right) = 3\left(\frac{x\frac{d}{dx}\ln x - \ln x \frac{d}{dx}x}{x^2}\right) = 3\left(\frac{x\left(\frac{1}{x}\right) - \ln x}{x^2}\right) = 3\left(\frac{1-\ln x}{x^2}\right)$$

5. $\frac{d}{dx}\left(e^x \ln x\right)$

Use the product rule.

$$\frac{d}{dx}\left(e^x \ln x\right) = \ln x \frac{d}{dx}e^x + e^x \frac{d}{dx}\ln x = e^x \ln x + e^x\left(\frac{1}{x}\right) = e^x\left(\ln x + \frac{1}{x}\right)$$

6. $$\frac{d}{dx}e^{1+\ln x} = e^{1+\ln x}\frac{d}{dx}(1+\ln x) = e^{1+\ln x}\left(\frac{1}{x}\right) = \frac{e^{1+\ln x}}{x}$$

7. $\frac{d}{dx}\frac{\ln x}{\sqrt{x}} = \frac{d}{dx}\frac{\ln x}{x^{1/2}}$

Use the quotient rule.

$$\frac{d}{dx}\frac{\ln x}{\sqrt{x}} = \frac{d}{dx}\frac{\ln x}{x^{1/2}} = \frac{x^{1/2}\frac{d}{dx}\ln x - \ln x\frac{d}{dx}\left(x^{1/2}\right)}{\left(x^{1/2}\right)^2} = \frac{x^{1/2}\left(\frac{1}{x}\right) - \frac{1}{2}\ln x\left(x^{-1/2}\right)}{x} = \frac{x^{-1/2}}{x} - \frac{\ln x\left(x^{-1/2}\right)}{2x} = \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x\sqrt{x}} = \frac{1}{x\sqrt{x}}\left(1 - \frac{\ln x}{2}\right)$$

8. $$\frac{d}{dx}\frac{1}{2+3\ln x} = \frac{d}{dx}(2+3\ln x)^{-1} = -(2+3\ln x)^{-2}\frac{d}{dx}(2+3\ln x) = -(2+3\ln x)^{-2}\left(\frac{3}{x}\right) = -\frac{3}{x(2+3\ln x)^2}$$

9. $\frac{d}{dx}\ln x^2 = \frac{d}{dx}(2\ln x) = 2\frac{d}{dx}\ln x = \frac{2}{x}$

10. $$\frac{d}{dx}\ln\sqrt{x} = \frac{d}{dx}\ln x^{1/2} = \frac{d}{dx}\left(\frac{1}{2}\ln x\right) = \frac{1}{2}\frac{d}{dx}\ln x = \frac{1}{2}\left(\frac{1}{x}\right) = \frac{1}{2x}$$

11. $$\frac{d}{dx}\ln\frac{1}{x} = \frac{d}{dx}\ln x^{-1} = \frac{d}{dx}(-\ln x) = -\frac{d}{dx}\ln x = -\frac{1}{x}$$

12. $$\frac{d}{dx}\ln\frac{1}{x^2} = \frac{d}{dx}\ln x^{-2} = \frac{d}{dx}(-2\ln x) = -2\frac{d}{dx}\ln x = -\frac{2}{x}$$

13. $$\frac{d}{dx}\ln\left(3x^4 - x^2\right) = \frac{1}{3x^4 - x^2}\frac{d}{dx}\left(3x^4 - x^2\right) = \frac{1}{3x^4 - x^2}\left(12x^3 - 2x\right) = \frac{12x^3 - 2x}{3x^4 - x^2} = \frac{12x^2 - 2}{3x^3 - x}$$

14. $\frac{d}{dx}\left[\ln\left(e^x+e^{-x}\right)\right]=\frac{1}{e^x+e^{-x}}\left(e^x+e^{-x}(-1)\right)$
$=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

15. $\frac{d}{dx}\left(\frac{1}{\ln x}\right)=\frac{d}{dx}(\ln x)^{-1}=-(\ln x)^{-2}\frac{d}{dx}(\ln x)$
$=-(\ln x)^{-2}\left(\frac{1}{x}\right)=-\frac{1}{x(\ln x)^2}$

16. $\frac{d}{dx}(\ln x\cdot\ln 2x)$
Use the product rule.
$\frac{d}{dx}(\ln x\cdot\ln 2x)=\ln 2x\frac{d}{dx}\ln x+\ln x\frac{d}{dx}\ln 2x$
$=\ln 2x\left(\frac{1}{x}\right)+\ln x\left(\frac{1}{2x}(2)\right)$
$=\frac{\ln 2x+\ln x}{x}$

17. $\frac{d}{dx}\left(\frac{\ln x}{\ln 2x}\right)$
Use the quotient rule
$\frac{d}{dx}\left(\frac{\ln x}{\ln 2x}\right)=\frac{\ln 2x\frac{d}{dx}\ln x-\ln x\frac{d}{dx}\ln 2x}{(\ln 2x)^2}$
$=\frac{\ln 2x\left(\frac{1}{x}\right)-\ln x\left(\frac{2}{2x}\right)}{(\ln 2x)^2}$
$=\frac{\ln 2x-\ln x}{x(\ln 2x)^2}$

18. $\frac{d}{dx}(\ln x)^2=2(\ln x)\frac{d}{dx}\ln x=2\ln x\left(\frac{1}{x}\right)$
$=\frac{2\ln x}{x}$

19. $\frac{d}{dx}\left[\left(x^3+1\right)\ln\left(x^3+1\right)\right]$
Use the product rule.
$\frac{d}{dx}\left[\left(x^3+1\right)\ln\left(x^3+1\right)\right]$
$=\ln\left(x^3+1\right)\frac{d}{dx}\left(x^3+1\right)+\left(x^3+1\right)\frac{d}{dx}\ln\left(x^3+1\right)$
$=3x^2\ln\left(x^3+1\right)+\left(x^3+1\right)\frac{1}{x^3+1}\left(3x^2\right)$
$=3x^2\ln\left(x^3+1\right)+3x^2$
$=3x^2\left[\ln\left(x^3+1\right)+1\right]$

20. $\frac{d}{dx}\left[\frac{\ln\left(x^2+1\right)}{x^2+1}\right]$
Use the quotient rule
$\frac{d}{dx}\left[\frac{\ln\left(x^2+1\right)}{x^2+1}\right]$
$=\frac{\left(x^2+1\right)\frac{d}{dx}\ln\left(x^2+1\right)-\ln\left(x^2+1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$
$=\frac{\left(x^2+1\right)\left(\frac{1}{x^2+1}\right)(2x)-2x\ln\left(x^2+1\right)}{\left(x^2+1\right)^2}$
$=\frac{2x-2x\ln\left(x^2+1\right)}{\left(x^2+1\right)^2}=\frac{2x\left[1-\ln\left(x^2+1\right)\right]}{\left(x^2+1\right)^2}$

21. $\frac{d}{dt}\left[t^2\ln t\right]=t^2\left(\frac{1}{t}\right)+2t\ln t=t+2t\ln t$
$\frac{d^2}{dt^2}\left[t^2\ln t\right]=\frac{d}{dt}\left[t+2t\ln t\right]$
$=1+2t\left(\frac{1}{t}\right)+2\ln t$
$=3+2\ln t$

22. $\frac{d}{dt}\ln(\ln t)=\frac{1}{t\ln t}$
$\frac{d^2}{dt^2}\ln(\ln t)=\frac{d}{dt}\left(\frac{1}{t\ln t}\right)$
$=\frac{t\ln t(0)-(1)(1+\ln t)}{(t\ln t)^2}=\frac{-1-\ln t}{(t\ln t)^2}$

23. $f'(x)=\frac{\sqrt{x}\cdot\frac{1}{x}-(\ln x)\left(\frac{1}{2}x^{-1/2}\right)}{(\sqrt{x})^2}$
$=\frac{x^{-1/2}-\frac{1}{2}x^{-1/2}\ln x}{x}=\frac{2-\ln x}{2x^{3/2}}$
$f'(x)=0$ when $2-\ln x=0$ or
$x=e^2\approx 7.389$.
From the graph, we see that this value gives a maximum.
$f(e^2)=\frac{\ln e^2}{\sqrt{e^2}}=\frac{2}{e}$
The coordinates of the maximum point are $\left(e^2,\frac{2}{e}\right)$.

