Chapter 5 Applications of the Exponential and Natural Logarithm Functions

5.1 Exponential Growth and Decay

For exercises 1–10, refer to Theorem 1 and Example 1 in your text.

1. $y' = y \Rightarrow k = 1$
Then, $y = Ce^{t}$.

2. $y' = .4y \Rightarrow k = .4$
Then, $y = Ce^{0.4t}$.

3. $y' = 1.7y \Rightarrow k = 1.7$
Then, $y = Ce^{1.7t}$.

4. $y' = \dfrac{y}{4} \Rightarrow k = \dfrac{1}{4} = .25$
Then, $y = Ce^{0.25t}$.

5. $y' - \dfrac{y}{2} = 0 \Rightarrow y' = \dfrac{y}{2} \Rightarrow k = \dfrac{1}{2}$
Then, $y = Ce^{0.5t}$.

6. $y' - 6y = 0 \Rightarrow y' = 6y \Rightarrow k = 6$
Then, $y = Ce^{6t}$.

7. $2y' - \dfrac{y}{2} = 0 \Rightarrow 2y' = \dfrac{y}{2} \Rightarrow y' = \dfrac{y}{4} \Rightarrow k = \dfrac{1}{4}$
Then, $y = Ce^{0.25t}$.

8. $y = 1.6y' \Rightarrow y' = \dfrac{1}{1.6}y \Rightarrow y' = .625y \Rightarrow$
$k = .625$
Then, $y = Ce^{0.625t}$.

9. $\dfrac{y}{3} = 4y' \Rightarrow \dfrac{y}{12} = y' \Rightarrow k = \dfrac{1}{12}$
Then, $y = Ce^{t/12}$.

10. $5y' - 6y = 0 \Rightarrow 5y' = 6y \Rightarrow y' = \dfrac{6}{5}y \Rightarrow$
$k = \dfrac{6}{5} = 1.2$
Then, $y = Ce^{1.2t}$.

For exercises 11–18, refer to Theorem 2 and Example 3 in your text.

11. $y' = 3y,\ y(0) = 1$
Then, $k = 3$ and $y = P(t) = P_0e^{kt} \Rightarrow y = e^{3t}$.

12. $y' = 4y,\ y(0) = 0$
Then, $k = 4$ and $y = P(t) = P_0e^{kt} \Rightarrow y = 0$.

13. $y' = 2y,\ y(0) = 2$
Then, $k = 2$ and $y = P(t) = P_0e^{kt} \Rightarrow y = 2e^{2t}$.

14. $y' = y,\ y(0) = 4$
Then, $k = 1$ and $P(t) = P_0e^{kt} \Rightarrow P(t) = 4e^{t}$.

15. $y' - .6y = 0 \Rightarrow y' = .6y,\ y(0) = 5$
Then, $k = .6$ and $y = P(t) = P_0e^{kt} \Rightarrow$
$y = 5e^{0.6t}$.

16. $y' - \dfrac{y}{7} = 0 \Rightarrow y' = \dfrac{y}{7},\ y(0) = 6$
Then, $k = \frac{1}{7}$ and $y = P(t) = P_0e^{kt} \Rightarrow$
$y = 6e^{t/7}$.

17. $6y' = y \Rightarrow y' = \dfrac{y}{6},\ y(0) = 12$
Then, $k = \frac{1}{6}$ and $y = P(t) = P_0e^{kt} \Rightarrow$
$y = 12e^{t/6}$.

18. $5y = 3y' \Rightarrow y' = \dfrac{5}{3}y,\ y(0) = 7$
Then, $k = \frac{5}{3}$ and $y = P(t) = P_0e^{kt} \Rightarrow$
$y = 7e^{5t/3}$.

19. **a.** $P' = .01P(t),\ P(0) = 2$
$k = .01$, so $P(t) = P_0e^{kt} \Rightarrow$
$P(t) = 2e^{0.01t}$.

b. The initial population in 2015 was 2 million.

c. The year 2019 corresponds to $t = 4$.
$P(4) = 2e^{0.01(4)} \approx 2.0816$ million.

20. **a.** $k = .08 \Rightarrow P'(t) = .08P(t);\ P(0) = 500$

b. $P(t) = P_0e^{kt} \Rightarrow P(t) = 500e^{0.08t}$

c. $P(5) = 500e^{0.08(5)} \approx 746$ fruit flies

21. a. $P(t) = P_0 e^{kt}$
$P(2) = 4P_0 \Rightarrow 4P_0 = P_0 e^{2k} \Rightarrow 4 = e^{2k} \Rightarrow$
$\ln 4 = 2k \Rightarrow \ln(2^2) = 2k \Rightarrow 2\ln 2 = 2k \Rightarrow$
$k = \ln 2$
So, $P(t) = P_0 e^{t \ln 2}$.

b. $P(0.5) = 20000e^{0.5 \ln 2} \approx 28,284$ bacteria

22. a. $P(t) = P_0 e^{kt}$; $P(0) = 10000$
$P(1) = 15000 = 10000e^k \Rightarrow 1.5 = e^k \Rightarrow$
$k = \ln 1.5$
So, $P(t) = 10,000e^{t \ln 1.5}$.

b. $P(t) = 20000 \Rightarrow 10000e^{t \ln 1.5} = 20000 \Rightarrow$
$e^{t \ln 1.5} = 2 \Rightarrow t \ln 1.5 = \ln 2 \Rightarrow$
$t = \dfrac{\ln 2}{\ln 1.5} \approx 1.7$ days

23. $P'(t) = 0.03P(t)$, $P(0) = 4$

a. $P(t) = 5$
$P'(t) = 0.03P(t) = 0.03(5) = 0.15$
When the population reaches 5 million people, it is growing at the rate of 0.15 million people per year.

b. $P'(t) = 0.03P(t)$
$400000 = 0.4 \text{ million} = 0.03P(t)$
$P(t) = \dfrac{0.4}{0.03} \approx 13.33$ million
The population is about 13.33 million when it is growing at the rate of 400,000 people per year.

c. $k = 0.03$, so $P(t) = P_0 e^{0.03t} = 4e^{0.03t}$.

24. a. $P(0) = 10,000$

b. $P(t) = Ce^{0.55t}$ and since
$10,000 = P(0) = Ce^{(0.55)(0)} = C$, we have
$P(t) = 10,000e^{0.55t}$.

c. $P(5) = 10,000e^{(0.55)(5)} \approx 156{,}426$ bacteria

d. $k = .55$

e. $P'(t) = .55P(t) = (.55)(100,000)$
$= 55,000$ bacteria per hour

f. Solve $P'(t) = 0.55P(t)$ for $P(t)$ when
$P'(t) = 34,000$.
$.55P(t) = 34,000 \Rightarrow P(t) \approx 61{,}818$ bacteria

25. a. $P(0) = 5000e^{(0.2)(0)} = 5000$, so 5000 cells were present initially.

b. $k = 0.2$, so a differential equation is $P'(t) = .2P(t)$.

c. $10,000 = 5000e^{0.2t} \Rightarrow 2 = e^{0.2t} \Rightarrow$
$\ln 2 = 0.2t \Rightarrow t = \dfrac{\ln 2}{0.2} \approx 3.5$ hours

d. $20,000 = 5000e^{0.2t} \Rightarrow 4 = e^{0.2t} \Rightarrow$
$\ln 4 = 0.2t \Rightarrow t = \dfrac{\ln 4}{0.2} \approx 6.9$ hours

26. a. $P(0) = 300e^{(0.01)(0)} = 300$, so 300 cells were present initially.

b. $k = .01$, so a differential equation is $P'(t) = .01P(t)$.

c. $600 = 300e^{0.01t} \Rightarrow 2 = e^{0.01t} \Rightarrow$
$\ln 2 = .01t \Rightarrow t = \dfrac{\ln 2}{.01} \approx 69.3$ days

d. $1200 = 300e^{0.01t} \Rightarrow 4 = e^{0.01t} \Rightarrow$
$\ln 4 = .01t \Rightarrow t = \dfrac{\ln 4}{.01} \approx 138.6$ days

27. Let $P(t)$ be the population after t days, $P(t) = P_0 e^{kt}$. It is given that $P(40) = 2P(0)$, so
$P_0 e^{40k} = 2P_0 e^{0(k)} = 2P_0 \Rightarrow e^{40k} = 2 \Rightarrow$
$\ln e^{40k} = \ln 2 \Rightarrow k = \dfrac{\ln 2}{40} \approx .017$

