Chapter 12 Probability and Calculus

12.1 Discrete Random Variables

1. $E(X) = 0\left(\frac{1}{5}\right) + 1\left(\frac{4}{5}\right) = \frac{4}{5}$

$V(X) = \left(0 - \frac{4}{5}\right)^2\left(\frac{1}{5}\right) + \left(1 - \frac{4}{5}\right)^2\left(\frac{4}{5}\right) = \frac{20}{125} = 0.16$

Standard deviation $= \sqrt{0.16} = 0.4$

2. $E(X) = 1\left(\frac{4}{9}\right) + 2\left(\frac{4}{9}\right) + 3\left(\frac{1}{9}\right) = \frac{15}{9} = \frac{5}{3}$

$V(X) = \left(1 - \frac{5}{3}\right)^2\left(\frac{4}{9}\right) + \left(2 - \frac{5}{3}\right)^2\left(\frac{4}{9}\right) + \left(3 - \frac{5}{3}\right)^2\left(\frac{1}{9}\right) = \frac{36}{81} = \frac{4}{9}$

Standard deviation $= \sqrt{\frac{36}{81}} = \frac{6}{9} = \frac{2}{3}$

3. a. $E(X) = 4(0.5) + 6(0.5) = 5$
$V(X) = (4-5)^2(0.5) + (6-5)^2(0.5) = 1$

b. $E(X) = 3(0.5) + 7(0.5) = 5$
$V(X) = (3-5)^2(0.5) + (7-5)^2(0.5) = 4$

c. $E(X) = 1(0.5) + 9(0.5) = 5$
$V(X) = (1-5)^2(0.5) + (9-5)^2(0.5) = 16$

As the difference between maximum and minimum values increases, so does the variance.

4. a. $E(X) = 20(0.1) + 4(0.4) + 6(0.4) + 8(0.1) = 5$
$V(X) = (2-5)^2(.1) + (4-5)^2(.4) + (6-5)^2(.4) + (8-5)^2(.1) = 2.6$

b. $E(X) = 2(0.3) + 4(0.2) + 6(0.2) + 8(0.3) = 5$
$V(X) = (2-5)^2(0.3) + (4-5)^2(0.2) + (6-5)^2(0.2) + (8-5)^2(0.3) = 5.8$

Values of (b) farther from $E(X)$ have higher probabilities. Thus the variance is larger.

5. a.

Outcome	0	1	2	3
Probability	$\frac{11}{52}$	$\frac{26}{52}$	$\frac{13}{52}$	$\frac{2}{52}$

b. $E(X) = 0\left(\frac{11}{52}\right) + 1\left(\frac{26}{52}\right) + 2\left(\frac{13}{52}\right) + 3\left(\frac{2}{52}\right)$
$= \frac{58}{52} = \frac{29}{26} \approx 1.12$

c. $E(X)$ is the average number of accidents per week in the given year.

6. a.

Outcome	0	1	2
Probability	$\frac{30}{60}$	$\frac{20}{60}$	$\frac{10}{60}$

b. $E(X) = 0\left(\frac{30}{60}\right) + 1\left(\frac{20}{60}\right) + 2\left(\frac{10}{60}\right) = \frac{2}{3}$

c. $E(X)$ is the average number of calls coming into the switchboard each minute.

7. a. $\dfrac{\text{area within } \left(\frac{1}{2}\right) \text{ unit of center}}{\text{Total area}} = \dfrac{\pi\left(\frac{1}{2}\right)^2}{\pi(1)^2} = \dfrac{1}{4}$
$= 0.25$

Thus 25% of the points in the circle are within $\frac{1}{2}$ unit of the center.

b. $100 \times \dfrac{\pi c^2}{\pi(1)^2} = 100c^2\%$

8. a. $\frac{1}{2}$ **b.** $\frac{1}{4}$

c. $\frac{1}{100}$ **d.** 0

9. Let X be the profit that the grower makes if he does not protect the fruit. Then
$E(X) = 100{,}000(0.75) + 60{,}000(0.25)$
$= 90{,}000 < 95{,}000.$
Therefore he *should* spend the \$5000 to protect the fruit.

10. Let X be the random variable defined in Exercise 9. In this case,
$E(X) = 100{,}000(0.80) + 85{,}000(0.1) + 75{,}000(0.1)$
$= 96{,}000 > 95{,}000$, so the grower should *not* spend the money to protect the fruit.

12.2 Continuous Random Variables

1. I. $\frac{1}{18}x \ge 0$ for all $0 \le x \le 6$

II. $\int_0^6 \frac{1}{18}x\,dx = \frac{1}{18}\left(\frac{x^2}{2}\right)\Big|_0^6 = 1 - 0 = 1$

2. I. $2(x-1) = 2x - 2 \ge 0$ for all $1 \le x \le 2$

II. $\int_1^2 (2x-2)dx = x^2 - 2x\Big|_1^2 = 0 + 1 = 1$

3. I. $\frac{1}{4} \ge 0$

II. $\int_1^5 \frac{1}{4}dx = \frac{1}{4}x\Big|_1^5 = \frac{5}{4} - \frac{1}{4} = 1$

4. I. $\frac{8}{9}x \ge 0$ for all $0 \le x \le \frac{3}{2}.$

II. $\int_0^{3/2} \frac{8}{9}x\,dx = \frac{4}{9}x^2\Big|_0^{3/2} = 1 - 0 = 1$

5. I. $5x^4 \ge 0$ for all $0 \le x \le 1.$

II. $\int_0^1 5x^4 dx = x^5\Big|_0^1 = 1 - 0 = 1$

6. I. $\frac{3}{2}x - \frac{3}{4}x^2 = \frac{3}{2}x\left(1 - \frac{1}{2}x\right) \ge 0$ for all $0 \le x \le 2.$

II. $\int_0^2 \left(\frac{3}{2}x - \frac{3}{4}x^2\right)dx = \frac{3}{4}x^2 - \frac{1}{4}x^3\Big|_0^2$
$= 1 - 0 = 1$

7. $\int_1^3 kx\,dx = \frac{k}{2}x^2\Big|_1^3 = \frac{9k}{2} - \frac{k}{2} = 4k = 1;$ so $k = \frac{1}{4}.$

8. $\int_0^2 kx^2 dx = \frac{k}{3}x^3\Big|_0^2 = \frac{8}{3}k = 1;$ so $k = \frac{3}{8}.$

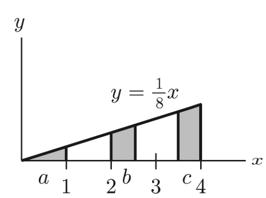
9. $\int_5^{20} k\,dx = kx\Big|_5^{20} = 20k - 5k = 15k = 1;$ so
$k = \frac{1}{15}.$

10. $\int_1^4 kx^{-1/2}dx = 2kx^{1/2}\Big|_1^4 = 4k - 2k = 2k = 1;$ so
$k = \frac{1}{2}.$

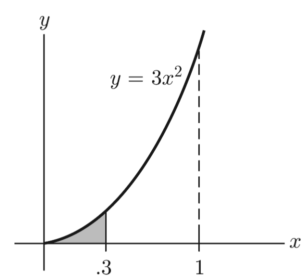
11. $\int_0^1 kx^2(1-x)dx = \int_0^1 (kx^2 - kx^3)dx$
$= \frac{k}{3}x^3 - \frac{k}{4}x^4\Big|_0^1 = \frac{k}{12} = 1;$
so $k = 12.$

12. $\int_0^3 k(3x - x^2)dx = \frac{3}{2}kx^2 - \frac{k}{3}x^3\Big|_0^3 = \frac{27}{2}k - 9k$
$= \frac{9}{2}k = 1;$
so $k = \frac{2}{9}.$

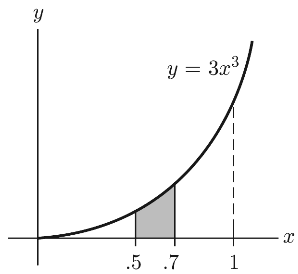
13.



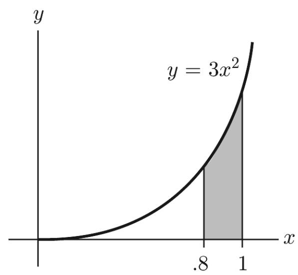
14. a.



b.



c.



