Chapter 7 Functions of Several Variables

7.1 Examples of Functions of Several Variables

1. $f(x,\ y) = x^2 - 3xy - y^2$

$f(5,0) = 5^2 - 3(5)(0) - 0^2 = 25$

$f(5,-2) = 5^2 - 3(5)(-2) - (-2)^2 = 51$

$f(a,\ b) = a^2 - 3ab - b^2$

2. $g(x,\ y) = \sqrt{x^2 + 2y^2}$

$g(1,\ 1) = \sqrt{1^2 + 2\left(1^2\right)} = \sqrt{3}$

$g(0,\ -1) = \sqrt{0^2 + 2\left(-1^2\right)} = \sqrt{2}$

$g(a,\ b) = \sqrt{a^2 + 2b^2}$

3. $g(x,\ y,\ z) = \dfrac{x}{y - z}$

$g(2,3,4) = \dfrac{2}{3-4} = -2$

$g(7,\ 46,\ 44) = \dfrac{7}{46 - 44} = \dfrac{7}{2}$

4. $f(x,\ y,\ z) = x^2 e^{\sqrt{y^2 + z^2}}$

$f(1,-1,1) = \left(1^2\right)e^{\sqrt{(-1)^2 + 1^2}} = e^{\sqrt{2}}$

$f(2,\ 3,\ -4) = \left(2^2\right)e^{\sqrt{3^2 + (-4)^2}} = 4e^5$

5. $f(x,y) = xy \Rightarrow$

$f(2+h,3) = (2+h)3 = 6 + 3h$

$f(2,3) = (2)3 = 6$

$f(2+h,3) - f(2,3) = (6+3h) - 6 = 3h$

6. $f(x,y) = xy \Rightarrow$

$f(2,3+k) = 2(3+k) = 6 + 2k$

$f(2,3) = (2)3 = 6$

$f(2+h,3) - f(2,3) = (6+2k) - 6 = 2k$

7. $C(x, y, z)$ is the cost of materials for the rectangular box with dimensions x, y, z in feet. The area of the top and the bottom together is $2xy$, so the cost is $3(2xy) = 6xy$. The area of the front and back together is $2xz$, so the cost is $5(2xz) = 10xz$. The area of the right and left side together is $2yz$, so the cost is $5(2yz) = 10yz$.
Thus, $C(x, y, z) = 6xy + 10xz + 10yz$.

8. $C(x, y, z)$ is the cost of material.
Using the same reasoning as in exercise 7, we have $C(x, y, z) = 3xy + 5xz + 10yz$.

9. $f(x,\ y) = 20x^{1/3}y^{2/3}$

$f(8,1) = 20\left(8^{1/3}\right)\left(1^{2/3}\right) = 40$

$f(1,27) = 20\left(1^{1/3}\right)\left(27^{2/3}\right) = 180$

$f(8,27) = 20\left(8^{1/3}\right)\left(27^{2/3}\right) = 360$

$f(8k,\ 27k)$
$= 20(8k)^{1/3}(27k)^{2/3}$
$= 20\left(8^{1/3}\right)\left(k^{1/3}\right)\left(27^{2/3}\right)\left(k^{2/3}\right)$
$= k(20)\left(8^{1/3}\right)\left(27^{2/3}\right) = kf(8,\ 27)$

10. $f(x,\ y) = 10x^{2/5}y^{3/5}$

$f(3a,\ 3b) = 10(3a)^{2/5}(3b)^{3/5}$
$= 10\left(3^{2/5}\right)\left(a^{2/5}\right)\left(3^{3/5}\right)\left(b^{3/5}\right)$
$= 3(10)\left(a^{2/5}\right)\left(b^{3/5}\right)$
$= 3f(a,b)$

11. $P(A,\ t) = Ae^{-0.05t}$

$P(100,\ 13.8) = 100e^{-0.05(13.8)} = 100e^{-0.69}$
≈ 50.16

$50 invested at 5% continuously compounded interest will yield $100 in 13.8 years.

12. $C(x, y)$ is the cost of utilizing x units of labor and y units of capital. $C(x, y) = 100x + 200y$

13. $T = f(r,v,x) = \dfrac{r}{100}(0.40v - x)$

a. $v = 200{,}000$, $x = 5000$, $r = 2.5$

$T = \dfrac{r(0.4v - x)}{100}$
$= \dfrac{2.5(0.4(200,000) - 5000)}{100}$
$T = \$1875$

b. If $v = 200{,}000$, $x = 5000$, $r = 3$:

$T = \dfrac{r(0.4v - x)}{100} = \dfrac{3(0.4(200,000) - 5000)}{100}$
$= \$2250$

The tax due also increases by 20% since $1875 + (0.2)(1875) = \$2250$.

14. **a.** $v = 100{,}000$, $x = 5000$, $r = 2.2$

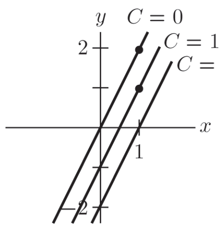
$T = \dfrac{r(0.4v - x)}{100}$
$= \dfrac{2.2(0.4(100,000) - 5000)}{100} = \$770$

b. If $v = 120{,}000$, $x = 5000$, $r = 2.2$:

$$T = \frac{r(0.4v - x)}{100} = \frac{2.2(0.4(120{,}000) - 5000)}{100} = \$946$$

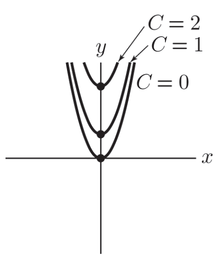
20% of \$770 is \$154, so tax due does not increase by 20%.

15. $C = 2x - y$, so $y = 2x - C$
The level curves are $y = 2x$, $y = 2x - 1$, and $y = 2x - 2$.

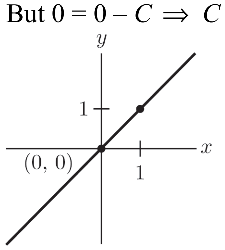


16. $C = -x^2 + 2y$, so $y = \frac{x^2}{2} + \frac{C}{2}$.
The level curves are

$$-x^2 + 2y = 0 \Rightarrow y = \frac{x^2}{2}$$
$$-x^2 + 2y = 1 \Rightarrow y = \frac{x^2}{2} + \frac{1}{2}$$
$$-x^2 + 2y = 2 \Rightarrow y = \frac{x^2}{2} + 1$$

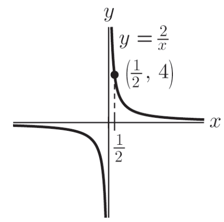


17. $C = x - y$, $y = x - C$
But $0 = 0 - C \Rightarrow C = 0$, so $y = x$.



18. $C = xy \Rightarrow y = \frac{C}{x}$

But $4 = \frac{C}{1/2} \Rightarrow C = 2$, thus $y = \frac{2}{x}$.



19. $y = 3x - 4 \Rightarrow y - 3x = -4$, so $y - 3x = C \Rightarrow$ $f(x, y) = y - 3x$.

20. $y = \frac{2}{x^2} \Rightarrow yx^2 = 2$

Thus, $yx^2 = C \Rightarrow f(x, y) = x^2 y$.

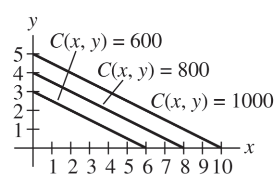
21. They correspond to the points having the same altitude above sea level.

22. $C(x, y) = 100x + 200y$ is the cost of using x units of labor and y units of capital.
If $C(x, y) = 600$, then $100x + 200y = 600 \Rightarrow$ $y = 3 - \frac{1}{2}x$.

If $C(x, y) = 800$, then $y = 4 - \frac{1}{2}x$.

If $C(x, y) = 1000$, then $y = 5 - \frac{1}{2}x$.

Points on the same level curve correspond to production amounts that have the same total cost.



23. (d)

24. (b)

25. (c)

26. (a)

7.2 Partial Derivatives

1. $f(x, y) = 5xy$

$$\frac{\partial f}{\partial x} = 5(1)y = 5y; \quad \frac{\partial f}{\partial y} = 5(1)x = 5x$$

2. $f(x, y) = x^2 - y^2$

$$\frac{\partial f}{\partial x} = 2x; \quad \frac{\partial f}{\partial y} = -2y$$

3. $f(x, y) = 2x^2 e^y$

$$\frac{\partial f}{\partial x} = 2(2x)e^y = 4xe^y$$
$$\frac{\partial f}{\partial y} = 2x^2 e^y(1) = 2x^2 e^y$$

4. $f(x, y) = xe^{xy}$

$\frac{\partial f}{\partial x} = xe^{xy}(y) + (1)e^{xy} = xe^{xy}y + e^{xy}$

$\frac{\partial f}{\partial y} = xe^{xy}(x) + (0)e^{xy} = x^2e^{xy}$

5. $f(x, y) = \frac{x}{y} + \frac{y}{x} = xy^{-1} + x^{-1}y$

$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{y}{x^2}$; $\frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{x}$

6. $f(x, y) = \frac{1}{x+y} = (x+y)^{-1}$

$\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2}$; $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2}$

7. $f(x, y) = (2x - y + 5)^2$

$\frac{\partial f}{\partial x} = 2(2x - y + 5)(2) = 4(2x - y + 5)$

$\frac{\partial f}{\partial y} = 2(2x - y + 5)(-1) = -2(2x - y + 5)$

8. $f(x, y) = \frac{e^x}{1 + e^y}$

$\frac{\partial f}{\partial x} = \frac{e^x}{1+e^y}$; $\frac{\partial f}{\partial y} = \frac{-e^x e^y}{(1+e^y)^2} = \frac{-e^{x+y}}{(1+e^y)^2}$

9. $f(x, y) = xe^{x^2y^2}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= x\left(\frac{\partial}{\partial x}e^{x^2y^2}\right) + e^{x^2y^2}\left(\frac{\partial}{\partial x}x\right)\\ &= x\left(e^{x^2y^2}\frac{\partial}{\partial x}x^2y^2\right) + e^{x^2y^2}\\ &= 2x^2y^2e^{x^2y^2} + e^{x^2y^2}\\ &= e^{x^2y^2}\left(2x^2y^2 + 1\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= x\left(\frac{\partial}{\partial y}e^{x^2y^2}\right) = xe^{x^2y^2}\left(\frac{\partial}{\partial y}x^2y^2\right)\\ &= xe^{x^2y^2}\left(2x^2y\right) = 2x^3ye^{x^2y^2}\end{aligned}$$

10. $f(x, y) = \ln(xy)$

$\frac{\partial f}{\partial x} = \frac{1}{xy} \cdot y = \frac{1}{x}$

$\frac{\partial f}{\partial y} = \frac{1}{xy} \cdot x = \frac{1}{y}$

11. $f(x, y) = \frac{x-y}{x+y}$

$\frac{\partial f}{\partial x} = \frac{1(x+y) - (x-y)(1)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$\frac{\partial f}{\partial y} = \frac{(-1)(x+y) - (x-y)(1)}{(x+y)^2} = -\frac{2x}{(x+y)^2}$

12. $f(x, y) = \sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{1/2}$

$\frac{\partial f}{\partial x} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$

13. $f(L, K) = 3\sqrt{LK}$

$\frac{\partial f}{\partial L} = 3\left(\frac{1}{2}\right)(KL)^{-1/2}(K) = \frac{3}{2}\sqrt{\frac{K}{L}}$