28. Let $P(t)$ be the population after t years, $P(t) = P_0 e^{kt}$. It is given that $P(10) = 3P(0)$, so
$P_0 e^{10k} = 3P_0 e^{(0)k} = 3P_0 \Rightarrow e^{10k} = 3 \Rightarrow$
$\ln e^{10k} = \ln 3 \Rightarrow k = \dfrac{\ln 3}{10} \approx .11$

29. Let $P(t)$ be the population after t years.
$P(t) = P_0 e^{.05t} \Rightarrow 3P_0 = P_0 e^{.05t} \Rightarrow 3 = e^{.05t} \Rightarrow$
$\ln 3 = .05t \Rightarrow t = \dfrac{\ln 3}{.05} \approx 22$ years

30. Let $P(t)$ be the population after t years.
$P(t) = P_0 e^{0.04t} \Rightarrow 2P_0 = P_0 e^{0.04t} \Rightarrow$
$2 = e^{0.04t} \Rightarrow \ln 2 = 0.04t \Rightarrow$
$t = \dfrac{\ln 2}{0.04} \approx 17.3$ years

31. Let $P(t)$ be the cell population (in millions) after t hours, $P(t) = P_0 e^{kt}$. It is given that $P(0) = 1$ and that $P(10) = 9$. Thus,
$9 = 1 \cdot e^{k(10)} \Rightarrow \ln 9 = \ln e^{10k} \Rightarrow k = \frac{\ln 9}{10} \approx .22$.
$P(t) = e^{.22t}$ so $P(15) = e^{.22(15)} = e^{3.3}$
≈ 27 (million) cells.

32. Let $P(t)$ be the population (in billions) t years after January 1, 1993.
$P(t) = 5.51e^{kt}$ since $P(0) = 5.51$.
Solve $P(5) = 5.88$ for k.
$5.51e^{k(5)} = 5.88 \Rightarrow 5k = \ln \frac{5.88}{5.51} \Rightarrow$
$k = \frac{1}{5}\ln\left(\frac{5.88}{5.51}\right) \Rightarrow P(t) = 5.51e^{(1/5)\ln(5.88/5.51)t}$
Solve $P(t) = 7$ for t.
$$5.51e^{(1/5)\ln(5.88/5.51)t} = 7$$
$$e^{(1/5)\ln(5.88/5.51)t} = \frac{7}{5.51}$$
$$\frac{1}{5}\ln\left(\frac{5.88}{5.51}\right)t = \ln\left(\frac{7}{5.51}\right)$$
$$t = \frac{5\ln\left(\frac{7}{5.51}\right)}{\ln\left(\frac{5.88}{5.51}\right)} \approx 18.4 \text{ years}$$
The world's population will reach 7 billion in about 18.4 years (the year 2011).

33. Let $P(t)$ be the population (in millions) t years after the beginning of 1990.
$P(0) = 20.2$, so $P(t) = 20.2e^{kt}$.
Solve $P(5) = 23$ for k.
$20.2e^{k(5)} = 23 \Rightarrow 5k = \ln\left(\frac{23}{20.2}\right) \Rightarrow$
$k = \frac{1}{5}\ln\left(\frac{23}{20.2}\right) \Rightarrow P(t) = 20.2e^{(1/5)\ln(23/20.2)t}$
$P(20) = 20.2e^{(1/5)\ln(23/20.2)(20)} \approx 34.0$ million

34. a. From the graph, $P(10) = 10.5$ million.

b. From the graph, $P(t) = 10$ when $t = 8$ years, so the population was 10 million in 2018.

c. $P'(t) = .025P(t)$
$$P'(10) = .025P(10) = (.025)(10.5) = 0.2625 = 262{,}500 \text{ people per year}$$

d. $.275 = .025P(t) \Rightarrow P(t) = \frac{.275}{.025} = 11$
$P(t) = 11$ million when $t = 11$.
The population is growing at the rate of 275,000 people per year in 2021.

35. a. $P(t) = 8e^{-0.021t}$

b. $P(0) = 8$ g

c. .021

d. $P(10) = 8e^{(-0.021)(10)} \approx 6.5$ grams

e. $P'(t) = (-.021)(1) = -.021$
The sample is disintegrating at a rate of .021 gram per year

f. $-.105 = -.021P(t) \Rightarrow$
$P(t) = \frac{-.105}{-.021} = 5$ grams remaining

g. 4 grams will remain after 33 years.
2 grams will remain after 66 years.
1 gram will remain after 99 years.

36. a. $P(t) = 12e^{-0.00043t}$

b. 12 grams

c. .00043

d. $P(943) = 12e^{(-0.00043)(943)} \approx 8$ grams

e. $P'(t) = (-.00043)(1) = -.00043$
The sample is disintegrating at a rate of .00043 gram per year

f. $-.00043P(t) = -.004$
$$P(t) = \frac{-.004}{-.00043} \approx 9.3 \text{ grams}$$

g. 6 grams will remain after 1612 years.
3 grams will remain after 3224 years.
1.5 grams will remain after 4836 years.

37. a. $f'(t) = -.6f(t)$

b. $f(5) = 300e^{(-0.6)(5)} \approx 14.9$ mg

c. $150 = 300e^{-0.6t} \Rightarrow .5 = e^{-0.6t} \Rightarrow$
$\ln .5 = -.6t \Rightarrow t = \frac{\ln .5}{-.6} \approx 1.2$ hours

38. a. $P(t) = 10e^{-0.04t}$

b. $P'(t) = -.04P(t)$

c. $P(5) = 10e^{(-0.04)(5)} \approx 8.2$ grams

d. $5 = 10e^{-0.04t} \Rightarrow .5 = e^{-0.04t} \Rightarrow$
$\ln .5 = -.04t \Rightarrow t = \dfrac{\ln .5}{-.04} \approx 17.3$ days

39. $.5 = e^{-0.023t} \Rightarrow \ln .5 = -.023t \Rightarrow$
$t = \dfrac{\ln 5}{-.023} \approx 30.1$ years

40. $.5 = e^{k(5.3)} \Rightarrow \ln .5 = 5.3k \Rightarrow$
$k = \dfrac{\ln .5}{5.3} \approx -.13$

41. $.5 = e^{8k} \Rightarrow \ln .5 = 8k \Rightarrow$
$k = \dfrac{\ln .5}{8} \approx -.0866$
$P(t) = 10e^{-0.0866t} \Rightarrow 1 = 10e^{-0.0866t} \Rightarrow$
$.1 = e^{-0.0866t} \Rightarrow \ln .1 = -.0866t \Rightarrow$
$t = \dfrac{\ln .1}{-.0866} \approx 26.6$ days

42. $P(t) = 10e^{kt} \Rightarrow 3 = 10e^{k(5)} \Rightarrow .3 = e^{5k} \Rightarrow$
$\ln .3 = 5k \Rightarrow k = \dfrac{\ln .3}{5}$
$5 = 10e^{kt} \Rightarrow .5 = e^{kt} \Rightarrow \ln .5 = kt \Rightarrow$
$t = \dfrac{\ln .5}{k} = \dfrac{5 \ln .5}{\ln .3} \approx 2.9$ years

43. $f'(t) = kf(t)$ so $f(t) = Ce^{kt}$ where $C = 8$
$f(50) = 4 = 8e^{k(50)} \Rightarrow 0.5 = e^{50k} \Rightarrow$
$\ln .5 = 50k \Rightarrow k = \dfrac{\ln .5}{50} \approx -.014$
So, $.5 = e^{50k}$. Thus, $f(t) = 8e^{-0.014t}$.

44. $P(t) = 40e^{kt}$
$P(220) = 16 = 40e^{k(220)} \Rightarrow .4 = e^{220k} \Rightarrow$
$\ln .4 = 220k \Rightarrow k = \dfrac{\ln .4}{220} \approx -.0042$
Thus, $P(t) = 40e^{-0.0042t}$.
$P(300) = 40e^{-0.0042(300)} \approx 11.466$ grams

45. a. From the graph $P(1) = 8$ grams.

b. From the graph, the half-life appears to be about 3.5 hours.

c. $P'(6) = -.2P(6) = (-.2)(3) = -.6$
The sample was decaying at the rate of .6 grams per hour.

d. $-.4 = -.2P(t) \Rightarrow P(t) = \dfrac{-.4}{-.2} = 2$
$P(t) = 2$ when $t = 8$ hours

46. $P'(t) = -.25P(t) = (-.25)(8) = -2$
It will be disintegrating at the rate of 2 grams per hour when the sample size is 8 grams.
2 grams per day $= \dfrac{1}{12}$ gram per hour.
$-\dfrac{1}{12} = -.25P(t) \Rightarrow P(t) = \dfrac{-\frac{1}{12}}{-.25} = \dfrac{1}{3}$ gram
The sample size will be $\dfrac{1}{3}$ gram when decreasing at the rate of 2 grams per day.