15. $\int_1^2 \frac{1}{18} x\,dx = \left.\frac{x^2}{36}\right|_1^2 = \frac{1}{12}$

16. $\int_{1.5}^{1.7} (2x-2)dx = x^2 - 2x\Big|_{1.5}^{1.7} = 0.24$

17. $\int_1^3 \frac{1}{4} dx = \frac{1}{2}$

18. $\int_1^{3/2} \frac{8}{9} x\,dx = \left.\frac{4}{9}x^2\right|_1^{3/2} = \frac{5}{9}$

19. $\int_{35}^{50} \frac{1}{20} dx = \left.\frac{x}{20}\right|_{35}^{50} = \frac{15}{20} = \frac{3}{4}$

20. $\int_0^4 \frac{11}{10}(x+1)^{-2}\,dx = -\left.\frac{11}{10}(x+1)^{-1}\right|_0^4$

$= -\frac{11}{50} + \frac{11}{10} = \frac{44}{50} = \frac{22}{25}$

21. $f(x) = F'(x) = \frac{1}{4}(x-1)^{-1/2}$

22. $f(x) = F'(x) = \frac{8}{3}x^{-3}$

23. $F(x) = \frac{1}{5}x + C;\ F(2) = \frac{1}{5}(2) + C = 0;$ so

$C = -\frac{2}{5}$ and $F(x) = \frac{1}{5}x - \frac{2}{5}.$

24. $F(x) = \frac{3}{2}x - \frac{x^2}{4} + C;\ F(1) = \frac{5}{4} + C = 0;$ so

$C = -\frac{5}{4}$ and $F(x) = \frac{3}{2}x - \frac{1}{4}x^2 - \frac{5}{4}.$

25. a. $\int_2^3 \frac{1}{21}x^2 dx = \left.\frac{x^3}{63}\right|_2^3 = \frac{19}{63}$

b. $F(1) = \frac{x^3}{63} + C;\ F(1) = \frac{1}{63} + C = 0;$

$F(x) = \frac{x^3}{63} - \frac{1}{63}$

c. $F(3) - F(2) = \frac{19}{63}$

26. a. $\int_3^4 \left(\frac{4}{9}x - \frac{1}{9}x^2\right)dx = \left.\frac{2}{9}x^2 - \frac{1}{27}x^3\right|_3^4$

$= \frac{32}{27} - 1 = \frac{5}{27}$

b. $F(x) = \frac{2}{9}x^2 - \frac{1}{27}x^3 + C;$

$F(1) = \frac{2}{9} - \frac{1}{27} + C = 0;\ C = \frac{-5}{27}$

$F(x) = \frac{2}{9}x^2 - \frac{1}{27}x^3 - \frac{5}{27}$

c. $F(4) - F(3) = \frac{5}{27}$

27. Points whose largest coordinate has value $\le x$ lie within the square with vertices (0, 0), (0, x),
(x, x), and (x, 0). The area of this square is x^2. Thus $F(x)$ = the probability that a randomly selected point has maximum coordinate $\le x$ $= \frac{x^2}{4}.$

28. $f(x) = F'(x) = \frac{x}{2},\ p \le x \le 2$

29. Points whose coordinates sum to a value $\le x$ lie within the triangle with vertices (0, 0), (0, x), and (x, 0). The area of this triangle is $\frac{x^2}{2}$. Thus $F(x)$ = the probability that a randomly selected point has coordinates summing to a value $\le x = \frac{\left(\frac{x^2}{2}\right)}{2} = \frac{x^2}{4}.$

30. $f(x) = F'(x) = \frac{x}{2},\ 0 \le x \le 2$

31. $\int_0^5 2ke^{-kx}dx = -2e^{-kx}\Big|_0^5 = -2e^{-5k} + 2$
$= -2(e^{\ln 2})^{-1/2} + 2 = -\frac{2}{\sqrt{2}} + 2$
$= 2 - \sqrt{2} \approx 0.59$

32. $\int_{10-M}^{10} 2ke^{-kx}dx = -2e^{-kx}\Big|_{10-M}^{10}$
$= -2e^{-10k} + 2e^{-10k+Mk}$
$k = \frac{\ln 2}{10} = -1 + 2^{M/10} = 0.1$
$2^{M/10} = 1.1;\ \frac{M}{10}\ln 2 = \ln 1.1;$
$M = 10\frac{\ln 1.1}{\ln 2} \approx 1.37504$ days

33. $\int_0^b \frac{1}{3}dx = \frac{1}{3}b = 0.6$, so $b = 1.8$.

34. $\int_a^2 \frac{2}{3}x\,dx = \frac{1}{3}x^2\Big|_a^2 = \frac{4}{3} - \frac{a^2}{3} = \frac{1}{3}$, so $a = \sqrt{3}$.

35. $F(b) = \frac{b^2}{4} = .09$, so $b = .6$.

36. $F(b) = (b-1)^2 = \frac{1}{4}$, so $b - 1 = \frac{1}{2}$; $b = \frac{3}{2}$.

37. a. I. $4x^{-5} \ge 0$ for all $x \ge 1$.
II. $\int_1^\infty 4x^{-5}dx = \lim_{b\to\infty}\int_1^b 4x^{-5}dx$
$= \lim_{b\to\infty}\left[-x^{-4}\Big|_1^b\right]$
$= \lim_{b\to\infty}\left[1 - \frac{1}{b^4}\right] = 1$

b. $F(x) = -x^{-4} + C$; $F(1) = -1 + C = 0$; so $C = 1$ and $F(x) = 1 - x^{-4}$.

c. $\Pr(1 \le X \le 2) = F(2) - F(1) = 1 - \frac{1}{16} = \frac{15}{16}$
$\Pr(2 \le X) = 1 - \Pr(1 \le X \le 2) = \frac{1}{16}$

38. a. I. $2(x+1)^{-3} \ge 0$ for all $x \ge 0$.
II. $\int_0^\infty 2(x+1)^{-3}dx = \lim_{b\to\infty}\int_0^b 2(x+1)^{-3}dx$
$= \lim_{b\to\infty}\left[-(x+1)^{-2}\Big|_0^b\right]$
$= \lim_{b\to\infty}\left[1 - \frac{1}{(b+1)^2}\right] = 1$

b. $F(x) = -\frac{1}{(x+1)^2} + C$, $F(0) = -1 + C = 0$;
so $C = 1$ and $F(x) = 1 - \frac{1}{(x+1)^2}$.

c. $\Pr(1 \le X \le 2) = F(2) - F(1) = \frac{8}{9} - \frac{3}{4} = \frac{5}{36}$
$\Pr(3 \le X) = 1 - \Pr(0 \le X < 3) = 1 - F(3)$
$= 1 - \left(1 - \frac{1}{16}\right) = \frac{1}{16}$

12.3 Expected Value and Variance

1. $E(X) = \int_0^6 \frac{1}{18}x^2dx = \frac{x^3}{54}\Big|_0^6 = 4;$

$V(X) = \int_0^6 \frac{1}{18}x^3dx - 4^2 = \frac{x^4}{72}\Big|_0^6 - 16 = 2$

2. $E(X) = \int_1^2 (2x^2 - 2x)dx = \frac{2}{3}x^3 - x^2\Big|_1^2 = \frac{5}{3}$

$V(X) = \int_1^2 (2x^3 - 2x^2)dx - \left(\frac{5}{3}\right)^2$
$= \frac{1}{2}x^4 - \frac{2}{3}x^3\Big|_1^2 - \frac{25}{9} = \frac{17}{6} - \frac{25}{9} = \frac{1}{18}$

3. $E(X) = \int_1^5 \frac{1}{4}x\,dx = \frac{x^2}{8}\Big|_1^5 = 3$

$V(X) = \int_1^5 \frac{1}{4}x^2dx - 3^2 = \frac{x^3}{12}\Big|_1^5 - 9$
$= \frac{31}{3} - 9 = \frac{4}{3}$

4. $E(X) = \int_0^{3/2} \frac{8}{9}x^2dx = \frac{8}{27}x^3\Big|_0^{3/2} = 1$

$V(X) = \int_0^{3/2} \frac{8}{9}x^3dx - 1^2 = \frac{8}{36}x^4\Big|_0^{3/2} - 1$
$= \frac{9}{8} - 1 = \frac{1}{8}$

5. $E(X) = \int_0^1 5x^5dx = \frac{5}{6}x^6\Big|_0^1 = \frac{5}{6}$

$V(X) = \int_0^1 5x^6dx - \left(\frac{5}{6}\right)^2$
$= \frac{5}{7}x^7\Big|_0^1 - \left(\frac{5}{6}\right)^2 = \frac{5}{252}$