14. $f(p, q) = 1 - p(1 + q) = 1 - p - pq$

$\frac{\partial f}{\partial q} = -p$; $\frac{\partial f}{\partial p} = -1 - q$

15. $f(x, y, z) = \frac{(1 + x^2y)}{z} = z^{-1} + x^2yz^{-1}$

$\frac{\partial f}{\partial x} = 0 + 2xyz^{-1} = \frac{2xy}{z}$

$\frac{\partial f}{\partial y} = 0 + x^2z^{-1} = \frac{x^2}{z}$

$$\begin{aligned}\frac{\partial f}{\partial z} &= -z^{-2} + (-z^{-2}x^2y) = -\frac{1}{z^2} - \frac{x^2y}{z^2}\\ &= -\frac{1 + x^2y}{z^2}\end{aligned}$$

16. $f(x, y, z) = ze^{x/y} = ze^{xy^{-1}}$

$\frac{\partial f}{\partial x} = ze^{xy^{-1}}y^{-1} = \frac{ze^{x/y}}{y}$

$\frac{\partial f}{\partial y} = ze^{xy^{-1}}\left(-xy^{-2}\right) = \frac{-xze^{x/y}}{y^2}$

$\frac{\partial f}{\partial z} = e^{x/y}$

17. $f(x, y, z) = xze^{yz}$

$\frac{\partial f}{\partial x} = (1)ze^{yz} = ze^{yz}$

$\frac{\partial f}{\partial y} = e^{yz}(z)xz = xz^2e^{yz}$

$\frac{\partial f}{\partial z} = (1)xe^{yz} + e^{yz}(y)xz = xe^{yz}(1+yz)$

18. $f(x, y, z) = \frac{xy}{z} = xyz^{-1}$

$\frac{\partial f}{\partial x} = \frac{y}{z}; \quad \frac{\partial f}{\partial y} = \frac{x}{z}$

$\frac{\partial f}{\partial z} = -xyz^{-2} = -\frac{xy}{z^2}$

19. $f(x, y) = x^2 + 2xy + y^2 + 3x + 5y$

$\frac{\partial f}{\partial x} = 2x + 2(y) + 0 + 3 + 0 = 2x + 2y + 3$

$\frac{\partial f}{\partial x}(2, -3) = 2(2) + 2(-3) + 3 = 1$

$\frac{\partial f}{\partial y} = 0 + 2x(1) + 2y + 0 + 5 = 2x + 2y + 5$

$\frac{\partial f}{\partial y}(2, -3) = 2(2) + 2(-3) + 5 = 3$

20. $f(x, y) = \left(x + y^2\right)^3$

$\frac{\partial f}{\partial x} = 3\left(x + y^2\right)^2(1) = 3\left(x + y^2\right)^2$

$\frac{\partial f}{\partial x}(1, 2) = 3(1 + 2^2)^2 = 75$

$\frac{\partial f}{dy} = 3\left(1 + y^2\right)^2(2y) = 6y\left(1 + y^2\right)^2$

$\frac{\partial f}{\partial y}(1, 2) = 6(2)\left(1 + 2^2\right)^2 = 300$

21. $f(x, y) = xy^2 + 5$

$\frac{\partial f}{\partial y} = 2xy$

$\frac{\partial f}{\partial y}(2, -1) = 2(2)(-1) = -4$

This means that if x is kept constant at 2 and y is allowed to vary near -1, then $f(x, y)$ changes at a rate of -4 times the change in y.

22. $f(x, y) = \frac{x}{y - 6}$

$\frac{\partial f}{\partial y} = -\frac{x}{(y-6)^2}$

$\frac{\partial f}{\partial y}(2, 1) = -\frac{2}{(1-6)^2} = -\frac{2}{25}$

This means that if x is kept constant at 2 and y is allowed to vary near 1, then $f(x, y)$ changes at a rate of $-\frac{2}{25}$ times the change in y.

23. $f(x, y) = x^3y + 2xy^2$

$\frac{\partial f}{\partial x} = 3x^2y + 2y^2 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy$

$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4y$

$\frac{\partial f}{\partial y} = x^3 + 4xy \Rightarrow \frac{\partial^2 f}{\partial y^2} = 4x$

$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4y$

24. $f(x, y) = xe^y + x^4y + y^3$

$\frac{\partial f}{\partial x} = e^y + 4x^3y + 0 = e^y + 4x^3y$

$\frac{\partial^2 f}{\partial x^2} = 0 + 12x^2y = 12x^2y$

$\frac{\partial^2 f}{\partial y \partial x} = e^y(1) + 4x^3 = e^y + 4x^3$

$\frac{\partial f}{\partial y} = xe^y(1) + x^4 + 3y^2 = xe^y + x^4 + 3y^2$

$\frac{\partial^2 f}{\partial y^2} = xe^y(1) + 0 + 6y = xe^y + 6y$

$\frac{\partial^2 f}{\partial x \partial y} = e^y + 4x^3 + 0 = e^y + 4x^3$

25. $f(x, y) = 200\sqrt{6x^2 + y^2}$

a. $\frac{\partial f}{\partial x}$ is the marginal productivity of labor.

$\frac{\partial f}{\partial x} = 200\left(\frac{1}{2}\right)(6x^2 + y^2)^{-1/2}(12x)$

$= 1200x(6x^2 + y^2)^{-1/2} = \frac{1200x}{\sqrt{6x^2 + y^2}}$

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When $x = 10$ and $y = 5$, the marginal productivity of labor is

$$\frac{\partial f}{\partial x}(10,\ 5) = \frac{1200(10)}{\sqrt{6(10)^2 + 5^2}} = 480 .$$

$\frac{\partial f}{\partial y}$ is the marginal productivity of capital.

$$\frac{\partial f}{\partial y} = 200\left(\frac{1}{2}\right)(6x^2 + y^2)^{-1/2}(2y)$$
$$= 200y(6x^2 + y^2)^{-1/2} = \frac{200y}{\sqrt{6x^2 + y^2}}$$

When $x = 10$ and $y = 5$, the marginal productivity of capital is

$$\frac{\partial f}{\partial y}(10,\ 5) = \frac{200(5)}{\sqrt{6(10)^2 + 5^2}} = 40 .$$

b. $f(10 + h, 5) - f(10, 5) \approx \frac{\partial f}{\partial x}(10,\ 5) \cdot h$
$= 480h$

c. Using part b, if $h = -.5$ then $f(9.5, 5) \approx 480(-.5) = -240$. So, if capital is fixed at 5 units and labor decreased by .5 unit from 10 to 9.5 units, the number of goods produced will decrease by approximately 240 units.

26. $f(x,\ y) = 300x^{2/3}y^{1/3}$ is the productivity of a country, where x and y are the amounts of labor and capital.

a. $\frac{\partial f}{\partial x}$ is the marginal productivity of labor.

$$\frac{\partial f}{\partial x} = 300\left(\frac{2}{3}\right)x^{-1/3}y^{1/3} = \frac{200\sqrt[3]{y}}{\sqrt[3]{x}}$$

When $x = 125$ and $y = 64$, the marginal productivity of labor is

$$\frac{\partial f}{\partial x}(125,\ 64) = \frac{200\sqrt[3]{64}}{\sqrt[3]{125}} = 160 .$$

$\frac{\partial f}{\partial y}$ is the marginal productivity of capital.

$$\frac{\partial f}{\partial y} = 300x^{2/3}\left(\frac{1}{3}\right)y^{-2/3} = \frac{100\sqrt[3]{x^2}}{\sqrt[3]{y^2}}$$

When $x = 125$ and $y = 64$, the marginal productivity of capital is

$$\frac{\partial f}{\partial y}(125,\ 64) = \frac{100\sqrt[3]{125^2}}{\sqrt[3]{64^2}} = 156.25 .$$

b. $f(125, 66) = f(125, 64 + 66)$
$\approx \frac{\partial f}{\partial y}(125,\ 64) \cdot 2$
$= 156.25 \cdot 2 = 312.5$

So if labor is fixed at 125 units and capital is increased from 64 to 66 units, then productivity increases by 312.5 units.

c. $f(124, 64) = f(125 - 1, 64)$
$\approx \frac{\partial f}{\partial x}(125,\ 64)(-1)$
$= 160(-1) = -160$

Thus, if capital is fixed at 64 units and labor is decreased by one unit, then productivity will decrease by 160 units.

27. As the price of a bus ride increases, fewer people will ride the bus if the train fare remains constant. An increase in train ticket prices, coupled with constant bus fare, should cause more people to ride the bus.

28. $g(p_1,\ p_2)$ is the number of people who will take the train when p_1 is the price of the bus ride and p_2 is the price of the train ride.

$\frac{\partial g}{\partial p_1}$ is positive (an increase in bus fare would mean more people would take the train).

$\frac{\partial g}{\partial p_2}$ is negative (an increase in the train fare would mean fewer people taking the train).

29. If the average price of MP3 players remains constant and the average price of audio files increases, people will purchase fewer MP3 players. An increase in the average price of the MP3 players, coupled with constant audio file prices, should cause a decline in the number of audio files purchased.

30. When the gasoline price is constant, an increase in the price of the car will decrease the demand for the car. If the price of the car is constant and the price of the gasoline increases, the demand for the car will decrease.

31. $V = .08\left(\frac{T}{P}\right) \Rightarrow \frac{\partial V}{\partial P} = \frac{-.08T}{P^2}$

When $P = 20$, $T = 300$,

$\frac{\partial V}{\partial P} = \frac{-.08(300)}{400} = -.06$.

At this level, increasing the pressure by one unit will decrease the volume by approximately .06 unit.

$\frac{\partial V}{\partial T} = \frac{.08}{P}$

When $P = 20$, $T = 300$, $\frac{\partial V}{\partial T} = \frac{.08}{20} = .004$.

At this level, increasing the temperature by one unit will increase the volume by approximately .004 unit.

32. Assuming $m, p, r, s > 0$, all first partial derivatives are positive except

$\frac{\partial f}{\partial p} \approx -.769m^{1.136}r^{0.914}s^{0.816}p^{-1.727} < 0.$

Thus increases in aggregate income, retail prices of the other goods or the strength of the beer (holding the other quantities constant) should cause an increase in the amount of beer consumed; while an increase in the price of beer itself should cause the amount consumed to decrease.

33. Assuming $m, p, r > 0$, $\frac{\partial f}{\partial m} > 0, \frac{\partial f}{\partial r} > 0$ and

$\frac{\partial f}{\partial p} \approx -1.187m^{.595}r^{.922}p^{-1.543} < 0$.

Thus increases in aggregate income or retail prices of other goods (holding the other quantities constant) should cause an increase in the amount of food consumed; while an increase in the price of the food itself should cause the amount consumed to decrease.

34. $f(x, y) = 60x^{3/4}y^{1/4} \Rightarrow \frac{\partial f}{\partial x} = 45y^{1/4}x^{-1/4}$

$\frac{\partial f}{\partial y} = 15x^{3/4}y^{-3/4}$

$$\begin{aligned} f(a, b) &= 60a^{3/4}b^{1/4} \\ &= a\left[45b^{1/4}a^{-1/4}\right] + b\left[15a^{3/4}b^{-3/4}\right] \\ &= a\left[\frac{\partial f}{\partial x}(a, b)\right] + b\left[\frac{\partial f}{\partial y}(a, b)\right] \end{aligned}$$

35. $f(x, y) = 60x^{3/4}y^{1/4}$

$\frac{\partial f}{\partial x} = 45y^{1/4}x^{-1/4} \Rightarrow \frac{\partial^2 f}{\partial x^2} = -\frac{45}{4}y^{1/4}x^{-5/4} < 0$

for all $x, y > 0$. The fact that $\frac{\partial^2 f}{\partial x^2} < 0$ confirms the *law of diminishing returns*, which says that as additional units of a given productive input are added (holding other factors constant) production increases at a decreasing rate. In other words, the marginal productivity of labor is decreasing.

