Chapter 11 Taylor Polynomials and Infinite Series

11.1 Taylor Polynomials

1. $p_3(x) = \sin 0 + \cos(0)x + \dfrac{-\sin 0}{2!}x^2 + \dfrac{-\cos 0}{3!}x^3 = x - \dfrac{1}{6}x^3$

2. $p_3(x) = e^{-0/2} - \dfrac{1}{2}e^{-0/2}x + \dfrac{1}{4}e^{-0/2}\dfrac{x^2}{2!} - \dfrac{1}{8}e^{-0/2}\dfrac{x^3}{3!} = 1 - \dfrac{1}{2}x + \dfrac{1}{8}x^2 - \dfrac{1}{48}x^3$

3. $p_3(x) = 5e^{2(0)} + 10e^{2(0)}x + 20e^{2(0)}\dfrac{x^2}{2!} + 40e^{2(0)}\dfrac{x^3}{3!} = 5 + 10x + 10x^2 + \dfrac{20}{3}x^3$

4. $p_3(x) = \cos\pi + (5\sin\pi)x - (25\cos\pi)\dfrac{x^2}{2!} - (125\sin\pi)\dfrac{x^3}{3!} = -1 + \dfrac{25}{2}x^2$

5. $p_3(x) = 1 + 2x - \dfrac{4x^2}{2!} + 24\dfrac{x^3}{3!} = 1 + 2x - 2x^2 + 4x^3$

6. $p_3(x) = \dfrac{1}{2} - \dfrac{1}{2^2}x + \dfrac{2}{2^3}\dfrac{x^2}{2!} - \dfrac{6}{2^4}\dfrac{x^3}{3!} = \dfrac{1}{2} - \dfrac{1}{4}x + \dfrac{x^2}{8} - \dfrac{x^3}{16}$

7. $f(x) = xe^{3x};\ f'(x) = e^{3x} + 3xe^{3x}$

$f''(x) = 3e^{3x} + 3e^{3x} + 9xe^{3x} = 6e^{3x} + 9xe^{3x}$

$f'''(x) = 18e^{3x} + 9e^{3x} + 27xe^{3x} = 27e^{3x} + 27xe^{3x}$

$p_3(x) = 0 + (1+0)x + (6+0)\dfrac{x^2}{2!} + (27+0)\dfrac{x^3}{3!} = x + 3x^2 + \dfrac{9}{2}x^3$

8. $p_3(x) = 1 - \dfrac{1}{2}x - \dfrac{1}{4}\dfrac{x^2}{2!} - \dfrac{3}{8}\dfrac{x^3}{3!} = 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2 - \dfrac{1}{16}x^3$

9. $p_4(x) = e^0 + e^0x + e^0\dfrac{x^2}{2!} + e^0\dfrac{x^3}{3!} + e^0\dfrac{x^4}{4!} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$

$e^{0.01} \approx 1 + .01 + \dfrac{(.01)^2}{2} + \dfrac{(.01)^3}{6} + \dfrac{(.01)^4}{24} \approx 1.01005$

10. $f(x) = \ln(1-x);\ f'(x) = -\dfrac{1}{(1-x)};\ f''(x) = -\dfrac{1}{(1-x)^2};\ f'''(x) = -\dfrac{2}{(1-x)^3};\ f^{(4)}(x) = -\dfrac{6}{(1-x)^4}$

$p_4(x) = 0 - x - \dfrac{x^2}{2!} - \dfrac{2x^3}{3!} - \dfrac{6x^4}{4!} = -x - \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3 - \dfrac{1}{4}x^4$

$\ln .9 = \ln(1-.1) \approx -.1 - \dfrac{(.1)^2}{2} - \dfrac{(.1)^3}{3} - \dfrac{(.1)^4}{4} = -.10536$

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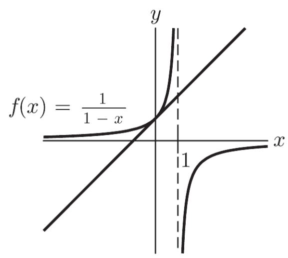
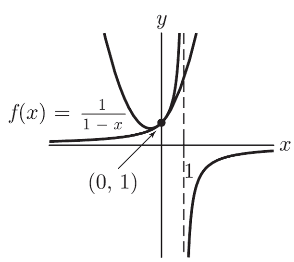
11. See Example 3 on page 515 in the text for the derivation of the Taylor polynomials at $x = 0$.

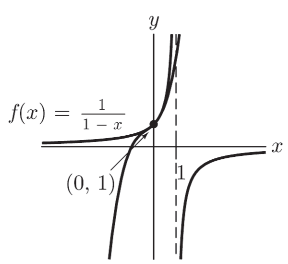
$f(x)$ and $p_1(x) = 1 + x$



$f(x)$ and $p_2(x) = 1 + x + x^2$

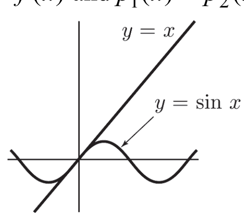


$f(x)$ and $p_3(x) = 1 + x + x^2 + x^3$

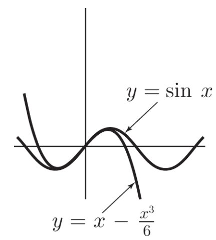


12. $f(x) = \sin x, f(0) = 1$
$f'(x) = \cos x,\ f'(0) = 1$
$f''(x) = -\sin x,\ f''(0) = 0$
$f'''(x) = -\cos x,\ f'''(0) = -1$

$f(x)$ and $p_1(x) = p_2(x) = x$



$f(x)$ and $p_3(x) = x - \frac{x^3}{6}$



13. $$p_n(x) = e^0 + e^0 x + e^0 \frac{x^2}{2!} + \cdots + e^0 \frac{x^n}{n!}$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

14. $p_0(x) = 0^2 + 2(0) + 1 = 1$
$p_1(x) = 1 + (2(0) + 2)x = 1 + 2x$
$$p_2(x) = 1 + 2x + 2\frac{x^2}{2!} = 1 + 2x + x^2$$
Since $f^n(x) = 0$ for all $n \geq 3$,
$p_n(x) = p_2(x) = f(x)$ for all $n \geq 3$.

15. $$f(x) = \ln(1 + x^2);\ f'(x) = \frac{2x}{1 + x^2};$$
$$f''(x) = \frac{2(1 + x^2) - 4x^2}{(1 + x^2)^2} = \frac{-2x^2 + 2}{(1 + x^2)^2}$$
$$p_2(x) = \ln(1) + (0)x + 2\frac{x^2}{2!} = x^2$$
$$\int_0^{1/2} \ln(1 + x^2)dx \approx \int_0^{1/2} x^2 dx = \left.\frac{x^3}{3}\right|_0^{1/2}$$
$$= \frac{1}{24} \approx .0417$$

16. $$f(x) = \sqrt{\cos x};\ f'(x) = \frac{-\sin x}{2\sqrt{\cos x}};$$
$$f''(x) = \frac{-2\cos x\sqrt{\cos x} + \sin^2 x(\cos x)^{-1/2}}{4\cos x}$$
$$p_2(x) = 1 + \frac{0}{2(1)}x - \frac{1}{2}\frac{x^2}{2!} = 1 - \frac{1}{4}x^2$$
$$\int_{-1}^{1} \sqrt{\cos x}\, dx \approx \int_{-1}^{1}\left(1 - \frac{1}{4}x^2\right)dx$$
$$= \left. x - \frac{1}{12}x^3 \right|_{-1}^{1} = \frac{11}{6}$$

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17. $f(x)=\dfrac{1}{5-x};\ f'(x)=\dfrac{1}{(5-x)^2};\ f''(x)=\dfrac{2}{(5-x)^3};\ f'''(x)=\dfrac{6}{(5-x)^4}$

$$p_3(x)=\frac{1}{5-4}+\frac{1}{(5-4)^2}(x-4)-\frac{2}{(5-4)^3}\frac{(x-4)^2}{2!}+\frac{6}{(5-4)^4}\frac{(x-4)^3}{3!}$$
$$=1+(x-4)+(x-4)^2+(x-4)^3$$

18. $f(x)=\ln x;\ f'(x)=\dfrac{1}{x};\ f''(x)=\dfrac{-1}{x^2};\ f'''(x)=\dfrac{2}{x^3};\ f^{(4)}(x)=\dfrac{-6}{x^4}$

$$p_4(x)=\ln(1)+\frac{1}{1}(x-1)-\frac{1}{1^2}\frac{(x-1)^2}{2!}+\frac{2}{1^3}\frac{(x-1)^3}{3!}+\frac{6}{1^4}\frac{(x-1)^4}{4!}=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$$

19. $f(x)=\cos x;\ f'(x)=-\sin x;\ f''(x)=-\cos x;\ f'''(x)=\sin x;\ f^{(4)}(x)=\cos x$

$$p_3(x)=\cos\pi-\sin\pi(x-\pi)-\cos\pi\frac{(x-\pi)^2}{2!}+\sin\pi\frac{(x-\pi)^3}{3!}=-1+\frac{1}{2}(x-\pi)^2$$

$$p_4(x)=p_3(x)+\cos\pi\frac{(x-\pi)^4}{4!}=-1+\frac{1}{2}(x-\pi)^2-\frac{1}{24}(x-\pi)^4$$

20. $f(x)=x^3+3x-1;\ f'(x)=3x^2+3;\ f''(x)=6x;\ f'''(x)=6;\ f^{(n)}(x)=0$ for all $n>3$.

$$p_3(x)=(-1)^3+3(-1)-1+(3(-1)^2+3)(x+1)+6(-1)\frac{(x+1)^2}{2!}+6\frac{(x+1)^3}{3!}=-5+6(x+1)-3(x+1)^2+(x+1)^3$$

$$p_4(x)=p_3(x)$$

21. $f(x)=\sqrt{x};\ f'(x)=\dfrac{1}{2\sqrt{x}};\ f''(x)=-\dfrac{1}{4}x^{-3/2}$

$$p_2(x)=\sqrt{9}+\frac{1}{2\sqrt{9}}(x-9)-\frac{1}{4}9^{-3/2}\frac{(x-9)^2}{2!}=3+\frac{1}{6}(x-9)-\frac{1}{216}(x-9)^2$$

$$p_2(9.3)=3+\frac{.3}{6}-\frac{(.3)^2}{216}\approx 3.04958$$

22. $f(x)=\ln x;\ f'(x)=\dfrac{1}{x};\ f''(x)=-\dfrac{1}{x^2}$

$$p_2(x)=\ln(1)+\frac{1}{1}(x-1)-\frac{1}{1^2}\frac{(x-1)^2}{2!}=(x-1)-\frac{1}{2}(x-1)^2\Rightarrow p_2(.8)=-.2-\frac{1}{2}(-.2)^2=-.22$$