24. $f'(x)=\dfrac{(\ln x+x)(1)-x\left(\frac{1}{x}+1\right)}{(\ln x+x)^2}$
$=\dfrac{\ln x+x-1-x}{(\ln x+x)^2}=\dfrac{\ln x-1}{(\ln x+x)^2}$

$f'(x)=0$ when $\ln x-1=0$ or $x=e\approx 2.7183$

From the graph, we see that this value gives a minimum.

$f(e)=\dfrac{e}{\ln e+e}=\dfrac{e}{1+e}$

The coordinates of the minimum point are $\left(e,\dfrac{e}{1+e}\right)$.

25. $\dfrac{dy}{dx}=\dfrac{1}{x^2+e}(2x)=\dfrac{2x}{x^2+e},\ \left.\dfrac{dy}{dx}\right|_{x=0}=\dfrac{0}{e}=0$

At $x=0$, $y=\ln(0^2+e)=1$. Thus the tangent line at $x=0$ is $y-1=0(x-0)$ or $y=1$.

26. $f(x)=\dfrac{\ln x+1}{x}$

$f'(x)=\dfrac{x\left(\frac{1}{x}\right)-(\ln x+1)(1)}{x^2}=\dfrac{-\ln x}{x^2}$

$f''(x)=\dfrac{x^2\left(-\frac{1}{x}\right)-(-\ln x)(2x)}{x^4}=\dfrac{-x+2x\ln x}{x^4}$

For $x>0$, the only critical point is at $x=1$, $f(1)=1$. Using the second derivative test,

$f''(1)=\dfrac{-1+2(1)\ln(1)}{1^4}=-1<0$, so (1, 1) is a relative maximum.

27. (a) $f(t)=\ln(\ln t)$

The domain of the inner function, $\ln t$, is $(0,\infty)$, so the domain of the outer function is $(1,\infty)$ or $t>1$.

(b) $f(t)=\ln(\ln(\ln t))$

From (a), we have the domain of the inner function $\ln(\ln t)$ is $(1,\infty)$, so the domain of the outer function is (e,∞) or $t>e$.

28. $\dfrac{dy}{dx}=\dfrac{1}{x};\ \left.\dfrac{dy}{dx}\right|_{x=1}=\dfrac{1}{1}=1.$

At $x=1$, $y=\ln|1|=0$. Thus the tangent line at $x=1$ is $y-0=1(x-1)$ or $y=x-1$.

Next, $\left.\dfrac{dy}{dx}\right|_{x=-1}=\dfrac{1}{-1}=-1.$

At $x=-1$, $y=\ln|-1|=0$. Thus the tangent line at $x=-1$ is $y-0=-1(x-(-1))$ or $y=-x-1$.

29. $y=x^2\ln x$

$\dfrac{dy}{dx}=x^2\left(\dfrac{1}{x}\right)+2x\ln x=x+2x\ln x$

$\dfrac{dy}{dx}=0\Rightarrow x+2x\ln x=x(1+2\ln x)=0\Rightarrow$
$x=0$ or

$1+2\ln x=0\Rightarrow \ln x=-\dfrac{1}{2}\Rightarrow x=e^{-1/2}$

$\dfrac{d^2y}{dx^2}=1+2x\left(\dfrac{1}{x}\right)+2\ln x=3+2\ln x$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=e^{-\frac{1}{2}}}=3+2\ln\left(e^{-1/2}\right)=3+2\left(-\dfrac{1}{2}\right)=2$

Since $\left.\dfrac{d^2y}{dx^2}\right|_{x=e^{-1/2}}>0$, there is a relative minimum at $x=e^{-1/2}$.

The relative minimum occurs at $\left(e^{-1/2},-\dfrac{1}{2}e^{-1}\right)$ or $\left(\dfrac{1}{\sqrt{e}},-\dfrac{1}{2e}\right)$.

30. $y=\sqrt{x}\ln x$

$\dfrac{dy}{dx}=\sqrt{x}\left(\dfrac{1}{x}\right)+\ln x\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)$
$=x^{-1/2}+\left(\dfrac{1}{2}x^{-1/2}\right)\ln x$
$=x^{-1/2}\left(1+\dfrac{1}{2}\ln x\right)$

$\dfrac{dy}{dx}=0\Rightarrow 1+\dfrac{1}{2}\ln x=0\Rightarrow \ln x=-2\Rightarrow x=e^{-2}$

$\dfrac{d^2y}{dx^2}$
$=-\dfrac{1}{2}x^{-3/2}+\dfrac{1}{2}x^{-1/2}\left(\dfrac{1}{x}\right)+\left(-\dfrac{1}{4}\right)x^{-3/2}\ln x$
$=-\dfrac{1}{4}x^{-3/2}\ln x$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=e^{-2}}=-\dfrac{1}{4}\left(e^{-2}\right)^{-3/2}\ln e^{-2}=\dfrac{1}{2}e^3$

Since $\left.\dfrac{d^2y}{dx^2}\right|_{x=e^{-2}}>0$, there is a relative minimum at $x=e^{-2}$.

The relative minimum occurs at $\left(e^{-2},-2e^{-1}\right)$.

31. a. $y = x + \ln x$

$y' = \frac{dy}{dx} = 1 + \frac{1}{x}$

The first derivative is positive for all values of $x > 0$, so the function is increasing.

$y = \ln 2x$

$y' = \frac{1}{2x}\frac{dy}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

The first derivative is positive for all values of $x > 0$, so the function is increasing.

b. $x + \ln x = \ln 2x \Rightarrow x = \ln 2x - \ln x \Rightarrow$

$x = \ln\frac{2x}{x} \Rightarrow x = \ln 2$

The graphs intersect at $x = \ln 2$. This is the point (ln 2, ln(2 ln 2)) or (0.6931, 0.3266).

32. a. $y = x + \ln x$

$y' = \frac{dy}{dx} = 1 + \frac{1}{x}$

The first derivative is positive for all values of $x > 0$, so the function is increasing.

$y = \ln 5x$

$y' = \frac{1}{5x}\frac{dy}{dx}(5x) = \frac{1}{5x}(5) = \frac{1}{x}$

The first derivative is positive for all values of $x > 0$, so the function is increasing.

b. $x + \ln x = \ln 5x \Rightarrow x = \ln 5x - \ln x \Rightarrow$

$x = \ln\frac{5x}{x} \Rightarrow x = \ln 5$

The graphs intersect at $x = \ln 2$. This is the point (ln 5, ln(5 ln 5)) or (1.6094, 2.0853).

33. $y = x^2 - \ln x;\ y' = 2x - \frac{1}{x}$

Solve $y' = 0$ to find the x-coordinate of the minimum point.

$2x - \frac{1}{x} = 0 \Rightarrow 2x = \frac{1}{x} \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Disregard the negative solution. The minimum occurs when $x = \frac{1}{\sqrt{2}} \approx 0.7071$.

$y = \left(\frac{1}{\sqrt{2}}\right)^2 - \ln\frac{1}{\sqrt{2}} = \frac{1}{2} + \frac{1}{2}\ln 2$

The minimum point is $\left(\frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{1}{2}\ln 2\right)$.

34. $y = 2x^2 - \ln 4x;\ y' = 4x - \frac{1}{x}$

Solve $y' = 0$ to find the x-coordinate of the minimum point.

$4x - \frac{1}{x} = 0 \Rightarrow 4x = \frac{1}{x} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{2}$

The minimum occurs when $x = \frac{1}{2}$.

35. The revenue function is $R(x) = x \cdot \frac{45}{\ln x} = \frac{45x}{\ln x}$.

The marginal revenue function is

$R'(x) = \frac{45\ln x - 45x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{45(\ln x - 1)}{(\ln x)^2}$.

When $x = 20$, $R'(20) = \frac{45(\ln 20 - 1)}{(\ln 20)^2} \Rightarrow$

$R'(20) \approx 10$.

36. We wish to maximize $R(x) - C(x)$.

$$\frac{d}{dx}[R(x) - C(x)] = \frac{d}{dx}[300\ln(x+1) - 2x] = 300\left(\frac{1}{x+1}\right) - 2 = \frac{300}{x+1} - 2$$

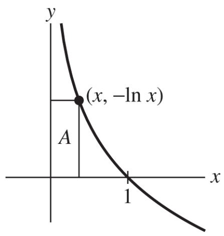
The only critical point occurs when $x = 149$. To show that this is a relative maximum, take the second derivative.

$$\frac{d^2}{dx^2}[R(x) - C(x)] = \frac{d}{dx}\left[\frac{300}{x+1} - 2\right] = \frac{-300}{(x+1)^2}$$

At $x = 149$, we have

$\frac{d^2}{dx^2}[R(x) - C(x)] = \frac{-300}{150^2} < 0$, so $x = 149$ is a relative maximum.