36. $f(x, y) = 60x^{3/4}y^{1/4}$

$\frac{\partial f}{\partial y} = 15x^{3/4}y^{-3/4} \Rightarrow$ for all $x, y > 0$.

$\frac{\partial^2 f}{\partial y^2} = -\frac{45}{4}x^{3/4}y^{-7/4} < 0$

The fact that $\frac{\partial^2 f}{\partial y^2} < 0$ confirms the law of diminishing returns, which says that as additional units of a given productive input are added (holding other factors constant) production increases at a decreasing rate.

37. $f(x, y) = 3x^2 + 2xy + 5y$

$$\begin{aligned} &f(1+h, 4) - f(1, 4) \\ &= \left[3(1+h)^2 + 2(1+h)(4) + 5(4)\right] \\ &\quad - \left[3(1)^2 + 2(1)(4) + 5(4)\right] \\ &= 3h^2 + 14h \end{aligned}$$

38. $A = (.007)W^{0.425}H^{0.725}$

$\frac{\partial A}{\partial W} = (.002975)W^{-0.575}H^{0.725}$

When $W = 54$, $H = 165$,

$\frac{\partial A}{\partial W} = .002975(54)^{-0.575}(165)^{0.725} \approx .01216$

If a person weighing 54 kg who is 165 cm tall increases his weight by 1 kg, the surface area of his body will increase by about $.012\text{ cm}^2$.

$\frac{\partial A}{\partial H} = (.005075)W^{0.425}H^{-0.275}$

When $W = 54$, $H = 165$, $\frac{\partial A}{\partial H} \approx .0067904$. If a person as above increases his height by 1 cm, his body surface will increase by approximately $.0068\text{ m}^2$.

7.3 Maxima and Minima of Functions of Several Variables

1. $f(x, y) = x^2 - 3y^2 + 4x + 6y + 8$

$\frac{\partial f}{\partial x} = 2x + 4; \frac{\partial f}{\partial y} = -6y + 6$

$\left.\begin{array}{r} 2x + 4 = 0 \\ -6y + 6 = 0 \end{array}\right\} \begin{array}{l} x = -2 \\ y = 1 \end{array}$

The only possible extreme point is (–2, 1).

2. $f(x, y) = \frac{1}{2}x^2 + y^2 - 3x + 2y - 5$

$\frac{\partial f}{\partial x} = x - 3; \frac{\partial f}{\partial y} = 2y + 2$

$\left.\begin{array}{r} x - 3 = 0 \\ 2y + 2 = 0 \end{array}\right\} \begin{array}{l} x = 3 \\ y = -1 \end{array}$

The only possible extreme point is (3, –1).

3. $f(x, y) = x^2 - 5xy + 6y^2 + 3x - 2y + 4$

$\frac{\partial f}{\partial x} = 2x - 5y + 3; \frac{\partial f}{\partial y} = -5x + 12y - 2$

$\left.\begin{array}{r} 2x - 5y + 3 = 0 \\ -5x + 12y - 2 = 0 \end{array}\right\} \begin{array}{l} x = 26 \\ y = 11 \end{array}$

The only possible extreme point is (26, 11).

4. $f(x, y) = -3x^2 + 7xy - 4y^2 + x + y$

$\frac{\partial f}{\partial x} = -6x + 7y + 1; \frac{\partial f}{\partial y} = 7x - 8y + 1$

$\left.\begin{array}{r} -6x + 7y + 1 = 0 \\ 7x - 8y + 1 = 0 \end{array}\right\} \begin{array}{l} x = -15 \\ y = -13 \end{array}$

The only possible extreme point is (–15, –13).

5. $f(x, y) = 3x^2 + 8xy - 3y^2 - 2x + 4y - 1$

$\frac{\partial f}{\partial x} = 6x + 8y - 2; \quad \frac{\partial f}{\partial y} = 8x - 6y + 4$

$\left.\begin{array}{r} 6x + 8y - 2 = 0 \\ 8x - 6y + 4 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{1}{5} \\ y = \frac{2}{5} \end{array}$

The only possible extreme point is $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

6. $f(x, y) = 4x^2 + 4xy - 3y^2 + 4y - 1$

$\frac{\partial f}{\partial x} = 8x + 4y; \quad \frac{\partial f}{\partial y} = 4x - 6y + 4$

$\left.\begin{array}{r} 8x + 4y = 0 \\ 4x - 6y + 4 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{1}{4} \\ y = \frac{1}{2} \end{array}$

The only possible extreme point is $\left(-\frac{1}{4}, \frac{1}{2}\right)$.

7. $f(x, y) = x^3 + y^2 - 3x + 6y$

$\frac{\partial f}{\partial x} = 3x^2 - 3; \frac{\partial f}{\partial y} = 2y + 6$

$\left.\begin{array}{r} 3x^2 - 3 = 0 \\ 2y + 6 = 0 \end{array}\right\} \begin{array}{l} x = \pm 1 \\ y = -3 \end{array}$

The possible extreme points are (1, –3) and (–1, –3).

8. $f(x, y) = x^2 - y^3 + 5x + 12y + 1$

$\frac{\partial f}{\partial x} = 2x + 5; \frac{\partial f}{\partial y} = -3y^2 + 12$

$\left.\begin{array}{r} 2x + 5 = 0 \\ -3y^2 + 12 = 0 \end{array}\right\} \begin{array}{l} x = -\frac{5}{2} \\ y = \pm 2 \end{array}$

The possible extreme points are $\left(-\frac{5}{2}, 2\right)$ and $\left(-\frac{5}{2}, -2\right)$.

9. $f(x, y) = -8y^3 + 4xy + 9y^2 - 2y$

$\frac{\partial f}{\partial x} = 4y; \quad \frac{\partial f}{\partial y} = -24y^2 + 4x + 18y - 2$

$\left.\begin{array}{r} 4y = 0 \\ -24y^2 + 18y + 4x - 2 = 0 \end{array}\right\} \begin{array}{l} x = \frac{1}{2} \\ y = 0 \end{array}$

The only possible extreme point is $\left(\frac{1}{2}, 0\right)$.

10. $f(x, y) = -8y^3 + 4xy + 4x^2 + 9y^2$

$\frac{\partial f}{\partial x} = 8x + 4y; \quad \frac{\partial f}{\partial y} = -24y^2 + 18y + 4x$

$\left.\begin{array}{r} 8x + 4y = 0 \\ -24y^2 + 18y + 4x = 0 \end{array}\right\} \left.\begin{array}{r} x = -\frac{1}{2}y \\ -24y^2 + 18y = -4x \end{array}\right\}$

$-24y^2 + 18y = 2y \Rightarrow y = 0, \frac{2}{3}$

The possible extreme points are $(0, 0)$ and $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

11. $f(x, y) = 2x^3 + 2x^2y - y^2 + y$

$\frac{\partial f}{\partial x} = 6x^2 + 4xy; \quad \frac{\partial f}{\partial y} = 2x^2 - 2y + 1$

$\left.\begin{array}{r} 6x^2 + 4xy = 0 \\ 2x^2 - 2y + 1 = 0 \end{array}\right\} \left.\begin{array}{r} 3x^2 + 2xy = 0 \\ y = x^2 + \frac{1}{2} \end{array}\right\}$

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$$3x^2 + 2x\left(x^2 + \frac{1}{2}\right) = 0 \Rightarrow 2x^3 + 3x^2 + x = 0 \Rightarrow$$
$$x\left(2x^2 + 3x + 1\right) = 0 \Rightarrow x(2x+1)(x+1) = 0 \Rightarrow$$
$$x = 0, -\frac{1}{2}, -1 \Rightarrow y = \frac{1}{2}, \frac{3}{4}, \frac{3}{2}$$

The possible extreme points are $\left(0, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{3}{4}\right)$, and $\left(-1, \frac{3}{2}\right)$.

12. $f(x, y) = \frac{15}{4}x^2 + 6xy - 3y^2 + 3x + 6y$

$$\frac{\partial f}{\partial x} = \frac{15}{2}x + 6y + 3;\ \frac{\partial f}{\partial y} = 6x - 6y + 6$$
$$\left.\begin{aligned} \frac{15}{2}x + 6y + 3 &= 0 \\ 6x - 6y + 6 &= 0 \end{aligned}\right\} \left.\begin{aligned} \frac{15}{2}x + 6y + 3 &= 0 \\ y &= x + 1 \end{aligned}\right\}$$
$$\frac{15}{2}x + 6(x+1) + 3 = 0 \Rightarrow$$
$$\frac{27}{2}x = -9 \Rightarrow x = -\frac{2}{3} \Rightarrow y = -\frac{2}{3} + 1 = \frac{1}{3}$$

The only possible extreme point is $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

13. $f(x, y) = \frac{1}{3}x^3 - 2y^3 - 5x + 6y - 5$

$$\frac{\partial f}{\partial x} = x^2 - 5; \frac{\partial f}{\partial y} = -6y^2 + 6$$
$$\left.\begin{aligned} x^2 - 5 &= 0 \\ -6y^2 + 6 &= 0 \end{aligned}\right\} \begin{aligned} x &= \pm\sqrt{5} \\ y &= \pm 1 \end{aligned}$$

There are four possible extreme points: $\left(\sqrt{5}, 1\right), \left(\sqrt{5}, -1\right), \left(-\sqrt{5}, 1\right)$, and $\left(-\sqrt{5}, -1\right)$.

14. $f(x, y) = x^4 - 8xy + 2y^2 - 3$

$$\frac{\partial f}{\partial x} = 4x^3 - 8y; \frac{\partial f}{\partial y} = -8x + 4y$$
$$\left.\begin{aligned} 4x^3 - 8y &= 0 \\ -8x + 4y &= 0 \end{aligned}\right\} \left.\begin{aligned} y &= \frac{1}{2}x^3 \\ y &= 2x \end{aligned}\right\} \Rightarrow$$
$$2x = \frac{1}{2}x^3 \Rightarrow x^3 - 4x = 0 \Rightarrow$$
$$x(x-2)(x+2) = 0 \Rightarrow x = 0,\ \pm 2 \Rightarrow$$
$$y = 0, 4, -4$$

The possible extreme points are (0, 0), (2, 4), and (–2, –4).

15. $f(x, y) = x^3 + x^2y - y$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy;\ \frac{\partial f}{\partial y} = x^2 - 1$$
$$\left.\begin{aligned} 3x^2 + 2xy &= 0 \\ x^2 - 1 &= 0 \end{aligned}\right\} \left.\begin{aligned} y &= \frac{-3x^2}{2x} = -\frac{3}{2}x \\ x^2 &= 1 \end{aligned}\right\} \Rightarrow$$
$$x = \pm 1 \Rightarrow y = \mp\frac{3}{2}$$

The possible extreme points are $\left(-1, \frac{3}{2}\right)$ and $\left(1, -\frac{3}{2}\right)$.