23. $f(x)=x^4+x+1;\ f'(x)=4x^3+1;\ f''(x)=12x^2;\ f'''(x)=24x;\ f^{(4)}(x)=24;$

$f^{(n)}(x)=0$ all $n>4$.

$$p_0(x)=2^4+2+1=19$$

$$p_1(x)=19+(4(2)^3+1)(x-2)=19+33(x-2)$$

$$p_2(x)=19+33(x-2)+12(2)^2\frac{(x-2)^2}{2!}=19+33(x-2)+24(x-2)^2$$

$$p_3(x)=19+33(x-2)+24(x-2)^2+24(2)\frac{(x-2)^3}{3!}=19+33(x-2)+24(x-2)^2+8(x-2)^3$$

$$p_4(x)=p_3(x)+24\frac{(x-2)^4}{4!}=19+33(x-2)+24(x-2)^2+8(x-2)^3+(x-2)^4;\ p_n(x)=p_4(x)\ \text{for all } n\geq 4.$$

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24. $f(x)=\frac{1}{x};\ f'(x)=-\frac{1}{x^2};\ f''(x)=\frac{2}{x^3};\ f'''(x)=-\frac{3\cdot 2}{x^4};\ f^{(n)}(x)=\frac{(-1)^{n+1}n!}{x^n}$

$$p_n(x)=1-(x-1)+\frac{2(x-1)^2}{2!}-\frac{3!(x-1)^3}{3!}+\cdots+(-1)^n n!\frac{(x-1)^n}{n!}$$
$$=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n$$

25. The expression on the right must be the Taylor expansion of $f(x)$ at $x = 0$. Therefore, $f''(0)=-5$ and $f'''(0)=7$.

26. The expression on the right must be the Taylor expansion of $f(x)$ at $x = 1$. Therefore, $f''(1)=3$ and $f'''(1)=-5$.

27. a. If $f(x) = \cos x$, then $f^{(4)}(x)=\cos x$ as well, so $\left|f^{(4)}(c)\right|\le|\cos(c)|\le 1$ for all c.

b. $R_3(0.12)=\frac{\cos(c)}{4!}(0.12)^4$ for some c between 0 and 0.12. From part (a), it follows that
$$|R_3(0.12)|=|\text{error in the approximation}|$$
$$\le\frac{1}{4!}(.12)^4=8.64\times10^{-6}$$

28. $|R_4(0.1)|=|\text{error in the approximation}|$
$$=\left|\frac{f^{(5)}(c)}{5!}(.1)^5\right|$$
for some c, $0\le c\le .1$.
Now $f^{(5)}(c)=e^c\le e^{0.1}$ for all $0\le c\le .1$, since e^x is increasing on [0, .1]. Therefore
$$|\text{error in approximation}|\le\frac{e^{.1}}{5!}(.1)^5\approx 9.21\times10^{-8}$$
$$<\frac{3}{5!}(.1)^5\approx 2.5\times10^{-7}.$$

29. a. $f'''(x)=\frac{3}{8}x^{-5/2}$
$$R_2(x)=\frac{f'''(c)}{3!}(x-9)^3$$
$$=\frac{\frac{3}{8}c^{-5/2}}{3!}(x-9)^3=\frac{c^{-5/2}}{16}(x-9)^3$$
for some c between 9 and x.

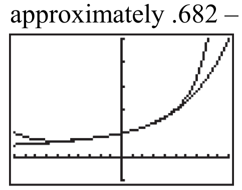
b. The function $f^{(3)}(c)=\frac{3}{8}c^{-5/2}$ is positive and decreasing for $c > 0$. Thus for all $c\ge 9$, $\left|f^{(3)}(c)\right|\le f^{(3)}(9)=\frac{3}{8\cdot 3^5}=\frac{1}{648}$.

c. $|\text{error}|=R_2(9.3)\le\frac{\frac{1}{648}}{3!}(.3)^3$ (from part b)
$$=\frac{1}{144}\times10^{-3}<7\times10^{-6}$$

30. a. $f^{(3)}(c)=\frac{2}{c^3}\le\frac{2}{(.8)^3}<4$ for all $c\ge .8$, since $f^{(3)}(x)=\frac{2}{x^3}$ is positive and decreasing for $x > 0$.

b. $R_2(x)=\frac{\frac{2}{c^3}}{3!}(x-1)^3=\frac{1}{3c^3}(x-1)^3$ for some c between 1 and x. By part (a),
$$|R_2(.8)|\le\left|\frac{4}{3!}(.8-1)^3\right|=\frac{16}{3}\times10^{-3}<.0054.$$

31. $y_1=\frac{1}{1-x}$, $y_2=1+x+x^2+x^3+x^4$
When $b = .55$, the difference is approximately $2.22 - 2.11 = .11$.
When $b = -.68$, the difference is approximately $.682 - .595 = .087$.



$[-1, 1]$ *by* $[-1, 5]$

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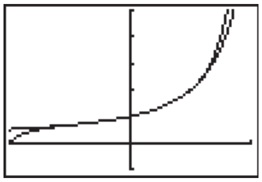
32. $y_1 = \dfrac{1}{1-x}$

$y_2 = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$

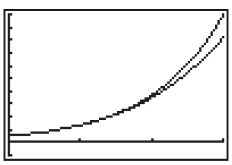
When $b = .68$, the difference is approximately $3.125 - 2.982 = .143$.
When $b = -.77$, the difference is approximately $.565 - .495 = .07$.



[−1, 1] by [−1, 5]

33. $y_1 = e^x,\ y_2 = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$

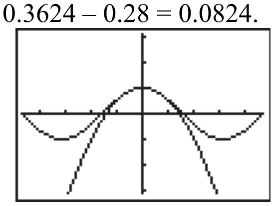
Choose $b = 1.85$. Then the difference is approximately $6.3598 - 6.1046 = .2552$. When $x = 3$, the difference is approximately $20.0855 - 16.375 = 3.7105$.



[0, 3] *by* [−2, 20]

34. $y_1 = \cos x,\ y_2 = 1 - \dfrac{1}{2}x^2$

Choose $b = 1.2$. Greatest difference over the interval $[-1.2, 1.2]$ is approximately $0.3624 - 0.28 = 0.0824$.



[−4.7, 4.7] by [−3.1, 3.1]

11.2 The Newton-Raphson Algorithm

1. Let $f(x) = x^2 - 5,\ f'(x) = 2x.$

$$x_0 = 2;\ \ x_1 = 2 - \frac{2^2 - 5}{2(2)} = 2.25$$

$$x_2 = 2.25 - \frac{(2.25)^2 - 5}{2(2.25)} \approx 2.2361$$

$$x_3 \approx 2.2361 - \frac{(2.2361)^2 - 5}{2(2.2361)} \approx 2.23607$$

2. Let $f(x) = x^2 - 7;\ f'(x) = 2x.\ \ x_0 = 3;$

$$x_1 = 3 - \frac{3^2 - 7}{2(3)} \approx 2.667$$

$$x_2 \approx 2.667 - \frac{2.667^2 - 7}{2(2.667)} \approx 2.6458$$

$$x_3 \approx 2.6458 - \frac{2.6458^2 - 7}{2(2.6458)} \approx 2.64575$$

3. Let $f(x) = x^3 - 6;\ f'(x) = 3x^2.$

$$x_0 = 2;\ x_1 = 2 - \frac{2^3 - 6}{3(2^2)} \approx 1.8333$$

$$x_2 \approx 1.8333 - \frac{1.8333^3 - 6}{3(1.8333)^2} \approx 1.81726$$

$$x_3 = 1.81726 - \frac{1.81726^3 - 6}{3(1.81726)^2} \approx 1.81712$$

4. Let $f(x) = x^3 - 11;\ f'(x) = 3x^2.$

$$x_0 = 2;\ x_1 = 2 - \frac{2^3 - 11}{3(2)^2} = 2.25$$

$$x_2 = 2.25 - \frac{2.25^3 - 11}{3(2.25)^2} \approx 2.2243$$

$$x_3 \approx 2.2243 - \frac{2.2243^3 - 11}{3(2.2243)^2} \approx 2.22398$$

5. $f(x) = x^2 - x - 5;\ f'(x) = 2x - 1$

$$x_0 = 2;\ x_1 = \frac{2^2 - 2 - 5}{2(2) - 1} = 3$$

$$x_2 = 3 - \frac{3^2 - 3 - 5}{2(3) - 1} = 2.8$$

$$x_3 = 2.8 - \frac{2.8^2 - 2.8 - 5}{2(2.8) - 1} \approx 2.7913$$

6. $f(x) = x^2 + 3x - 11;\ f'(x) = 2x + 3$

$$x_0 = -5;\ x_1 = -5 - \frac{(-5)^2 + 3(-5) - 11}{2(-5) + 3} \approx -5.1429$$

$$x_2 \approx -5.1429 - \frac{(-5.1429)^2 + 3(-5.1429) - 11}{2(-5.1429) + 3} \approx -5.14006$$

$$x_3 \approx -5.14006 - \frac{(-5.14006)^2 + 3(-5.14006) - 11}{2(-5.14006) + 3} \approx -5.14005$$

7. $f(x) = \sin x + x^2 - 1;\ f'(x) = \cos x + 2x$

$x_0 = 0,\ x_1 = 1,\ x_2 \approx .66875,\ x_3 \approx .63707$

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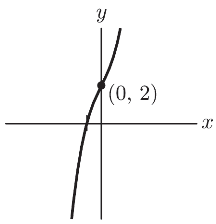
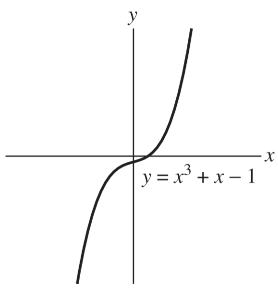
8. $f(x)=e^x+10x-3;\ f'(x)=e^x+10$
$x_0=0,\ x_1\approx .18182,\ x_2\approx .18025,$
$x_3\approx .18025$

9.