6. $E(X)=\int_0^2\left(\frac{3}{2}x^2-\frac{3}{4}x^3\right)dx$

$$=\frac{x^3}{2}-\frac{3}{16}x^4\Big|_0^2=1$$

$$V(X)=\int_0^2\left(\frac{3}{2}x^3-\frac{3}{4}x^4\right)dx-1^2$$

$$=\frac{3}{8}x^4-\frac{3}{20}x^5\Big|_0^2-1=\frac{6}{5}-1=\frac{1}{5}$$

7. $E(X)=\int_0^1 12x^2(1-x)^2\,dx$

$$=\int_0^1(12x^4-24x^3+12x^2)dx$$

$$=\frac{12}{5}x^5-6x^4+4x^3\Big|_0^1=\frac{2}{5}$$

$$V(X)=\int_0^1(12x^5-24x^4+12x^3)dx-\frac{4}{25}$$

$$=2x^6-\frac{24}{5}x^5+3x^4\Big|_0^1-\frac{4}{25}=\frac{1}{25}$$

8. $E(X)=\int_0^4\frac{3}{16}x^{3/2}dx=\frac{3}{40}x^{5/2}\Big|_0^4=\frac{12}{5}$

$$V(X)=\int_0^4\frac{3}{16}x^{5/2}dx-\frac{144}{25}=\frac{3}{56}x^{7/2}\Big|_0^4-\frac{144}{25}$$

$$=\frac{48}{7}-\frac{144}{25}=\frac{192}{175}$$

9. a. $f(x)=30x^2(1-x)^2=30x^4-60x^3+30x^2,$
so $F(x)=6x^5-15x^4+10x^3+C;$
$F(0)=C=0$, so $F(x)=6x^5-15x^4+10x^3.$

b. $F(0.25)=\frac{53}{512}$

c. $E(X)=\int_0^1(30x^5-60x^4+30x^3)dx$

$$=5x^6-12x^5+\frac{15}{2}x^4\Big|_0^1=\frac{1}{2}$$

On average, the newspaper devotes $\frac{1}{2}$ of its space to advertising.

d. $V(X)=\int_0^1(30x^6-60x^5+30x^4)dx-\frac{1}{4}$

$$=\frac{30}{7}x^7-10x^6+6x^5\Big|_0^1-\frac{1}{4}=\frac{1}{28}$$

10. $f(x)=20x^3(1-x)=-20x^4+20x^3$

a. $E(X)=\int_0^1-20x^5+20x^4dx$

$$=-\frac{10}{3}x^6+4x^5\Big|_0^1=\frac{2}{3}$$

On average, $\frac{2}{3}$ of the new restaurants make a profit during their first year of operation.

b. $V(X)=\int_0^1(-20x^6+20x^5)dx-\frac{4}{9}$

$$=-\frac{20}{7}x^7+\frac{10}{3}x^6\Big|_0^1-\frac{4}{9}=\frac{2}{63}$$

11. a. Since $F(x)=\frac{1}{9}x^2,\ 0\le x\le 3,\ f(x)=\frac{2}{9}x$ and $E(X)=\int_0^3\frac{2}{9}x^2dx=\frac{2}{27}x^3\Big|_0^3=2.$

This means that the average useful life of the component is 200 hrs.

b. $V(X)=\int_0^3\frac{2}{9}x^3dx-4=\frac{1}{18}x^4\Big|_0^3-4=\frac{1}{2}$

12. a. Since $F(x)=\frac{1}{125}x^3,\ 0\le x\le 5,$

$f(x)=\frac{3}{125}x^2$, and

$$E(X)=\int_0^5\frac{3}{125}x^3dx=\frac{3}{4\cdot125}x^4\Big|_0^5=\frac{15}{4}.$$

On average, the assembly takes $\frac{15}{4}$ minutes to complete.

b. $V(X)=\int_0^5\frac{3}{125}x^4dx-\left(\frac{15}{4}\right)^2$

$$=\frac{3}{5^4}x^5\Big|_0^5-\left(\frac{15}{4}\right)^2=15-\frac{225}{16}=\frac{15}{16}$$

13. $E(X)=\int_0^{12}\frac{1}{72}x^2dx=\frac{x^3}{72\cdot3}\Big|_0^{12}=8$ (minutes)

14. $E(X)=\int_0^{10}\frac{-6x^3+60x^2}{1000}dx$

$$=\frac{1}{1000}\left[-\frac{3}{2}x^4+20x^3\Big|_0^{10}\right]=5$$

15. a. $f(x)=\dfrac{6x-x^2}{18},\ 3\le x\le 6$, so

$$F(x)=\frac{x^2}{6}-\frac{x^3}{54}+C.$$

$F(3)=\dfrac{3}{2}-\dfrac{1}{2}+C=0$, so $C=-1$ and

$$F(x)=\frac{x^2}{6}-\frac{x^3}{54}-1.$$

b. $F(5)=\dfrac{25}{6}-\dfrac{125}{54}-1=\dfrac{23}{27}$

c.
$$\begin{aligned}E(X)&=\frac{1}{18}\int_3^6(6x^2-x^3)dx\\&=\frac{1}{18}\left(2x^3-\frac{x^4}{4}\right)\Bigg|_3^6=\frac{1}{18}\left(108-\frac{135}{4}\right)\\&=4.125\end{aligned}$$

Thus the mean completion time is 412.5 worker-hrs.

d.
$$\begin{aligned}V(X)&=\frac{1}{18}\int_3^6(6x^3-x^4)dx-(4.125)^2\\&=\frac{1}{18}\left(\frac{3}{2}x^4-\frac{x^5}{5}\right)\Bigg|_3^6-(4.125)^2\\&\approx .5344\end{aligned}$$

16. $f(x)=4(x-1)^3,\ 1\le x\le 2$

a.
$$\begin{aligned}\Pr(X>1.5)&=1-\Pr(X\le 1.5)\\&=1-\int_1^{1.5}4(x-1)^3\,dx\\&=1-\left[(x-1)^4\Big|_1^{1.5}\right]\\&=1-\frac{1}{16}=\frac{15}{16}\end{aligned}$$

b.
$$\begin{aligned}E(X)&=\int_1^2 4x(x-1)^3\,dx\\&=\int_1^2(4x^4-12x^3+12x^2-4x)dx\\&=\frac{4}{5}x^5-3x^4+4x^3-2x^2\Big|_1^2=\frac{9}{5}\\&=1.8\text{ thousand gallons}\end{aligned}$$

17.
$$\begin{aligned}E(X)&=\int_1^\infty 4x^{-4}dx=\lim_{b\to\infty}\left[-\frac{4}{3}x^{-3}\Big|_1^b\right]\\&=\lim_{b\to\infty}\left[\frac{4}{3}-\frac{4}{3}b^{-3}\right]=\frac{4}{3}\end{aligned}$$

$$\begin{aligned}V(X)&=\int_1^\infty 4x^{-3}dx-\left(\frac{4}{3}\right)^2\\&=\lim_{b\to\infty}\left[-2x^{-2}\Big|_1^b\right]-\frac{16}{9}\\&=\lim_{b\to\infty}[2-2b^{-2}]-\frac{16}{9}=\frac{2}{9}\end{aligned}$$

18. $f(x)=3x^{-4},\ x\ge 1$

$$\begin{aligned}E(X)&=\int_1^\infty 3x^{-3}dx=\lim_{b\to\infty}\left[-\frac{3}{2}x^{-2}\Big|_1^b\right]\\&=\lim_{b\to\infty}\left[\frac{3}{2}-\frac{3}{2}b^{-2}\right]=\frac{3}{2}\end{aligned}$$

$$\begin{aligned}V(X)&=\int_1^\infty 3x^{-2}dx-\left(\frac{3}{2}\right)^2\\&=\lim_{b\to\infty}\left[-3x^{-1}\Big|_1^b\right]-\frac{9}{4}=\frac{3}{4}\end{aligned}$$

19. $\displaystyle\int_0^M\frac{1}{18}x\,dx\Rightarrow\frac{x^2}{36}\Bigg|_0^M=\frac{1}{2}\Rightarrow$

$\dfrac{M^2}{36}=\dfrac{1}{2}\Rightarrow M=\sqrt{18}=3\sqrt{2}$

20. $\displaystyle\int_1^M(2x-2)dx\Rightarrow x^2-2x\Big|_1^M=\frac{1}{2}\Rightarrow$

$M^2-2M+1=\dfrac{1}{2}\Rightarrow(M-1)^2=\dfrac{1}{2}\Rightarrow$

$M=1+\dfrac{\sqrt{2}}{2}$

21. $F(M)=\dfrac{1}{9}x^2=\dfrac{1}{2}\Rightarrow$

$x=\dfrac{3\sqrt{2}}{2}$ (hundred hours)

22. $F(T)=\dfrac{1}{125}T^3=\dfrac{1}{2}\Rightarrow T=\dfrac{5}{\sqrt[3]{2}}$

23. $\displaystyle\int_0^T\frac{11}{10(x+1)^2}dx=\frac{1}{2}\Rightarrow-\frac{11}{10}(x+1)^{-1}\Big|_0^T=\frac{1}{2}\Rightarrow$

$\dfrac{11}{10}-\dfrac{11}{10(T+1)}=\dfrac{1}{2}\Rightarrow T=\dfrac{5}{6}$ minutes