37.



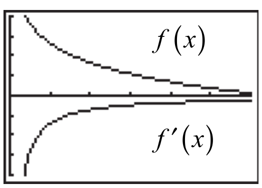
From the graph we see that area $= A = x(-\ln x) = -x \ln x$.
To maximize the area, take the first derivative.

$$A' = -x\left(\frac{1}{x}\right) + (\ln x)(-1) = -1 - \ln x$$

Now set $A' = 0$:
$-1 - \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow$
$e^{\ln x} = e^{-1} \Rightarrow x = e^{-1} \approx .36788$. Thus, area is maximized when $x \approx .36788$, and the maximum area is

$$-\left(e^{-1}\right)\ln\left(e^{-1}\right) = \frac{1}{e} \approx .36788 .$$

38. a. $f(x) = 26.48 - 14.09\ln x;\ f'(x) = \dfrac{-14.09}{x}$



[0, 6] by [−40, 40]

b. $f(3.25) \approx 10$ bites

c. $f(x) = 15$ when $x \approx 2.26\%$.
Algebraically: $15 = 26.48 - 14.09 \ln x$

$$\ln x = \frac{11.48}{14.09}$$

$$x = e^{11.48/14.09} \approx 2.26$$

d. $f'(2.75) \approx -5.12$ bites per percentage increase in concentration

e. $f'(x) = -10$ when $x = 1.409\%$
Algebraically:

$$-10 = \frac{-14.09}{x} \Rightarrow x = 1.409$$

4.6 Properties of the Natural Logarithm Function

1. $\ln 5 + \ln x = \ln(5x)$

2. $\ln x^5 - \ln x^3 = \ln\left(\dfrac{x^5}{x^3}\right) = \ln x^2 = 2\ln x$

3. $\dfrac{1}{2}\ln 9 = \ln 9^{1/2} = \ln 3$

4. $3\ln\dfrac{1}{2} + \ln 16 = \ln\left(\dfrac{1}{2}\right)^3 + \ln 16 = \ln\dfrac{1}{8} + \ln 16$
$= \ln\left(\dfrac{16}{8}\right) = \ln 2$

5. $\ln 4 + \ln 6 - \ln 12 = \ln\left(\dfrac{4\cdot 6}{12}\right) = \ln 2$

6. $\ln 2 - \ln x + \ln 3 = \ln\left(\dfrac{2\cdot 3}{x}\right) = \ln\dfrac{6}{x} = \ln 6 - \ln x$

7. $e^{2\ln x} = e^{\ln x^2} = x^2$

8. $\dfrac{3}{2}\ln 4 - 5\ln 2 = \ln 4^{3/2} - \ln 2^5 = \ln 2^3 - \ln 2^5$
$= \ln\left(\dfrac{2^3}{2^5}\right) = \ln 2^{-2} = -\ln 4$

9. $5\ln x - \dfrac{1}{2}\ln y + 3\ln z = \ln x^5 - \ln y^{1/2} + \ln z^3$
$= \ln\left(\dfrac{x^5 z^3}{y^{1/2}}\right)$

10. $e^{\ln x^2 + 3\ln y} = e^{\ln x^2 + \ln y^3} = e^{\ln(x^2 y^3)} = x^2 y^3$

11. $\ln x - \ln x^2 + \ln x^4 = \ln\left(\dfrac{x\cdot x^4}{x^2}\right) = \ln x^3 = 3\ln x$

12. $\dfrac{1}{2}\ln xy + \dfrac{3}{2}\ln\dfrac{x}{y} = \ln(xy)^{1/2} + \ln\left(\dfrac{x}{y}\right)^{3/2}$
$= \ln\left((xy)^{1/2}\cdot\left(\dfrac{x}{y}\right)^{3/2}\right)$
$= \ln\left(\dfrac{x^2}{y}\right)$

13. $2\ln 5 = \ln 5^2 = \ln 25, 3\ln 3 = \ln 3^3 = \ln 27$
The natural log function increases as x increases so $3 \ln 3 > 2 \ln 5$.

14. $\dfrac{1}{2}\ln 16 = \ln 16^{1/2} = \ln 4, \dfrac{1}{3}\ln 27 = \ln 27^{1/3} = \ln 3$
$\ln 4 > \ln 3$ so $\dfrac{1}{2}\ln 16 > \dfrac{1}{3}\ln 27$.

15. a. $\ln 4 = \ln 2^2 = 2\ln 2 = 2(.69) = 1.38$

b. $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3 = .69 + 1.1 = 1.79$

c. $\ln 54 = \ln(2\cdot 3^3) = \ln 2 + \ln 3^3$
$= \ln 2 + 3\ln 3 = .69 + 3(1.1) = 3.99$

16. a. $\ln 12 = \ln(3\cdot 2^2) = \ln 3 + \ln 2^2$
$= \ln 3 + 2\ln 2 = 1.1 + 2(.69) = 2.48$

b. $\ln 16 = \ln(2^4) = 4\ln 2 = 4(.69) = 2.76$

c. $\ln(9\cdot 2^4) = \ln(3^2\cdot 2^4) = 2\ln 3 + 4\ln 2$
$= 2(1.1) + 4(.69) = 4.96$

17. a. $\ln\frac{1}{6} = \ln 6^{-1} = -\ln 6 = -\ln(2\cdot 3)$
$= -(\ln 2 + \ln 3) = -(.69 + 1.1) = -1.79$

b. $\ln\frac{2}{9} = \ln 2 - \ln 9 = \ln 2 - \ln 3^2$
$= \ln 2 - 2\ln 3 = .69 - 2(1.1) = -1.51$

c. $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2 = -\frac{1}{2}(.69)$
$= -.345$

18. a. $\ln 100 - 2\ln 5 = \ln 100 - \ln 5^2 = \ln\left(\frac{100}{5^2}\right)$
$= \ln 4 = \ln 2^2 = 2\ln 2$
$= 2(.69) = 1.38$

b. $\ln 10 + \ln\frac{1}{5} = \ln\left(10\cdot\frac{1}{5}\right) = \ln 2 = .69$

c. $\ln\left(\sqrt{108}\right) = \ln 108^{\frac{1}{2}} = \frac{1}{2}\ln 108$
$= \frac{1}{2}\ln\left(2^2 3^3\right) = \frac{1}{2}\left(\ln 2^2 + \ln 3^3\right)$
$= \frac{1}{2}(2\ln 2 + 3\ln 3)$
$= \frac{1}{2}(2(.69) + 3(1.1)) = 2.34$

19. (d) $4\ln 2x = \ln(2x)^4 = \ln 16x^4$

20. (d) $\ln 9x - \ln 3x = \ln\left(\frac{9x}{3x}\right) = \ln 3$

21. (d) none of these

22. (c) $\ln 9x^2 = \ln(3x)^2 = 2\ln 3x$

23. $\ln x - \ln x^2 + \ln 3 = 0 \Rightarrow \ln\left(\frac{3x}{x^2}\right) = 0 \Rightarrow$
$\ln\left(\frac{3}{x}\right) = 0 \Rightarrow e^0 = \frac{3}{x} \Rightarrow 1 = \frac{3}{x} \Rightarrow x = 3$

24. $\ln\sqrt{x} - 2\ln 3 = 0 \Rightarrow \ln\sqrt{x} - \ln 3^2 = 0 \Rightarrow$
$\ln\left(\frac{\sqrt{x}}{3^2}\right) = 0 \Rightarrow e^0 = \frac{\sqrt{x}}{9} \Rightarrow 1 = \frac{\sqrt{x}}{9} \Rightarrow x = 81$

25. $\ln x^4 - 2\ln x = 1 \Rightarrow 4\ln x - 2\ln x = 1 \Rightarrow$
$2\ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$

26. $\ln x^2 - \ln 2x + 1 = 0 \Rightarrow \ln x^2 - \ln 2x = -1 \Rightarrow$
$\ln\left(\frac{x^2}{2x}\right) = -1 \Rightarrow \ln\left(\frac{x}{2}\right) = -1 \Rightarrow e^{-1} = \frac{x}{2} \Rightarrow$
$x = 2e^{-1} = \frac{2}{e}$