$x_0=-1;\ x_1=-.8;\ x_2\approx -.77143;$
$x_3\approx -.77092$

10.



If $x_0=0$, then $x_1=1$, $x_2=.75$, and $x_3=.68605$. The other two zeros are complex.

11. $f(x)=e^{-x}-x^2;\ f'(x)=-e^{-x}-2x$
$x_0=1,\ x_1\approx .73304,\ x_2\approx .70381,$
$x_3\approx .70347$

12. $f(x)=e^{5-x}+x-10;\ f'(x)=-e^{5-x}+1$
There are 2 roots. Starting with $x_0=3$ gives $x_1\approx 3.06089,\ x_2\approx 3.06315,\ x_3\approx 3.06315.$
Starting with $x_0=9$ gives $x_1=10,$
$x_2\approx 9.99322,\ \ x_3\approx 9.99322.$

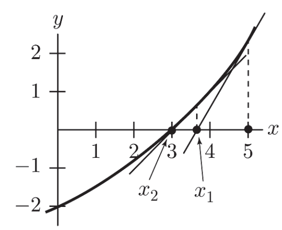
13. The internal rate of return i satisfies the equation
$500(1+i)^3-100(1+i)^2-200(1+i)-300=0.$
Putting $x = 1 + i$,
$f(x)=500x^3-100x^2-200x-300;$
$f'(x)=1500x^2-200x-200$
If $x_0=1.1$, we have
$x_1\approx 1.08244,\ x_2\approx 1.08208,\ x_3\approx 1.08208.$
Thus $i \approx .0821$ or 8.21% per month.

14. The internal rate of return satisfies the equation $1000(1+i)^2-10(1+i)-1050=0.$
Putting
$x = 1 + i,\ f(x)=1000x^2-10x-1050,$
$f'(x)=2000x-10,\ \ x_0=1.1,$ we have
$x_1\approx 1.03196,\ x_2\approx 1.02971,\ x_3\approx 1.02971.$
Thus, $i \approx .02971$ or 2.971% per month.

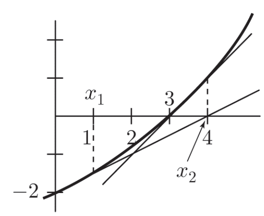
15. The interest rate i satisfies the equation $f(i) = 0$, where $f(i)=563i+116((1+i)^{-5}-1);$
$f'(i)=563-580(1+i)^{-6}$. Starting with
$i_0=0.02$ gives $i_1\approx 0.01323,\ i_2\approx 0.01062,$
$i_3\approx 0.01003.$
Thus the monthly interest rate is about 1%.

16. The interest rate i satisfies $f(i) = 0$, where
$f(i)=100,050i+900((1+i)^{-240}-1).$
$f'(i)=100,050-216,000(1+i)^{-241}$
Starting with $i_0=.02$ gives
$i_1\approx .00871,\ i_2\approx .00757,\ i_3\approx .00750.$ Thus the monthly interest rate is about .75%.

17. $x_1\approx 3.5, x_2\approx 3.0$



18.



19. $m = 4(\text{slope}) = f'(x)$ at $x = 3, f(3) = 17.$
$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=3-\frac{17}{4}=-\frac{5}{4}$$

20. $m = -2(\text{slope}) = f'(x)$ at $x = 1, f(1) = 2;$
$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{2}{(-2)}=2$$

21. $x_0>0$

22. $\sqrt{12};\ 0;\ \sqrt{12}$

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23. Given any x_0, x_1 will be the zero of $f(x)$.

$$x_1 = x_0 - \frac{mx_0 + b}{m},$$

$$f(x_1) = m\left(x_0 - \frac{mx_0 + b}{m}\right) + b = 0$$

24. If the first approximation is a zero of $f(x)$, then $x_1 = x_2 = \cdots = x_0$, since

$$x_1 = x_0 - \frac{0}{f'(x_0)} = x_0.$$

25. $f(x) = x^{1/3}$; $f'(x) = \frac{1}{3}x^{-2/3}$

$$x_0 = 1;\ x_1 = 1 - \frac{1}{\left(\frac{1}{3}\right)} = -2;$$

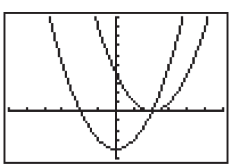
$$x_2 = -2 - \frac{\sqrt[3]{-2}}{\left(\frac{1}{3}\right)(-2)^{-2/3}} = 4;\ x_3 = 4 - 12 = -8$$

The iterates diverge.

26. $x_0 = 1;\ x_1 = 1 - \dfrac{\sqrt{1}}{\left(\frac{1}{2}\right)(1)} = -1;$

$$x_2 = -1 - \frac{-\sqrt{1}}{\left(\frac{1}{2}\right)} = 1;\ x_3 = -1;\ \text{etc.}$$

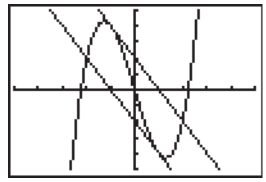
27. Assume that the calculator displays up to 10 digits. For $f(x) = x^2 - 4$, $x_n = 2$ when $n = 4$. For $g(x) = (x-2)^2$, $x_n = 2$ when $n = 31$. This difference is due to the derivatives: $f'(x) = 2x$, whereas $g'(x) = 2(x-2) = 2x - 4$.



$[-6, 6]$ *by* $[-5, 10]$

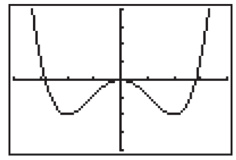
28. $f(x) = x^3 - 5x$, $x_0 = 1$

The values of x_n alternate between –1 and 1. The tangent line at $x = 1$ is $y = -2x - 2$ and at $x = -1$ is $y = -2x + 2$. The Newton-Raphson algorithm starts at $x_0 = 1$, calculates the tangent line at (1, –4) that leads to $x_1 = -1$. Then the algorithm calculates the tangent line at (–1, 4) that leads to $x_2 = 1$. This repeats indefinitely, so $x_3 = -1$, $x_4 = 1$, etc.



$[-5, 5]$ by $[-5, 5]$

29.



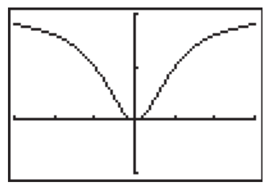
$[-2, 2]$ *by* $[-2, 2]$

a. $x_0 = 1.1$ leads to the zero at $x = \sqrt{2}$.

b. $x_0 = 0.95$ leads to the zero at $x = -\sqrt{2}$.

c. $x_0 = 0.9$ leads to the zero at $x = 0$.

30. $f(x) = \dfrac{x^2}{1 + x^2}$



$[-3, 3]$ by $[-0.5, 1]$

Choose $x_0 = 1$ or $x_0 = -1$. Then $x_1 = 0$.

11.3 Infinite Series

1. The series is geometric with $a = 1$, $r = \frac{1}{6}$, so the sum is $\dfrac{1}{1 - \frac{1}{6}} = \dfrac{6}{5}$.

2. The series is geometric with $a = 1$, $r = \frac{3}{4}$, so the sum is $\dfrac{1}{1 - \frac{3}{4}} = 4$.

3. $a = 1$, $r = -\frac{1}{9}$; sum $= \dfrac{1}{1 + \frac{1}{9}} = \dfrac{9}{10}$

4. $a = 1$, $r = \frac{1}{8}$; sum $= \dfrac{1}{1 - \frac{1}{8}} = \dfrac{8}{7}$

5. $a = 2$, $r = \frac{1}{3}$; sum $= \dfrac{2}{1 - \frac{1}{3}} = 3$

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6. $a = 3,\ r = \frac{2}{5};\ \text{sum} = \frac{3}{1-\frac{2}{5}} = 5$

7. $a = \frac{1}{5},\ r = \frac{\frac{1}{5^4}}{\frac{1}{5}} = \frac{1}{5^3} = \frac{1}{125}$

$\text{sum} = \frac{\frac{1}{5}}{1-\frac{1}{125}} = \frac{25}{124}$

8. $a = \frac{1}{3^2} = \frac{1}{9};\ r = -\frac{\frac{1}{3^3}}{\frac{1}{3^2}} = -\frac{1}{3}$

$\text{sum} = \frac{\frac{1}{9}}{1+\frac{1}{3}} = \frac{1}{12}$

9. $a = 3;\ r = \frac{-\frac{3^2}{7}}{3} = -\frac{3}{7}$

$\text{sum} = \frac{3}{1+\frac{3}{7}} = \frac{21}{10}$

10. $a = 6;\ r = \frac{-1.2}{6} = -.2$

$\text{sum} = \frac{6}{1+.2} = 5$

11. $a = \frac{2}{5^4} = \frac{2}{625};\ r = \frac{\frac{-2^4}{5^5}}{\frac{2}{5^4}} = \frac{-2^3}{5} = -\frac{8}{5}$

Since $|r| > 1,$ the series diverges.

12. $a = \frac{3^2}{2^5} = \frac{9}{32};\ r = \frac{\frac{3^4}{2^8}}{\frac{3^2}{2^5}} = \frac{9}{8}$

Since $\left|\frac{9}{8}\right| > 1,$ the series diverges.