24. $\displaystyle\int_1^M 4(x-1)^3\,dx=\frac{1}{2}\Rightarrow(x-1)^4\Big|_1^M=\frac{1}{2}\Rightarrow$

$(M-1)^4=\dfrac{1}{2}\Rightarrow M=\sqrt[4]{\dfrac{1}{2}}+1$

25. By definition, $E(X) = \int_A^B xf(x)\,dx$ and $F'(x) = f(x)$. Using integration by parts,

$$\int_A^B xf(x)\,dx = xF(x)\Big|_A^B - \int_A^B F(x)\,dx$$
$$= BF(B) - AF(A) - \int_A^B F(x)\,dx$$
$$= B(1) - A(0) - \int_A^B F(x)\,dx$$
$$= B - \int_A^B F(x)\,dx$$

26. $5 - \int_0^5 \frac{1}{125}x^3 dx = 5 - \frac{5^4}{4(125)} + 0 = 3.75$

12.4 Exponential and Normal Random Variables

1. $E(X) = \frac{1}{3}$; $V(X) = \frac{1}{9}$

2. $E(X) = 4$; $V(X) = 16$

3. $E(X) = 5$; $V(X) = 25$

4. $E(X) = \frac{2}{3}$; $V(X) = \frac{4}{9}$

5. The probability density function is $f(x) = 2e^{-2x}$. Thus

$$\Pr\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^1 2e^{-2x}dx = -e^{-2x}\Big|_{1/2}^1$$
$$= e^{-1} - e^{-2}$$

6. $\int_0^{1/3} 2e^{-2x}dx = -e^{-2x}\Big|_0^{1/3} = 1 - e^{-2/3}$

7. The probability density function is $f(x) = \frac{1}{3}e^{-(1/3)x}$. Thus

$$\Pr(X < 2) = \int_0^2 \frac{1}{3}e^{-(1/3)x}dx = -e^{-(1/3)x}\Big|_0^2$$
$$= 1 - e^{-2/3}$$

8. $\Pr(X > 5) = 1 - \Pr(X \le 5)$

$$= 1 - \int_0^5 \frac{1}{3}e^{-(1/3)x}dx$$
$$= 1 + \left[e^{-(1/3)x}\Big|_0^5\right] = e^{-5/3}$$

9. The probability density function is $f(x) = \frac{1}{20}e^{-(1/20)x}$. Thus

$\Pr(X > 60) = 1 - \Pr(X \le 60)$

$$= 1 - \int_0^{60} \frac{1}{20}e^{-(1/20)x}dx$$
$$= 1 + \left[e^{-(1/20)x}\Big|_0^{60}\right] = e^{-3}$$

10. $\Pr(10 < X < 30) = \int_{10}^{30} \frac{1}{20}e^{-(1/20)x}dx$

$$= -e^{-(1/20)x}\Big|_{10}^{30} = e^{-1/2} - e^{-3/2}$$

11. The probability density function is $f(x) = \frac{1}{2}e^{-(1/2)x}$. Thus

$$\Pr(X < 4) = \int_0^4 \frac{1}{2}e^{-(1/2)x}dx = -e^{-(1/2)x}\Big|_0^4$$
$$= 1 - e^{-2}$$

12. $\Pr(X \ge 5) = 1 - \Pr(X < 5) = 1 - \int_0^5 \frac{1}{2}e^{-(1/2)x}dx$

$$= 1 + \left[e^{-(1/2)x}\Big|_0^5\right] = e^{-5/2}$$

13. The probability density function is $f(x) = \frac{1}{72}e^{-(1/72)x}$.

a. $\Pr(X > 24) = 1 - \Pr(X \le 24)$

$$= 1 - \int_0^{24} \frac{1}{72}e^{-(1/72)x}dx$$
$$= 1 + \left[e^{-(1/72)x}\Big|_0^{24}\right] = e^{-(1/3)}$$

b. $r(t) = \Pr(X > t) = 1 - \Pr(X \le t)$

$$= 1 - \int_0^t \frac{1}{72}e^{-(1/72)x}dx$$
$$= 1 + \left[e^{-(1/72)x}\Big|_0^t\right] = e^{-t/72}$$

14. a. $S(x) = 1 - F(x)$ since

$$S(x) = \Pr(X \ge x) = 1 - \Pr(X < x)$$
$$= 1 - F(x)$$
$$= 1 - \int_0^x ke^{-kt}dt$$
$$= 1 + \left[e^{-kt}\Big|_0^x\right] = e^{-kx}$$

b. $S(t) = e^{-5k} = 0.9 \Rightarrow -5k = \ln(0.9) \Rightarrow$

$$k = \frac{-\ln(0.9)}{5} \approx 0.02107$$

15. $\mu = 4$, $\sigma = 1$

16. $\mu = -5$, $\sigma = 1$

17. $\mu = 0$, $\sigma = 3$

18. $\mu = 3$, $\sigma = 5$

19. $f(x) = e^{-x^2/2}$; $f'(x) = -xe^{-x^2/2}$
Thus $f'(0) = 0$.
$f''(x) = -\left[e^{-x^2/2} - x^2 e^{-x^2/2}\right]$ so
$f''(0) = -1 < 0$.
Therefore $f(x)$ has a relative maximum at $x = 0$.

20. $f(x) = e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$

$f'(x) = \dfrac{\mu - x}{\sigma^2} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$
Thus $f'(\mu) = 0$.

$f''(x) = -\dfrac{1}{\sigma^2} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} + \left(\dfrac{\mu - x}{\sigma^2}\right)^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$

so $f''(\mu) = -\dfrac{1}{\sigma^2} < 0$. Therefore, $f(x)$ has a relative maximum at $x = \mu$.

21. $f(x) = e^{-x^2/2}$
$f''(x) = x^2 e^{-x^2/2} - e^{-x^2/2}$ (see Exercise 19)
$f''(\pm 1) = 0$, but $f'(\pm 1) = \pm 1 e^{-1/2} > 0$, so $f(x)$ has inflection points at $x = \pm 1$.

22. $f(x) = e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$ (see Exercise 20)

$f''(x) = \left(\dfrac{\mu - x}{\sigma^2}\right)^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} - \dfrac{1}{\sigma^2} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$

$f''(\mu \pm \sigma) = 0$, but

$f'(\mu \pm \sigma) = \dfrac{\mu - (\mu \pm \sigma)}{\sigma} e^{-1/2\left(\frac{\mu \pm \sigma - \mu}{\sigma}\right)^2}$

$= \pm \dfrac{1}{\sigma} e^{-1/2} \neq 0$, so $f(x)$ has inflection points at $x = \mu \pm \sigma$.

23. a. $\Pr(-1.3 \le Z \le 0) = A(1.3) = 0.4032$

b. $\Pr(0.25 \le Z) = 1 - \Pr(Z \le 0.25)$
$= 1 - (0.5 + A(0.25)) = 0.4013$

c. $\Pr(-1 \le Z \le 2.5) = A(1) + A(2.5)$
$= 0.3413 + 0.4938 = 0.8351$

d. $\Pr(Z \le 2) = 0.5 + A(2) = 0.9772$

24. a. $A(1.5) - A(0.5) = 0.2417$

b. $2A(.75) = 0.5468$

c. $1 - (A(0.3) + 0.5) = 0.3821$

d. $A(1) + 0.5 = 0.8413$

25. $\mu = 6$, $\sigma = \dfrac{1}{2}$

a. $\Pr(6 \le x \le 7) = A\left(\dfrac{7-6}{\left(\frac{1}{2}\right)}\right) = A(2) = 0.4772$

So 47.72% of births occur between 6 and 7 months.

b. $\Pr(5 \le x \le 6) = A\left(\dfrac{5-6}{\left(\frac{1}{2}\right)}\right)$
$= A(-2) = A(2) = 0.4772$
So 47.72% of births occur between 5 and 6 months.