47. Let the original amount be 1, $P(t) = e^{-0.00012t}$.
Then for some $t_0, P(t_0) = .20 = e^{-0.00012t_0}$.
$\dfrac{\ln .20}{-.00012} = t_0 \Rightarrow t_0 \approx 13{,}412$ years
That was the estimated age of the cave paintings more than 65 years ago. The estimated age now is about 13,500 years.

48. $P(t) = P_0 e^{-\lambda t}$ for carbon-14, $\lambda = .00012$
$P(t) = .91P_0$, so $.91 = e^{-0.00012t} \Rightarrow$
$-.09431 = -.00012t \Rightarrow t \approx 786$. So the table dates from approximately 1190, and thus could not have belonged to King Arthur.

49. Let the original amount be 1. Then
$P(t) = 1 \cdot e^{-0.00012t}$.
$P(4500) = e^{-0.00012(4500)} = e^{-0.54} \approx .583$.
Thus about 58.3% remains.

50. Let the original amount be 1, $P(t) = e^{-0.00012t}$.
$P(t_0) = .34 = e^{-0.00012t_0} \Rightarrow$
$t_0 = \dfrac{\ln .34}{-.00012} \approx 8990$ years old

51. $P(t) = e^{-0.00012t}$; $P(t_0) = 0.27 = e^{-0.00012t_0}$;
$\dfrac{\ln .27}{-.00012} = t_0$; $t_0 \approx 10{,}911$ years ago.

52. Following the hint,

(i) In Fig. 5 the slope of the tangent line is $-\frac{C}{T}$.

(ii) $\frac{dy}{dt} = Ce^{-\lambda t}(-\lambda) = -\lambda Ce^{-\lambda t}$ so at $t = 0$, the slope is $-\lambda Ce^{-\lambda(0)} = -\lambda C$.

Since (i) and (ii) both express the slope at $t = 0$, we may write $-\lambda C = -\frac{C}{T}$ or $T = \frac{1}{\lambda}$.

53. a-D, b-G, c-E, d-B, e-H, f-F, g-A, h-C

54. $y = P_0 e^{-\lambda t}$

From exercise 52, we have $T = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{T}$.

If $t = T$, then $y = P_0 e^{-\frac{1}{T}\cdot T} = \frac{P_0}{e} \approx \frac{P_0}{2.718} \approx \frac{P_0}{3}$.

Thus, the function $P(t)$ decays to about one-third of its initial size, which is more than one-half of its initial size. In other words, y is decreasing and $\frac{1}{3}P_0 < \frac{1}{2}P_0$. Thus, $T > \lambda$.

55. $y'(t) = -.5y(t);\; y(0) = 10$

a. $P(0) = 10$ and $P'(0) = -.5(10) = -5$. Thus, the slope of the tangent is -5 and the y-intercept of the tangent is 10, so the equation of the tangent is $y = -5t + 10$.

b. $P(t) = 10e^{-.5t}$

c. $T = \frac{1}{\lambda} = \frac{1}{.5} = 2$

56. $y = P_0 e^{-\lambda t} \Rightarrow y = P_0 e^{-\frac{1}{T}\cdot t}$

$.01P_0 = P_0 e^{-t/T} \Rightarrow .01 = e^{-t/T} \Rightarrow$

$\ln .01 = -\frac{t}{T} \Rightarrow -\ln .01 = \frac{t}{T} \Rightarrow 4.6 \approx \frac{t}{T} \Rightarrow$

$t \approx 4.6T$

5.2 Compound Interest

1. a. \$5000

b. $.04 = 4\%$

c. $A(10) = 5000e^{(0.04)(10)} \approx \$7459.12$

d. $A'(t) = .04A(t)$

e. $A'(10) = (.04)(7459.12) \approx \$298.36$ per year

f. $280 = .04A(t) \Rightarrow A(t) = \frac{280}{.04} = \$7000$

2. a. \$3000

b. $.045 = 4.5\%$

c. $A(t) = 3000e^{0.045t}$

d. $A(5) = 3000e^{(0.045)(5)} \approx \$3756.97$

e. $A'(5) = .045A(5) \approx \$169.06$ per year.

f. $270 = .045A(t)$

$A(t) = \frac{270}{.045} \approx \$6000$

3. a. $r = .035,\; P = 4000$

$A(t) = 4000e^{0.035t}$

b. $A'(t) = .035A(t)$

c. $A(2) = 4000e^{(0.035)(2)} \approx \$4290.03$

d. $5000 = 4000e^{0.035t} \Rightarrow 1.25 = e^{0.035t} \Rightarrow$

$\ln 1.25 = .035t \Rightarrow t = \frac{\ln 1.25}{.035} \approx 6.4$ years

e. $A'(t) = .035A(t) = (.035)(5000) = \$175$ per year

4. a. $A'(t) = .046A(t)$

b. $A(t) = 10{,}000e^{0.046t}$

c. $A(3) = 10{,}000e^{(0.046)(3)} \approx \$11{,}479.76$

d. $30{,}000 = 10{,}000e^{0.046t} \Rightarrow 3 = e^{0.046t} \Rightarrow$

$\ln 3 = .046t \Rightarrow t = \frac{\ln 3}{.046} \approx 23.9$ years

e. $A'(t) = .046(30{,}000) = \$1380$ per year

5. $r = .042;\; A'(t) = .042A(t)$

$A'(t) = (.042)(9000) = \$378$ per year

6. $r = .051;\; A'(t) = .051A(t)$

$765 = .051A(t) \Rightarrow A(t) = \frac{765}{.051} = \$15{,}000$

7. $A(t) = 1000e^{0.06t} \Rightarrow 2500 = 1000e^{0.06t} \Rightarrow$
$2.5 = e^{0.06t} \Rightarrow \ln 2.5 = .06t \Rightarrow$
$t = \dfrac{\ln 2.5}{.06} \approx 15.3$ years

8. $A(t) = 10,000e^{0.065t} \Rightarrow$
$41,787 = 10,000e^{0.065t} \Rightarrow 4.1787 = e^{0.065t} \Rightarrow$
$\ln 4.1787 = .065t \Rightarrow$
$t = \dfrac{\ln 4.1787}{.065} \approx 22$ years

9. $A(t) = 1200e^{rt} \Rightarrow 12,500 = 1200e^{r(8)} \Rightarrow$
$\dfrac{125}{12} = e^{8r} \Rightarrow \ln\dfrac{125}{12} = 8r \Rightarrow$
$r = \dfrac{1}{8}\ln\dfrac{125}{12} \approx 0.293 = 29.3\%$

10. $A(t) = 22,220e^{rt} \Rightarrow$
$29,100,000 = 22,220e^{r(49)} \Rightarrow$
$\dfrac{1,455,000}{1111} = e^{49r} \Rightarrow \ln\dfrac{1,455,000}{1111} = 49r \Rightarrow$
$r = \dfrac{1}{49}\ln\dfrac{1,455,000}{1111} \approx .146 = 14.6\%$

11. $2 = e^{0.04t} \Rightarrow \ln 2 = 0.04t \Rightarrow$
$t = \dfrac{\ln 2}{.04} \approx 17.3$ years

12. $2 = e^{r(10)} \Rightarrow \ln 2 = 10r \Rightarrow$
$r = \dfrac{\ln 2}{10} \approx .069 = 6.9\%$

13. $3 = e^{r(15)} \Rightarrow \ln 3 = 15r \Rightarrow$
$r = \dfrac{\ln 3}{15} \approx .073 = 7.3\%$

14. $3 = e^{0.15t} \Rightarrow \ln 3 = .15t \Rightarrow$
$t = \dfrac{\ln 3}{.15} = 7.3$ years

15. $A(t) = P_0 e^{-rt} = 10000e^{-0.009t}$
$A(2) = 10000e^{-0.009(2)} \approx 9822$ SFr

16. $A'(t) = -rA(t)$
$A'(t) = -.009(9500) = 85.5$
The account is decreasing by 85.5 SFr per year.

17. $A(t) = e^{rt}$ (in millions) where t is the number of years since 2000.
$3 = e^{r(10)} \Rightarrow \ln 3 = 10r \Rightarrow r = \dfrac{\ln 3}{10} \approx .11$
$10 = e^{[(\ln 3)/10]t} \Rightarrow \ln 10 = \dfrac{\ln 3}{10}t \Rightarrow$
$t = \dfrac{10\ln 10}{\ln 3} \approx 21$ years
It will be worth \$10 million in 2021.