13. $a = 5;\ r = \frac{4}{5};\ \text{sum} = \frac{5}{1-\frac{4}{5}} = 25$

14. $a = \frac{5^3}{3} = \frac{125}{3};\ r = \frac{-\frac{5^5}{3^4}}{\frac{5^3}{3}} = -\frac{25}{27}$

$\text{sum} = \frac{\frac{125}{3}}{1+\frac{25}{27}} = \frac{1125}{52}$

15. $.272727\overline{27} = \frac{27}{100} + \frac{27}{100^2} + \frac{27}{100^3} + \cdots$

This is a geometric series with $a = \frac{27}{100};$

$r = \frac{1}{100}.$ Therefore $.2727\overline{27} = \frac{\frac{27}{100}}{1-\frac{1}{100}} = \frac{3}{11}.$

16. $0.173\overline{173} = \frac{173}{1000} + \frac{173}{1000^2} + \cdots$

This is a geometric series with $a = \frac{173}{1000};$

$r = \frac{1}{1000}.$ Therefore,

$.173\overline{173} = \frac{\frac{173}{1000}}{1-\frac{1}{1000}} = \frac{173}{999}.$

17. $0.222\overline{2} = \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3} + \cdots$

This is a geometric series with $a = \frac{1}{5};\ r = \frac{1}{10}.$

Therefore, $.22\overline{2} = \frac{\frac{1}{5}}{1-\frac{1}{10}} = \frac{2}{9}.$

18. $0.1515\overline{15} = \frac{15}{100} + \frac{15}{100^2} + \frac{15}{100^3} + \cdots$

This is a geometric series with $a = \frac{3}{20};$

$r = \frac{1}{100}.$ Therefore, $0.1515\overline{15} = \frac{\frac{3}{20}}{1-\frac{1}{100}} = \frac{5}{33}.$

19. $0.011\overline{011} = \frac{11}{1000} + \frac{11}{1000^2} + \cdots$

This is a geometric series with $a = \frac{11}{1000}$ and

$r = \frac{1}{1000}.$ Its sum is $\frac{\frac{11}{1000}}{1-\frac{1}{1000}} = \frac{11}{999}.$

Therefore, $4.011\overline{011} = 4 + \frac{11}{999} = \frac{4007}{999}.$

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20. $0.44\overline{4} = \frac{4}{10} + \frac{4}{10^2} + \frac{4}{10^3} + \cdots$

This is a geometric series with $a = \frac{2}{5}$, $r = \frac{1}{10}$.

Its sum is $\frac{\frac{2}{5}}{1-\frac{1}{10}} = \frac{4}{9}$. Therefore,

$5.44\overline{4} = 5 + \frac{4}{9} = \frac{49}{9}$.

21. $0.99\overline{9} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots$

This is a geometric series with

$a = \frac{9}{10}$, $r = \frac{1}{10}$. Therefore, $.99\overline{9} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$.

22. $0.1212\overline{1212} = \frac{0.1212}{1-0.0001} = \frac{1212}{9999} = \frac{4}{33}$

23. The additional spending would be

$$10(0.95) + 10(0.95)^2 + 10(0.95)^3 + \cdots$$
$$= \frac{10(0.95)}{1-0.95}$$
$$= 190 \text{ billion dollars}$$

24. The effect on spending would be

$$20(0.98) + 20(0.98)^2 + 20(0.98)^3 + \cdots$$
$$= \frac{20(0.98)}{1-0.98}$$
$$= 980 \text{ billion dollars}$$

In this case, the "multiplier" is $\frac{0.98}{1-0.98} = 49$.

25. a. $\sum_{k=0}^{\infty} 100(1.01)^{-k}$

b. $a = 100$; $r = \frac{1}{1.01}$

$$\text{sum} = \frac{100}{1-\frac{100}{101}} = \$10{,}100$$

26. The capital value is

$$\sum_{i=1}^{\infty} P(1+r)^{-i} = \sum_{i=0}^{\infty} P(1+r)^{-i} - P$$
$$= \frac{P}{1-\frac{1}{1+r}} - P = \frac{P}{r}$$

27. $$\text{Amount} = 1{,}000{,}000[1 + (0.396) + (0.396)^2 + (0.396)^3 + \ldots + (0.396)^n + \ldots]$$
$$= 1{,}000{,}000\left(\frac{1}{1-0.396}\right) \approx \$1{,}655{,}629$$

28. Total distance

$$= 6 + 0.7(6) + 0.7(6) + 0.7(0.7(6)) + \cdots$$
$$= 6 + 2(6)(0.7)[1 + 0.7 + (0.7)^2 + (0.7)^3 + \cdots]$$
$$= 6 + 8.4\left(\frac{1}{1-.7}\right) = 34 \text{ feet}$$

29. $$6 + 6(0.7) + 6(0.7)^2 + 6(0.7)^3 + \cdots = \frac{6}{1-0.7} = 20 \text{ mg}$$

30. $$2 + 2(0.8) + 2(0.8)^2 + 2(0.8)^3 + \cdots = 2\left(\frac{1}{1-0.8}\right) = 10 \text{ mg}$$

Since we wish to know before a dose is given, 10 – 2 = 8 mg.

31. $M + \frac{3}{4}M + \left(\frac{3}{4}\right)^2 M + \cdots = M\left(\frac{1}{1-0.75}\right) = 4M$

$4M = 20$, so $M = 5$ mg.

32. $$M + M(1-q) + M(1-q)^2 + \cdots = \frac{M}{1-(1-q)} = \frac{M}{q} \text{ mg}$$

33. a. $S_{10} = 3 - \frac{5}{10} = 2.5$

b. Yes, since $\lim_{n\to\infty}\left(3 - \frac{5}{n}\right) = 3$.

34. a. $S_{10} = 10 - \frac{1}{10} = 9.9$

b. No, since $\lim_{n\to\infty}\left(n - \frac{1}{n}\right)$ does not exist

35. $\frac{1}{1-\frac{5}{6}} = 6$

36. $\frac{7}{1-\frac{1}{10}} = \frac{70}{9}$

37. $$\sum_{j=1}^{\infty} 5^{-2j} = \sum_{j=1}^{\infty}\left(\frac{1}{25}\right)^j = \frac{1}{25}\sum_{j=0}^{\infty}\left(\frac{1}{25}\right)^j = \frac{\frac{1}{25}}{1-\frac{1}{25}} = \frac{1}{24}$$

38. $\frac{1}{1+\frac{1}{3}} = \frac{3}{4}$

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39. $\sum_{k=0}^{\infty}(-1)^k \frac{3^{k+1}}{5^k} = 3\sum_{k=0}^{\infty}\left(-\frac{3}{5}\right)^k = \frac{3}{1+\frac{3}{5}} = \frac{15}{8}$

40. $\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{2k} = \sum_{k=1}^{\infty}\left(\frac{1}{9}\right)^k = \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{1}{8}$

41. a. $(1-r)(a+ar+ar^2+\cdots+ar^n) = a+ar+ar^2+\cdots+ar^n-ar-ar^2-\cdots-ar^n-ar^{n+1} = a-ar^{n+1}$

Thus $a+ar+ar^2+\cdots+ar^n = \frac{a-ar^{n+1}}{1-r} = \frac{a}{1-r}-\frac{ar^{n+1}}{1-r}$.

b. As $n\to\infty$, $\frac{ar^{n+1}}{1-r}$ approaches 0 if $|r|<1$. Hence, in this case,

$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty}\sum_{k=0}^{n} ar^k = \lim_{n\to\infty}\left[\frac{a}{1-r}-\frac{ar^{n+1}}{1-r}\right] = \frac{a}{1-r}$.

c. If $|r|>1$, $\left|\frac{ar^{n+1}}{1-r}\right|$ increases without bound as $n\to\infty$. Thus in this case, the series diverges.

d. If $r = 1$, then the series is $a + a + a + \cdots$ which clearly diverges. If $r = -1$, then the expression in part (a) is $\frac{a}{2}-\frac{a(-1)^n}{2}$. Thus the partial sums alternate between 0 and a and $\lim_{n\to\infty}\sum_{k=0}^{\infty} ar^k$ does not exist.

42. From the hint it follows that $\sum_{k=1}^{\infty}\frac{1}{k} > \sum_{k=1}^{\infty}\frac{1}{2}$ which diverges.

43. $a=1,\ r=\frac{2}{3}$

$\sum_{x=0}^{\infty}\left(\frac{2}{3}\right)^x = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$

44. $\sum_{x=1}^{\infty}\frac{2}{3^{2x}} = \frac{2}{3^{2\cdot1}}+\frac{2}{3^{2\cdot2}}+\frac{2}{3^{2\cdot3}}+\cdots = \frac{2}{9}+\frac{2}{81}+\frac{2}{729}+\cdots$

$a=\frac{2}{9},\ r=\frac{1}{9}$

$\sum_{x=1}^{\infty}\frac{2}{3^{2x}} = \frac{\frac{2}{9}}{1-\frac{1}{9}} = \frac{\frac{2}{9}}{\frac{8}{9}} = \frac{1}{4}$

45. $\sum_{x=1}^{5}\frac{(-1)^{2x}}{2^{(x+1)}} = \frac{(-1)^2}{2^2}+\frac{(-1)^4}{2^3}+\frac{(-1)^3}{2^4}+\frac{(-1)^8}{2^5}+\frac{(-1)^{10}}{2^6} = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

$\sum_{x=1}^{\infty}\frac{(-1)^{2x}}{2^{(x+1)}} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots-\left(1+\frac{1}{2}\right) = 2-\left(\frac{3}{2}\right) = \frac{1}{2}$

46. $\sum_{x=1}^{5}\frac{2^{(3x+1)}}{9^{x+1}} = \frac{2^4}{9^2}+\frac{2^7}{9^3}+\frac{2^{10}}{9^4}+\frac{2^{13}}{9^5}+\frac{2^{16}}{9^6}$

$\sum_{x=1}^{\infty}\frac{2^{(3x+1)}}{9^{x+1}} = \sum_{x=1}^{\infty}\frac{2}{9}\left(\frac{8}{9}\right)^x = \frac{\frac{16}{81}}{1-\frac{8}{9}} = \frac{16}{9}$

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47. $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$

$\sum_{x=1}^{10} x = 55 \Rightarrow \frac{10(10+1)}{2} = 55$

$\sum_{x=1}^{50} x = 1275 \Rightarrow \frac{50(50+1)}{2} = 1275$

$\sum_{x=1}^{100} x = 5050 \Rightarrow \frac{100(101)}{2} = 5050$

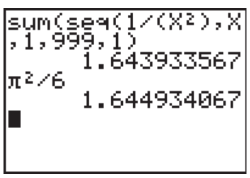
48. $\sum_{x=1}^{n} (2x-1) = n^2$

$\sum_{x=1}^{5} (2x-1) = 1+3+5+7+9 = 25$

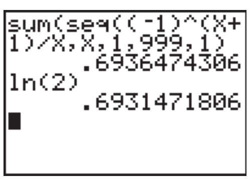
$$\sum_{x=1}^{10} (2x-1) = \sum_{x=1}^{5} (2x-1) + \sum_{x=6}^{10} (2x-1)$$
$$= 25 + 11 + 13 + 15 + 17 + 19$$
$$= 25 + 75 = 100$$

$\sum_{x=1}^{25} (2x-1) = 625$ (using a calculator)

49.