26. $\mu = 25{,}000$, $\sigma = 2000$

a. $\Pr(28{,}000 < X < 30{,}000)$
$= A\left(\dfrac{30{,}000 - 25{,}000}{2000}\right) - A\left(\dfrac{28{,}000 - 25{,}000}{2000}\right)$
$= A(2.5) - A(1.5) = 0.0606$

b. $\Pr(X > 29{,}000) = 1 - \Pr(X \le 29{,}000)$
$= 1 - \left[0.5 + A\left(\dfrac{29{,}000 - 25{,}000}{2000}\right)\right]$
$= 1 - [0.5 + A(2)] = 0.0228$

27. $M = 128.2$, $\sigma = .2$
$\Pr(X < 128) = 1 - \left[0.5 + A\left(\dfrac{128.2 - 128}{0.2}\right)\right]$
$= 1 - [0.5 + A(1)] = 0.1587$

28. $M = 43$, $\sigma = 1.5$
$\Pr(X < 40) = 1 - \left[0.5 + A\left(\dfrac{43 - 40}{1.5}\right)\right]$
$= 1 - [0.5 + A(2)] = 0.0228$

29. Let B be the amount of time the Beltway route takes and let L be the time it takes on the local route. Then $\mu_B = 25$, $\sigma_B = 5$, $\mu_L = 28$, $\sigma_L = 3$ and

$\Pr(B < 30) = 0.5 + A\left(\dfrac{30 - 25}{5}\right) = 0.5 + A(1)$
$= 0.8413$

$\Pr(L < 30) = 0.5 + A\left(\dfrac{30 - 28}{3}\right) = 0.5 + A\left(\dfrac{2}{3}\right)$
$= 0.7475$

Therefore the student should take the Beltway route.

30. (See Exercise 29)

$$\Pr(B<34)=0.5+A\left(\frac{34-25}{5}\right)=0.5+A\left(\frac{9}{5}\right)$$

$$\Pr(L<34)=0.5+A\left(\frac{34-28}{3}\right)=0.5+A(2)$$

Since $2>\frac{9}{5}$, in this case $\Pr(L<34)>\Pr(B<34)$, so she should take the local route.

31. Let X be the diameter of a randomly selected bolt.

$$\Pr(X>20)=1-\Pr(X\le 20)$$
$$=1-\left[0.5+A\left(\frac{20-18.2}{.8}\right)\right]$$
$$=1-[0.5+A(2.25)]=0.0122$$

Therefore about 1.22% of the bolts will be discarded.

32. a. $\Pr(500<X<600)$

$$=A\left(\frac{600-535}{100}\right)+A\left(\frac{535-500}{100}\right)$$
$$=A(0.65)+A(0.35)=0.2422+0.1368$$
$$=0.379$$

Thus approximately 37.9% of the scores were between 500 and 600.

b. $\Pr(X<t)=0.5+A\left(\frac{t-535}{100}\right)=0.9$

$$A\left(\frac{t-535}{100}\right)=0.4$$

In the table $A(1.28)=0.3997\approx 0.4$, so

$$\frac{t-535}{100}=1.28,\ t=663.$$

Thus the top 10% consists of scores 663 and above.

33. If X has density $f(x)=ke^{-kx}$, then

$$\frac{\Pr(a\le X\le a+b)}{\Pr(a\le X)}=\frac{\int_a^{a+b}ke^{-kx}dx}{1-\int_0^a ke^{-kx}dx}$$
$$=\frac{-e^{-kx}\Big|_a^{a+b}}{1+e^{-kx}\Big|_0^a}$$
$$=\frac{e^{-ak}-e^{-ak-bk}}{e^{-ak}}=1-e^{-bk}$$
$$=-e^{-kx}\Big|_0^b=\int_0^b ke^{-kx}dx$$
$$=\Pr(0\le X\le b)$$

34. $\Pr(X\le M)=\int_0^M ke^{-kx}dx=-e^{-kx}\Big|_0^M$

$$=1-e^{-Mk}=0.5$$
$$e^{-Mk}=0.5$$
$$-Mk=\ln\left(\frac{1}{2}\right)=-\ln 2$$
$$M=\frac{\ln 2}{k}$$

35. Let X be the lifetime of a light bulb. $\Pr(0\le X\le 100)=0.8$ so solve the following for k:

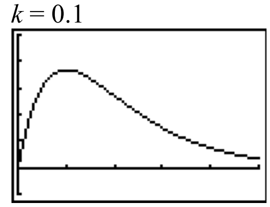
$$0.8=\int_0^{100}ke^{-kx}dx=-e^{-kx}\Big|_0^{100}=-e^{-100k}+1$$

So $e^{-100k}=0.2\Rightarrow\ln(e^{-100k})=\ln(0.2)\Rightarrow$
$-100k\approx -1.609\Rightarrow k\approx 0.0160944$

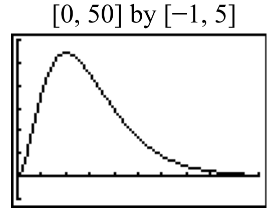
Then the average lifetime is

$$\mathrm{E}(X)=\frac{1}{k}\approx 62.13\text{ weeks.}$$

36. $k=0.1$

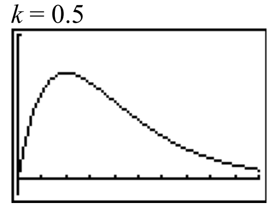


[0, 50] by [−1, 5]

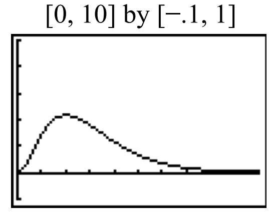


[0, 100] by [−10, 60]

$k=0.5$



[0, 10] by [−.1, 1]

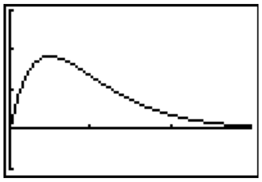


[0, 20] by [−1, 5]

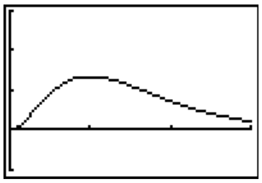
(*continued on next page*)

(*continued*)

$k = 2$



[0, 3] by [−0.1, .3]

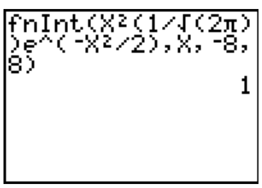


[0, 3] by [−0.1, .3]

37. The screen below shows that $\int_{-8}^{8} x^2 f(x)\,dx = 1,$ where $f(x)$ is the standard normal density function. By equation (2) on page 571, we have

$$Var(X) = \int_{-\infty}^{\infty} [x-0]^2 f(x)\,dx$$
$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1$$

standard deviation $= \sqrt{Var(X)} = \sqrt{1} = 1$



12.5 Poisson and Geometric Random Variables

1. $p_6 = \frac{(3)^6}{6!} e^{-3} \approx 0.0504$

$p_7 = \frac{(3)^7}{7!} e^{-3} \approx 0.0216$

$p_8 = \frac{(3)^8}{8!} e^{-3} \approx 0.0081$

2. $p_0 = e^{-5} \approx 0.0067, p_1 = \frac{5}{1!} e^{-5} \approx 0.0337$

$p_2 = \frac{(5)^2}{2!} e^{-5} \approx 0.0842$

$p_3 = \frac{(5)^3}{3!} e^{-5} \approx 0.1404$

$p_4 = \frac{(5)^4}{4!} e^{-5} \approx 0.1755$

$p_5 = \frac{(5)^5}{5!} e^{-5} \approx 0.1755$

$p_6 = \frac{(5)^6}{6!} e^{-5} \approx 0.1462$

3. $p_0 = e^{-0.75} \approx 0.4724,$

$p_1 = \frac{0.75}{1!} e^{-0.75} \approx 0.3543$

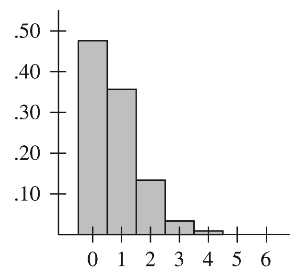
$p_2 = \frac{(0.75)^2}{2!} e^{-0.75} \approx 0.1329$

$p_3 = \frac{(0.75)^3}{3!} e^{-0.75} \approx 0.0332$

$p_4 = \frac{(0.75)^4}{4!} e^{-0.75} \approx 0.0062$

$p_5 = \frac{(0.75)^5}{5!} e^{-0.75} \approx 0.0009$

$p_6 = \frac{(0.75)^6}{6!} e^{-0.75} \approx 0.0001$



4. $p_0 = e^{-2.5} \approx 0.0821,\ p_1 = \frac{2.5}{1!} e^{-2.5} \approx 0.2052$

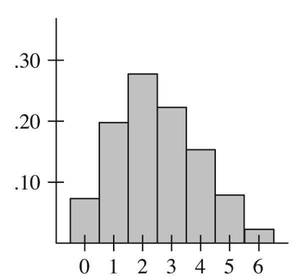
$p_2 = \frac{(2.5)^2}{2!} e^{-2.5} \approx 0.2565$