18. $A(t) = 10,000e^{rt}$ where t is the number of years since 1990.
$16,000 = 10,000e^{r(5)} \Rightarrow 1.6 = e^{5r} \Rightarrow$
$\ln 1.6 = 5r \Rightarrow r = \dfrac{\ln 1.6}{5} \approx .094$
$45,000 = 10,000e^{[(\ln 1.6)/5]t} \Rightarrow$
$4.5 = e^{[(\ln 1.6)/5]t} \Rightarrow \ln 4.5 = \dfrac{\ln 1.6}{5}t \Rightarrow$
$t = \dfrac{5\ln 4.5}{\ln 1.6} \approx 16$ years
It will be worth \$45,000 approximately 16 years (in 2006).

19. $A(t) = Pe^{0.08t}$
$1000 = Pe^{(0.08)(3)} \Rightarrow P = \dfrac{1000}{e^{0.24}} \approx \$786.63$

20. $A(t) = Pe^{0.08t}$
$2000 = Pe^{(0.08)(10)} \Rightarrow P = \dfrac{2000}{e^{0.8}} \approx \$898.66$

21. $A(t) = Pe^{0.045t}$
$10,000 = Pe^{(0.045)(5)} \Rightarrow$
$P = \dfrac{10,000}{e^{0.225}} \approx \$7985.16$

22. $A(t) = 559.9e^{rt}$
$1000 = 559.9e^{r(5)} \Rightarrow \dfrac{1000}{559.9} = e^{5r} \Rightarrow$
$\ln\dfrac{1000}{559.9} = 5r \Rightarrow r = \dfrac{1}{5}\ln\dfrac{1000}{559.9} \approx .116 = 11.6\%$

23. $A(t) = 70,200e^{0.13t}; B(t) = 60,000e^{0.14t}$
Set $A(t) = B(t)$ and solve for t.
$70,200e^{0.13t} = 60,000e^{0.14t} \Rightarrow$
$\dfrac{70,200}{60,000} = \dfrac{e^{0.14t}}{e^{0.13t}} \Rightarrow 1.17 = e^{0.01t} \Rightarrow$
$\ln 1.17 = .01t \Rightarrow t = \dfrac{\ln 1.17}{.01} \approx 15.7$ years

24. $A(t) = 10,000e^{0.08t}$
$A(2) - A(1) = 10,000e^{(0.08)(2)} - 10,000e^{(0.08)(1)}$
$\approx \$902.24$

25. a-B, b-D, c-G, d-A, e-F, f-E, g-H, h-C

26. a. From the graph, the balance will be \$600.

b. The slope of the tangent line at $t = 20$ is given as 54, so the rate is \$54 per year.

c. $A'(20) = rA(20) \Rightarrow 54 = r(600) \Rightarrow$

$$r = \frac{54}{600} = .09 = 9\%$$

27. a. From the graph in Fig. 2(a), the balance will be \$200.

b. From the graph in Fig. 2(b), the rate is \$8 per year.

c. $A'(20) = rA(200) \Rightarrow 8 = r(200) \Rightarrow$

$$r = \frac{8}{200} = .04 = 4\%$$

d. From the graph in Fig. 2(a), $A(t) = 300$ when $t = 30$ years.

e. From the graph in Fig. 2(b), $A'(t) = 12$ when $t = 30$ years.

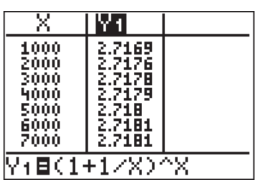
f. $A'(t) = rA(t)$, so the graph of $A'(t)$ is a constant multiple of the graph of $A(t)$.

28. a. From the graph, the balance will be \$2000.

b. From the graph, when $f(r) = 3000$, $r = 11$ so the interest rate will be 11%.

c. From the graph, $f(9) = 2500$
Growth rate = (.09)(2500) = \$225 per unit increase in interest

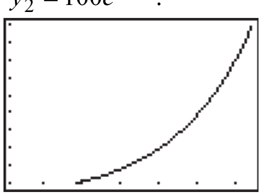
29.



X	Y1
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

Y1=(1+1/X)^X

30. Graph $y_1 = 100(1 + (.05/360))^{360x}$ and $y_2 = 100e^{.05x}$.

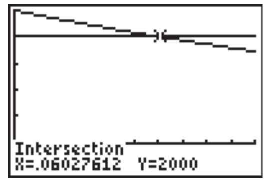


[0, 64] by [250, 2500]

When $x = 32$, $|y_1 - y_2| = .055$.

When $x = 64$, $|y_1 - y_2| = .545$.

31. Use a graphing calculator to graph $y_1 = 1200e^{-3x} + 800e^{-4x} + 500e^{-5x}$ and $y_2 = 2000$ and find the intersection point.



[0, .1] by [−500, 2500]

This occurs when $x \approx .06$, so $r \approx .06$.

5.3 Applications of the Natural Logarithm Function to Economics

1. $f(t) = t^2;\ \dfrac{f'(t)}{f(t)} = \dfrac{2t}{t^2} = \dfrac{2}{t}$

$$\frac{f'(10)}{f(10)} = \frac{2}{10} = \frac{1}{5} = 20\%$$
$$\frac{f'(50)}{f(50)} = \frac{2}{50} = \frac{1}{25} = 4\%$$

2. $f(t) = t^{10};\ \dfrac{f'(t)}{f(t)} = \dfrac{10t^9}{t^{10}} = \dfrac{10}{t}$

$$\frac{f'(10)}{f(10)} = \frac{10}{10} = 1 = 100\%$$
$$\frac{f'(50)}{f(50)} = \frac{10}{50} = 20\%$$

3. $f(t) = e^{0.3x};\ \dfrac{f'(t)}{f(t)} = \dfrac{0.3e^{0.3x}}{e^{0.3x}} = 0.3 = 30\%$

Thus for all x, the percentage rate of change is 30%.

4. $f(x) = e^{-0.05x};$

$$\frac{f'(t)}{f(t)} = \frac{-0.05e^{-0.05x}}{e^{-0.05x}} = -0.05 = -5\%$$

Thus for all x, the percentage rate of change is –5%.

5. $f(t) = e^{0.3t^2};\ \dfrac{f'(t)}{f(t)} = \dfrac{.6te^{.3t^2}}{e^{0.3t^2}} = .6t$

$$\frac{f'(1)}{f(1)} = .6 = 60\%;\quad \frac{f'(5)}{f(5)} = 3 = 300\%$$

6. $G(s) = e^{-0.05e^2}$; $\dfrac{G'(s)}{G(s)} = \dfrac{-0.1se^{-0.05s^2}}{e^{-0.05s^2}} = -.1s$

$\dfrac{G'(1)}{G(1)} = -.1(1) = -10\%$

$\dfrac{G'(10)}{G(10)} = -.1(10) = -100\%$

7. $f(p) = \dfrac{1}{p+2}$; $\dfrac{f'(p)}{f(p)} = -\dfrac{\frac{1}{(p+2)^2}}{\frac{1}{p+2}} = -\dfrac{1}{p+2}$

$\dfrac{f'(2)}{f(2)} = -\dfrac{1}{2+2} = -.25 = -25\%$

$\dfrac{f'(8)}{f(8)} = -\dfrac{1}{8+2} = -.1 = -10\%$

8. $g(p) = \dfrac{5}{2p+3}$; $\dfrac{f'(p)}{f(p)} = \dfrac{-\frac{5(2)}{(2p+3)^2}}{\frac{5}{2p+3}} = -\dfrac{2}{2p+3}$

$\dfrac{g'(1)}{g(1)} = -\dfrac{2}{2+3} = -\dfrac{2}{5} = -40\%$

$\dfrac{g'(11)}{g(11)} = -\dfrac{2}{22+3} = -\dfrac{2}{25} = -8\%$

9. $S = 50,000\sqrt{e^{\sqrt{t}}} = 50,000\left(e^{t^{1/2}}\right)^{1/2}$

$\ln S = \ln\left[50,000\left(e^{t^{1/2}}\right)^{1/2}\right]$

$= \ln 50,000 + \dfrac{1}{2}\ln e^{t^{1/2}} = \ln 50,000 + \dfrac{t^{1/2}}{2}$

Differentiating both sides,

$\dfrac{S'}{S} = 0 + \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)t^{-1/2} = \dfrac{1}{4\sqrt{t}}$

At $t = 4$, the percentage rate of growth is

$\dfrac{1}{4\sqrt{4}} = \dfrac{1}{8} = 12.5\%.$

10. $\dfrac{f'(t)}{f(t)} = \dfrac{.001 - .01e^{-t}}{4 + .001t + .01e^{-t}}$

$\dfrac{f'(0)}{f(0)} = \dfrac{.001 - .01}{4 + .01} \approx -.00224 = -.224\%$

$\dfrac{f'(1)}{f(1)} = \dfrac{0.001 - 0.01e^{-1}}{4 + 0.001 + 0.01e^{-1}}$

$\approx -0.00067 = -0.067\%$

$\dfrac{f'(2)}{f(2)} = \dfrac{0.001 - (0.01)e^{-2}}{4 + 0.002 + (0.01)e^{-2}} \approx -0.000088$

$= -0.0088\%$

11. $f(t) = 3.08 + .57t - .1t^2 + .01t^3$

$f'(t) = .57 - .2t + .03t^2$

a. $f(1) = 3.08 + .57(1) - .1(1)^2 + .01(1)^3$

$= 3.56$

In 2011, the wholesale price of one pound of ground beef was \$3.56.