16. $f(x, y) = x^4 - 2xy - 7x^2 + y^2 + 3$

$$\frac{\partial f}{\partial x} = 4x^3 - 2y - 14x;\ \frac{\partial f}{\partial y} = -2x + 2y$$
$$\left.\begin{aligned} 4x^3 - 2y - 14x &= 0 \\ -2x + 2y &= 0 \end{aligned}\right\} \left.\begin{aligned} y &= 2x^3 - 7x \\ y &= x \end{aligned}\right\} \Rightarrow$$
$$x = 2x^3 - 7x \Rightarrow 2x^3 - 8x = 0 \Rightarrow$$
$$2x\left(x^2 - 4\right) = 0 \Rightarrow x = 0, \pm 2 \Rightarrow y = 0, \pm 2$$

The possible extreme points are (–2, –2), (0, 0), and (2, 2).

17. $f(x, y) = 2x + 3y + 9 - x^2 - xy - y^2$

$$\frac{\partial f}{\partial x} = 2 - 2x - y; \frac{\partial f}{\partial y} = 3 - x - 2y$$
$$\left.\begin{aligned} 2 - 2x - y &= 0 \\ 3 - x - 2y &= 0 \end{aligned}\right\} \left.\begin{aligned} y &= 2 - 2x \\ x &= \frac{1}{3} \end{aligned}\right\} \begin{aligned} x &= \frac{1}{3} \\ y &= \frac{4}{3} \end{aligned}$$

$\left(\frac{1}{3}, \frac{4}{3}\right)$ is the only point at which $f(x, y)$ can have a maximum, so the maximum value must occur at this point.

18. $f(x, y) = \frac{1}{2}x^2 + 2xy + 3y^2 - x + 2y$

$$\frac{\partial f}{\partial x} = x + 2y - 1; \frac{\partial f}{\partial y} = 2x + 6y + 2$$
$$\left.\begin{aligned} x + 2y - 1 &= 0 \\ 2x + 6y + 2 &= 0 \end{aligned}\right\} \left.\begin{aligned} x &= 1 - 2y \\ y &= -2 \end{aligned}\right\} \begin{aligned} x &= 5 \\ y &= -2 \end{aligned}$$

Thus (5, –2) is the only point at which $f(x, y)$ can have a minimum, so the minimum value must occur at this point.

19. $f(x, y) = 3x^2 - 6xy + y^3 - 9y$

$$\frac{\partial f}{\partial x} = 6x - 6y; \frac{\partial f}{\partial y} = -6x + 3y^2 - 9$$

$$\frac{\partial^2 f}{\partial x^2} = 6; \frac{\partial^2 f}{\partial y^2} = 6y; \frac{\partial^2 f}{\partial x \partial y} = -6$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6 \cdot 6y - (-6)^2 = 36y - 36$$

$D(3, 3) = 36 \cdot 3 - 36 > 0$, and $\frac{\partial^2 f}{\partial x^2}(3, 3) > 0$ so (3, 3) is a relative minimum of $f(x, y)$. $D(-1, -1) = 36(-1) - 36 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at (–1, –1).

20. $f(x, y) = 6xy^2 - 2x^3 - 3y^4$

$$\frac{\partial f}{\partial x} = 6y^2 - 6x^2; \frac{\partial f}{\partial y} = 12xy - 12y^3$$

$$\frac{\partial^2 f}{\partial x^2} = -12x; \frac{\partial^2 f}{\partial y^2} = 12x - 36y^2; \frac{\partial^2 f}{\partial x \partial y} = 12y$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (-12x)\left(12x - 36y^2\right) - (12y)^2 = -144x^2 + 432xy^2 - 144y^2$$

$D(0, 0) = 0$, so the test is inconclusive.

$D(1, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 1) < 0$, so $f(x, y)$ has a relative maximum at (1, 1).

$D(1, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, -1) < 0$, so $f(x, y)$ has a relative maximum at (1, –1).

21. $f(x, y) = 2x^2 - x^4 - y^2$

$$\frac{\partial f}{\partial x} = 4x - 4x^3; \frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \left(4 - 12x^2\right)(-2) - (0)^2 = -8 + 24x^2$$

$D(-1, 0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 0) < 0$, so $f(x, y)$ has a relative maximum at (–1, 0). $D(0, 0) < 0$, so $f(x, y)$ has neither a relative minimum nor a relative maximum at (0, 0). $D(1, 0) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 0) < 0$, so $f(x, y)$ has a relative maximum at (1, 0).

22. $f(x, y) = x^4 - 4xy + y^4$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y; \frac{\partial f}{\partial y} = -4x + 4y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2; \frac{\partial^2 f}{\partial y^2} = 12y^2; \frac{\partial^2 f}{\partial x \partial y} = -4$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12x^2\left(12y^2\right) - (-4)^2 = 144x^2y^2 - 16$$

$D(0, 0) < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at (0, 0).

$D(1, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 1) > 0$, so $f(x, y)$ has a relative minimum at (1, 1).

$D(-1, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, -1) > 0$, so $f(x, y)$ has a relative minimum at (–1, –1).

23. $f(x, y) = ye^x - 3x - y + 5$

$$\frac{\partial f}{\partial x} = ye^x - 3; \frac{\partial f}{\partial y} = e^x - 1$$

$$\frac{\partial^2 f}{\partial x^2} = ye^x; \frac{\partial^2 f}{\partial y^2} = 0; \frac{\partial^2 f}{\partial x \partial y} = e^x$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = ye^x(0) - \left(e^x\right)^2 = -e^{2x}$$

$D(0, 3) < 0$, thus $f(x, y)$ has neither a maximum nor a minimum at (0, 3).

24. $f(x, y) = \frac{1}{x} + \frac{1}{y} + xy$;

$\frac{\partial f}{\partial x} = \frac{-1}{x^2} + y; \frac{\partial f}{\partial y} = \frac{-1}{y^2} + x$

$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}; \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}; \frac{\partial^2 f}{\partial x \partial y} = 1$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{2}{x^3} \cdot \frac{2}{y^3} - 1^2 = \frac{4}{x^3 y^3} - 1$$

$D(1, 1) > 0$, $\frac{\partial^2 f}{\partial x^2}(1, 1) > 0$, so $f(x, y)$ has a relative minimum at (1, 1).

25. $f(x, y) = -5x^2 + 4xy - 17y^2 - 6x + 6y + 2$;

$\frac{\partial f}{\partial x} = -10x + 4y - 6; \ \frac{\partial f}{\partial y} = -34y + 4x + 6$

$\frac{\partial^2 f}{\partial x^2} = -10; \ \frac{\partial^2 f}{\partial y^2} = -34; \ \frac{\partial^2 f}{\partial x \partial y} = 4$

$$\left.\begin{aligned} -10x + 4y - 6 = 0 \\ 4x - 34y + 6 = 0 \end{aligned}\right\} x = -\frac{5}{9}, \ y = \frac{1}{9}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -10(-34) - 4^2 = 324$$

$D\left(-\frac{5}{9}, \frac{1}{9}\right) = -10(-34) - 4^2 = 324 > 0$ and $\frac{\partial^2 f}{\partial x^2}\left(-\frac{5}{9}, \frac{1}{9}\right) < 0$, so $f(x, y)$ has a relative maximum at $\left(-\frac{5}{9}, \frac{1}{9}\right)$.

26. $f(x, y) = -2x^2 + 6xy - 17y^2 - 4x + 6y$;

$\frac{\partial f}{\partial x} = -4x + 6y - 4; \ \frac{\partial f}{\partial y} = -34y + 6x + 6$

$\frac{\partial^2 f}{\partial x^2} = -4; \ \frac{\partial^2 f}{\partial y^2} = -34; \ \frac{\partial^2 f}{\partial x \partial y} = 6$

$$\left.\begin{aligned} -4x + 6y - 4 = 0 \\ 6x - 34y + 6 = 0 \end{aligned}\right\} x = -1, \ y = 0$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4(-34) - 6^2 = 100$$

$D(-1, 0) = -4(-34) - 6^2 = 100 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 0) < 0$, so $f(x, y)$ has a relative maximum at $(-1, 0)$.

27. $f(x, y) = 3x^2 + 8xy - 3y^2 + 2x + 6y$;

$\frac{\partial f}{\partial x} = 6x + 8y + 2; \ \frac{\partial f}{\partial y} = -6y + 8x + 6$

$\frac{\partial^2 f}{\partial x^2} = 6; \ \frac{\partial^2 f}{\partial y^2} = -6; \ \frac{\partial^2 f}{\partial x \partial y} = 8$

$$\left.\begin{aligned} 6x + 8y + 2 = 0 \\ 8x - 6y + 6 = 0 \end{aligned}\right\} x = -\frac{3}{5}, \ y = \frac{1}{5}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6(-6) - 8^2 = -100$$

$D\left(-\frac{3}{5}, \frac{1}{5}\right) = 6(-6) - 8^2 = -100 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{3}{5}, \frac{1}{5}\right)$.

28. $f(x, y) = 8xy + 8y^2 - 2x + 2y - 1$;

$\frac{\partial f}{\partial x} = 8y - 2; \ \frac{\partial f}{\partial y} = 16y + 8x + 2$

$\frac{\partial^2 f}{\partial x^2} = 0; \ \frac{\partial^2 f}{\partial y^2} = 16; \ \frac{\partial^2 f}{\partial x \partial y} = 8$

$$\left.\begin{aligned} 8y - 2 = 0 \\ 8x + 16y + 2 = 0 \end{aligned}\right\} x = -\frac{3}{4}, \ y = \frac{1}{4}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0(16) - 8^2 = -64 < 0$$

The potential relative extreme point is $\left(-\frac{3}{4}, \frac{1}{4}\right)$. However, because $D(x, y) < 0$, it is neither a relative maximum nor a relative minimum.

29. $f(x, y) = x^4 - x^2 - 2xy + y^2 + 1$;

$\frac{\partial f}{\partial x} = 4x^3 - 2x - 2y$; $\frac{\partial f}{\partial y} = 2y - 2x$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 2$; $\frac{\partial^2 f}{\partial y^2} = 2$; $\frac{\partial^2 f}{\partial x \partial y} = -2$

Solving the system

$$\left.\begin{aligned} 4x^3 - 2x - 2y &= 0 \\ -2x + 2y &= 0 \end{aligned}\right\}$$

yields the solutions $(0, 0)$, $(-1, -1)$, and $(1, 1)$.

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= \left(12x^2 - 2\right)2 - (-2)^2 \\ &= 24x^2 - 8 \end{aligned}$$

$D(0, 0) = -8 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(0, 0)$.

$D(-1, -1) = 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 10 > 0$ so $f(x, y)$ has a relative minimum at $(-1, -1)$.

$D(1, 1) = 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 10 > 0$ so $f(x, y)$ has a relative minimum at $(1, 1)$.

30. $f(x, y) = x^2 + 2xy + 10y^2$;

$\frac{\partial f}{\partial x} = 2x + 2y$; $\frac{\partial f}{\partial y} = 20y + 2x$

$\frac{\partial^2 f}{\partial x^2} = 2$; $\frac{\partial^2 f}{\partial y^2} = 20$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

Solving the system

$$\left.\begin{aligned} 2x + 2y &= 0 \\ 2x + 20y &= 0 \end{aligned}\right\}$$

yields the solution $(0, 0)$.

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 2(20) - 2^2 = 36 \end{aligned}$$

$D(0, 0) = 38 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, so $f(x, y)$ has a relative minimum at $(0, 0)$.