$p_3 = \frac{(2.5)^3}{3!} e^{-2.5} \approx 0.2138$

$p_4 = \frac{(2.5)^4}{4!} e^{-2.5} \approx 0.1336$

$p_5 = \frac{(2.5)^5}{5!} e^{-2.5} \approx 0.0668$

$p_6 = \frac{(2.5)^6}{6!} e^{-2.5} \approx 0.0278$



5. a. $p_0 = e^{-10} \approx 0.0000454$

b. $p_0 + p_1 + p_2 = e^{-10} + 10e^{-10} + \frac{(10)^2}{2}e^{-10} \approx 0.0027694$

c. $1 - [p_0 + p_1 + p_2] \approx 0.9972306$

6. a. $p_0 + p_1 = e^{-6.5} + 6.5e^{-6.5} \approx 0.0112758$

b. $p_0 + p_1 + p_2 + p_3 + p_4$

$$= e^{-6.5} + 6.5e^{-6.5} + \frac{(6.5)^2}{2}e^{-6.5} + \frac{(6.5)^3}{6}e^{-6.5} + \frac{(6.5)^4}{24}e^{-6.5}$$

≈ 0.22367

c. $1 - [p_0 + p_1 + p_2 + p_3 + p_4]$

$$\approx .8881504 - \frac{(6.5)^4}{24}e^{-6.5}$$

≈ 0.7763282

7. a. $p_0 = e^{-1.5} \approx 0.2231302$

b. $p_2 + p_3 = \frac{(1.5)^2}{2}e^{-1.5} + \frac{(1.5)^3}{6}e^{-1.5} \approx .3765321$

c. $1 - [p_0 + p_1 + p_2 + p_3]$

$$= 1 - \left(e^{-1.5} + 1.5e^{-1.5} + \frac{(1.5)^2}{2}e^{-1.5} + \frac{(1.5)^3}{6}e^{-1.5}\right)$$

≈ 0.0656425

8. a. $1 - [p_0 + p_1 + p_2 + p_3 + p_4]$

$$= 1 - \left(e^{-5} + 5e^{-5} + \frac{25}{2}e^{-5} + \frac{125}{6}e^{-5} + \frac{625}{24}e^{-5}\right)$$

≈ 0.5595067

b. $\Pr\left(0 \le Y \le \frac{1}{5}\right) = \int_0^{1/5} 5e^{-5x}dx$

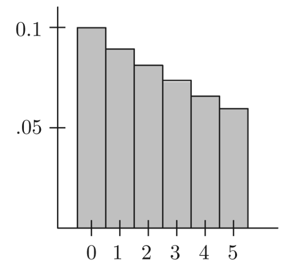
$$= -e^{-5x}\Big|_0^{1/5} = -e^{-1} + e^0$$

$$= 1 - \frac{1}{e} \approx 0.63212$$

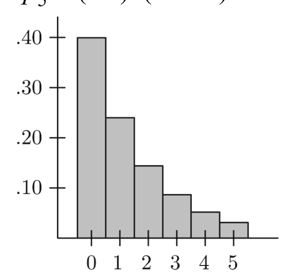
9. If r is the number of raisins used in the batter and X is the number of raisins in a particular cookie, then $E(X) = \frac{r}{4800}$ and $p_0 = e^{-r/4800} = 0.01$. So

$$-\frac{r}{4800} = \ln(0.01) \Rightarrow r \approx 22{,}105.$$

10. $p_0 = (0.9)^0(1-0.9) = 0.1$
$p_1 = (0.9)^1(1-0.9) = .09$
$p_2 = (0.9)^2(1-0.9) = 0.081$
$p_3 = (0.9)^3(1-0.9) = 0.0729$
$p_4 = (0.9)^4(1-0.9) = 0.06561$
$p_5 = (0.9)^5(1-0.9) = 0.059049$



11. $p_0 = (0.6)^0(1-0.6) = 0.4$
$p_1 = (0.6)^1(1-0.6) = 0.24$
$p_2 = (0.6)^2(1-0.6) = 0.144$
$p_3 = (0.6)^3(1-0.6) = 0.0864$
$p_4 = (0.6)^4(1-0.6) = 0.05184$
$p_5 = (0.6)^5(1-0.6) = 0.031104$



12. a. $\Pr(X = n) = p_n = p^n(1-p)$,

$p = \frac{39}{40} = 0.975$ so

$\Pr(X = n) = 0.975^n(0.025)$.

b. $\Pr(X = 4) = 0.975^4(0.025) \approx 0.02259$

13. a. $\Pr(X = n) = \left(\frac{3}{4}\right)^n\left(\frac{1}{4}\right)$

b. $\Pr(x \geq 3) = 1 - \Pr(x=0) - \Pr(x=1) - \Pr(x=2) = 1 - \left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) \approx 0.4219$

c. $E(X) = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$

14. $\Pr(X = n) = \left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)$, so $\Pr(X=2) = \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) \approx 0.07407$

15. $\Pr(x < n) = (1-p)\left(1 + p + p^2 + \cdots + p^{n-1}\right) = (1-p)\left(\frac{1-p^n}{1-p}\right) = 1 - p^n$

16. a. $E(X) = \frac{0.995}{1-0.995} = 199$

b. From Exercise 15,
$\Pr(x \geq 100) = 1 - \Pr(x < 100) = 1 - (1-p^{100}) = p^{100} = (0.995)^{100} = 0.6058$

17. $\Pr(X = 4) = (0.95)^4(1-0.95) \approx 0.04073$

18. a. $\Pr(X = n) = \left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{n+1}$

b. $E(X) = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

c.
$$\begin{aligned}\mathrm{Var}(x) &= (0-1)^2 \cdot p_0 + (1-1)^2 \cdot p_1 + (2-1)^2 \cdot p_2 + (3-1)^2 \cdot p_3 + (4-1)^2 \cdot p_4 + (5-1)^2 \cdot p_5 + \cdots \\ &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{4}{2}\left(\frac{1}{2}\right)^3 + \frac{9}{2}\left(\frac{1}{2}\right)^4 + \frac{16}{2}\left(\frac{1}{2}\right)^5 + \cdots \\ &= \frac{1}{2} + \left(\frac{1}{2}\right)^3\left[1 + 4\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right)^3 + \cdots\right] = \frac{1}{2} + \left(\frac{1}{2}\right)^3\left[\frac{1-\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^4}\right] = 2\end{aligned}$$

19. First derivative:
$\lambda e^{-\lambda} - \frac{\lambda^2}{2}e^{-\lambda} = 0$ when $\lambda e^{-\lambda} = \frac{\lambda^2}{2}e^{-\lambda}$ so $\lambda = 0$ is a possibility, then $2\lambda = \lambda^2 \Rightarrow 2 = \lambda$.

Second derivative: $e^{-\lambda} - 2\lambda e^{-\lambda} + \frac{1}{2}\lambda^2 e^{-\lambda}$

When $\lambda = 2$, $e^{-2} - 2(2)e^{-2} + \frac{1}{2}(2)^2 e^{-2} = -e^{-2}$, which is negative, and when $\lambda = 0$, $e^0 - 0 + 0 = 1$.

Therefore, the probability has a maximum at $\lambda = 2$.

20. First derivative: $5x^4 - 6x^5 = (5-6x)x^4 = 0$ when $x = 0$ or when $x = \frac{5}{6}$.

Second derivative: $20x^3 - 30x^4$

When $x = \frac{5}{6}$, $20\left(\frac{5}{6}\right)^3 - 30\left(\frac{5}{6}\right)^4$ is negative and when $x = 0$, $20(0)^3 - 30(0)^3 = 0$.

Therefore the probability has a maximum at $x = \frac{5}{6}$.

21. $E(X) = 0\cdot p_0 + 1\cdot p_1 + 2\cdot p_2 + 3\cdot p_3 + \cdots = p(1-p) + 2p^2(1-p) + 3p^3(1-p) + \cdots$

$$= p(1-p)[1 + 2p + 3p^2 + \cdots] = p(1-p)\left(\frac{1}{(1-p)^2}\right) = \frac{p}{1-p}$$

22. The probability that X is any even integer is $p_0 + p_2 + p_4 + p_6 + \cdots$, that is $\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda}$.

We want to how that $\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda} = e^{-\lambda}\left(\frac{1}{2}\right)(e^{\lambda} + e^{-\lambda}).$

Using the Taylor series expansion of e^x we have

$$(e^{\lambda} + e^{-\lambda}) = 1 + \lambda + \frac{1}{2!}\lambda^2 + \frac{1}{3!}\lambda^3 + \frac{1}{4!}\lambda^4 + \cdots + 1 - \lambda + \frac{1}{2!}\lambda^2 - \frac{1}{3!}\lambda^3 + \frac{1}{4!}\lambda^4 - \cdots$$

$$= 2 + 2\left(\frac{\lambda^2}{2!}\right) + 2\left(\frac{\lambda^4}{4!}\right) + \cdots = 2\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda}$$

Therefore $\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} = \frac{1}{2}(e^{\lambda} + e^{-\lambda})$ and so $\sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda} = e^{-\lambda}\frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right) = e^{-\lambda}\cosh\lambda$

23. $[p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8] - [p_0 + p_1] \approx 0.887058$

24. **a.** $p_8 = 0.139586532$

b. $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 \approx 0.592547341$

25. **a.** $p_7 - p_6$ is negative, so $p_6 > p_7$ and the answer is no.

b. $\Pr(X \le 15) = 0.9979061$

26. **a.** $p_5 - p_4$ is negative, so $p_4 > p_5$ and the answer is no.

b. $1 - [p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8] \approx 0.0558169$

Chapter 12 Fundamental Concept Check Exercises

1. A probability table is a table that lists the possible outcomes of an experiment and the probability of each outcome occurring.