27. $(\ln x)^2 - 1 = 0 \Rightarrow (\ln x)^2 = 1 \Rightarrow \ln x = \pm 1 \Rightarrow$
$x = e^1$ or $x = e^{-1}$

28. $3\ln x - \ln 3x = 0 \Rightarrow \ln x^3 - \ln 3x = 0 \Rightarrow$
$\ln\left(\frac{x^3}{3x}\right) = 0 \Rightarrow \ln\left(\frac{x^2}{3}\right) = 0 \Rightarrow e^0 = \frac{x^2}{3} \Rightarrow$
$1 = \frac{x^2}{3} \Rightarrow x = \sqrt{3}$

29. $\ln\sqrt{x} = \sqrt{\ln x} \Rightarrow \ln\left(x^{1/2}\right) = \sqrt{\ln x} \Rightarrow$
$\frac{1}{2}\ln x = \sqrt{\ln x}$

Let $u = \ln x$. Then $\frac{1}{2}\ln x = \sqrt{\ln x} \Rightarrow$
$\frac{1}{2}u = \sqrt{u} \Rightarrow \frac{1}{4}u^2 = u \Rightarrow \frac{1}{4}u^2 - u = 0 \Rightarrow$
$u\left(\frac{1}{4}u - 1\right) = 0 \Rightarrow u = 0$ or $u = 4$
If $u = 0$, then $0 = \ln x \Rightarrow x = 1$. If $u = 4$, then $4 = \ln x \Rightarrow x = e^4$.

30. $2(\ln x)^2 + \ln x - 1 = 0;$ Let $X = \ln x.$

$2X^2 + X - 1 = 0 \Rightarrow X = \dfrac{-1 \pm \sqrt{1-4(2)(-1)}}{4} \Rightarrow X = -1,\ X = \dfrac{1}{2} \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \dfrac{1}{e}$ or

$\ln x = \dfrac{1}{2} \Rightarrow x = e^{1/2}$

31. $\ln(x+1) - \ln(x-2) = 1 \Rightarrow \ln\left(\dfrac{x+1}{x-2}\right) = 1 \Rightarrow e = \dfrac{x+1}{x-2} \Rightarrow e(x-2) = x+1 \Rightarrow ex - x = 2e+1 \Rightarrow$

$x(e-1) = 2e+1 \Rightarrow x = \dfrac{2e+1}{e-1}$

32. $\ln[(x-3)(x+2)] - \ln(x+2)^2 - \ln 7 = 0 \Rightarrow \ln\left[\dfrac{(x-3)(x+2)}{(x+2)^2}\right] = \ln 7 \Rightarrow \ln\left(\dfrac{x-3}{x+2}\right) = \ln 7 \Rightarrow$

$\dfrac{x-3}{x+2} = 7 \Rightarrow x - 3 = 7x + 14 \Rightarrow x = -\dfrac{17}{6}$

33. $\dfrac{d}{dx}\ln\left[(x+5)(2x-1)(4-x)\right] = \dfrac{d}{dx}\left[\ln(x+5) + \ln(2x-1) + \ln(4-x)\right] = \dfrac{1}{x+5}(1) + \dfrac{1}{2x-1}(2) + \dfrac{1}{4-x}(-1)$

$= \dfrac{1}{x+5} + \dfrac{2}{2x-1} - \dfrac{1}{4-x}$

34. $\dfrac{d}{dx}\ln\left[(x+1)(2x+1)(3x+1)\right] = \dfrac{d}{dx}\left[\ln(x+1) + \ln(2x+1) + \ln(3x+1)\right]$

$= \dfrac{1}{x+1} + \dfrac{1}{2x+1}(2) + \dfrac{1}{3x+1}(3) = \dfrac{1}{x+1} + \dfrac{2}{2x+1} + \dfrac{3}{3x+1}$

35. $\dfrac{d}{dx}\ln\left[(1+x)^2(2+x)^3(3+x)^4\right] = \dfrac{d}{dx}\left[\ln(1+x)^2 + \ln(2+x)^3 + \ln(3+x)^4\right]$

$= \dfrac{d}{dx}\left[2\ln(1+x) + 3\ln(2+x) + 4\ln(3+x)\right] = \dfrac{2}{1+x} + \dfrac{3}{2+x} + \dfrac{4}{3+x}$

36. $\dfrac{d}{dx}\ln\left[e^{2x}(x^3+1)(x^4+5x)\right] = \dfrac{d}{dx}\left[\ln e^{2x} + \ln(x^3+1) + \ln(x^4+5x)\right] = \dfrac{d}{dx}\left[2x + \ln(x^3+1) + \ln(x^4+5x)\right]$

$= 2 + \dfrac{1}{x^3+1}(3x^2) + \dfrac{1}{x^4+5x}(4x^3+5) = 2 + \dfrac{3x^2}{x^3+1} + \dfrac{4x^3+5}{x^4+5x}$

37. $\dfrac{d}{dx}\ln\left[\sqrt{xe^{x^2+1}}\right] = \dfrac{d}{dx}\left[\ln\left(xe^{x^2+1}\right)^{1/2}\right] = \dfrac{d}{dx}\left[\dfrac{1}{2}\ln\left(xe^{x^2+1}\right)\right]$

$= \dfrac{1}{2}\cdot\dfrac{1}{xe^{x^2+1}}\cdot\left(2x^2e^{x^2+1} + e^{x^2+1}\right) = x + \dfrac{1}{2x}$

38. $\dfrac{d}{dx}\ln\left[\dfrac{x+1}{x-1}\right] = \dfrac{d}{dx}\left[\ln(x+1) - \ln(x-1)\right] = \dfrac{1}{x+1} - \dfrac{1}{x-1} = \dfrac{x-1-(x+1)}{x^2-1} = -\dfrac{2}{x^2-1}$

39. $\dfrac{d}{dx}\ln\left[\dfrac{(x+1)^4}{e^{x-1}}\right] = \dfrac{d}{dx}\left[\ln(x+1)^4 - \ln\left(e^{x-1}\right)\right] = \dfrac{d}{dx}\left[4\ln(x+1) - (x-1)\right] = \dfrac{4}{x+1} - 1$

40. $\dfrac{d}{dx}\ln\left[\dfrac{(x+1)^4\left(x^3+2\right)}{x-1}\right] = \dfrac{d}{dx}\left[4\ln(x+1) + \ln\left(x^3+2\right) - \ln(x-1)\right] = \dfrac{4}{x+1} + \dfrac{3x^2}{x^3+2} - \dfrac{1}{x-1}$

41. $\frac{d}{dx}\left[\ln(3x+1)\ln(5x+1)\right]=\ln(3x+1)\left(\frac{5}{5x+1}\right)+\ln(5x+1)\left(\frac{3}{3x+1}\right)=\frac{5\ln(3x+1)}{5x+1}+\frac{3\ln(5x+1)}{3x+1}$

42. $\frac{d}{dx}\left[(\ln 4x)(\ln 2x)\right]=(\ln 4x)\left(\frac{2}{2x}\right)+(\ln 2x)\left(\frac{4}{4x}\right)=\frac{\ln 4x}{x}+\frac{\ln 2x}{x}$

43. $f(x)=(x+1)^4(4x-1)^2 \Rightarrow \ln f(x)=\ln\left[(x+1)^4(4x-1)^2\right]=4\ln(x+1)+2\ln(4x-1)$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\frac{4}{x+1}+\frac{8}{4x-1}\Rightarrow f'(x)=(x+1)^4(4x-1)^2\left(\frac{4}{x+1}+\frac{8}{4x-1}\right).$

44. $f(x)=e^x(3x-4)^8 \Rightarrow \ln f(x)=\ln\left[e^x(3x-4)^8\right]=x+8\ln(3x-4)$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=1+\frac{24}{3x-4}\Rightarrow f'(x)=\left[e^x(3x-4)^8\right]\left(1+\frac{24}{3x-4}\right).$

45. $f(x)=\frac{(x+1)(2x+1)(3x+1)}{\sqrt{4x+1}}\Rightarrow$

$\ln f(x)=\ln\left(\frac{(x+1)(2x+1)(3x+1)}{\sqrt{4x+1}}\right)=\ln(x+1)+\ln(2x+1)+\ln(3x+1)-\frac{1}{2}\ln(4x+1)$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\frac{1}{x+1}+\frac{2}{2x+1}+\frac{3}{3x+1}-\frac{2}{4x+1}\Rightarrow f'(x)=\frac{(x+1)(2x+1)(3x+1)}{\sqrt{4x+1}}\cdot\left(\frac{1}{x+1}+\frac{2}{2x+1}+\frac{3}{3x+1}-\frac{2}{4x+1}\right).$