31. $f(x, y) = 6xy - 3y^2 - 2x + 4y - 1$;

$\frac{\partial f}{\partial x} = 6y - 2$; $\frac{\partial f}{\partial y} = -6y + 6x + 4$

$\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y^2} = -6$; $\frac{\partial^2 f}{\partial x \partial y} = 6$

Solving the system

$$\left.\begin{aligned} 6y - 2 &= 0 \\ 6x - 6y + 4 &= 0 \end{aligned}\right\}$$

yields the solution $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 0(-6) - 6^2 = -36 \end{aligned}$$

$D\left(-\frac{1}{3}, \frac{1}{3}\right) = -36 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

32. $f(x, y) = 2xy + y^2 + 2x - 1$

$\frac{\partial f}{\partial x} = 2y + 2$; $\frac{\partial f}{\partial y} = 2x + 2y$

$\frac{\partial^2 f}{\partial x^2} = 0$; $\frac{\partial^2 f}{\partial y^2} = 2$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

Solving the system

$$\left.\begin{aligned} 2y + 2 &= 0 \\ 2x + 2y &= 0 \end{aligned}\right\}$$

yields the solution $(1, -1)$.

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= 0(2) - 2^2 = -4 \end{aligned}$$

$D(1, -1) = -4 < 0$, so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(1, -1)$.

33. $f(x, y) = -2x^2 + 2xy - 25y^2 - 2x + 8y - 1$

$\frac{\partial f}{\partial x} = -4x + 2y - 2$; $\frac{\partial f}{\partial y} = 2x - 50y + 8$

$\frac{\partial^2 f}{\partial x^2} = -4$; $\frac{\partial^2 f}{\partial y^2} = -50$; $\frac{\partial^2 f}{\partial x \partial y} = 2$

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Solving the system

$$\left.\begin{array}{r}-4x+2y-2=0\\ 2x-50y+8=0\end{array}\right\}$$

yields the solution $\left(-\frac{3}{7}, \frac{1}{7}\right)$.

$$\begin{aligned}D(x,y)&=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\&=-4(-50)-2^2=196\end{aligned}$$

$D\left(-\frac{3}{7}, \frac{1}{7}\right)=196>0$ and $\frac{\partial^2 f}{\partial x^2}\left(-\frac{3}{7}, \frac{1}{7}\right)=-4<0,$ so $f(x, y)$ has a relative maximum at $\left(-\frac{3}{7}, \frac{1}{7}\right)$.

34. $f(x, y)=3x^2+8xy-3y^2-2x+4y+1$

$\frac{\partial f}{\partial x}=6x+8y-2;\ \frac{\partial f}{\partial y}=8x-6y+4$

$\frac{\partial^2 f}{\partial x^2}=6;\ \frac{\partial^2 f}{\partial y^2}=-6;\ \frac{\partial^2 f}{\partial x\partial y}=8$

Solving the system

$$\left.\begin{array}{r}6x+8y-2=0\\ 8x-6y+4=0\end{array}\right\}$$

yields the solution $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

$$\begin{aligned}D(x,y)&=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\&=6(-6)-8^2=-100\end{aligned}$$

$D\left(-\frac{1}{5}, \frac{2}{5}\right)=-100<0,$ so $f(x, y)$ has neither a relative maximum nor a relative minimum at $\left(-\frac{1}{5}, \frac{2}{5}\right)$.

35. $f(x, y)=x^4-12x^2-4xy-y^2+16$

$\frac{\partial f}{\partial x}=4x^3-24x-4y;\ \frac{\partial f}{\partial y}=-4x-2y$

$\frac{\partial^2 f}{\partial x^2}=12x^2-24;\ \frac{\partial^2 f}{\partial y^2}=-2;\ \frac{\partial^2 f}{\partial x\partial y}=-4$

Solving the system

$$\left.\begin{array}{r}4x^3-24x-4y=0\\ -4x-2y=0\end{array}\right\}$$

yields the solutions $(0, 0)$, $(-2, 4)$, and $(2, -4)$.

$$\begin{aligned}D(x,y)&=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\&=\left(12x^2-24\right)(-2)-(-4)^2\\&=-24x^2+32\end{aligned}$$

$D(0, 0)=32>0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0)=-24<0,$ so $f(x, y)$ has a relative maximum at $(0, 0)$.

$D(-2, 4)=-24(-2)^2+32=-64<0,$ so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(-2, 4)$.

$D(2, -4)=-24(2)^2+32=-64<0,$ and $\frac{\partial^2 f}{\partial x^2}=24>0$ so $f(x, y)$ has neither a relative maximum nor a relative minimum at $(2, -4)$.

36. $f(x, y)=\frac{17}{4}x^2+2xy+5y^2+5x-2y+2$

$\frac{\partial f}{\partial x}=\frac{17}{2}x+2y+5;\ \frac{\partial f}{\partial y}=2x+10y-2$

$\frac{\partial^2 f}{\partial x^2}=\frac{17}{2};\ \frac{\partial^2 f}{\partial y^2}=10;\ \frac{\partial^2 f}{\partial x\partial y}=2$

Solving the system

$$\left.\begin{array}{r}\frac{17}{2}x+2y+5=0\\ 2x+10y-2=0\end{array}\right\}$$

yields the solution $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

$$\begin{aligned}D(x,y)&=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\\&=\frac{17}{2}(10)-2^2=81\end{aligned}$$

(*continued on next page*)

(*continued*)

$D\left(-\frac{2}{3}, \frac{1}{3}\right) = 81 > 0$ and

$\frac{\partial^2 f}{\partial x^2}\left(-\frac{2}{3}, \frac{1}{3}\right) = \frac{17}{2} > 0$, so $f(x, y)$ has a relative minimum at $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

37. $f(x, y) = x^2 - 2xy + 4y^2$

$\frac{\partial f}{\partial x} = 2x - 2y; \frac{\partial f}{\partial y} = -2x + 8y$

$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 8; \frac{\partial^2 f}{\partial x \partial y} = -2$

$$\left.\begin{array}{r} 2x - 2y = 0 \\ -2x + 8y = 0 \end{array}\right\} \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2 \cdot 8 - (-2)^2 = 12$$

$D(0, 0) = 2 \cdot 8 - (-2)^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0) > 0$, so $f(x, y)$ has a relative minimum at (0, 0).

38. $f(x, y) = 2x^2 + 3xy + 5y^2$

$\frac{\partial f}{\partial x} = 4x + 3y; \frac{\partial f}{\partial y} = 3x + 10y$

$\frac{\partial^2 f}{\partial x^2} = 4; \frac{\partial^2 f}{\partial y^2} = 10; \frac{\partial^2 f}{\partial x \partial y} = 3$

$$\left.\begin{array}{r} 4x + 3y = 0 \\ 3x + 10y = 0 \end{array}\right\} \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4 \cdot 10 - 3^2 = 31$$

$D(0, 0) = 4 \cdot 10 - 3^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, 0) > 0$, so $f(x, y)$ has a relative minimum at (0, 0).

39. $f(x, y) = -2x^2 + 2xy - y^2 + 4x - 6y + 5$

$\frac{\partial f}{\partial x} = -4x + 2y + 4; \frac{\partial f}{\partial y} = 2x - 2y - 6$

$\frac{\partial^2 f}{\partial x^2} = -4; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 2$

$$\left.\begin{array}{r} -4x + 2y + 4 = 0 \\ 2x - 2y - 6 = 0 \end{array}\right\} \begin{array}{l} x = -1 \\ y = -4 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4(-2) - 2^2 = 4$$

$D(-1, -4) = (-4)(-2) - 2^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, -4) < 0$, so $f(x, y)$ has a relative maximum at (–1, –4).

40. $f(x, y) = -x^2 - 8xy - y^2$

$\frac{\partial f}{\partial x} = -2x - 8y; \frac{\partial f}{\partial y} = -8x - 2y$

$\frac{\partial^2 f}{\partial x^2} = -2; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = -8$

$$\left.\begin{array}{r} -2x - 8y = 0 \\ -8x - 2y = 0 \end{array}\right\} \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -2(-2) - (-8)^2 = -60$$

$D(0, 0) = (-2)(-2) - (-8)^2 < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (0, 0).

41. $f(x, y) = x^2 + 2xy + 5y^2 + 2x + 10y - 3$

$\frac{\partial f}{\partial x} = 2x + 2y + 2; \frac{\partial f}{\partial y} = 2x + 10y + 10$

$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 10; \frac{\partial^2 f}{\partial x \partial y} = 2$

$$\left.\begin{array}{r} 2x + 2y + 2 = 0 \\ 2x + 10y + 10 = 0 \end{array}\right\} \begin{array}{l} x = 0 \\ y = -1 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(10) - 2^2 = 16$$

$D(0, -1) = (2)(10) - 2^2 > 0, \frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (0, –1).

42. $f(x, y) = x^2 - 2xy + 3y^2 + 4x - 16y + 22$

$$\frac{\partial f}{\partial x} = 2x - 2y + 4; \frac{\partial f}{\partial y} = -2x + 6y - 16$$

$$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 6; \frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\left.\begin{array}{r} 2x - 2y + 4 = 0 \\ -2x + 6y - 16 = 0 \end{array}\right\} \begin{array}{l} x = 1 \\ y = 3 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(6) - (-2)^2 = 8$$

$D(1, 3) = 2 \cdot 6 - (-2)^2 > 0, \frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (1, 3).

43. $f(x, y) = x^3 - y^2 - 3x + 4y$

$$\frac{\partial f}{\partial x} = 3x^2 - 3; \frac{\partial f}{\partial y} = -2y + 4$$

$$\frac{\partial^2 f}{\partial x^2} = 6x; \frac{\partial^2 f}{\partial y^2} = -2; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\left.\begin{array}{r} 3x^2 - 3 = 0 \\ -2y + 4 = 0 \end{array}\right\} \begin{array}{l} x = \pm 1 \\ y = 2 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6x(-2) - 0^2 = -12x$$

$D(1, 2) < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (1, 2).

$D(-1, 2) > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1, 2) = -6 < 0$, so $f(x, y)$ has a relative maximum at (–1, 2).

44. $f(x, y) = x^3 - 2xy + 4y$

$$\frac{\partial f}{\partial x} = 3x^2 - 2y; \frac{\partial f}{\partial y} = -2x + 4$$

$$\frac{\partial^2 f}{\partial x^2} = 6x; \frac{\partial^2 f}{\partial y^2} = 0; \frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\left.\begin{array}{r} 3x^2 - 2y = 0 \\ -2x + 4 = 0 \end{array}\right\} \begin{array}{l} x = 2 \\ y = 6 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 6x(0) - (-2)^2 = -4$$

$D(2, 6) < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (2, 6).

45. $f(x, y) = 2x^2 + y^3 - x - 12y + 7$

$$\frac{\partial f}{\partial x} = 4x - 1; \frac{\partial f}{\partial y} = 3y^2 - 12$$

$$\frac{\partial^2 f}{\partial x^2} = 4; \frac{\partial^2 f}{\partial y^2} = 6y; \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\left.\begin{array}{r} 4x - 1 = 0 \\ 3y^2 - 12 = 0 \end{array}\right\} \begin{array}{l} x = 1/4 \\ y = \pm 2 \end{array}$$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4(6y) - 0^2 = 24y$$

$D\left(\frac{1}{4}, 2\right) = 48 > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at $\left(\frac{1}{4}, 2\right)$.

$D\left(\frac{1}{4}, -2\right) = -48 < 0$, so $f(x, y)$ has neither a maximum nor a minimum at $\left(\frac{1}{4}, -2\right)$.