2. A discrete random variable is a random variable whose values are countable.

3. Answers will vary. Sample answer:

Outcome (X)	2	3	6
Probability	0.2	0.5	0.3

$\mathrm{E}(X) = 2(0.2) + 3(0.5) + 6(0.3) = 3.7$

$$\mathrm{Var}(x) = (2-3.7)^2(0.2) + (3-3.7)^2(0.5) + (6-3.7)^2(0.3) = 2.41$$

Standard deviation $= \sqrt{2.41} \approx 1.55$

4. List the outcomes of the random variable on one axis. Over each outcome, draw a rectangle with height equal to the probability of the outcome. All rectangles have equal width.

5. The outcomes of a discrete random variable are countable. The outcomes of a continuous random variable take values in a continuous interval.

6. A probability density function $f(x)$ satisfies the following properties:

(i) $f(x) \ge 0$ for $A \le x \le B$.

(ii) $\int_A^B f(x) = 1.$

7. The probability that the outcomes of X lie in the interval $[a, b]$ is given by $\int_a^b f(x)\,dx$.

8. If the probability density function $f(x)$ is defined for all $A \le x \le B$, then the cumulative distribution function $F(x) = \int_A^x f(t)\,dt$.

9. The expected value of a continuous random variable $\mathrm{E}(X) = \int_A^B xf(x)\,dx$.

10. $\mathrm{Var}(X) = \int_A^B [x - \mathrm{E}(X)]^2 f(x)\,dx$ and $\mathrm{Var}(X) = \int_A^B x^2 f(x)\,dx - \mathrm{E}(X)^2$.

11. An exponential density function $f(x)$ is defined as $f(x) = ke^{-kx}$, $k > 0$, $x \ge 0$. For example $f(x) = 2e^{-2x}$ is an exponential density function.

12. The expected value of an exponential random variable is given by $\mathrm{E}(X) = \frac{1}{k}$.

13. The density function for a normal random variable with mean μ and standard deviation σ is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$.

14. A normal random variable with expected value $\mu = 0$ and standard deviation $\sigma = 1$ is called a standard normal random variable and is often denoted by the letter Z. The density function for Z is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)z^2}$.

15. Convert an integral involving a normal density function to an integral involving a standard normal density function using the change of variable $z = \frac{x-\mu}{\sigma}$.

16. For a Poisson random variable with parameter λ,

$$\Pr(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$
$$\mathrm{E}(X) = \lambda$$

17. For a geometric random variable with parameter p (the probability of success),

$$\Pr(X = n) = p^n (1 - p)$$
$$\mathrm{E}(X) = \frac{p}{1-p}$$

Chapter 12 Review Exercises

1. $f(x) = \frac{3}{8}x^2,\ 0 \le x \le 2$

a. $\Pr(X \le 1) = \int_0^1 \frac{3}{8}x^2\,dx = \frac{1}{8}x^3\Big|_0^1 = \frac{1}{8}$

$\Pr(1 \le X \le 1.5) = \frac{1}{8}x^3\Big|_1^{1.5} = \frac{19}{64}$

b. $E(X) = \int_0^2 \frac{3}{8}x^3\,dx = \frac{3}{32}x^4\Big|_0^2 = \frac{3}{2}$

$V(X) = \int_0^2 \frac{3}{8}x^4\,dx - \frac{9}{4} = \frac{3}{40}x^5\Big|_0^2 - \frac{9}{4}$
$= \frac{12}{5} - \frac{9}{4} = \frac{3}{20}$

2. $f(x) = 2x - 6,\ 3 \le x \le 4$

a. $\Pr(3.2 \le X) = \int_{3.2}^4 (2x - 6)\,dx$
$= x^2 - 6x\Big|_{3.2}^4 = 0.96$

$\Pr(3 \le x) = 1$ since the random variable is defined for $3 \le x \le 4$.

b. $E(X) = \int_3^4 (2x^2 - 6x)\,dx = \left(\frac{2}{3}x^3 - 3x^2\right)\Big|_3^4$
$= \frac{11}{3}$

$\int_3^4 (2x^3 - 6x^2)\,dx = \left(\frac{1}{2}x^4 - 2x^3\right)\Big|_3^4 = \frac{27}{2}$

$V(X) = \frac{27}{2} - \left(\frac{11}{3}\right)^2 = \frac{1}{18}$

3. I. $e^{A-x} \ge 0$ for all x.

II. $\int_A^\infty e^{A-x}\,dx = \lim_{b\to\infty}\left[-e^{A-x}\Big|_A^b\right]$
$= \lim_{b\to\infty}[1 - e^{A-b}] = 1$

Thus $f(x) = e^{A-x}$, $x \ge A$ is a density function.

$F(x) = \int_A^x e^{A-t}\,dt = -e^{A-t}\Big|_A^x = 1 - e^{A-x}$

4. $f(x) = \dfrac{kA^k}{x^{k+1}}$, $k > 0, A > 0, x \ge A$

I. Since k and A are > 0, $f(x) \ge 0$ for all $x \ge A$.

II. $$\int_A^\infty \left(\frac{kA^k}{x^{k+1}}\right) dx = \lim_{b\to\infty}\left[\left.\frac{-A^k}{x^k}\right|_A^b\right] = \lim_{b\to\infty}\left[1 - \frac{A^k}{b^k}\right] = 1$$

Thus $f(x)$ is a density function.

$$F(x) = \int_A^x \left(\frac{kA^k}{t^{k+1}}\right) dt = -\left.\frac{A^k}{t^k}\right|_A^x = 1 - \frac{A^k}{x^k}$$

5. For $n \ge 2$, any choice of $c_n > 0$ will ensure $f_n(x) \ge 0$ for all $x \ge 0$. Thus we need only $\int_0^\infty c_n x^{(n-2)/2} e^{-x/2} dx = 1$. If $n = 2$ this becomes $c_2 \int_0^\infty e^{-x/2} dx = 1$

$$c_2 \lim_{b\to\infty}\left[\left.-2e^{-x/2}\right|_0^b\right] = 1 \Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2}.$$

For $n = 4$, we have $c_4 \int_0^\infty xe^{-x/2} dx = 1$.
Integrating by parts twice gives
$\int_0^b xe^{-x/2} dx = \left. e^{-x/2}(-4-2x)\right|_0^b$.

Therefore, $c_4 \lim_{b\to\infty}\left[\left. e^{-x/2}(-4-2x)\right|_0^b\right] = 1 \Rightarrow$

$4c_4 = 1 \Rightarrow c_4 = \dfrac{1}{4}$.

6. I. If $k > 0$, $\dfrac{1}{2k^3} x^2 e^{-x/k} \ge 0$ for all x.

II. Integrating by parts twice,
$$\int_0^b x^2 e^{-x/k} dx = \left.-e^{x/k}[kx^2 + 2k^2 x + 2k^3]\right|_0^b.$$

Thus $\displaystyle\int_0^\infty \frac{1}{2k^3} x^2 e^{-x/k} dx = \frac{1}{2k^3}(2k^3) = 1$.

7. **a.** $E(X) = 1(.599) + 11(.401) = 5.01$

b. 200 samples is 20 batches of 10. Thus they can expect to run $20(5.01) \approx 100$ tests.

8. **a.** $E(X) = 1(.774) + 6(.226) = 2.13$

b. 200 samples is 40 batches of 5. Thus they can expect to run $40(2.13) \approx 85$ tests.

9. $F(x) = 1 - \dfrac{1}{4}(2-x)^2$, $0 \le x \le 2$

a. $\Pr(X \le 1.6) = F(1.6) = 0.96$

b. $\Pr(X \le t) = 1 - \dfrac{1}{4}(2-t)^2 = 0.99$
$t = 1.8$ (thousand gal.)

c. $f(x) = F'(x) = \dfrac{1}{2}(2-x)$, $0 \le x \le 2$

10. $$E(X) = \frac{1}{625}\int_0^5 x(x-5)^4 dx = \frac{1}{625}\left[\frac{x}{5}(x-5)^5 - \frac{1}{30}(x-5)^6\right]_0^5$$
(Integration by parts)
$$= \frac{(-5)^6}{30 \cdot 5^4} = \frac{5}{6} = 0.8333 \text{ (hundred dollars)}$$
Thus on average the manufacturer can expect to make $100 - 83.33 = \$16.67$ on each service contract sold.