$f'(1) = 0.57 - .2 + .03 = .4$

In 2011, the wholesale price of one pound of ground beef was changing at \$0.40 per year.

b. $\dfrac{f'(1)}{f(1)} = \dfrac{.4}{3.56} \approx .11 = 11\%$

In 2011, the price was rising at about 11% per year.

c. $f(6) = 3.08 + .57(6) - .1(6)^2 + .01(6)^3$

$= 5.06$

In 2016, the wholesale price of one pound of ground beef was \$5.06.

$f'(6) = .57 - .2(6) + .03(6)^2 = .45$

In 2016, the wholesale price of one pound of ground beef was changing at \$0.45 per year.

$\dfrac{f'(6)}{f(6)} = \dfrac{.45}{5.06} \approx .089 \approx 9\%$

In 2016, the price was rising at about 9% per year.

12. $f(t) = 1.4 + .26t - .1t^2 + .01t^3$

$f'(t) = .26 - .2t + .03t^2$

a. $f(2) = 1.4 + .26(2) - .1(2)^2 + .01(2)^3$

$= 1.60$

$f'(2) = .26 - .2(2) + .03(2)^2 = -.02$

$\dfrac{f'(2)}{f(2)} = -\dfrac{.02}{1.60} = -.0125 = -1.25\%$

In 2012, the price of one pound of pork was \$1.60. The price was decreasing at the rate of 1.25% per year.

b. $f(7) = 1.4 + .26(7) - .1(7)^2 + .01(7)^3$

$= 1.75$

$f'(7) = .26 - .2(7) + .03(7)^2 = .33$

$\dfrac{f'(7)}{f(7)} = \dfrac{.33}{1.75} = .188 \approx 19\%$

In 2017, the price of one pound of pork was \$1.75. The price was increasing at the rate of 19% per year.

For exercises 13−23, the elasticity of demand $E(p)$ at price p for the demand function $q = f(p)$ is $E(p) = \frac{-pf'(p)}{f(p)}$. Demand is elastic if $E(p_0) > 1$ and inelastic if $E(p_0) < 1$.

13. $E(p) = \frac{-p(-5)}{700-5p} = \frac{5p}{700-5p} = \frac{p}{140-p}$

$E(80) = \frac{80}{140-80} = \frac{4}{3} > 1$ so demand is elastic.

14. $E(p) = \frac{-p\left(-120e^{-0.2p}\right)}{600e^{-0.2p}} = \frac{pe^{-0.2p}}{5e^{-0.2p}} = \frac{p}{5}$;

$E(10) = \frac{10}{5} = 2 > 1$ so demand is elastic.

15. $E(p) = \frac{-p(-800p)}{400(116-p^2)} = \frac{2p^2}{116-p^2}$;

$E(6) = \frac{2\cdot 36}{116-36} = \frac{72}{80} < 1$ so demand is inelastic.

16. $E(p) = \frac{-p\left(-2\frac{77}{p^3}\right)}{\frac{77}{p^2}+3} = \frac{\frac{2(77)}{p^2}}{\frac{77}{p^2}+3} = \frac{154}{77+3p^2}$;

$E(1) = \frac{154}{77+3} > 1$ so demand is elastic.

17. $q' = p^2e^{-(p+3)}(-1) + e^{-(p+3)}2p$
$= pe^{-(p+3)}(2-p)$

$E(p) = \frac{-p\left(pe^{-(p+3)}\right)(2-p)}{p^2e^{-(p+3)}}$
$= -(2-p) = p-2$

$E(4) = 4 - 2 = 2 > 1$ so demand is elastic.

18. $E(p) = \frac{-p\left(-\frac{700}{(p+5)^2}\right)}{\frac{700}{p+5}} = \frac{p}{p+5}$;

$E(15) = \frac{15}{15+5} < 1$ so demand is inelastic.

19. a. $q = 3000 - 600p^{1/2}; q' = -300p^{-1/2}$

$E(p) = \frac{-p(-300p^{-1/2})}{3000-600p^{1/2}}$
$= \frac{300p^{1/2}}{3000-600p^{1/2}} = \frac{p^{1/2}}{10-2p^{1/2}}$

$E(4) = \frac{4^{1/2}}{10-2\cdot 4^{1/2}} = \frac{2}{6} = \frac{1}{3} < 1$

The demand is inelastic at $p = 4$.

b. Since demand is inelastic, to increase revenue, the ticket price should be raised.

20. a. $q' = 10,000\left[-1(p+50)^{-2}\right] = -\frac{10,000}{(p+50)^2}$

$E(p) = \frac{-pf'(p)}{f(p)} = \frac{-pq'}{q} = \frac{p\left(\frac{10,000}{(p+50)^2}\right)}{\frac{10,000}{p+50}-30}$

$E(150) = \frac{150\left(\frac{10,000}{(150+50)^2}\right)}{\frac{10,000}{150+50}-30} = \frac{\frac{150}{4}}{20} = \frac{15}{8} > 1$

The demand is elastic.

b. Since $q = \frac{1}{p+50} - 30$, a decrease in p increases q. Therefore, revenue will increase.

21. a. $E(p) = \frac{-p\left(-\frac{18,000}{p^2}\right)}{\frac{18,000}{p}-1500} = \frac{\frac{18,000}{p}}{\frac{18,000}{p}-1500}$

$E(6) = \frac{3000}{3000-1500} = \frac{3000}{1500} > 1$

Thus, demand is elastic.

b. If the price is lowered, revenue will increase.

22. a. $q = 2000(90-p)^{1/2}$;

$q' = 2000\left(\frac{1}{2}\right)(90-p)^{-1/2}(-1)$
$= -\frac{1000}{(90-p)^{1/2}}$

$E(p) = \frac{p\left(\frac{1000}{(90-p)^{1/2}}\right)}{2000(90-p)^{1/2}} = \frac{p}{2(90-p)}$

$E(65) = \frac{65}{2(90-65)} = \frac{65}{50} > 1$

Thus, demand is elastic.

b. The price of a ride should be lowered in order to increase the amount of money taken in by the subway.

23. a. $q = \dfrac{1000}{p^2};\ q' = -\dfrac{2000}{p^3}$

$$E(p) = \frac{\frac{2000}{p^2}}{\frac{1000}{p^2}} = 2$$

b. Yes, the country succeed in raising its revenue since the demand is always elastic.

24. $q = \dfrac{a}{p^m} \Rightarrow q' = -\dfrac{ma}{p^{m+1}}$

$$E(p) = \frac{-p\left(-\frac{ma}{p^{m+1}}\right)}{\frac{a}{p^m}} = \frac{\frac{ma}{p^m}}{\frac{a}{p^m}} = m$$

25. The relative rate of change of cost $C(x)$, with respect to x, is $\dfrac{C'(x)}{C(x)}$. The relative rate of change of quantity x with respect to x is $\dfrac{x'}{x} = \dfrac{1}{x}$. $E_c(x)$ is the ratio of these two:

$$E_c(x) = \frac{\frac{C'(x)}{C(x)}}{\frac{1}{x}} = \frac{x \cdot C'(x)}{C(x)}$$

26. The average cost (AC) is $\dfrac{C(x)}{x}$. The marginal cost (MC) is $C'(x)$. Thus,

$$\frac{MC}{AC} = \frac{C'(x)}{\frac{C(x)}{x}} = \frac{x \cdot C'(x)}{C(x)} = E_c \text{ (from ex. 25).}$$

27. $C(x) = \dfrac{1}{10}x^2 + 5x + 300$

$$C'(x) = \frac{2}{10}x + 5 = \frac{1}{5}x + 5$$

$$E_c(x) = \frac{x\left(\frac{1}{5}x + 5\right)}{\frac{1}{10}x^2 + 5x + 300} = \frac{\frac{1}{5}x^2 + 5x}{\frac{1}{10}x^2 + 5x + 300} = \frac{2x^2 + 50x}{x^2 + 50x + 3000}$$

$$E_c(50) = \frac{500 + 250}{250 + 250 + 300} = \frac{750}{800} = \frac{15}{16} < 1$$

28. $C(x) = 1000e^{0.02x} \Rightarrow C'(x) = 20e^{0.02x}$

$$E_c(x) = \frac{20xe^{0.02x}}{1000e^{0.02x}} = \frac{x}{50}$$

$$E_c(60) = \frac{60}{50} > 1.$$

An increase in the production level will result in a greater percentage increase in cost.