50.



11.4 Series with Positive Terms

1. $\int_1^b \frac{3}{\sqrt{x}}\,dx = 6x^{1/2}\Big|_1^b = 6b^{1/2} - 6$

Thus $\int_1^\infty \frac{3}{\sqrt{x}}\,dx$ diverges so $\sum_{k=1}^{\infty} \frac{3}{\sqrt{k}}$ diverges.

2. $\int_1^b \frac{5}{x^{3/2}}\,dx = -10x^{-1/2}\Big|_1^b = 10 - 10b^{-1/2}$

Thus $\int_1^\infty \frac{5}{x^{3/2}}\,dx = 10$ and $\sum_{k=1}^{\infty} \frac{5}{k^{3/2}}$ converges.

3. $\int_2^b \frac{1}{(x-1)^3}\,dx = -\frac{1}{2}(x-1)^{-2}\Big|_2^b = \frac{1}{2} - \frac{1}{2}(b-1)^{-2}$

Thus $\int_2^\infty \frac{1}{(x-1)^3}\,dx = \frac{1}{2}$ and $\sum_{k=2}^{\infty} \frac{1}{(k-1)^3}$ converges.

4. $$\int_0^b \frac{7}{x+100}\,dx = 7\ln(x+100)\Big|_0^b$$
$$= 7\ln(b+100) - 7\ln 100$$

Thus $\int_0^\infty \frac{7}{x+100}\,dx$ diverges, so $\sum_{k=0}^{\infty} \frac{7}{k+100}$ diverges.

5. $$\int_1^b \frac{2}{5x-1}\,dx = \frac{2}{5}\ln(5x-1)\Big|_1^b$$
$$= \frac{2}{5}\ln(5b-1) - \frac{2}{5}\ln 4$$

Thus $\int_1^\infty \frac{2}{5x-1}\,dx$ diverges, so $\sum_{k=1}^{\infty} \frac{2}{5k-1}$ diverges.

6. $\int_2^b \frac{1}{x\sqrt{\ln x}}\,dx = 2\sqrt{\ln x}\Big|_2^b = 2\sqrt{\ln b} - 2\sqrt{\ln 2}$, so $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ diverges.

7. $$\int_2^\infty \frac{x}{(x^2+1)^{3/2}}\,dx = -(x^2+1)^{-1/2}\Big|_2^b$$
$$= \frac{1}{\sqrt{5}} - (b^2+1)^{-1/2}$$

$\sum_{k=2}^{\infty} \frac{k}{(k^2+1)^{3/2}}$ converges.

8. $$\int_1^b \frac{1}{(2x+1)^3}\,dx = -\frac{1}{4}(2x+1)^{-2}\Big|_1^b$$
$$= \frac{1}{36} - \frac{1}{4}(2b+1)^{-2}$$

so $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^3}$ converges.

9. $\int_2^b \frac{1}{x(\ln x)^2}\,dx = -(\ln x)^{-1}\Big|_2^b = \frac{1}{\ln 2} - \frac{1}{\ln b}$

so $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}dx$ converges.

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10. $\int_1^b \frac{1}{9x^2}dx = -\frac{1}{9}x^{-1}\Big|_1^b = \frac{1}{9} - \frac{1}{9b}$

so $\sum_{k=1}^{\infty} \frac{1}{(3k)^2}$ converges.

11. $\int_1^b e^{3-x}dx = -e^{3-x}\Big|_1^b = e^2 - e^{3-b}$

so $\sum_{k=1}^{\infty} e^{3-k}$ converges.

12. $\int_1^b e^{-2x-1}dx = -\frac{1}{2}e^{-2x-1}\Big|_1^b = \frac{1}{2}e^{-3} - \frac{1}{2}e^{-2b-1}$

so $\sum_{k=1}^{\infty} e^{-2k-1}$ converges.

13. $\int_1^b xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_1^b = \frac{1}{2}e^{-1} - \frac{1}{2}e^{-b^2}$

so $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

14. $\int_1^b x^{-3/4}dx = 4x^{1/4}\Big|_1^b = 4b^{1/4} - 4$

so $\sum_{k=1}^{\infty} k^{-3/4}$ diverges.

15. $\int_1^b \frac{2x+1}{x^2+x+2}dx = \ln(x^2+x+2)\Big|_1^b$

$= \ln(b^2+b+2) - \ln 4$

so $\sum_{k=1}^{\infty} \frac{2k+1}{k^2+k+2}$ diverges.

16. $\int_2^b \frac{x+1}{(x^2+2x+1)^2}dx = -\frac{1}{2}(x^2+2x+1)^{-1}\Big|_2^b$

$= \frac{1}{18} - \frac{1}{2}(b^2+2b+1)^{-1}$

so $\sum_{k=2}^{\infty} \frac{k+1}{(k^2+2k+1)^2}$ converges.

17. The series is $\sum_{k=0}^{\infty} \frac{3}{9+k^2}$. Let $f(x) = \frac{3}{9+x^2}$. Then $f(x) > 0$ for all $x \ge 0$, $f(x)$ is continuous, $f'(x) = -6x(9+x^2)^{-2} < 0$ for all $x > 0$, so $f(x)$ is decreasing. Therefore, the series converges.

18. Let $f(x) = \frac{e^{1/x}}{x^2}$. Then $f(x) > 0$ for all $x \ge 1$.

$f'(x) = \frac{-\frac{1}{x^2}e^{1/x}x^2 - 2xe^{1/x}}{x^4} = \frac{-e^{1/x} - 2xe^{1/x}}{x^4}$

< 0 for all $x > 0$, so $f(x)$ is decreasing, positive, and continuous for $x \ge 1$

$\int_1^b \frac{e^{1/x}}{x^2}dx = -e^{1/x}\Big|_1^b = e - e^{1/b}$, so $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$ converges.

19. Let $f(x) = \frac{x}{e^x} = xe^{-x}$. Then $f(x) > 0$ for all $x \ge 1$, $f(x)$ is continuous and

$f'(x) = \frac{e^x - xe^x}{e^{2x}} < 0$ for all $x > 1$, so $f(x)$ is decreasing for $x \ge 1$.

Integrating by parts,

$\int_1^b xe^{-x}dx = -xe^{-x} - e^{-x}\Big|_1^b$

$= 2e^{-1} - be^{-b} - e^{-b}$.

Thus $\sum_{k=1}^{\infty} ke^{-k}$ converges.

20. The series is convergent, since it is geometric with $r = \frac{3}{4} < 1$. The integral $\int_1^{\infty} \frac{3^x}{4^x}dx$ must also converge, since $f(x) = \left(\frac{3}{4}\right)^x = e^{\ln(3/4)x}$ is continuous, positive, and decreasing, since $f'(x) = \ln\left(\frac{3}{4}\right)e^{\ln(3/4)x}$ and $\ln\left(\frac{3}{4}\right) < 0$. Therefore the integral test applies.

21. For all $k \ge 2$, $\frac{1}{k^2+5} < \frac{1}{k^2}$. The series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is shown in the text to be convergent. Thus $\sum_{k=2}^{\infty} \frac{1}{k^2+5}$ converges by the comparison test.

22. For $k \ge 2$, $\frac{1}{\sqrt{k^2-1}} > \frac{1}{\sqrt{k^2}} = \frac{1}{k}$. $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2-1}}$ diverges.

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23. For $k \ge 1$, $\dfrac{1}{2^k + k} < \dfrac{1}{2^k}$. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges. It is geometric with $r = \dfrac{1}{2}$, so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k + k}$ converges.

24. For $k \ge 1$, $\dfrac{1}{k3^k} \le \dfrac{1}{3^k}$. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k}$ converges $\left(\text{geometric with } r = \dfrac{1}{3}\right)$ so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k3^k}$ converges.

25. For $k \ge 1$, $\dfrac{1}{5^k} \cos^2\left(\dfrac{k\pi}{4}\right) \le \dfrac{1}{5^k}(1)$. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges (geometric with $r = 1$), so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k} \cos^2\left(\frac{k\pi}{4}\right)$ converges.

26. For $k \ge 0$,

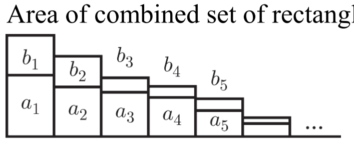
$$\frac{1}{\left(\frac{3}{4}\right)^k + \left(\frac{5}{4}\right)^k} < \frac{1}{\left(\frac{5}{4}\right)^k} = \left(\frac{4}{5}\right)^k. \ \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^k$$

converges $\left(\text{geometric with } r = \dfrac{4}{5}\right)$, so $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\left(\frac{3}{4}\right)^k + \left(\frac{5}{4}\right)^k}$ converges.

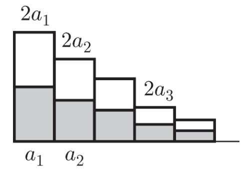
27. No; in order for the comparison test to yield any information we would need $\dfrac{1}{k \ln k} > \dfrac{1}{k}$ for $k \ge 2$, which is false.

28. Yes; the first series converges, since for $k \ge 3$, $\dfrac{1}{k^2 \ln k} < \dfrac{1}{k^2}$. (Actually to apply the test, we need to start the series at $k = 3$, since $\ln 2 < 1$).

29. Area of top set of rectangles: T
Area of bottom set of rectangles: S
Area of combined set of rectangles: $S + T$



30. When the height of each rectangle is doubled the area is doubled. Hence the area of the set of all the taller rectangles is twice the area of the set of all the original rectangles.