46. $f(x)=\frac{(x-2)^3(x-3)^4}{(x+4)^5}\Rightarrow \ln f(x)=\ln\left(\frac{(x-2)^3(x-3)^4}{(x+4)^5}\right)=3\ln(x-2)+4\ln(x-3)-5\ln(x+4)$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\frac{3}{x-2}+\frac{4}{x-3}-\frac{5}{x+4}\Rightarrow f'(x)=\frac{(x-2)^3(x-3)^4}{(x+4)^5}\left(\frac{3}{x-2}+\frac{4}{x-3}-\frac{5}{x+4}\right).$

47. $f(x)=2^x \Rightarrow \ln f(x)=\ln 2^x=x\ln 2$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\ln 2\Rightarrow f'(x)=2^x\ln 2.$

48. $f(x)=\sqrt[x]{3}\Rightarrow$

$\ln f(x)=\ln\sqrt[x]{3}=\ln\left(3^{1/x}\right)=\frac{1}{x}\ln 3$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=-\frac{\ln 3}{x^2}\Rightarrow f'(x)=-\frac{3^{1/x}\ln 3}{x^2}.$

49. $f(x)=x^x \Rightarrow \ln f(x)=\ln x^x=x\ln x$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=x\left(\frac{1}{x}\right)+\ln x=1+\ln x\Rightarrow$

$f'(x)=x^x\left[1+\ln x\right]$

50. $f(x)=x^{1/x}\Rightarrow \ln f(x)=\ln x^{1/x}=\frac{1}{x}\ln x$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\frac{1}{x}\cdot\frac{1}{x}+(\ln x)\left(-\frac{1}{x^2}\right)$

$=\frac{1}{x^2}-\frac{1}{x^2}\ln x=\frac{1-\ln x}{x^2}\Rightarrow$

$f'(x)=x^{1/x}\left[\frac{1-\ln x}{x^2}\right].$

51. $\ln y-k\ln x=\ln c;$

$\ln y=\ln c+k\ln x=\ln c+\ln x^k=\ln(cx^k)\Rightarrow$

$e^{\ln y}=e^{\ln(cx^k)}\Rightarrow y=cx^k$

52. $\ln(1-y)-\ln y = C - rt \Rightarrow$

$\ln\left(\frac{1-y}{y}\right) = C - rt \Rightarrow$

$e^{\ln\left(\frac{1-y}{y}\right)} = e^{C-rt} \Rightarrow \frac{1-y}{y} = e^{C-rt} \Rightarrow$

$\frac{1}{y} - 1 = e^{C-rt} \Rightarrow \frac{1}{y} = e^{C-rt} + 1 \Rightarrow$

$y = \frac{1}{e^{C-rt}+1}$

53. $y = he^{kx}$

$$\left.\begin{matrix} 6 = he^{k} \\ 48 = he^{4k} \end{matrix}\right\} \left.\begin{matrix} \ln\left(\frac{6}{h}\right) = k \\ \ln\left(\frac{48}{h}\right) = 4k \end{matrix}\right\} 4\ln\left(\frac{6}{h}\right) = \ln\left(\frac{48}{h}\right) \Rightarrow$$

$\ln\left(\frac{6}{h}\right)^4 = \ln\left(\frac{48}{h}\right) \Rightarrow \frac{6^4}{h^4} = \frac{48}{h} \Rightarrow h^3 = \frac{6^4}{48} \Rightarrow$

$h^3 = 27 \Rightarrow h = 3$

$6 = 3e^{k} \Rightarrow e^{k} = 2 \Rightarrow k = \ln 2$

54. $y = kx^r$

$$\left.\begin{matrix} 3 = k(2)^r \\ 15 = k(4)^r \end{matrix}\right\} \left.\begin{matrix} 3 = k(2)^r \\ 15 = k(2)^{2r} \end{matrix}\right\} \frac{15}{3} = \frac{k(2)^{2r}}{k(2)^r} \Rightarrow$$

$5 = 2^r = e^{r\ln 2} \Rightarrow r = \frac{\ln 5}{\ln 2} \Rightarrow$

$3 = k(2)^{\ln 5/\ln 2} = k(e^{\ln 2})^{\ln 5/\ln 2} = k(5) \Rightarrow k = \frac{3}{5}$

Chapter 4 Fundamental Concepts Check Exercises

1. (i) $b^x \cdot b^y = b^{x+y}$ (ii) $b^{-x} = \frac{1}{b^x}$

(iii) $\frac{b^x}{b^y} = b^x \cdot b^{-y} = b^{x-y}$ (iv) $\left(b^y\right)^x = b^{xy}$

(v) $a^x b^x = (ab)^x$ (vi) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

2. $e \approx 2.71828$

3. $y' = ky$

4. $k > 0$: graph is increasing; y-intercept (0, 1); graph tends to ∞ as x tends to ∞; graph tends to 0 as x tends to $-\infty$.
$k < 0$: graph is decreasing; y-intercept (0, 1); graph tends to 0 as x tends to ∞; graph tends to ∞ as x tends to $-\infty$.

5. The coordinates of the reflection of the point (a, b) in the line $x = y$ are (b, a).

6. A logarithm is the inverse of an exponential. For example, the natural logarithm is the inverse of the exponential function e^x.

7. The x-intercept of the graph of the natural logarithm function is (1, 0).

8. The domain is $x > 0$. The graph is always increasing. Its x-intercept is (1, 0). The graph tends to ∞ as x tends to ∞; graph tends to 0 as x tends to $-\infty$. The graph is the reflection of the graph of $y = e^x$ through the line $y = x$.

9. $e^{\ln x} = \ln\left(e^x\right) = 1$

10. $\ln x$ is the inverse of e^x. $\log x$ is the inverse of 10^x.

11. $b^x = e^{x\ln b}$

12. **a.** $f'(x) = ke^{kx}$

b. $f'(x) = g'(x)e^{g(x)}$

c. $f'(x) = \frac{g'(x)}{g(x)}$

13. Let $x, y > 0$, b be any number. Then

(i) $\ln(xy) = \ln x + \ln y$

(ii) $\ln\left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

(iv) $\ln\left(x^b\right) = b\ln x$

14. See Section 4.6, Example 5.
Another example is to differentiate $f(x) = x^{x+2}$.

$$\ln f(x) = \ln\left(x^{x+2}\right) = (x+2)\ln x$$

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)} = \frac{1}{x}(x+2) + (1)\ln x = 1 + \frac{2}{x} + \ln x = \frac{x+2}{x} + \ln x$$

$$f'(x) = f(x)\left(\frac{x+2}{x} + \ln x\right) = \left(x^{x+2}\right)\left(\frac{x+2}{x} + \ln x\right)$$

Chapter 4 Review Exercises

1. $27^{4/3} = (3^3)^{4/3} = 3^4 = 81$

2. $4^{1.5} = (2^2)^{3/2} = 2^3 = 8$

3. $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

4. $16^{-.25} = 16^{-1/4} = \frac{1}{(2^4)^{1/4}} = \frac{1}{2}$

5. $(2^{5/7})^{14/5} = 2^{14/7} = 2^2 = 4$

6. $8^{1/2} \cdot 2^{1/2} = (2^3)^{1/2} \cdot 2^{1/2} = 2^{3/2} \cdot 2^{1/2} = 2^{4/2} = 4$

7. $\frac{9^{5/2}}{9^{3/2}} = \frac{(3^2)^{5/2}}{(3^2)^{3/2}} = \frac{3^5}{3^3} = 3^2 = 9$

8. $4^{.2} \cdot 4^{.3} = 4^{.5} = 4^{1/2} = 2$

9. $(e^{x^2})^3 = e^{3x^2}$

10. $e^{5x} \cdot e^{2x} = e^{7x}$

11. $\frac{e^{3x}}{e^x} = e^{3x-x} = e^{2x}$

12. $2^x \cdot 3^x = (2 \cdot 3)^x = 6^x$

13. $(e^{8x} + 7e^{-2x})e^{3x} = e^{11x} + 7e^x$

14. $\frac{e^{5x/2} - e^{3x}}{\sqrt{e^x}} = (e^{5x/2} - e^{3x})e^{(-1/2)x} = e^{4x/2} - e^{5x/2} = e^{2x} - e^{5x/2}$