29. a. $q' = -30{,}000e^{-.5p}$

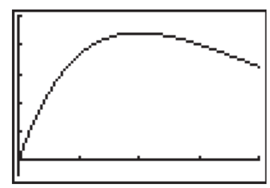
$$E(p) = \frac{-pq'}{q} = \frac{-p(-30{,}000e^{-.5p})}{60{,}000e^{-.5p}} = \frac{p}{2}$$

$$\frac{p}{2} < 1 \Rightarrow p < 2$$

Demand is inelastic for $p < 2$

b. $R(p) = q \cdot p = 60{,}000pe^{-.5p}$

Graph $y_1 = 60{,}000xe^{-.5x}$



[0, 4] by [−5000, 50,000]

The graph is increasing for $x < 2$, so revenue is increasing for $p < 2$.

5.4 Further Exponential Models

1. $f(x) = 5(1 - e^{-2x}), x \geq 0$

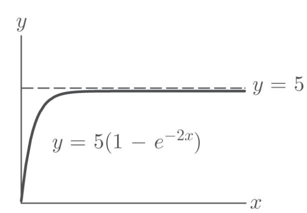
a. $f'(x) = 5\left[-e^{-2x}(-2)\right] = 10e^{-2x}$

$f''(x) = -20e^{-2x}$

Since $f'(x) > 0$ for all $x \geq 0$, $f(x)$ is increasing for all $x \geq 0$. Furthermore, since $f''(x) < 0$ for all $x \geq 0$, $f(x)$ is concave down for all $x \geq 0$.

b. Since $\lim_{x\to\infty} e^{-2x} = 0$, then for very large values of x, $f(x) \approx 5(1 - 0) = 5$.

c.

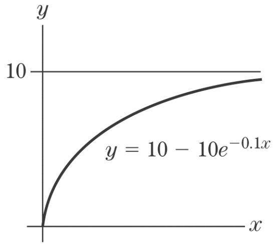


2. $g(x) = 10 - 10e^{-0.1x}, x \ge 0$

a. $g'(x) = e^{-0.1x}, g''(x) = -.1e^{-0.1x}$
Since $g'(x) > 0$ and $g''(x) < 0$ for all $x \ge 0$, it follows that $g(x)$ is increasing and concave down for all $x \ge 0$.

b. Since $\lim_{x\to\infty} 10e^{-0.1x} = 0$, then for very large values of x we have $g(x) \approx 10$.

c.



3. $y' = -2e^{-x}(-1) = 2e^{-x}$

$2 - y = 2 - 2(1 - e^{-x}) = 2 - 2 + 2e^{-x} = 2e^{-x} \Rightarrow$
$2 - y = y'$

4. $y' = -5e^{-2x}(-2) = 10e^{-2x}$

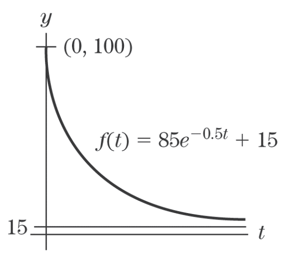
$10 - 2y = 10 - 2(5 - 5e^{-2x}) = 10e^{-2x} = y'$

5. $y' = f'(x) = -3e^{-10x}(-10) = 30e^{-10x}$

$10(3 - y) = 10\left[3 - \left(3 - 3e^{-10x}\right)\right]$
$= 30e^{-10x} = y'$

Also, $f(0) = 3(1 - 1) = 0$.

6.



7. The number of people who have heard about the indictment by time t is given by $f(t) = P(1 - e^{-kt})$. It is given that

$f(1) = \frac{1}{4}P = P\left(1 - e^{k(1)}\right) \Rightarrow \frac{1}{4} = 1 - e^{-k} \Rightarrow$

$e^{-k} = \frac{3}{4} \Rightarrow -k = \ln\frac{3}{4} \Rightarrow k \approx .29$

Thus, $f(t) = P\left(1 - e^{-.29t}\right)$.

Now, solve $\frac{3}{4}P = P\left(1 - e^{-.29t}\right)$ for t.

$\frac{3}{4} = 1 - e^{-.29t} \Rightarrow e^{-.29t} = \frac{1}{4} \Rightarrow$

$-.29t = \ln .25 \Rightarrow \frac{\ln .25}{-.29} = t \approx 4.8$ hours

8. Formula (8) is

$A(t) = M\left(1 - e^{-\lambda t}\right) = \frac{r}{\lambda}\left(1 - e^{-\lambda t}\right)$. If r were doubled, then $A(t) = \frac{2r}{\lambda}\left(1 - e^{-\lambda t}\right)$, and the amount of glucose will stabilize at $\frac{2r}{\lambda} = 2M$.

9. a. $f(7) = \frac{10{,}000}{1 + 50e^{-0.4(7)}} \approx 2475 \approx 2{,}500$

b. $f'(t) = \frac{200{,}000e^{-0.4t}}{\left(1 + 50e^{-0.4t}\right)^2}$

$f'(14) = \frac{200{,}000e^{-0.4(14)}}{\left(1 + 50e^{-0.4(14)}\right)^2} \approx 526$

≈ 500 people per day

c. $f(t) = 7000 \Rightarrow \frac{10{,}000}{1 + 50e^{-0.4t}} = 7000 \Rightarrow$

$\frac{10}{7} = 1 + 50e^{-0.4t} \Rightarrow \frac{3}{7} = 50e^{-0.4t} \Rightarrow$

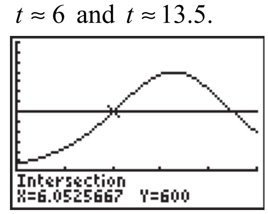
$\frac{3}{350} = e^{-0.4t} \Rightarrow \ln\frac{3}{350} = -.4t \Rightarrow$

$t = \frac{\ln\frac{3}{350}}{-.4} \approx 12$

7000 will have heard the news on day 12.

d. $f'(t) = 600 \Rightarrow \frac{200{,}000e^{-0.4t}}{\left(1 + 50e^{-0.4t}\right)^2} = 600$

Using a graphing calculator, we find that $t \approx 6$ and $t \approx 13.5$.



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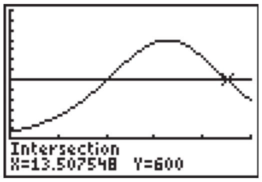
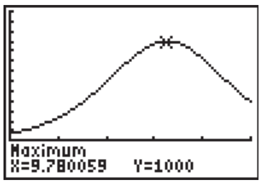


The news will be spreading at the rate of 600 people per day on day 6 and day 14.

e. f' has a maximum at approximately $t = 9.78$, so the news will be spreading at the greatest rate on day 10.



f. Using Equation (9), $y = \dfrac{M}{1+Be^{-Mkt}}$, we have $y = \dfrac{M}{1+Be^{-Mkt}} = \dfrac{10,000}{1+50e^{-0.4t}} \Rightarrow$
$M = 10,000$, $B = 50$, and $Mk = .4$ so $k = .00004$.
From Equation (10), we have
$y' = ky(M-y) \Rightarrow$
$f'(t) = .00004 f(t)(10,000 - f(t))$.

g. When $f(t) = 5000$,
$f'(t) = 0.00004(5000)(10,000-5000)$
$= 1000$ people per day.

10. $\dfrac{dC}{dt} = \dfrac{A'(t)}{V} = \dfrac{\lambda(M - A(t))}{V} = \lambda\left(\dfrac{M}{V} - \dfrac{A(t)}{V}\right)$
$= \lambda\left(\dfrac{M}{V} - C\right)$

11. a. $f(10) = 50,000\left(1 - e^{-0.3(10)}\right) \approx 47,510$

b. $f'(t) = 15,000e^{-0.3t}$
$f'(0) = 15,000e^{-0.3(0)} = 15,000$
Initially, the news is spreading at 15,000 people/day.

c. $f(t) = 22,500 \Rightarrow$
$50,000\left(1-e^{-0.3t}\right) = 22,500 \Rightarrow$
$1 - e^{-0.3t} = 0.5 \Rightarrow e^{-0.3t} = 0.5 \Rightarrow$
$-0.3t = \ln 0.5 \Rightarrow t = \dfrac{\ln 0.5}{-0.3} \approx 2.3$
22,500 people will have heard the news on about day 2.