31.
$$\sum_{k=0}^{\infty} \frac{8^k + 9^k}{10^k} = \sum_{k=0}^{\infty} \left(\frac{8}{10}\right)^k + \sum_{k=0}^{\infty} \left(\frac{9}{10}\right)^k = \frac{1}{1 - \frac{8}{10}} + \frac{1}{1 - \frac{9}{10}} = 5 + 10 = 15$$

32. We know from the text that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. By Exercise 30, $\displaystyle\sum_{k=1}^{\infty} 3\left(\frac{1}{k^2}\right)$ is also convergent. Since $e^{1/k} \le e < 3$ for all $k \ge 1$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$ is convergent, by the comparison test.

11.5 Taylor Series

1. $f(x) = \dfrac{1}{2x + 3}$; $f'(x) = -\dfrac{2}{(2x+3)^2}$;

$f''(x) = \dfrac{2^2 \cdot 2}{(2x+3)^3}$; $f'''(x) = \dfrac{-2^3 \cdot 2 \cdot 3}{(2x+3)^4}$

The Taylor series at $x = 0$ is

$$\begin{aligned} f(x) &= \frac{1}{3} - \frac{2}{9}x + \frac{2^2 \cdot 2}{3^3 \cdot 2!}x^2 - \frac{2^3 \cdot 3!}{3^4 \cdot 3!}x^3 + \cdots \\ &= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{2^3}{3^4}x^3 + \frac{2^4}{3^5}x^4 - \cdots \end{aligned}$$

2. $f(x) = \ln(1 - 3x)$; $f'(x) = -\dfrac{3}{1 - 3x}$;

$f''(x) = -\dfrac{3^2}{(1-3x)^2}$; $f'''(x) = \dfrac{-3^3 \cdot 2}{(1-3x)^3}$;

$f^{(4)}(x) = \dfrac{-3^4 \cdot 3!}{(1-3x)^4}$.

The Taylor series at $x = 0$ is

$$f(x) = 0 - 3x - \frac{9x^2}{2!} - \frac{3^3 \cdot 2!}{3!}x^3 - \frac{3^4 \cdot 3!}{4!}x^4 + \cdots$$

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3. $f(x)=(1+x)^{1/2}$; $f'(x)=\frac{1}{2}(1+x)^{-1/2}$; $f''(x)=-\frac{1}{2^2}(1+x)^{-3/2}$; $f'''(x)=\frac{3}{2^3}(1+x)^{-5/2}$;

$f^{(4)}(x)=\frac{-3\cdot 5}{2^4}(1+x)^{-7/2}$.

The Taylor series at $x=0$ is

$$f(x)=1+\frac{1}{2}x-\frac{1}{2^2\cdot 2!}x^2+\frac{1\cdot 3}{2^3\cdot 3!}x^3-\frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4+\cdots$$

4. $f(x)=(1+x)^3$; $f'(x)=3(1+x)^2$; $f''(x)=6(1+x)$; $f'''(x)=6$; $f^{(n)}(x)=0$ for all $n\geq 4$. The Taylor series at $x=0$ is

$$f(x)=1+3x+\frac{6}{2!}x^2+\frac{6}{3!}x^3=1+3x+3x^2+x^3$$

5. $\frac{1}{1-3x}=1+3x+(3x)^2+(3x)^3+\cdots$

6. $\frac{1}{1+x}=1+(-x)+(-x)^2+(-x)^3+(-x)^4+\cdots=1-x+x^2-x^3+x^4-\ldots$

7. $\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\cdots$

(using Exercise 6)

8. $\frac{x}{1+x^2}=x\left(\frac{1}{1+x^2}\right)=x-x^3+x^5-x^7+x^9-\cdots$

(using Exercise 7)

9. $\frac{1}{(1+x)^2}=-\frac{d}{dx}\left[\frac{1}{1+x}\right]=1-2x+3x^2-4x^3+\cdots$

(using Exercise 6)

10.
$$\frac{x}{(1-x)^3}=\frac{x}{2}\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right]=\frac{x}{2}\left(0+0+2+6x+4\cdot 3x^2+5\cdot 4x^3+6\cdot 5x^4+\cdots\right)$$
$$=x+\frac{3\cdot 2}{2}x^2+\frac{4\cdot 3}{2}x^3+\frac{5\cdot 4}{2}x^4+\frac{6\cdot 5}{2}x^5+\cdots$$

11. $5e^{x/3}=5+5\left(\frac{x}{3}\right)+\frac{5}{2!}\left(\frac{x}{3}\right)^2+\frac{5}{3!}\left(\frac{x}{3}\right)^3+\frac{5}{4!}\left(\frac{x}{3}\right)^4+\cdots$

12. $x^3e^{x^2}=x^3\left[1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots\right]=x^3+x^5+\frac{x^7}{2!}+\frac{x^9}{3!}+\frac{x^{11}}{4!}+\cdots$

13. $1-e^{-x}=1-\left[1-x+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\cdots\right]=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\cdots$

14. $3(e^{-2x}-2)=\left[3-6x+\frac{3(-2x)^2}{2!}+\frac{3(-2x)^3}{3!}+\frac{3(-2x)^4}{4!}+\cdots\right]-6=-3-6x+\frac{3\cdot 2^2x^2}{2!}-\frac{3\cdot 2^3x^3}{3!}+\frac{3\cdot 2^4x^4}{4!}-\cdots$

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15. $\ln(1+x) = \int \frac{1}{(1+x)}dx = \int (1 - x + x^2 - x^3 + x^4 - \cdots)dx = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

For $x = 0$, $\ln(1 + x) = 0 = C + 0 + 0 + \cdots$; so $C = 0$ and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

16. $\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \cdots$ (from Exercise 15)

17. $\cos 3x = 1 - \frac{1}{2!}(3x)^2 + \frac{1}{4!}(3x)^4 - \cdots$

18. $\cos x^2 = 1 - \frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \cdots$

19. $\sin 3x = 3x - \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 - \cdots$

20. $x \sin x^2 = x\left[x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots\right] = x^3 - \frac{1}{3!}x^7 + \frac{1}{5!}x^{11} - \cdots$

21. $xe^{x^2} = x\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right] = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \frac{x^9}{4!} + \cdots$

22. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right] = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$

23. **a.** $f(x) = \frac{1}{2}(e^x + e^{-x})$; $f'(x) = \frac{1}{2}(e^x - e^{-x})$; $f''(x) = \frac{1}{2}(e^x + e^{-x})$

The Taylor expansion at $x = 0$ is

$\cosh x = \frac{1}{2}(2) + \frac{1}{2}(2)\frac{x^2}{2!} + \frac{1}{2}(2)\frac{x^4}{4!} + \frac{1}{2}(2)\frac{x^6}{6!} + \cdots = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

b. $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right] + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right]\right)$

$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

24. **a.** $f(x) = \frac{1}{2}(e^x - e^{-x})$; $f'(x) = \frac{1}{2}(e^x + e^{-x})$; $f''(x) = f(x)$

The Taylor expansion of $f(x)$ at $x = 0$ is

$\sinh x = \frac{1}{2}(0) + \frac{1}{2}(2)x + \frac{1}{2}(0)\frac{x^2}{2!} + \frac{1}{2}(2)\frac{x^3}{3!} + \frac{1}{2}(0)\frac{x^4}{4!} + \frac{1}{2}(2)\frac{x^5}{5!}\cdots = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

b. $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right] - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right]\right)$

$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

25. Substituting $-x$ for x in the given series yields

$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \cdots$ at $x = 0$.

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26. Substituting x^2 for x in the given series of Exercise 25 gives
$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \cdots.$$

27. Substituting x^2 for x in the given series in the statement of Exercise 25 gives
$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 - \cdots.$$
Since $\ln\left(x+\sqrt{1+x^2}\right) + C = \int \frac{1}{\sqrt{1+x^2}}\,dx$, it follows that
$$\ln\left(x+\sqrt{1+x^2}\right) + C = x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}x^9 - \cdots.$$
Since $\ln(0+1) + C = 0 + C = 0 + 0 + \cdots$, $C = 0$.

28. $\frac{x}{(1-x)^2} = x\frac{d}{dx}\left[\frac{1}{1-x}\right] = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$

Thus $\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1-x^2}{(1-x)^4} = \frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + \cdots$

29. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\frac{d}{dx}[e^x] = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

30. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$

Substituting $-x$ for x,

$\cos(-x) = 1 - \frac{1}{2!}(-x)^2 + \frac{1}{4!}(-x)^4 - \frac{1}{6!}(-x)^6 + \cdots = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \cos x.$

31. The coefficient of x^5 in the series must equal $\frac{f^{(5)}(0)}{5!}$. Thus $f^{(5)}(0) = 5!\left(\frac{2}{5}\right) = 48.$

32. The coefficient of x^4 in the series must equal $\frac{f^{(4)}(0)}{4!}$. Thus $f^{(4)}(0) = 4!\left(\frac{5}{24}\right) = 5.$

33. The coefficient x^4 in the series must equal $\frac{f^{(4)}(0)}{4!}$. Thus $f^{(4)}(0) = 4!(0) = 0.$

34. The coefficient of x^2 in the given series will be the coefficient of x^4 in the expansion of $f(x)$. Thus $\frac{f^{(4)}(0)}{4!} = 2;\ f^{(4)}(0) = 48.$

35. $\int e^{-x^2}\,dx = \int\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right]dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right] + C$

36. $\int xe^{x^3}\,dx = \int\left[x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + \cdots\right]dx = \left[\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{8\cdot 2!} + \frac{x^{11}}{11\cdot 3!} + \cdots\right] + C$

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37. $\int \frac{1}{1+x^3} = \int [1-x^3+x^6-x^9+\cdots]dx$

$$= \left[x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots\right] + C$$

38. $\int_0^1 \sin x^2 dx = \int_0^1 \left[x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots\right]dx$

$$= \left[\frac{x^3}{3} - \frac{1}{7\cdot 3!}x^7 + \frac{1}{11\cdot 5!}x^{11} - \cdots\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \cdots$$

39. $\int_0^1 e^{-x^2} dx = \int_0^1 \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right]dx$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \cdots$$

40. $\int_0^1 xe^{x^3} dx = \int_0^1 \left[x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + \cdots\right]dx$

$$= \left[\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^8}{8\cdot 2!} + \frac{x^{11}}{11\cdot 3!} + \cdots\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{8\cdot 2!} + \frac{1}{11\cdot 3!} + \cdots$$

41. a. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Since the above expansion is valid for all x and all of the terms are positive for $x > 0$, it follows that $e^x > 1 + x + \frac{x^2}{2!} > \frac{x^2}{2}$.

b. For $x > 0$, e^x and $\frac{x^2}{2}$ are both positive.