15. $e^{-3x} = e^{-12} \Rightarrow \ln e^{-3x} = \ln e^{-12} \Rightarrow -3x = -12 \Rightarrow x = 4$

16. $e^{x^2-x} = e^2 \Rightarrow \ln e^{x^2-x} = \ln e^2 \Rightarrow x^2 - x = 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

17. $(e^x \cdot e^2)^3 = e^{-9} \Rightarrow e^{3x+6} = e^{-9} \Rightarrow \ln e^{3x+6} = \ln e^{-9} \Rightarrow 3x + 6 = -9 \Rightarrow x = -5$

18. $e^{-5x} \cdot e^4 = e \Rightarrow e^{-5x+4} = e \Rightarrow \ln e^{-5x+4} = \ln e \Rightarrow -5x + 4 = 1 \Rightarrow x = \frac{3}{5}$

19. $\frac{d}{dx}[10e^{7x}] = 10e^{7x}(7) = 70e^{7x}$

20. $\frac{d}{dx}e^{\sqrt{x}} = \frac{d}{dx}e^{x^{1/2}} = e^{x^{1/2}} \cdot \frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

21. $\frac{d}{dx}[xe^{x^2}] = xe^{x^2}(2x) + e^{x^2}(1) = e^{x^2}(2x^2 + 1)$

22. $\frac{d}{dx}\left[\frac{e^x + 1}{x - 1}\right] = \frac{(x-1)e^x - (e^x + 1)(1)}{(x-1)^2} = \frac{-2e^x + xe^x - 1}{(x-1)^2} = \frac{(x-2)e^x - 1}{(x-1)^2}$

23. $\frac{d}{dx}[e^{e^x}] = e^{e^x}(e^x) = e^{x+e^x}$

24. $\frac{d}{dx}\left[\left(\sqrt{x} + 1\right)e^{-2x}\right] = \left(\sqrt{x} + 1\right)e^{-2x}(-2) + e^{-2x}\left(\frac{1}{2}x^{-1/2}\right) = e^{-2x}\left(\frac{1}{2\sqrt{x}} - 2\sqrt{x} - 2\right)$

25. $\frac{d}{dx}\left[\frac{x^2 - x + 5}{e^{3x} + 3}\right] = \frac{(e^{3x} + 3)(2x - 1) - (x^2 - x + 5)3e^{3x}}{(e^{3x} + 3)^2} = \frac{(2x - 1)(e^{3x} + 3) - 3e^{3x}(x^2 - x + 5)}{(e^{3x} + 3)^2}$

26. $\frac{d}{dx}x^e = ex^{e-1}$

27. $f(x)=e^{x^2}-4x^2;\ f'(x)=2xe^{x^2}-8x$

Solve $f'(x)=0$ to find the first coordinates of the relative extreme point.s

$2xe^{x^2}-8x=0 \Rightarrow 2x\left(e^{x^2}-4\right) \Rightarrow$

$2x=0$ or $e^{x^2}-4=0 \Rightarrow$

$x=0$ or $x^2=\ln 4 \Rightarrow x=\pm\sqrt{\ln 4}$

The relative extreme points occur at $x = 0$ and $x=\pm\sqrt{\ln 4}\approx \pm 1.1774$.

28. $f(x)=e^{x^2}-4x^2;\ f'(x)=2xe^{x^2}-8x$

$$\begin{aligned}f''(x)&=2e^{x^2}\frac{d}{dx}x+2x\frac{d}{dx}e^{x^2}-\frac{d}{dx}(8x)\\&=2e^{x^2}+(2x)^2e^{x^2}-8\\&=2e^{x^2}\left(1+2x^2\right)-8\end{aligned}$$

Evaluate $f''(0)$ to determine the concavity at $x = 0$.

$f''(0)=2e^{0^2}\left(1+2(0)^2\right)-8=2(1)-8=-6$

Since $f''(0)<0$, the graph is concave down at $x = 0$ and there is a relative maximum there.

29. $4e^{0.03t}-2e^{0.06t}=0 \Rightarrow$

$2e^{0.03t}\left(2-e^{0.03t}\right)=0 \Rightarrow$

$2e^{0.03t}=0$ (not possible) or $2-e^{0.03t}=0 \Rightarrow$

$2=e^{0.03t} \Rightarrow \ln 2=.03t \Rightarrow t=\dfrac{\ln 2}{.03}\approx 23.1049$

30. $e^t-8e^{0.02t}=0 \Rightarrow e^t=8e^{0.02t} \Rightarrow$

$t=\ln 8+.02t \Rightarrow .98t=\ln 8 \Rightarrow$

$t=\dfrac{\ln 8}{.98}\approx 2.1219$

31. $4\cdot 2^x=e^x \Rightarrow 2^2\cdot 2^x=e^x \Rightarrow 2^{2+x}=e^x \Rightarrow$

$(2+x)\ln 2=x \Rightarrow 2\ln 2+x\ln 2=x \Rightarrow$

$2\ln 2=x-x\ln 2 \Rightarrow 2\ln 2=x(1-\ln 2) \Rightarrow$

$x=\dfrac{2\ln 2}{1-\ln 2}=\dfrac{\ln 2^2}{1-\ln 2}=\dfrac{\ln 4}{1-\ln 2}\approx 4.52$

32. $3^x=2e^x \Rightarrow x\ln 3=\ln 2+x \Rightarrow$

$x\ln 3-x=\ln 2 \Rightarrow x(\ln 3-1)=\ln 2 \Rightarrow$

$x=\dfrac{\ln 2}{\ln 3-1}\approx 7.0290$

33. Solve $y'=4$.

$y=e^x;\ y'=e^x$

$e^x=4 \Rightarrow x=\ln 4$

The tangent line has slope 4 when $x = \ln 4$.

34. When the tangent line is horizontal, its slope is 0. Solve $y'=0$.

$y=e^x+e^{-2x};\ y'=e^x-2e^{-2x}$

$e^x-2e^{-2x}=0 \Rightarrow e^x=2e^{-2x} \Rightarrow$

$x=\ln 2-2x \Rightarrow 3x=\ln 2 \Rightarrow x=\dfrac{\ln 2}{3}$

35. $f(x)=\ln\left(x^2+1\right)$

First, determine the relative extreme and inflection points using the first derivative test.

$f'(x)=\dfrac{2x}{x^2+1}$

$f'(x)=\dfrac{2x}{x^2+1}=0 \Rightarrow x=0$

The only relative extreme occurs at $x = 0$. Now use the second derivative test to determine if the graph is increasing or decreasing.

$$f''(x)=\frac{2\left(x^2+1\right)-2x(2x)}{\left(x^2+1\right)^2}=\frac{-2x^2+2}{\left(x^2+1\right)^2}$$

$$f''(0)=\frac{-2(0)^2+2}{\left(0^2+1\right)^2}=2$$

Since $f''(0)>0$, the graph is concave up and there is a relative minimum at $x = 0$. Thus, the graph is decreasing for $x < 0$ and increasing for $x > 0$.

36. $f(x)=x\ln x, x>0$

First, determine the relative extreme and inflection points using the first derivative test.

$f'(x)=\ln x+x\left(\dfrac{1}{x}\right)=\ln x+1$

$f'(x)=\ln x+1=0 \Rightarrow \ln x=-1 \Rightarrow$

$x=e^{-1}=\dfrac{1}{e}$

The only relative extreme occurs at $x=\frac{1}{e}$.

Now use the second derivative test to determine if the graph is increasing or decreasing.

$f''(x)=\dfrac{1}{x}$

$f''\left(\dfrac{1}{e}\right)=e$

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(*continued*)

$f''(0) > 0$, so the graph is concave up and there is a relative minimum at $x = \frac{1}{e}$. Thus, the graph is decreasing for $0 < x < \frac{1}{e}$ and increasing for $x > \frac{1}{e}$.