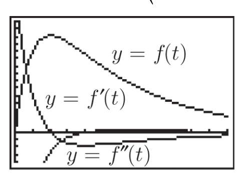
d. $f'(t) = 2500 \Rightarrow 15,000e^{-0.3t} = 2500 \Rightarrow$
$e^{-0.3t} = \dfrac{1}{6} \Rightarrow -0.3t = \ln\left(\dfrac{1}{6}\right) \Rightarrow$
$t = \dfrac{\ln\left(\frac{1}{6}\right)}{-0.3} \approx 6$
The news will be spreading at the rate of 2500 people per day on about day 6.

e. Using Equation (4), we have
$f(t) = P\left(1-e^{-kt}\right) \Rightarrow P = 50,000$ and $k = .3$. From Equation (3), we have
$f'(t) = k[P - f(t)] \Rightarrow$
$f'(t) = .3(50,000 - f(t))$.

f. When $f(t) = 25,000$,
$f'(t) = .3(50,000 - 25,000)$
$= 7500$ people per day.

12. Since the body uses up excess glucose at a rate proportional to the amount of excess glucose, we have $A(t) = A_0e^{-\lambda t}$ which gives the amount of excess glucose above equilibrium at any time, t, after an initial injection of A_0 milligrams. Thus, the doctor should give the injection and (after some known time interval) measure the amount of excess remaining. From this information the velocity constant of elimination, λ, can be calculated from the equation above.

13. a. $f(t) = 122\left(e^{-0.2t} - e^{-t}\right)$
$f'(t) = 122\left(-.2e^{-0.2t} + e^{-t}\right)$
$f''(t) = 122\left(.04e^{-0.2t} - e^{-t}\right)$

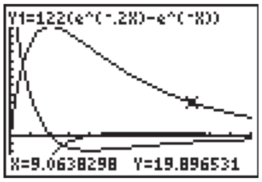


[0, 12] *by* [−20, 75]

b. $f(7) = 122\left(e^{-0.2(7)} - e^{-7}\right) \approx 30$
There are about 30 units of the drug in the bloodstream after 7 hours.

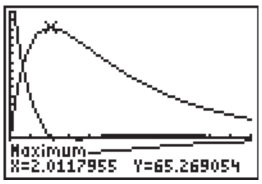
c. $f'(1) = 122\left(-.2e^{-0.2} + e^{-1}\right) \approx 25$
After 1 hour, the level of the drug is increasing at about 25 units per hour.

d. Use a graphing calculator to find when $f(t) = 20$ for $f(t)$ decreasing.

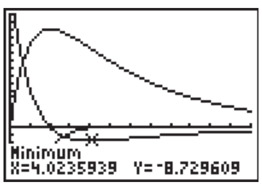


The level of the drug in the bloodstream is 20 units at about 9 hours

e. Use a graphing calculator to find the maximum of $f(t)$. The greatest level is about 65.3 units after 2 hours.



f. Use a graphing calculator to find the minimum of $f'(t)$.

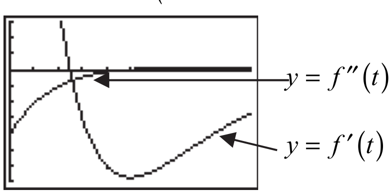


The level of the drug is decreasing the fastest after 4 hours.

14. a. $f(t) = 5000\left(20 + te^{-0.04t}\right)$

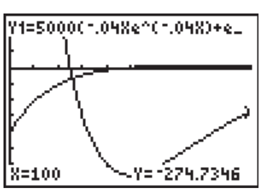
$f'(t) = 5000\left(-.04te^{-0.04t} + e^{-0.04t}\right)$

$f''(t) = 5000\left(.0016te^{-0.04t} - .08e^{-0.04t}\right)$



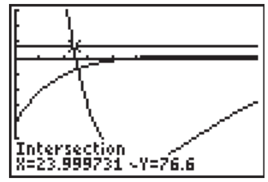
[0, 100] by [−700, 300]

b.



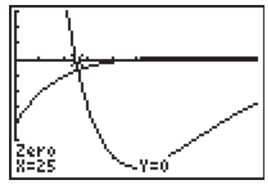
$f'(100) \approx -274.7$ bacteria per day.

c.



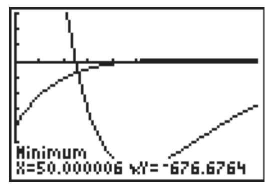
$f'(t) = 76.6$ at $t \approx 24$ days.

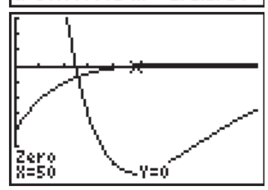
d. Find where $f'(t) = 0$.



The size of the culture is greatest at $t = 25$ days

e. Find where $f'(t)$ is a minimum (or where $f''(t) = 0$).





The size of the culture is decreasing the fastest at $t \approx 50$ days.

Chapter 5 Fundamental Concept Check Exercises

1. The differential equation that is key to solving exponential growth and decay problems is $y' = ky$. The solution to this differential equation is given by $y = Ce^{kt}$.

2. In the exponential growth equation $y = Ce^{kt}$, k is the growth constant. In the exponential decay equation $y = Pe^{-\lambda t}$, λ is the decay constant.

3. The half-life of a radioactive element is the time needed for a given quantity of that element to decay to one-half of its original mass.

4. See Section 5.1, Example 9. The most common substance used for radioactive-dating is radiocarbon, ^{14}C. Carbon 14 is produced in the upper atmosphere when cosmic rays react with atmospheric nitrogen. Because the ^{14}C eventually decays, the concentration of ^{14}C cannot rise above certain levels. An equilibrium is reached when ^{14}C is produced at the same rate as it decays. Scientists usually assume that the total amount of ^{14}C in the biosphere has remained constant over the past 50,000 years. Consequently, it is assumed that the ratio of ^{14}C to ordinary nonradioactive carbon 12, ^{12}C, has been constant during this same period. All living vegetation and most forms of animal life contain ^{14}C and ^{12}C in the same proportion. When an organism dies, it stops replacing its carbon; therefore, the amount of ^{14}C begins to decrease through radioactive decay, but the ^{12}C in the dead organism remains constant. The ratio of ^{14}C to ^{12}C can be later measured to determine when the organism died.

5. **a.** $A = Pe^{rt}$ **b.** $P = Ae^{-rt}$

6. The relative rate of change of $f(t)$ per unit change of t is given by $\frac{f'(t)}{f(t)}$. This quantity compares the rate of change of $f(t)$ (that is, $f'(t)$) with $f(t)$ itself. The *percentage rate of change* is the relative rate of change of $f(t)$ expressed as a percentage.

7. Let $q = f(p)$ denote a demand function where p is the price and q is the number of items sold at this price. The elasticity of demand is given by $E(p) = \frac{-pf'(p)}{f(p)}$.
Revenue changes in the opposite direction as price when demand is elastic, and in the same direction when demand is inelastic. If $E(p) > 1$, then demand is elastic. If $E(p) < 1$, then demand is inelastic.

8. An application of the differential equation $y' = k(M - y)$ is the diffusion of information by mass media. See Section 5.4, Example 2.

9. An application of the differential equation $y' = ky(M - y)$ is the spread of an epidemic. See Section 5.4, Example 4.

Chapter 5 Review Exercises

1. $P(x) = Ce^{-0.2x}$
$P(0) = 29.92 = Ce^{-0.2(0)} = C$
Thus, $P(x) = 29.92e^{-0.2x}$.