Thus from $\frac{x^2}{2} < e^x$, it follows that

$\frac{1}{\left(\frac{x^2}{2}\right)} > \frac{1}{e^x}$; or $\frac{2}{x^2} > e^{-x}$.

c. For $x > 0$, from (b) we have

$xe^{-x} < \frac{2x}{x^2} = \frac{2}{x}$. Now $xe^{-x} > 0$ for all $x > 0$. Thus $0 < xe^{-x} < \frac{2}{x}$ for all $x > 0$.

Since $\frac{2}{x} \to 0$ as $x \to \infty$, it follows that

$\lim_{x\to\infty} xe^{-x} = 0$.

42. a. The expansion

$e^{kx} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \cdots$ is valid for all x. Since k and x are both positive, all terms of the series are positive. Hence for $x > 0$, $e^{kx} > 1 + kx + \frac{k^2x^2}{2!} > \frac{k^2x^2}{2}$.

b. For $x > 0$, e^{kx} and $\frac{k^2x^2}{2}$ are both positive. Thus from $e^{kx} > \frac{k^2x^2}{2}$, we may conclude $\frac{1}{e^{kx}} < \frac{1}{\left(\frac{k^2x^2}{2}\right)}$; or $e^{-kx} < \frac{2}{k^2x^2}$.

c. For $x > 0$, from (b) we have

$xe^{-kx} < \frac{2x}{k^2x^2} = \frac{2}{k^2x}$. Now $xe^{-kx} > 0$ for all $x > 0$. Thus $0 < xe^{-kx} < \frac{2}{k^2x}$. Since $\frac{2}{k^2x} \to 0$ and $x \to \infty$, it follows that

$\lim_{x\to\infty} xe^{-kx} = 0$.

43. The expression

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ is valid for all x. For $x > 0$, all terms in the series are positive. Therefore, for $x > 0$,

$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6} > \frac{x^3}{6}$. Thus for $x > 0$,

$\frac{1}{e^x} < \frac{1}{\left(\frac{x^3}{6}\right)}$; or $e^{-x} < \frac{6}{x^3}$, which implies

$x^2e^{-x} < \frac{6x^2}{x^3} = \frac{6}{x}$. Therefore, for $x > 0$,

$0 < x^2e^{-x} < \frac{6}{x}$. Since $\lim_{x\to\infty} \frac{6}{x} = 0$, it follows that $\lim_{x\to\infty} x^2e^{-x} = 0$.

44. Replace x by kx in the solution to Exercise 43.

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45. For any fixed value of x there exists a value c such that $|R_n(x)| = \frac{|\sin(c)|}{(n+1)!}|x|^{n+1}$ when n is even and $|R_n(x)| = \frac{|\cos(c)|}{(n+1)!}|x|^{n+1}$ when n is odd. For any value c, $|\cos(c)| \le 1$ and $|\sin(c)| \le 1$ so in either case we have $|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$. Now as $n \to \infty$, $\frac{|x|^{n+1}}{(n+1)!} \to 0$ and therefore $|R_n(x)| \to 0$.

46. For any number c, the $n + 1$ derivative of e^x evaluated at c is e^c. By the Remainder Formula there exists a number c between 0 and x such that $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$. Then, since $|c| < |x|$ we have

$$|R_n(x)| = \frac{e^{|c|}}{(n+1)!}|x|^{n+1} \le \frac{e^{|x|}}{(n+1)!}|x|^{n+1}.$$

Since $\lim_{x\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ we can conclude that $\lim_{x\to\infty} |R_n(x)| = 0$ as well.

Chapter 11 Fundamental Concept Check Exercises

1. The nth Taylor polynomial of $f(x)$ at $x = a$ is

$$p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

2. The values of $f(x)$ and $p_n(x)$ and their derivatives of order up to n agree at $x = a$. That is, $f(a) = p_n(a)$, $f'(a) = p_n'(a)$, …, $f^{(n)}(a) = p_n^{(n)}(a)$.

3. $R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$, where c is a number between a and x.

4. The algorithm is defined by $x_{\text{new}} = x_{\text{old}} - \frac{f(x_{\text{old}})}{f'(x_{\text{old}})}$. To approximate f, a zero of $f(x)$, start by guessing a value x_0 that is close to r. Determine x_1 by substituting x_0 into the formula. Next, determine x_2 by substituting x_1 into the formula. Continue in this manner to obtain a sequence of approximations $x_0, x_1, x_2, \ldots,$ which usually approach as close to r as desired.

5. The nth partial sum of an infinite series is given by $S_n = a_1 + a_2 + \cdots + a_n$.

6. Form the partial sums S_n for $n = 1, 2, \ldots$. If the values of S_n approach a limit as n increases indefinitely, then the series is convergent. If S_n does not approach a limit as n increases indefinitely, then the series is divergent.

7. The sum of a convergent infinite series is the limit of its partial sums S_n.

8. The series $a + ar + ar^2 + \cdots$ is a geometric series with ratio r. The series converges if $|r| < 1$.

9. The sum of a convergent geometric series is given by $S = \frac{a}{1-r}$.

10. The Taylor series of $f(x)$ at $x = 0$ is given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots.$$

11. Given any power series $\sum_{k=0}^{\infty} a_k x^k$, we have:

Case 1: The series converges only for $x = a$ and diverges for all other values of x. Then, $R = 0$.
Case 2: There is a positive constant R such that the series converges for $|x| < R$ and diverges for $|x| > R$.
Case 3: The series converges for all x, so $R = \infty$.

Chapter 11 Review Exercises

1. $f(x) = x(x+1)^{3/2}$

$f'(x) = (x+1)^{3/2} + \frac{3}{2}x(x+1)^{1/2}$

$f''(x) = \frac{3}{2}(x+1)^{1/2} + \frac{3}{2}(x+1)^{1/2} + \frac{3}{4}x(x+1)^{-1/2}$

$= 3(x+1)^{1/2} + \frac{3}{4}x(x+1)^{-1/2}$

$p_2(x) = 0 + x + \frac{3}{2!}x^2 = x + \frac{3}{2}x^2$

2. $f(x) = (2x+1)^{3/2}$; $f'(x) = 3(2x+1)^{1/2}$

$f''(x) = 3(2x+1)^{-1/2}$;

$f'''(x) = -3(2x+1)^{-3/2}$;

$f^{(4)}(x) = 9(2x+1)^{-5/2}$

$p_4(x) = 1 + 3x + \frac{3x^2}{2!} - \frac{3x^3}{3!} + \frac{9x^4}{4!}$

$= 1 + 3x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{3}{8}x^4$

3. For all $n \geq 3$, $p_n(x) = x^3 - 7x^2 + 8$.

4. $f(x) = \frac{2}{2-x} = \frac{1}{1-\frac{x}{2}}$

$= 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots + \left(\frac{x}{2}\right)^n + \cdots$

So $p_n(x) = 1 + \frac{x}{2} + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots \frac{1}{2^n}x^n$.

5. $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$,

$f^{(n)}(x) = 0$ for $n \geq 3$.

$p_3(x) = 3^2 + 2(3)(x-3) + \frac{2}{2!}(x-3)^2 + 0$

$= 9 + 6(x-3) + (x-3)^2$

6. $f(x) = e^x = f^{(n)}(x)$ for all $n \geq 1$.

$p_3(x) = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3$

7. $f(t) = -\ln(\cos 2t)$, $f'(t) = \frac{2\sin 2t}{\cos 2t} = 2\tan 2t$,

$f''(t) = 4\sec^2 2t$

$p_2(t) = 0 + \frac{0}{1!}t + \frac{4}{2!}t^2 = 2t^2$

$\int_0^{1/2} f(t)dt \approx \int_0^{1/2} 2t^2 dt = \left(\frac{2}{3}t^3\right)\Big|_0^{\frac{1}{2}}$

$= \frac{2}{3}\left(\frac{1}{8}\right) = \frac{1}{12}$

8. $f(x) = \tan x$, $f'(x) = \sec^2 x$,

$f''(t) = (2\sec x)\sec x \tan x = 2\sec^2 x \tan x$

$p_2(x) = 0 + x + 0 = x$

$\tan(.1) \approx p_2(.1) = .1$

9. a. $f(x) = x^{1/2}$; $f'(x) = \frac{1}{2}x^{-1/2}$;

$f''(x) = -\frac{1}{4}x^{-3/2}$

$p_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$

b. $p_2(8.7) = 3 + \frac{1}{6}(-.3) - \frac{1}{216}(-.3)^2$

≈ 2.949583

c. $f(x) = x^2 - 8.7$; $f'(x) = 2x$

$x_0 = 3$, $x_1 = 3 - \frac{.3}{6} = 2.95$,

$x_2 = 2.95 - \frac{(2.95)^2 - 8.7}{2(2.95)} \approx 2.949576$

10. a. $f(x) = \ln(1-x)$; $f'(x) = -\frac{1}{1-x}$;

$f''(x) = -\frac{1}{(1-x)^2}$; $f'''(x) = -\frac{2}{(1-x)^3}$

$p_3(x) = 0 - (1)x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3$

$= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3$

$\ln(1.3) = f(-.3) \approx p_3(-.3)$

$= -(-.3) - \frac{1}{2}(-.3)^2 - \frac{1}{3}(-.3)^3$

$= .264$

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b. $f(x)=e^x-1.3;\ f'(x)=e^x;\ x_0=0$

$$x_1=0-\frac{e^0-1.3}{e^0}=.3$$
$$x_2=0.3-\frac{e^{0.3}-1.3}{e^{0.3}}\approx .2631$$

11. $f(x)=x^2-3x-2,\ f'(x)=2x-3$
$x_0=4$
$$x_1=4-\frac{4^2-3(4)-2}{2(4)-3}=\frac{18}{5}=3.6$$
$$x_2=3.6-\frac{3.6^2-3(3.6)-2}{2(3.6)-3}=\frac{374}{105}\approx 3.5619$$

12. $f(x)=e^{2x}-e^{-x}-1,\ f'(x)=2e^{2x}+e^{-x}$
$x_0=0$
$$x_1=0-\frac{e^{2(0)}-e^{-0}-1}{2e^{2(0)}+e^{-0}}=\frac{1}{3}$$
$x_2\approx .2832$

13. The series is geometric with $a=1,\ r=-\frac{3}{4}$.
The sum is $\frac{1}{1+\frac{3}{4}}=\frac{4}{7}$.

14. The series is geometric with
$a=\frac{5^2}{6}=\frac{25}{6},\ r=\frac{5}{6}$. The sum is $\frac{\frac{25}{6}}{1-\frac{5}{6}}=25$.