11. **a.** $$E(X) = \int_{20}^{25} \frac{1}{5} x\, dx = \left.\frac{1}{10}x^2\right|_{20}^{25} = 22.5$$
$$V(X) = \int_{20}^{25} \frac{1}{5} x^2 dx - 22.5^2 = \left.\frac{x^3}{15}\right|_{20}^{25} - 22.5^2 \approx 2.0833$$

b. $\Pr(X \le b) = \displaystyle\int_{20}^{b} \frac{1}{5} dx = 0.3 \Rightarrow$

$\dfrac{1}{5}b - 4 = 0.3 \Rightarrow b = 21.5$

12. $F(x) = \dfrac{(x^2-9)}{16}$, $3 \le x \le 5$

a. $f(x) = F'(x) = \dfrac{x}{8}$, $3 \le x \le 5$

b. $\Pr(a \le X) = \dfrac{1}{4}$, so

$F(a) = \dfrac{(a^2-9)}{16} = \dfrac{3}{4} \Rightarrow a = \sqrt{21}$

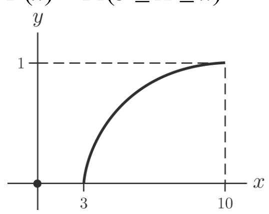
13. $f(x) = kx$, $5 \le x \le 25$

a. We need $k > 0$ and
$$\int_5^{25} kx\, dx = 1 \Rightarrow \left.\frac{k}{2}x^2\right|_5^{25} = 1 \Rightarrow$$
$$\frac{k}{2}(600) = 1 \Rightarrow k = \frac{1}{300}$$

b. $\Pr(X \ge 20) = \int_{20}^{25} \frac{1}{300} x\,dx = \frac{1}{600} x^2 \Big|_{20}^{25} = \frac{3}{8}$

c. $E(X) = \int_{5}^{25} \frac{1}{300} x^2 dx = \frac{1}{900} x^3 \Big|_{5}^{25}$
≈ 17.222 thousand dollars

14. a. $F(x) = \Pr(3 \le X \le x)$

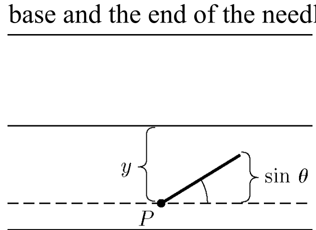


b. $F(7) - F(5) = \Pr(5 \le X \le 7)$

c. $\Pr(5 \le X \le 7) = \int_5^7 f(x)dx$

15. Points (θ, y) satisfying the given condition are precisely those points under the curve $y = \sin\theta, 0 \le \theta \le \pi$. This region has area $\int_0^{\pi} \sin\theta\, d\theta = -\cos\theta\big|_0^{\pi} = 2$. The area of the rectangle is π, so the probability that a randomly selected point falls in the region $y \le \sin\theta$ is $\frac{2}{\pi}$.

16. Since the length of the needle is 1 unit, $\sin\theta$ is the difference in the y-coordinates of the base and the end of the needle.



The needle will touch a ruled line if and only if this difference exceeds y, the vertical distance from the base to the next ruled line. To compute $\Pr(y \le \sin\theta)$, view dropping the needle as a random choice of a point (θ, y) from the square $0 \le \theta \le \pi$, $0 \le y \le 1$. Then Exercise 15 applies.

17. Let X be the lifetime of the computer monitor. Then

$$\Pr(Y = 0) = \Pr(X \le 3) = \int_0^3 \frac{1}{5} e^{-(1/5)x} dx$$
$$= -e^{-(1/5)x}\Big|_0^3 = 1 - e^{-3/5} \approx 0.45119$$

Thus $\Pr(Y = 100) \approx 0.54881$ and $E(Y) \approx \$54.88$.

18. Let Y be as in the hint and let X be the life span of the motor. Then

$$\Pr(Y = 300) = \Pr(X \le 1) = \int_0^1 \frac{1}{10} e^{-(1/10)x} dx$$
$$= -e^{-(1/10)x}\Big|_0^1 = 1 - e^{-1/10} \approx 0.09516$$

Thus $E(Y) \approx 300(0.09516) \approx \$28.55$. Since the insurance costs \$25 to buy, you should buy it for the first year.

19. $\Pr(X \le 4) = \int_0^4 ke^{-kx} dx = -e^{-kx}\Big|_0^4$
$= 1 - e^{-4k} = 0.75$

so $e^{-4k} = 0.25 \Rightarrow -4k = \ln(0.25) \Rightarrow$

$k = \frac{\ln(0.25)}{-4} \approx 0.35$

20. $E(X) = .01\int_0^{\infty} x^2 e^{-x/10} dx$

Integrating by parts twice,

$$\int_0^b x^2 e^{-x/10} dx = -e^{-x/10}(10x^2 + 200x + 2000)\Big|_0^b.$$

So $E(X) = 2000(0.01) = 20$ (thousand hours) and the expected additional earnings from the machine are $20(5000) = \$100,000$. Since this amount exceeds the price, the machine should be purchased.

21. $f(x)$ is the density of a normal random variable X with $\mu = 50$, $\sigma = 8$. Thus

$$\Pr(30 \le X \le 50) = A\left(\frac{50 - 30}{8}\right)$$
$$= A(2.5) = 0.4938$$

22. Let X be the length of a randomly selected part. Then $\Pr(79.95 \le X \le 80.05)$

$$= A\left(\frac{79.99 - 79.95}{0.02}\right) + A\left(\frac{80.05 - 79.99}{0.02}\right)$$

$= A(2) + A(3) = 0.4772 + 0.4987 = 0.9759$
Hence out of a lot of 1000 parts, $1000(.9759) = 975.9$ should be within the tolerance limits, leaving about 24 defective parts.

23. Let X be the height of a randomly selected man in the city.

$$\Pr(X \geq 69) = 0.5 + A\left(\frac{70-69}{2}\right) = 0.5 + A(0.5) = 0.6915$$

Thus about 69.15% of the men in the city are eligible.

24. Let Y be the height of a randomly selected woman from the city.

$$\Pr(Y \geq 69) = 0.5 - A\left(\frac{69-65}{1.6}\right) = 0.5 - A(2.5) = 0.0062$$

So only about 0.62% of the women are eligible.

25. $\Pr(a \leq Z) = 0.4$, then we must have $a > 0$ and $\Pr(0 \leq a \leq A) = A(a) = 0.5 - 0.4 = 0.1$.
From the table, $A(0.25) = 0.0987 \approx 0.1$, so $a \approx 0.25$.

26. Using the result of exercise 25, the cutoff grade t must satisfy $\dfrac{t-500}{100} = 0.25 \Rightarrow t = 525$.

27. **a.** $\Pr(-1 \leq Z \leq 1) = 2A(1) = 0.6826$

b. $\Pr(\mu - \sigma < X < \mu + \sigma) = \Pr(-1 < Z < 1) = 0.6826$

28. **a.** $\Pr(-2 \leq Z \leq 2) = 2A(2) = 0.9544$

b. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) = \Pr(-2 < Z < 2) = 0.9544$

29. **a.** Let X be an exponential random variable with density $f(x) = ke^{-kx}$. Then $E(X) = \mu = \dfrac{1}{k}$ and $V(X) = \sigma^2 = \dfrac{1}{k^2}$. Applying the inequality with $n = 2$ gives

$$\Pr\left(\frac{1}{k} - \frac{2}{k} \leq X \leq \frac{1}{k} + \frac{2}{k}\right) = \Pr\left(-\frac{1}{k} \leq X \leq \frac{3}{k}\right) = \Pr\left(0 \leq X \leq \frac{3}{k}\right) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$$

b. $$\Pr\left(0 \leq X \leq \frac{3}{k}\right) = \int_0^{3/k} ke^{-kx}\,dx = -e^{-kx}\Big|_0^{3/k} = 1 - e^{-3} \approx 0.9502$$

30. Let X be a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Applying the inequality with $n = 2$ gives $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \geq 1 - \dfrac{1}{2^2} = \dfrac{3}{4}$.

The exact value is $\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 2A(2) = 0.9544$

31. $p_4 = \dfrac{(4)^4}{4!}e^{-4} \approx 0.1953668$

32. $1 - [p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7] \approx 0.0511336$

33. $E(X) = 4$

34. $\left(\dfrac{2}{9}\right)\left(\dfrac{7}{9}\right)^n$

35. $E(X) = \dfrac{\frac{7}{9}}{1 - \frac{7}{9}} = \dfrac{7}{2}$

36. $1 - (p_0 + p_1 + p_2) \approx 0.4705075$