2. $P(x) = P_0e^{kt}$ (t in years, P_0 in herring gulls in 1990)
$P(13) = 2P_0 = P_0e^{(k)13} \Rightarrow \ln 2 = \ln e^{13k} \Rightarrow$
$\frac{\ln 2}{13} = k \Rightarrow k \approx 0.0533$
$P'(t) = .0533P(t)$

3. $10,000 = P_0e^{(0.12)5} = P_0e^{0.6}$
$P_0 = \frac{10,000}{e^{0.6}} \approx \$5488.12$

4. Solve $3000 = 1000e^{0.1t}$ for t.
$\ln 3 = \ln e^{0.1t} \Rightarrow \frac{\ln 3}{0.1} = t \Rightarrow t \approx 11$ years

5. $\frac{1}{2} = e^{-\lambda(12)}; \frac{\ln .5}{12} = -\lambda \Rightarrow \lambda \approx .058$

6. $.63 = e^{-0.00012t} \Rightarrow \frac{\ln .63}{-.00012} = t \Rightarrow t \approx 3850$ years old

7. **a.** $A(t) = 17e^{kt} \Rightarrow 19.3 = 17e^{k(7)} \Rightarrow$
$\frac{19.3}{17} = e^{7k} \Rightarrow \ln\frac{19.3}{17} = 7k \Rightarrow$
$k = \frac{1}{7}\ln\frac{19.3}{17} \approx .018$
$A(t) = 17e^{0.018t}$

b. $A(10) = 17e^{(0.018)(10)} \approx 20.4$ million

c. $25 = 17e^{0.018t} \Rightarrow \frac{25}{17} = e^{0.018t} \Rightarrow$
$\ln\frac{25}{17} = .018t \Rightarrow t = \frac{1}{.018}\ln\frac{25}{17} \approx 21$ years
The population will reach 25 million in the year 2011

8. $A(t) = 100{,}000e^{kt}$

$A(2) = 117{,}000 = 100{,}000e^{2k} \Rightarrow \dfrac{\ln \frac{11.7}{10}}{2} = k \Rightarrow$

$k \approx .0785$ so it earned 7.85%.

9. a. $A(t) = (10{,}000e^{.2(5)})e^{.06(5)} = (10{,}000e)e^{.3}$
$= 10{,}000e^{1.3} \approx \$36{,}693$

b. $A(t) = 10{,}000e^{.14(10)} = 10{,}000e^{1.4}$
$\approx \$40{,}552$
The alternative investment is superior by \$40,552 − \$36,693 = \$3859.

10. $P_1(t) = 1000e^{k_1 t}$

$P_1(21) = 2000 = 1000e^{k_1(21)} \Rightarrow \dfrac{\ln 2}{21} = k_1 \Rightarrow$

$k_1 \approx .033$

Thus, $P_1(t) = 1000e^{0.033t}$.

$P_2(t) = 710{,}000e^{k_2 t}$

$P_2(33) = 1{,}420{,}000 = 710{,}000e^{k_2(33)} \Rightarrow$

$\dfrac{\ln 2}{33} = k_2 \Rightarrow k_2 \approx .021$

Thus $P_2(t) = 710{,}000e^{0.021t}$.

Equating P_1 and P_2 and solving for t,

$1000e^{0.033t} = 710{,}000e^{0.021t} \Rightarrow$

$710 = e^{0.012t} \Rightarrow \dfrac{\ln 710}{.012} = t \Rightarrow t \approx 547$ min.

11. The growth constant is .02.
$y' = (.02)(3) = .06$
When the population reaches 3 million people, the population will be growing at the rate of 60,000 people per year.

$100{,}000 = .02y \Rightarrow y = \dfrac{100{,}000}{.02} = 5{,}000{,}000$

The population level at the growth rate of 100,000 people per year is 5 million people.

12. $y' = .4y \Rightarrow 200{,}000 = .4y \Rightarrow$

$y = \dfrac{200{,}000}{.4} = 500{,}000$

The size of the colony will be 500,000.
$y' = (.4)(1{,}000{,}000) = 400{,}000$
The colony will be growing at the rate of 400,000 bacteria per hour.

13. a–F, b–D, c–A, d–G, e–H, f–C, g–B, h–E

14. a. From the graph, $f(5) = 25$ grams.

b. From the graph, $f(t) = 10$ when $t = 9$ yr.

c. From the graph, $f(t) = 40$ when $t \approx 3$ yr, so the half-life is about 3 years.

d. From the graph, $f'(1) = -15$ grams/year.

e. From the graph, $f'(t) = -5$ when $t = 6$ yr.

15. $A'(t) = rA(t) \Rightarrow 60 = r(1000) \Rightarrow$

$r = \dfrac{60}{1000} = .06 = 6\%$

16. $A'(t) = rA(t) = (.045)(1230) = \$55.35$ per year

17. $\dfrac{f'(t)}{f(t)} = \dfrac{50e^{0.2t^2}(.4t)}{50e^{0.2t^2}} = .4t$

$\dfrac{f'(10)}{f(10)} = .4(10) = 4 = 400\%$

18. $E(p) = \dfrac{-p(-80p)}{4000 - 40p^2} = \dfrac{80p^2}{4000 - 40p^2} = \dfrac{2}{\frac{100}{p^2} - 1}$

$E(5) = \dfrac{2}{\frac{100}{25} - 1} = \dfrac{2}{4-1} = \dfrac{2}{3} < 1$ so demand is inelastic.

19. Since a price increase of \$0.16 represents a 2% increase in price, the quantity demanded will decrease 1.5(2%) = 3%. Demand is elastic, so revenue will decrease.

20. $f(p) = \dfrac{1}{3p+1},\ f'(p) = -\dfrac{3}{(3p+1)^2}$

$\dfrac{f'(p)}{f(p)} = \dfrac{-\frac{3}{(3p+1)^2}}{\frac{1}{3p+1}} = -\dfrac{3}{3p+1}$

$\dfrac{f'(1)}{f(1)} = -\dfrac{3}{3+1} = -\dfrac{3}{4} = -75\%$

21. $q = 1000p^2e^{-0.02(p+5)}$

$$q' = 1000p^2(e^{-0.02(p+5)})(-.02) + (e^{-0.02(p+5)})2000p$$
$$= -1000pe^{-0.02(p+5)}(.02p - 2)$$

$$E(p) = \frac{1000p^2e^{-0.02(p+5)}(.02p-2)}{1000p^2e^{-0.02(p+5)}} = .02p - 2$$

$E(200) = .02(200) - 2 = 4 - 2 = 2 > 1$
Thus, demand is elastic, so a decrease in price will increase revenue.

22. $E(p) = \dfrac{-p(ae^{-bp})(-b)}{ae^{-bp}} = pb$. Thus if $p = \dfrac{1}{b}$, $E(p) = \dfrac{1}{b} \cdot b = 1$.

23. Since for group A, $f'(t) = k(P - f(t))$, it follows that $f(t) = P(1 - e^{-kt})$.

$f(0) = 0 = 100\left(1 - e^{-k(0)}\right)$ and

$f(13) = 66 = 100\left(1 - e^{-k(13)}\right) \Rightarrow$ Thus,

$.66 = 1 - e^{-13k} \Rightarrow e^{-13k} = .34 \Rightarrow$

$\dfrac{\ln 0.34}{-13} = k \Rightarrow k \approx 0.083$

$f(t) = 100\left(1 - e^{-.083t}\right)$.

24. $f(t) = \dfrac{M}{1 + Be^{-Mkt}}$

Since 55 is the maximum height for the weed, $M = 55$.

$f(9) = 8 = \dfrac{55}{1 + Be^{-55(9)k}} \Rightarrow$

$1 + Be^{-55(9)k} = \dfrac{55}{8} \Rightarrow B = \dfrac{47}{8}e^{55(9)k}$

$f(25) = 48 = \dfrac{55}{1 + Be^{-55(25)k}} \Rightarrow$

$1 + Be^{-55(25)k} = \dfrac{55}{48} \Rightarrow B = \dfrac{7}{48}e^{55(25)k}$

So, $\dfrac{47}{8}e^{55(9)k} = \dfrac{7}{48}e^{55(25)k} \Rightarrow$

$\dfrac{47}{8} = \dfrac{7}{48}e^{880k} \Rightarrow \dfrac{\ln \frac{47 \cdot 48}{8 \cdot 7}}{880} = k \Rightarrow$

$k \approx .0042$ and $-Mk = -.231$

$B = \dfrac{47}{8}e^{55(9)k} \Rightarrow B = \dfrac{47}{8}e^{55(9)(0.0042)} \approx 46.98$

Thus, $f(t) = \dfrac{55}{1 + 46.98e^{-0.231t}}$.

25. a. From the graph, $f(11) = 400°$

b. From the graph $f'(6) = -100$. The temperature is decreasing at a rate of 100°/sec.

c. From the graph $f(t) = 200$ at $t = 17$ sec.

d. From the graph $f'(t) = -200$ at about $t = 2$ sec.

26. Since the culture grows at a rate proportional to its size, $500 = 10{,}000k \Rightarrow k = .05$. Then, $P' = .05(15{,}000) = 750$ bacteria per day.