46. $f(x, y) = x^2 + 4xy + 2y^4$

$$\frac{\partial f}{\partial x} = 2x + 4y; \frac{\partial f}{\partial y} = 4x + 8y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2} = 24y^2; \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\left.\begin{array}{r} 2x + 4y = 0 \\ 4x + 8y^3 = 0 \end{array}\right\} \begin{array}{l} x = -2y \\ 8y^3 - 8y = 0 \end{array} \Rightarrow$$

$$8y\left(y^2 - 1\right) = 0 \Rightarrow y = 0, \pm 1$$

Solutions: (0, 0), (–2, 1), (2, –1)

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2\left(24y^2\right) - 4^2 = 48y^2 - 16$$

$D(0, 0) < 0$, so $f(x, y)$ has neither a relative minimum nor a relative maximum at (0, 0).

$D(-2, 1) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (–2, 1).

$D(2, -1) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so $f(x, y)$ has a relative minimum at (2, –1).

47. $f(x, y, z) = 2x^2 + 3y^2 + z^2 - 2x - y - z$

$$\frac{\partial f}{\partial x} = 4x - 2; \frac{\partial f}{\partial y} = 6y - 1; \frac{\partial f}{\partial z} = 2z - 1$$

$$\left.\begin{aligned} 4x - 2 &= 0 \\ 6y - 1 &= 0 \\ 2z - 1 &= 0 \end{aligned}\right\} x = \frac{1}{2}; y = \frac{1}{6}; z = \frac{1}{2}$$

$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right)$ is the only point at which $f(x, y, z)$ can have a relative minimum.

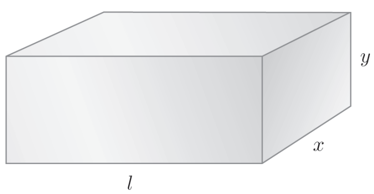
48. $f(x, y, z) = 5 + 8x - 4y + x^2 + y^2 + z^2$

$$\frac{\partial f}{\partial x} = 8 + 2x; \quad \frac{\partial f}{\partial y} = 2y - 4; \quad \frac{\partial f}{\partial z} = 2z$$

$$\left.\begin{aligned} 8 + 2x &= 0 \\ 2y - 4 &= 0 \\ 2z &= 0 \end{aligned}\right\} \begin{aligned} x &= -4 \\ y &= 2 \\ z &= 0 \end{aligned}$$

$(-4, 2, 0)$ is the only point at which $f(x, y, z)$ can have a relative minimum.

49.



Let x, y, and l be as shown in the figure.
Since $l = 84 - 2x - 2y$, the volume of the box may be written as

$$V(x, y) = xy(84 - 2x - 2y) = 84xy - 2x^2y - 2xy^2.$$

$$\frac{\partial V}{\partial x} = 84y - 4xy - 2y^2; \frac{\partial V}{\partial y} = 84x - 2x^2 - 4xy$$

$$\frac{\partial^2 V}{\partial x^2} = -4y; \frac{\partial^2 V}{\partial y^2} = -4x; \frac{\partial^2 V}{\partial x \partial y} = 84 - 4x - 4y$$

$$\left.\begin{aligned} 84y - 4xy - 2y^2 &= 0 \\ 84x - 2x^2 - 4xy &= 0 \end{aligned}\right\} \begin{aligned} x &= 0 \\ y &= 0 \end{aligned} \quad \text{or} \quad \left.\begin{aligned} 84 - 4x - 2y &= 0 \\ 84 - 4y - 2x &= 0 \end{aligned}\right\} \begin{aligned} x &= 14 \\ y &= 14 \end{aligned}$$

Obviously, (0, 0) does not give the maximum value of $V(x, y)$. To verify that (14, 14) is the maximum, check

$$D = \frac{\partial^2 V}{\partial x^2} \cdot \frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = -4y(-4x) - (84 - 4x - 4y)^2$$

$$D(14, 14) = -4(14)(-4)(14) - [84 - 4(14) - 4(14)]^2 > 0 \text{ and } \frac{\partial^2 V}{\partial x^2} = -4(14) < 0.$$

Thus, the dimensions that give the maximum volume are $x = 14$, $y = 14$, $l = 84 - 56 = 28$; or $14 \times 14 \times 28$ in.

50. Let x, y, and z be the dimensions of the box. Since the volume of the box is $1000\ \text{in}^3$, $x > 0$, $y > 0$, and $z = \dfrac{1000}{xy}$. The surface area is $S(x, y) = 2xy + 2x\left(\dfrac{1000}{xy}\right) + 2y\left(\dfrac{1000}{xy}\right) = 2xy + \dfrac{2000}{y} + \dfrac{2000}{x}.$

$$\frac{\partial S}{\partial x} = 2y - \frac{2000}{x^2}; \frac{\partial S}{\partial y} = 2x - \frac{2000}{y^2}; \frac{\partial^2 S}{\partial x^2} = \frac{4000}{x^3}; \frac{\partial^2 S}{\partial y^2} = \frac{4000}{y^3}; \frac{\partial^2 S}{\partial x \partial y} = 2$$

$$\left.\begin{aligned} 2y - \frac{2000}{x^2} &= 0 \\ 2x - \frac{2000}{y^2} &= 0 \end{aligned}\right\} x = y = 10$$

To verify that (10, 10) is a minimum, check

$$D = \frac{\partial^2 S}{\partial x^2} \cdot \frac{\partial^2 S}{\partial y^2} - \left(\frac{\partial^2 S}{\partial x \partial y}\right)^2 = \frac{4000}{x^3} \cdot \frac{4000}{y^3} - (2)^2 = \frac{4000^2}{x^3 y^3} - 4$$

$$D(10, 10) = \frac{4000}{1000^2}^2 - 2^2 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2}(10, 10) > 0.$$

Thus, the dimensions giving the smallest surface area are $10 \times 10 \times 10$ in.

51. The revenue is $10x + 9y$, so the profit function is

$$\begin{aligned} P(x, y) &= 10x + 9y - \left[400 + 2x + 3y + 0.01(3x^2 + xy + 3y^2)\right] \\ &= 8x + 6y - 0.03x^2 - 0.01xy - 0.03y^2 - 400 \end{aligned}$$

$$\frac{\partial P}{\partial x} = 8 - 0.06x - 0.01y; \frac{\partial P}{\partial y} = 6 - 0.01x - 0.06y; \quad \frac{\partial^2 P}{\partial x^2} = -0.06; \frac{\partial^2 P}{\partial y^2} = -0.06; \frac{\partial^2 P}{\partial x \partial y} = -0.01$$

$$\left.\begin{aligned} 8 - 0.06x - 0.01y &= 0 \\ 6 - 0.01x - 0.06y &= 0 \end{aligned}\right\} \begin{aligned} x &= 120 \\ y &= 80 \end{aligned} \Rightarrow \text{(120, 80) is a maximum. Verify this by checking}$$

$$D = \frac{\partial^2 P}{\partial x^2} \cdot \frac{\partial^2 P}{\partial y^2} - \left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 = -0.06(-0.06) - (-0.01)^2 = 0.0035 \Rightarrow D(120, 80) > 0.$$

$$\frac{\partial^2 P}{\partial x^2}(120, 80) = -0.06 < 0.$$

Thus profit is maximized by producing 120 units of product I and 80 units of product II.

52. The cost is $30x + 20y$, so the profit function is

$$P(x, y) = 98x + 112y - 0.04xy - 0.1x^2 - 0.2y^2 - 30x - 20y = 68x + 92y - 0.04xy - 0.1x^2 - 0.2y^2$$

$$\frac{\partial P}{\partial x} = 68 - 0.04y - 0.2x; \frac{\partial P}{\partial y} = 92 - 0.04x - 0.4y; \quad \frac{\partial^2 P}{\partial x^2} = -0.2; \frac{\partial^2 P}{\partial y^2} = -0.4; \frac{\partial^2 P}{\partial x \partial y} = -0.04$$

$$\left.\begin{aligned} 68 - 0.04y - 0.2x &= 0 \\ 92 - 0.04x - 0.4y &= 0 \end{aligned}\right\} \begin{aligned} x &= 300 \\ y &= 200 \end{aligned}$$

To verify that (300, 200) is a maximum, check

$$D = \frac{\partial^2 P}{\partial x^2} \cdot \frac{\partial^2 P}{\partial y^2} - \left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 = -0.2(-0.4) - (-0.04)^2 = 0.0784 \Rightarrow D(300, 200) > 0.$$

$$\frac{\partial^2 P}{\partial x^2} < 0.$$

Thus the profit is maximized by producing 300 units of product I and 200 units of product II.

53. Let $P(x, y)$ denote the company's profit from producing x units of product I and y units of product II. Then $P(x,\ y) = P_1x + P_2y - C(x,\ y)$. If (a, b) is the profit maximizing output combination, then
$$\frac{\partial P}{\partial x}(a,\ b) = \frac{\partial P}{\partial y}(a,\ b) = 0, \text{ so } P_1 - \frac{\partial C}{\partial x} = 0 \text{ and } P_2 - \frac{\partial C}{\partial y} = 0 \text{ or } \frac{\partial C}{\partial x} = P_1 \text{ and } \frac{\partial C}{\partial y} = P_2.$$

54. Let $P(x, y)$ denote the company's profit from producing x units of product I and y units of product II. Then $P(x,\ y) = R(x,\ y) - p_{\mathrm{I}}x - p_{\mathrm{II}}y$. If (a, b) is the profit maximizing output combination, then
$$\frac{\partial P}{\partial x}(a,\ b) = \frac{\partial P}{\partial y}(a,\ b) = 0, \text{ so } \frac{\partial R}{\partial x} - p_{\mathrm{I}} = 0 \text{ and } \frac{\partial R}{\partial y} - p_{\mathrm{II}} = 0 \text{ or } \frac{\partial R}{\partial x} = p_{\mathrm{I}} \text{ and } \frac{\partial R}{\partial y} = p_{\mathrm{II}}.$$

7.4 LaGrange Multipliers and Constrained Optimization

1. $F(x,\ y,\ \lambda) = x^2 + 3y^2 + 10 + \lambda(8 - x - y)$
$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x} = 2x - \lambda = 0\\[2ex] \dfrac{\partial F}{\partial y} = 6y - \lambda = 0\\[2ex] \dfrac{\partial F}{\partial \lambda} = 8 - x - y = 0\end{array}\right\} \left.\begin{array}{l}\lambda = 2x\\[2ex] \lambda = 6y\\[2ex] 8 - x - y = 0\end{array}\right\} \left.\begin{array}{l}2x = 6y \Rightarrow x = 3y\\[2ex] 8 - 3y - y = 0\end{array}\right\} \begin{array}{l}x = 6\\[2ex] y = 2\\ \lambda = 12\end{array}$$

The minimum value is $6^2 + 3\cdot 2^2 + 10 = 58.$

2. $F(x,\ y,\ \lambda) = x^2 - y^2 + \lambda(2x + y - 3)$
$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x} = 2x + 2\lambda = 0\\[2ex] \dfrac{\partial F}{\partial y} = -2y + \lambda = 0\\[2ex] \dfrac{\partial F}{\partial \lambda} = 2x + y - 3 = 0\end{array}\right\} \left.\begin{array}{l}\lambda = -x\\[2ex] \lambda = 2y\\[2ex] 2x + y - 3 = 0\end{array}\right\} \left.\begin{array}{l}x = -2y\\[2ex] -4y + y - 3 = 0\end{array}\right\} \begin{array}{l}x = 2\\[2ex] y = -1\\ \lambda = -2\end{array}$$

The maximum value is $2^2 - (-1)^2 = 3.$

3. $F(x,\ y,\ \lambda) = x^2 + xy - 3y^2 + \lambda(2 - x - 2y)$
$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x} = 2x + y - \lambda = 0\\[2ex] \dfrac{\partial F}{\partial y} = x - 6y - 2\lambda = 0\\[2ex] \dfrac{\partial F}{\partial \lambda} = 2 - x - 2y = 0\end{array}\right\} \left.\begin{array}{l}\lambda = 2x + y\\[2ex] \lambda = \dfrac{1}{2}x - 3y\\[2ex] 2 - x - 2y = 0\end{array}\right\} \left.\begin{array}{l}\dfrac{3}{2}x + 4y = 0\\[2ex] x + 2y = 2\end{array}\right\} \begin{array}{l}x = 8\\[2ex] y = -3\\ \lambda = 13\end{array}$$

The maximum value is $8^2 + 8(-3) - 3(-3)^2 = 13$.