37. $\frac{dy}{dx} = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$

$\left.\frac{dy}{dx}\right|_{x=0} = \frac{e^0}{(1+e^0)^2} = \frac{1}{4}$

The tangent line at (0, .5) is

$y - .5 = \frac{1}{4}(x-0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

38. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\frac{dy}{dx}$

$= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{\left(e^x+e^{-x}\right)^2}$

$= \frac{4}{\left(e^x+e^{-x}\right)^2}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{4}{\left(e^1+e^{-1}\right)^2}; \left.\frac{dy}{dx}\right|_{x=-1}$

$= \frac{4}{\left(e^{-1}+e^{-(-1)}\right)^2} = \frac{4}{\left(e^1+e^{-1}\right)^2}$

The tangent lines at $x = 1$ and $x = -1$ have the same slope, so they are parallel.

39. $e^{(\ln 5)/2} = e^{\ln\sqrt{5}} = \sqrt{5}$

40. $e^{\ln(x^2)} = x^2$

41. $\frac{\ln x^2}{\ln x^3} = \frac{2\ln x}{3\ln x} = \frac{2}{3}$

42. $e^{2\ln 2} = e^{\ln 2^2} = 2^2 = 4$

43. $e^{-5\ln 1} = e^{-5(0)} = 1$

44. $[e^{\ln x}]^2 = x^2$

45. $t^{\ln t} = e \Rightarrow \ln t^{\ln t} = \ln e \Rightarrow \ln t(\ln t) = 1 \Rightarrow$
$(\ln t)^2 = 1$

Taking the square root of both sides,

$|\ln t| = 1 \Rightarrow t = e$ or $t = \frac{1}{e}$.

46. $\ln(\ln 3t) = 0 \Rightarrow e^{\ln(\ln 3t)} = e^0 \Rightarrow \ln 3t = 1 \Rightarrow$
$e^{\ln 3t} = e \Rightarrow 3t = e \Rightarrow t = \frac{e}{3}$

47. $3e^{2t} = 15 \Rightarrow e^{2t} = 5 \Rightarrow \ln e^{2t} = \ln 5 \Rightarrow$
$2t = \ln 5 \Rightarrow t = \frac{1}{2}\ln 5$

48. $3e^{t/2} - 12 = 0 \Rightarrow 3(e^{t/2} - 4) = 0 \Rightarrow e^{t/2} = 4 \Rightarrow$
$\ln e^{t/2} = \ln 4 \Rightarrow t = 2\ln 4 \Rightarrow t = \ln 16$

49. $2\ln t = 5 \Rightarrow \ln t = \frac{5}{2} \Rightarrow e^{\ln t} = e^{5/2} \Rightarrow t = e^{5/2}$

50. $2e^{-0.3t} = 1 \Rightarrow e^{-0.3t} = \frac{1}{2} \Rightarrow \ln e^{-0.3t} = \ln\frac{1}{2} \Rightarrow$
$-.3t = \ln\frac{1}{2} \Rightarrow t = -\frac{1}{.3}\ln\frac{1}{2} = \frac{\ln 2}{.3} = \frac{10\ln 2}{3}$

51. $\frac{d}{dx}\ln(x^6 + 3x^4 + 1) = \frac{1}{x^6+3x^4+1}(6x^5 + 12x^3)$
$= \frac{6x^5+12x^3}{x^6+3x^4+1}$

52. $\frac{d}{dx}\left[\frac{x}{\ln x}\right] = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

53. $\frac{d}{dx}\ln(5x-7) = \frac{1}{5x-7}(5) = \frac{5}{5x-7}$

54. $\frac{d}{dx}\ln(9x) = \frac{1}{9x}(9) = \frac{1}{x}$

55. $\frac{d}{dx}[(\ln x)^2] = 2(\ln x)\frac{1}{x} = \frac{2\ln x}{x}$

56. $\frac{d}{dx}\left[(x\ln x)^3\right] = 3(x\ln x)^2\left(x\cdot\frac{1}{x} + \ln x\right)$
$= 3(x\ln x)^2(1+\ln x)$

57. $\dfrac{d}{dx}\ln\left[\dfrac{xe^x}{\sqrt{1+x}}\right]=\dfrac{d}{dx}\left[\ln\left(xe^x\right)-\ln\sqrt{1+x}\right]=\dfrac{1}{xe^x}(xe^x+e^x)-\dfrac{1}{\sqrt{1+x}}\cdot\dfrac{1}{2}(1+x)^{-1/2}=1+\dfrac{1}{x}-\dfrac{1}{2(1+x)}$

58. $\dfrac{d}{dx}\ln\left[e^{6x}(x^2+3)^5(x^3+1)^{-4}\right]=\dfrac{d}{dx}\left[6x+5\ln(x^2+3)-4\ln(x^3+1)\right]$

$=6+\dfrac{5}{x^2+3}(2x)-\dfrac{4}{x^3+1}(3x^2)=6+\dfrac{10x}{x^2+3}-\dfrac{12x^2}{x^3+1}$

59. $\dfrac{d}{dx}[x\ln x-x]=x\left(\dfrac{1}{x}\right)+\ln x-1=\ln x$

60. $\dfrac{d}{dx}\left[e^{2\ln(x+1)}\right]=\dfrac{d}{dx}\left[e^{\ln(x+1)^2}\right]=\dfrac{d}{dx}\left[(x+1)^2\right]=2(x+1)$

61. $\dfrac{d}{dx}\ln(\ln\sqrt{x})=\dfrac{1}{\ln\sqrt{x}}\cdot\dfrac{1}{\sqrt{x}}\cdot\dfrac{1}{2}x^{-1/2}=\dfrac{1}{2x\ln\sqrt{x}}=\dfrac{1}{x\ln x}$

62. $\dfrac{d}{dx}\left[\dfrac{1}{\ln x}\right]=\dfrac{d}{dx}\left[(\ln x)^{-1}\right]=-1(\ln x)^{-2}\left(\dfrac{1}{x}\right)=-\dfrac{1}{x(\ln x)^2}$

63. $\dfrac{d}{dx}[e^x\ln x]=e^x\left(\dfrac{1}{x}\right)+e^x\ln x=\dfrac{e^x}{x}+e^x\ln x$

64. $\dfrac{d}{dx}\ln(x^2+e^x)=\dfrac{1}{x^2+e^x}(2x+e^x)=\dfrac{2x+e^x}{x^2+e^x}$

65. $\dfrac{d}{dx}\ln\sqrt{\dfrac{x^2+1}{2x+3}}=\dfrac{d}{dx}\ln\left(\dfrac{x^2+1}{2x+3}\right)^{1/2}=\dfrac{d}{dx}\dfrac{1}{2}\left[\ln\left(x^2+1\right)-\ln(2x+3)\right]$

$=\dfrac{1}{2}\left[\dfrac{1}{x^2+1}(2x)-\dfrac{1}{2x+3}(2)\right]=\dfrac{x}{x^2+1}-\dfrac{1}{2x+3}$

66. $-2x+1>0$ gives us $x<\dfrac{1}{2}$. $-2x+1<0$ gives us $x>\dfrac{1}{2}$.

For $x<\dfrac{1}{2}$, we have $\dfrac{d}{dx}\ln|-2x+1|=\dfrac{d}{dx}\ln(-2x+1)=\dfrac{1}{-2x+1}(-2)=\dfrac{2}{2x-1}$.

For $x>\dfrac{1}{2}$, we have $\dfrac{d}{dx}\ln|-2x+1|=\dfrac{d}{dx}\ln(-(-2x+1))=\dfrac{d}{dx}\ln(2x-1)=\dfrac{1}{2x-1}(2)=\dfrac{2}{2x-1}$.

So, for $x\neq\dfrac{1}{2}$, $\dfrac{d}{dx}\ln|-2x+1|=\dfrac{2}{2x-1}$.

67. $\dfrac{d}{dx}\ln\left(\dfrac{e^{x^2}}{x}\right)=\dfrac{d}{dx}\left[\ln\left(e^{x^2}\right)-\ln x\right]=\dfrac{d}{dx}\left[x^2-\ln x\right]=2x-\dfrac{1}{x}$

68. $\dfrac{d}{dx}\ln\sqrt[3]{x^3+3x-2}=\dfrac{d}{dx}\left[\dfrac{1}{3}\ln(x^3+3x-2)\right]=\dfrac{1}{3}\dfrac{1}{x^3+3x-2}(3x^2+3)=\dfrac{x^2+1}{x^3+3x-2}$

69. $\dfrac{d}{dx}\ln(2^x)=\dfrac{d}{dx}(x\ln 2)=\ln 2$

70. $\dfrac{d}{dx}\left[\ln(3^{x+1})-\ln 3\right]=\dfrac{d}{dx}\left[(x+1)\ln 3-\ln 3\right]=\ln 3$

71. $x-1>0$ gives $x>1$. $x-1<0$ gives $x<1$.