15. The series is geometric with $a=\frac{1}{8},\ r=\frac{1}{8}$.
The sum is $\frac{\frac{1}{8}}{1-\frac{1}{8}}=\frac{1}{7}$.

16. The series is geometric with $a=\frac{4}{7},\ r=-\frac{8}{7}$,
so it diverges.

17. The series is geometric with
$a=\frac{1}{m+1},\ r=\frac{m}{m+1}$, so (since $m>0$) it
converges to $\frac{\frac{1}{m+1}}{1-\frac{m}{m+1}}=\frac{1}{m+1}\left(\frac{m+1}{1}\right)=1.$

18. The series is geometric with $a=\frac{1}{m},\ r=-\frac{1}{m}$,
so it converges if $m>1$. In this case, the sum
is $\frac{\frac{1}{m}}{1+\frac{1}{m}}=\frac{1}{m+1}$. It diverges if $m\le 1$.

19. This is the Taylor series for e^x with $x=2$.
Thus the sum is e^2.

20. This is the Taylor series for e^x with $x=\frac{1}{3}$.
Thus the sum is $e^{1/3}$.

21. Since $\sum_{k=0}^{\infty}\frac{1}{3^k}$ converges to $\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$ and
$\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$ converges to $\frac{1}{1-\frac{2}{3}}=3$,
$$\sum_{k=0}^{\infty}\left[\frac{1}{3^k}+\left(\frac{2}{3}\right)^k\right]=\sum_{k=0}^{\infty}\frac{1+2^k}{3^k}=\frac{3}{2}+3=\frac{9}{2}.$$

22.
$$\sum_{k=0}^{\infty}\frac{3^k+5^k}{7^k}=\sum_{k=0}^{\infty}\left(\frac{3}{7}\right)^k+\sum_{k=0}^{\infty}\left(\frac{5}{7}\right)^k=\frac{1}{1-\frac{3}{7}}+\frac{1}{1-\frac{5}{7}}=\frac{7}{4}+\frac{7}{2}=\frac{21}{4}$$

23.
$$\int_1^{\infty}\frac{1}{x^3}\,dx=\lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\Big|_1^b\right]=\lim_{b\to\infty}\left[\frac{1}{2}-\frac{1}{2b^2}\right]=\frac{1}{2}$$
The given series converges by the integral test.

24. The series is geometric with $a=\frac{1}{3}$ and $r=\frac{1}{3}$,
so it converges.

25.
$$\int_1^{\infty}\frac{\ln x}{x}\,dx=\lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\Big|_1^b\right]=\lim_{b\to\infty}\left[\frac{(\ln b)^2}{2}\right]=\infty$$
Thus the series diverges by the integral test.

26. $\frac{k^3}{(k^4+1)^2}=\frac{k^3}{k^8+2k^4+1}\le\frac{k^3}{k^8}=\frac{1}{k^5}$ for $k\ge 1$.
Thus since $\sum_{k=0}^{\infty}\frac{1}{k^5}$ converges by the integral
test, $\sum_{k=1}^{\infty}\frac{k^3}{(k^4+1)^2}$ converges by the
comparison test.

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27. The series converges if $\int_1^\infty \frac{1}{x^p}\,dx$ converges (by the integral test).

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\left.\frac{1}{-p+1}x^{-p+1}\right|_1^b\right] = \lim_{b\to\infty}\left[\frac{1}{1-p}(b^{1-p}-1)\right]$$

This limit is finite if $p > 1$.

28. This is a geometric series with $r = \frac{1}{p}$. Thus it converges when $\left|\frac{1}{p}\right| < 1$ or $|p| > 1$.

29. Replacing x by $-x^3$ in the series for $\frac{1}{1-x}$ gives $\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} - \cdots$

30. $\frac{d}{dx}[\ln(1+x^3)] = \frac{3x^2}{1+x^3}$, so

$$\ln\left(1+x^3\right) = \int\left[3x^2 - 3x^5 + 3x^8 - 3x^{11} + 3x^{14} - \cdots\right]dx = \left[x^3 - \frac{3}{6}x^6 + \frac{3}{9}x^9 - \frac{3}{12}x^{12} + \cdots\right] + C$$

$\left(\text{using the expansion of } \frac{1}{1+x^3} \text{ in Exercise 29.}\right)$

$\ln(1) = 0 = 0 + 0 + \cdots + C$; so $C = 0$.

$$\ln(1+x^3) = x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \frac{1}{4}x^{12} + \ldots,\ |x| < 1$$

31. $$\frac{1}{(1-3x)^2} = \frac{1}{3}\frac{d}{dx}\left[\frac{1}{1-3x}\right] = \frac{1}{3}\frac{d}{dx}[1 + 3x + 3^2x^2 + 3^3x^3 + \cdots] = 1 + 6x + 27x^2 + 108x^3 + \cdots$$

32. $$\frac{e^x - 1}{x} = \frac{1}{x}\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] = 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots$$

33. a. $$\cos 2x = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \cdots = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots$$

b. $$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots\right]$$
$$= \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \cdots = x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \cdots$$

34. a. $$\cos 3x = 1 - \frac{1}{2!}(3x)^2 + \frac{1}{4!}(3x)^4 - \frac{1}{6!}(3x)^6 + \cdots = 1 - \frac{3^2}{2!}x^2 + \frac{3^4}{4!}x^4 - \frac{3^6}{6!}x^6 + \cdots$$

b. Adding the first three terms above to the corresponding terms in the expansion of 3 cos x and multiplying by $\frac{1}{4}$ gives $p_4(x) = \frac{1}{4}\left[(1+3) - \left(\frac{3^2}{2!} + \frac{3}{2!}\right)x^2 + \left(\frac{3^4}{4!} + \frac{3}{4!}\right)x^4\right] = 1 - \frac{3}{2}x^2 + \frac{7}{8}x^4$.

35. $$\frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x} = [1 + x + x^2 + x^3 + \cdots] + [x + x^2 + x^3 + x^4 + \cdots] = 1 + 2x + 2x^2 + 2x^3 + \cdots$$

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36. Using Exercise 32,

$$\int_0^{1/2} \frac{e^x - 1}{x}\,dx = \int_0^{1/2}\left[1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots\right]dx = \left[x + \frac{1}{4}x^2 + \frac{1}{3!\cdot 3}x^3 + \frac{1}{4!\cdot 4}x^4 + \cdots\right]_0^{1/2}$$
$$= \frac{1}{2} + \frac{1}{4\cdot 2^2} + \frac{1}{3!\cdot 3\cdot 2^3} + \frac{1}{4!\cdot 4\cdot 2^4} + \cdots$$

37. **a.** x^2

b. 0

c. $\int_0^1 \sin x^2\,dx \approx \int_0^1\left(x^2 - \frac{1}{6}x^6\right)dx = \left.\frac{x^3}{3} - \frac{1}{42}x^7\right|_0^1 = \frac{1}{3} - \frac{1}{42} = \frac{13}{42} \approx .3095$

(exact value to four decimal places: .3103)

38. $p_4(x) = x + \frac{1}{3!}x^3$

39. **a.** $f'(x) = 2x + 4x^3 + 6x^5 + \cdots$

b. The series given for $f(x)$ is the Taylor series of $\frac{1}{1-x^2}$. Thus $f(x) = \frac{1}{1-x^2}$ and $f'(x) = \frac{2x}{(1-x^2)^2}$.

40. **a.** $\int f(x)\,dx = \int\left[x - 2x^3 + 4x^5 - 8x^7 + 16x^9 - \cdots\right]dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x^4 + \frac{2}{3}x^6 - x^8 + \frac{8}{5}x^{10} - \cdots\right] + C$

b. The series given for $f(x) = x[1 - 2x^2 + 2^2x^4 - 2^3x^6 + 2^4x^8 - \cdots]$ is the Taylor expansion of $\frac{x}{1+2x^2}$.

Thus $f(x) = \frac{x}{1+2x^2}$ and $\int f(x)\,dx = \frac{1}{4}\ln(1+2x^2) + C$.

41. $100 + 100(0.85) + 100(0.85)^2 + 100(0.85)^3 + \cdots = \frac{100}{1-0.85} \approx 666.666667$

The amount beyond the original 100 million dollars is \$566,666,667.

42. $100 + (0.85)(100) + (0.80)(0.85)^2(100) + (0.80)^2(0.85)^3(100) + \cdots$
$$= 100 + 85 + (0.80)(0.85)(85) + (0.80)^2(0.85)^2(85) + \cdots$$
$$= 100 + \frac{85}{1-(0.80)(0.85)} = 365.625 \text{ million dollars}$$

43. $\sum_{k=1}^{\infty} 10{,}000e^{-0.08k} = \sum_{k=1}^{\infty} 10{,}000\left(e^{-0.08}\right)^k = \frac{10{,}000(e^{-0.08})}{1-e^{-0.08}} \approx \$120{,}066.66$

44. $\sum_{k=1}^{\infty} 10{,}000(0.9)^k e^{-0.08k} = \sum_{k=1}^{\infty} 10{,}000\left(0.9e^{-0.08}\right)^k = \frac{10{,}000(0.9)e^{-0.08}}{1-0.9e^{-0.08}} \approx \$49{,}103.30$

45. $\sum_{k=1}^{\infty} 10{,}000(1.08)^k e^{-0.08k} = \sum_{k=1}^{\infty} 10{,}000\left(1.08e^{-0.08}\right)^k = \frac{10{,}000(1.08)e^{-0.08}}{1-1.08e^{-0.08}} \approx \$3{,}285{,}603.18$

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