4. $F(x,\ y,\ \lambda) = \frac{1}{2}x^2 - 3xy + y^2 + \frac{1}{2} + \lambda(3x - y - 1)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = x - 3y + 3\lambda = 0 \\ \frac{\partial F}{\partial y} = -3x + 2y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 3x - y - 1 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -\frac{1}{3}x + y \\ \lambda = -3x + 2y \\ 3x - y - 1 = 0 \end{array}\right\} \left.\begin{array}{l} \frac{8}{3}x - y = 0 \\ 3x - y = 1 \end{array}\right\} \begin{array}{l} x = 3 \\ y = 8 \\ \lambda = 7 \end{array}$$

The minimum value is $\frac{1}{2}\left(3^2\right) - 3(3)(8) + 8^2 + \frac{1}{2} = -3$.

5. $F(x,\ y,\ \lambda) = -2x^2 - 2xy - \frac{3}{2}y^2 + x + 2y + \lambda\left(x + y - \frac{5}{2}\right)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = -4x - 2y + 1 + \lambda = 0 \\ \frac{\partial F}{\partial y} = -2x - 3y + 2 + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = x + y - \frac{5}{2} = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = 4x + 2y - 1 \\ \lambda = 2x + 3y - 2 \\ x + y = \frac{5}{2} \end{array}\right\} \left.\begin{array}{l} 2x - y = -1 \\ x + y = \frac{5}{2} \end{array}\right\} \begin{array}{l} x = \frac{1}{2} \\ y = 2 \end{array}$$

6. $F(x,\ y,\ \lambda) = x^2 + xy + y^2 - 2x - 5y + \lambda(1 - x + y)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 2x + y - 2 - \lambda = 0 \\ \frac{\partial F}{\partial y} = x + 2y - 5 + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 1 - x + y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = 2x + y - 2 \\ \lambda = -x - 2y + 5 \\ 1 - x + y = 0 \end{array}\right\} \left.\begin{array}{l} 3x + 3y = 7 \\ -x + y = -1 \end{array}\right\} \begin{array}{l} x = \frac{5}{3} \\ y = \frac{2}{3} \end{array}$$

7. Minimize $xy + y^2 - x - 1$ subject to the constraint $x - 2y = 0$.

$F\left(x, y, \lambda\right) = xy + y^2 - x - 1 + \lambda\left(x - 2y\right)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = y - 1 + \lambda = 0 \\ \frac{\partial F}{\partial y} = x + 2y - 2\lambda = 0 \\ \frac{\partial F}{\partial \lambda} = x - 2y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -y + 1 \\ \lambda = \frac{x + 2y}{2} \\ x = 2y \end{array}\right\} \left.\begin{array}{l} x + 4y = 2 \\ x = 2y \end{array}\right\} \begin{array}{l} y = \frac{1}{3} \\ x = \frac{2}{3} \end{array}$$

8. Minimize $x^2 - 2xy + 2y^2$ subject to the constraint $2x - y + 5 = 0$.

$F\left(x, y, \lambda\right) = x^2 - 2xy + 2y^2 + \lambda\left(2x - y + 5\right)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 2x - 2y + 2\lambda = 0 \\ \frac{\partial F}{\partial y} = -2x + 4y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 2x - y + 5 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -x + y \\ \lambda = -2x + 4y \\ 2x - y = -5 \end{array}\right\} \left.\begin{array}{l} x = 3y \\ 2x - y = -5 \end{array}\right\} \begin{array}{l} x = -3 \\ y = -1 \end{array}$$

9. Minimize $2x^2 + xy + y^2 - y$ subject to the constraint $x + y = 0$.

$$F(x, y, \lambda) = 2x^2 + xy + y^2 - y + \lambda(x + y)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 4x + y + \lambda = 0 \\ \frac{\partial F}{\partial y} &= x + 2y - 1 + \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x + y = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -4x - y \\ \lambda &= -x - 2y + 1 \\ x &= -y \end{aligned}\right\} \left.\begin{aligned} -3x &= -y + 1 \\ x &= -y \end{aligned}\right\} \begin{aligned} x &= -\frac{1}{4} \\ y &= \frac{1}{4} \end{aligned}$$

10. Minimize $2x^2 - 2xy + y^2 - 2x + 1$ subject to the constraint $x - y = 3$.

$$F(x, y, \lambda) = 2x^2 - 2xy + y^2 - 2x + 1 + \lambda(x - y - 3)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 4x - 2y - 2 + \lambda = 0 \\ \frac{\partial F}{\partial y} &= -2x + 2y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x - y - 3 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -4x + 2y + 2 \\ \lambda &= -2x + 2y \\ x &= y + 3 \end{aligned}\right\} \begin{aligned} x &= 1 \\ y &= -2 \end{aligned}$$

11. Minimize $18x^2 + 12xy + 4y^2 + 6x - 4y + 5$ subject to the constraint $3x + 2y - 1 = 0$.

$$F(x, y, \lambda) = 18x^2 + 12xy + 4y^2 + 6x - 4y + 5 + \lambda(3x + 2y - 1)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 36x + 12y + 6 + 3\lambda = 0 \\ \frac{\partial F}{\partial y} &= 12x + 8y - 4 + 2\lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 3x + 2y - 1 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -12x - 4y - 2 \\ \lambda &= -6x - 4y + 2 \\ y &= \frac{-3x + 1}{2} \end{aligned}\right\} \left.\begin{aligned} x &= -\frac{2}{3} \\ y &= \frac{-3x + 1}{2} \end{aligned}\right\} \begin{aligned} x &= -\frac{2}{3} \\ y &= \frac{3}{2} \end{aligned}$$

12. Minimize $3x^2 - 2xy + x - 3y + 1$ subject to the constraint $x - 3y = 1$.

$$F(x, y, \lambda) = 3x^2 - 2xy + x - 3y + 1 + \lambda(x - 3y - 1)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 6x - 2y + 1 + \lambda = 0 \\ \frac{\partial F}{\partial y} &= -2x - 3 - 3\lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x - 3y - 1 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -6x + 2y - 1 \\ \lambda &= \frac{-2x - 3}{3} \\ y &= \frac{x - 1}{3} \end{aligned}\right\} \left.\begin{aligned} y &= \frac{8x}{3} \\ y &= \frac{x - 1}{3} \end{aligned}\right\} \begin{aligned} x &= -\frac{1}{7} \\ y &= -\frac{8}{21} \end{aligned}$$

13. Minimize $x - xy + 2y^2$ subject to the constraint $x - y + 1 = 0$.

$$F(x, y, \lambda) = x - xy + 2y^2 + \lambda(x - y + 1)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 1 - y + \lambda = 0 \\ \frac{\partial F}{\partial y} &= -x + 4y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= x - y + 1 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= y - 1 \\ \lambda &= -x + 4y \\ y &= x + 1 \end{aligned}\right\} \left.\begin{aligned} x &= 3y + 1 \\ y &= x + 1 \end{aligned}\right\} \begin{aligned} x &= -2 \\ y &= -1 \end{aligned}$$

14. Maximize xy subject to the constraint $x^2 - y = 3$.

$$F(x, y, \lambda) = xy + \lambda\left(x^2 - y - 3\right)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y + 2\lambda x = 0 \\ \dfrac{\partial F}{\partial y} = x - \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x^2 - y - 3 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -\dfrac{y}{2x} \\ \lambda = x \\ y = x^2 - 3 \end{array}\right\} \left.\begin{array}{l} y = -2x^2 \\ y = x^2 - 3 \end{array}\right\} \begin{array}{ll} x = -1 \text{ or } & x = 1 \\ y = -2 & y = -2 \end{array}$$

Now check both answers in the original function to see which pair maximizes xy. The pair that maximizes xy is $x = -1$, $y = -2$.

15. Minimize $xy + xz - yz$ subject to the constraint $x + y + z = 1$.

$$F(x, y, z, \lambda) = xy + xz - yz + \lambda(x + y + z - 1)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y + z + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = x - z + \lambda = 0 \\ \dfrac{\partial F}{\partial z} = x - y + \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x + y + z - 1 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -y - z \\ \lambda = -x + z \\ \lambda = -x + y \\ x + y + z = 1 \end{array}\right\} \left.\begin{array}{l} x - y = 2z \\ y = z \\ x + y + z = 1 \end{array}\right\} \left.\begin{array}{l} x = y + 2z \Rightarrow x = 3y \\ y = z \\ 3y + y + y = 1 \end{array}\right\} \begin{array}{l} x = \dfrac{3}{5} \\ y = \dfrac{1}{5} \\ z = \dfrac{1}{5} \end{array}$$

16. Minimize $xy + xz - 2yz$ subject to the constraint $x + y + z = 2$.

$$F(x, y, z, \lambda) = xy + xz - 2yz + \lambda(x + y + z - 2)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y + z + \lambda = 0 \\ \dfrac{\partial F}{\partial y} = x - 2z + \lambda = 0 \\ \dfrac{\partial F}{\partial z} = x - 2y + \lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x + y + z - 2 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = -y - z \\ \lambda = -x + 2z \\ \lambda = -x + 2y \\ x + y + z = 2 \end{array}\right\} \left.\begin{array}{l} x - y = 3z \\ y = z \\ x + y + z = 2 \end{array}\right\} \left.\begin{array}{l} x = y + 3z \Rightarrow x = 4y \\ y = z \\ 4y + y + y = 2 \end{array}\right\} \begin{array}{l} x = \dfrac{4}{3} \\ y = \dfrac{1}{3} \\ z = \dfrac{1}{3} \end{array}$$

17. We want to minimize the function $x + y$ subject to the constraint $xy = 25$ or $xy - 25 = 0$.

$$F(x, y, \lambda) = x + y + \lambda(xy - 25)$$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 1 + \lambda y = 0 \\ \dfrac{\partial F}{\partial y} = 1 + \lambda x = 0 \\ \dfrac{\partial F}{\partial \lambda} = xy - 25 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \dfrac{-1}{y} \\ \lambda = \dfrac{-1}{x} \\ xy - 25 = 0 \end{array}\right\} \left.\begin{array}{l} x - y = 0 \\ xy = 25 \end{array}\right\} x^2 = 25 \text{ or } x = \pm 5$$

so $x = 5$, $y = 5$ (the positive numbers).

18. Let x = length of the north side. Let y = length of the west side.
Cost = $2x(10) + 2y(15) = 20x + 30y = 480$
We want to maximize xy subject to $20x + 30y - 480 = 0$.
$F(x, y, \lambda) = xy + \lambda(20x + 30y - 480)$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = y + 20\lambda = 0 \\ \dfrac{\partial F}{\partial y} = x + 30\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = 20x + 30y - 480 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \dfrac{-y}{20} \\ \lambda = \dfrac{-x}{30} \\ 2x + 3y = 48 \end{array}\right\} \left.\begin{array}{l} 2x - 3y = 0 \\ 2x + 3y = 48 \end{array}\right\} \begin{array}{l} x = 12 \\ y = 8 \end{array}$$

The dimensions of the garden should be 12 ft × 8 ft.