For $x>1$, we have $\frac{d}{dx}\ln|x-1|=\frac{d}{dx}\ln(x-1)=\frac{1}{x-1}$.

For $x<1$, we have $\frac{d}{dx}\ln|x-1|=\frac{d}{dx}\ln(-(x-1))=\frac{d}{dx}\ln(-x+1)=\frac{1}{-x+1}(-1)=\frac{1}{x-1}$.

So, for $x\neq 1$, $\frac{d}{dx}\ln|x-1|=\frac{1}{x-1}$.

72. $\frac{d}{dx}e^{2\ln(2x+1)}=\frac{d}{dx}e^{\ln(2x+1)^2}=\frac{d}{dx}(2x+1)^2=2(2x+1)(2)=8x+4$

73. $\frac{d}{dx}\ln\left(\frac{1}{e^{\sqrt{x}}}\right)=\frac{d}{dx}\ln\left(e^{-\sqrt{x}}\right)=\frac{d}{dx}(-\sqrt{x})=-\frac{1}{2}x^{-1/2}=-\frac{1}{2\sqrt{x}}$

74. $\frac{d}{dx}\ln(e^x+3e^{-x})=\frac{1}{e^x+3e^{-x}}\left(e^x+3e^{-x}(-1)\right)=\frac{e^x-3e^{-x}}{e^x+3e^{-x}}$

75. $\ln f(x)=\ln\sqrt[5]{\frac{x^5+1}{x^5+5x+1}}=\frac{1}{5}\ln(x^5+1)-\frac{1}{5}\ln(x^5+5x+1)$

Differentiating both sides, we have $\frac{f'(x)}{f(x)}=\frac{1}{5}\frac{1}{x^5+1}(5x^4)-\frac{1}{5}\frac{1}{x^5+5x+1}(5x^4+5)\Rightarrow$

$f'(x)=\sqrt[5]{\frac{x^5+1}{x^5+5x+1}}\left(\frac{x^4}{x^5+1}-\frac{x^4+1}{x^5+5x+1}\right)$.

76. $\ln f(x)=\ln 2^x=x\ln 2$

Differentiating both sides, we have $\frac{f'(x)}{f(x)}=\ln 2\Rightarrow f'(x)=2^x\ln 2$.

77. $\ln f(x)=\ln x^{\sqrt{x}}=\sqrt{x}\ln x$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\sqrt{x}\left(\frac{1}{x}\right)+\left(\frac{1}{2}\right)x^{-1/2}\ln x=\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}\Rightarrow f'(x)=x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}\right)=x^{\sqrt{x}-1/2}\left(1+\frac{1}{2}\ln x\right)$

78. $\ln f(x)=\ln b^x=x\ln b$

Differentiating both sides, we have $\frac{f'(x)}{f(x)}=\ln b\Rightarrow f'(x)=b^x\ln b$.

79. $\ln f(x)=\ln\left[(x^2+5)^6(x^3+7)^8(x^4+9)^{10}\right]=6\ln(x^2+5)+8\ln(x^3+7)+10\ln(x^4+9)$

Differentiating both sides, we have

$\frac{f'(x)}{f(x)}=\frac{6}{x^2+5}(2x)+\frac{8}{x^3+7}(3x^2)+\frac{10}{x^4+9}(4x^3)\Rightarrow$

$f'(x)=(x^2+5)^6(x^3+7)^8(x^4+9)^{10}\left[\frac{12x}{x^2+5}+\frac{24x^2}{x^3+7}+\frac{40x^3}{x^4+9}\right]$

80. $\ln f(x)=\ln x^{1+x}=(1+x)\ln x=\ln x+x\ln x$

Differentiating both sides, we have $\frac{f'(x)}{f(x)}=\frac{1}{x}+x\left(\frac{1}{x}\right)+\ln x\Rightarrow f'(x)=x^{1+x}\left(\frac{1}{x}+1+\ln x\right)$.

81. $\ln f(x) = \ln 10^x = x\ln 10$

Differentiating both sides, we have $\dfrac{f'(x)}{f(x)} = \ln 10 \Rightarrow f'(x) = 10^x \ln 10.$

82. $\ln f(x) = \ln\left(\sqrt{x^2+5}e^{x^2}\right) = \dfrac{1}{2}\ln(x^2+5) + \ln e^{x^2} = \dfrac{1}{2}\ln(x^2+5) + x^2$

Differentiating both sides, we have

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\cdot\frac{1}{x^2+5}(2x) + 2x = \frac{x}{x^2+5} + 2x \Rightarrow f'(x) = \sqrt{x^2+5}\,e^{x^2}\left(\frac{x}{x^2+5} + 2x\right).$$

83. $\ln f(x) = \ln\sqrt{\dfrac{xe^x}{x^3+3}} = \dfrac{1}{2}\left[\ln x + \ln e^x - \ln(x^3+3)\right] = \dfrac{1}{2}\left[\ln x + x - \ln(x^3+3)\right]$

Differentiating both sides, we have $\dfrac{f'(x)}{f(x)} = \dfrac{1}{2}\left[\dfrac{1}{x} + 1 - \dfrac{1}{x^3+3}(3x^2)\right] \Rightarrow f'(x) = \dfrac{1}{2}\sqrt{\dfrac{xe^x}{x^3+3}}\left(\dfrac{1}{x} + 1 - \dfrac{3x^2}{x^3+3}\right).$

84. $\ln f(x) = \ln\left[\dfrac{e^x\sqrt{x+1}(x^2+2x+3)^2}{4x^2}\right] = x + \dfrac{1}{2}\ln(x+1) + 2\ln(x^2+2x+3) - 2\ln(2x)$

Differentiating both sides, we have $\dfrac{f'(x)}{f(x)} = 1 + \dfrac{1}{2(x+1)} + 2\dfrac{1}{x^2+2x+3}(2x+2) - 2\dfrac{1}{2x}(2) \Rightarrow$

$$f'(x) = \frac{e^x\sqrt{x+1}(x^2+2x+3)^2}{4x^2}\left[1 + \frac{1}{2x+2} + \frac{4x+4}{x^2+2x+3} - \frac{2}{x}\right].$$

85. $\ln f(x) = \ln\left[e^{x+1}(x^2+1)x\right]$

$= x + 1 + \ln(x^2+1) + \ln x$

Differentiating both sides, we have

$$\frac{f'(x)}{f(x)} = 1 + \frac{1}{x^2+1}(2x) + \frac{1}{x} \Rightarrow$$

$$f'(x) = e^{x+1}(x^2+1)x\left(1 + \frac{2x}{x^2+1} + \frac{1}{x}\right)$$

$$= e^{x+1}(x^3+3x^2+x+1)$$

86. $\ln f(x) = \ln(e^x x^2 2^x) = x + 2\ln x + x\ln 2$

Differentiating both sides, we have

$$\frac{f'(x)}{f(x)} = 1 + 2\left(\frac{1}{x}\right) + \ln 2 \Rightarrow$$

$$f'(x) = e^x x^2 2^x\left(1 + \ln 2 + \frac{2}{x}\right).$$

87. **a.** $f(2) \approx 800$ g/cm^2

Actual answer: 814.160 g/cm^2

b. $f(x) = 200$ when $x \approx 14$ km

c. $f'(8) \approx -50$ g/cm^2 per km

Actual answer:

$f'(x) = 1035e^{-0.12x}(-.12) = -124.2e^{-0.12x}$

$f'(8) = -124.2e^{-0.12(8)} = -47.6$

At an altitude of 8 km, the atmospheric pressure is dropping at the rate of 9.266 g/cm^2.

d. $f'(x) = -100$ when $x \approx 2$ km

88. **a.** $f(18) \approx 180$ billion dollars

Actual answer: $181.969 billion

b. $f'(12) \approx 10$ billion dollars per year

Actual answer:

$f'(t) = 27e^{0.106t}(.106) = 2.862e^{0.106t}$

$f'(12) = 2.862e^{0.106(12)} = 10.212$

Expenditures were rising at $10.212 billion per year.

c. $f(t) = 120$ when $t \approx 14$, so in 2004

d. $f'(t) = 20$ when $t \approx 18$, so expenditures were rising at the rate of $20 billion per year in 2008