19. Let x = length of a side of the base. Let y = height of the box.
Area = $x^2 + 4xy = 300$
Maximize the volume = x^2y subject to $x^2 + 4xy - 300 = 0$.
$F(x, y, \lambda) = x^2y + \lambda(x^2 + 4xy - 300)$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = 2xy + 2x\lambda + 4y\lambda = 0 \\ \dfrac{\partial F}{\partial y} = x^2 + 4x\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = x^2 + 4xy - 300 = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \dfrac{-2xy}{2(x+2y)} \\ \lambda = \dfrac{-x}{4x} = \dfrac{-x}{4} \\ x^2 + 4xy = 300 \end{array}\right\} \left.\begin{array}{l} x - 2y = 0 \\ x^2 + 4xy = 300 \end{array}\right\} \begin{array}{l} x = 10 \\ y = 5 \end{array}$$

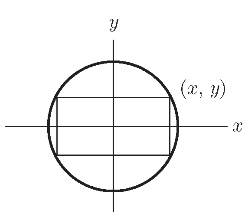
The sides of the base should be 10 in. and the height should be 5 in.

20. Let x = the number of units of labor and let y = the number of units of capital cost. The problem is to minimize $1000\sqrt{6x^2 + y^2}$ subject to $5000 - 480x - 40y = 0$.
$F(x, y, \lambda) = 1000\sqrt{6x^2 + y^2} + \lambda(5000 - 480x - 40y)$

$$\left.\begin{array}{l} \dfrac{\partial F}{\partial x} = \dfrac{6000x}{\sqrt{6x^2 + y^2}} - 480\lambda = 0 \\ \dfrac{\partial F}{\partial y} = \dfrac{1000y}{\sqrt{6x^2 + y^2}} - 40\lambda = 0 \\ \dfrac{\partial F}{\partial \lambda} = 5000 - 480x - 40y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = \dfrac{12.5x}{\sqrt{6x^2 + y^2}} \\ \lambda = \dfrac{25y}{\sqrt{6x^2 + y^2}} \\ 48x + 4y = 500 \end{array}\right\} \left.\begin{array}{l} x = 2y \\ 100y = 500 \end{array}\right\} \begin{array}{l} x = 10 \\ y = 5 \end{array}$$

There should be 10 units of labor and 5 units of capital to minimize the amount of space required.

21.



The length of the rectangle is $2x$, and the width of the rectangle is $2y$.
The problem is to maximize $4xy$ subject to $1 - x^2 - y^2 = 0$.
$F(x, y, \lambda) = 4xy + \lambda(1 - x^2 - y^2)$

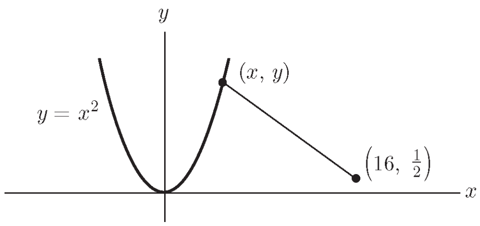
(*continued on next page*)

(*continued*)

$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x}=4y-2\lambda x=0\\ \dfrac{\partial F}{\partial y}=4x-2\lambda y=0\\ \dfrac{\partial F}{\partial \lambda}=1-x^2-y^2=0\end{array}\right\}\left.\begin{array}{l}\lambda=\dfrac{2y}{x}\\ \lambda=\dfrac{2x}{y}\\ x^2+y^2=1\end{array}\right\}\left.\begin{array}{l}\text{Assuming } x\ne 0,\ y\ne 0\\ \dfrac{2y}{x}=\dfrac{2x}{y}\\ x^2+y^2=1\end{array}\right\}\left.\begin{array}{l}x^2=y^2\\ 2x^2=1\end{array}\right\}\begin{array}{l}\text{Assuming } x>0,\ y>0,\\ x=\dfrac{\sqrt{2}}{2}\text{ and } y=\dfrac{\sqrt{2}}{2}\end{array}$$

The dimensions of the rectangle are $\sqrt{2}\times\sqrt{2}$.

22.



Following the hint, the problem is to minimize $(x-16)^2+\left(y-\frac{1}{2}\right)^2$ subject to $y-x^2=0$.

$$F(x,\ y,\ \lambda)=(x-16)^2+\left(y-\frac{1}{2}\right)^2+\lambda(y-x^2)$$

$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x}=2x-32-2\lambda x=0\\ \dfrac{\partial F}{\partial y}=2y-1+\lambda=0\\ \dfrac{\partial F}{\partial \lambda}=y-x^2=0\end{array}\right\}\left.\begin{array}{l}\lambda=1-\dfrac{16}{x}\\ \lambda=1-2y\\ y=x^2\end{array}\right\}\left.\begin{array}{l}x=0,\ y=0\text{ or}\\ \dfrac{16}{x}=2y\\ y=x^2\end{array}\right\}\begin{array}{c}x=0,\ y=0\\ \text{or}\\ x=2,\ y=4\end{array}$$

To decide which of (0, 0) or (2, 4) is the closer point, check that $(0-16)^2+\left(0-\frac{1}{2}\right)^2>(2-16)^2+\left(4-\frac{1}{2}\right)^2$.

Thus (2, 4) is the desired point.

23. The problem is to maximize $3x + 4y$ subject to $18,000-9x^2-4y^2=0$, $x\ge 0$, $y\ge 0$.

$$F(x,\ y,\ \lambda)=3x+4y+\lambda(18,000-9x^2-4y^2)$$

$$\left.\begin{array}{l}\dfrac{\partial F}{\partial x}=3-18\lambda x=0\\ \dfrac{\partial F}{\partial y}=4-8\lambda x=0\\ \dfrac{\partial F}{\partial \lambda}=18,000-9x^2-4y^2=0\end{array}\right\}\left.\begin{array}{l}\lambda=\dfrac{1}{6x}\\ \lambda=\dfrac{1}{2y}\\ 9x^2+4y^2=18,000\end{array}\right\}\left.\begin{array}{l}\text{If } x\ne 0\text{ and } y\ne 0\text{ then}\\ y=3x.\\ 9x^2+36x^2=18,000\end{array}\right\}\begin{array}{l}x=20\\ y=60\end{array}$$

Technically, we should also check the solutions $x=0,\ y=\sqrt{\frac{18,000}{9}}\approx 44.7$ and $y=0,\ x=\sqrt{\frac{18,000}{36}}\approx 22.4$.

These both give smaller values in the objective function $3x + 4y$ than does (20, 60).

24. We want to minimize the function $P = 2x + 10y$ subject to the constraint $4x^2 + 25y^2 = 50,000$.

$F(x, y, \lambda) = 2x + 10y + \lambda(4x^2 + 25y^2 - 50,000)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2 + 8\lambda x = 0 \\ \frac{\partial F}{\partial y} &= 10 + 50\lambda y = 0 \\ \frac{\partial F}{\partial \lambda} &= 4x^2 + 25y^2 - 50,000 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{-1}{4x} \\ \lambda &= \frac{-1}{5y} \\ 4x^2 &+ 25y^2 = 50,000 \end{aligned}\right\} \begin{aligned} x &= \frac{5}{4}y \\ 4x^2 &+ 25y^2 = 50,000 \end{aligned}$$

Solving gives $\frac{125}{4}y^2 = 50,000$ or $y = 40$, hence $x = 50$.

They should produce 50 units of product A and 40 units of product B.

25. a. $F(x, y, \lambda) = 96x + 162y + \lambda\left(3456 - 64x^{3/4}y^{1/4}\right)$

Note that $3456 = 64x^{3/4}y^{1/4}$ implies $x \neq 0, y \neq 0$.

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 96 - 48\lambda x^{-1/4}y^{1/4} \\ \frac{\partial F}{\partial y} &= 162 - 16\lambda x^{3/4}y^{-3/4} \\ \frac{\partial F}{\partial \lambda} &= 3456 - 64x^{3/4}y^{1/4} \end{aligned}\right\} \left.\begin{aligned} \lambda &= 2x^{1/4}y^{-1/4} \\ \lambda &= \frac{81}{8}x^{-3/4}y^{3/4} \\ 3456 &= 64x^{3/4}y^{1/4} \end{aligned}\right\} \begin{matrix}\text{Dividing 1st} \\ \text{equation by} \\ \text{2nd gives}\end{matrix} \left.\begin{aligned} \frac{16}{81}xy^{-1} &= 1 \\ y &= \frac{16}{81}x \\ 3456 &= 64x^{3/4}\left(\frac{16}{81}x\right)^{1/4} \end{aligned}\right\} \begin{aligned} x &= 81 \\ y &= 16 \end{aligned}$$

b. $\lambda = 2(81)^{1/4}(16)^{-1/4} = 3$

c. The production function is $f(x, y) = 64x^{3/4}y^{1/4}$. Thus

$$\frac{\text{marginal productivity of labor}}{\text{marginal productivity of capital}} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{48x^{-1/4}y^{1/4}}{16x^{3/4}y^{-3/4}} = \frac{48y}{16x}.$$

When $x = 81$ and $y = 16$, $\frac{48y}{16x} = \frac{48 \cdot 16}{16 \cdot 81} = \frac{96}{162}$, which is the ratio of the unit cost of labor and capital.

26. $F(x, y, \lambda) = 94x - \frac{x^2}{10} + 80y - \frac{y^2}{20} - 20,000 + \lambda\left(14 - \frac{x}{10} + \frac{y}{20}\right)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 94 - \frac{x}{5} - \frac{\lambda}{10} = 0 \\ \frac{\partial F}{\partial y} &= 80 - \frac{y}{10} + \frac{\lambda}{20} = 0 \\ \frac{\partial F}{\partial \lambda} &= 14 - \frac{x}{10} + \frac{y}{20} = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= 940 - 2x \\ \lambda &= -1600 - 2x \\ \frac{x}{10} &- \frac{y}{20} = 14 \end{aligned}\right\} \left.\begin{aligned} 2x + 2y &= 2540 \\ 2x - y &= 280 \end{aligned}\right\} \begin{aligned} x &= \frac{1550}{3} \\ y &= \frac{2260}{3} \end{aligned}$$

27. $F(x, y, z, \lambda) = xyz + \lambda(36 - x - 6y - 3z)$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= yz - \lambda = 0\\ \frac{\partial F}{\partial y} &= xz - 6\lambda = 0\\ \frac{\partial F}{\partial z} &= xy - 3\lambda = 0\\ \frac{\partial F}{\partial \lambda} &= 36 - x - 6y - 3z = 0\end{aligned}\right\} \left.\begin{aligned}\lambda &= yz\\ \lambda &= \frac{xz}{6}\\ \lambda &= \frac{xy}{3}\\ x + 6y + 3z &= 36\end{aligned}\right\} \left.\begin{aligned}y &= \frac{x}{6}\\ \frac{z}{6} &= \frac{y}{3}\\ x + 6y + 3z &= 36\end{aligned}\right\} \left.\begin{aligned}x &= 6y\\ 3z &= 6y\\ 3(6y) &= 36\end{aligned}\right\} \begin{aligned}x &= 12\\ y &= 2\\ z &= 4\end{aligned}$$

28. $F(x, y, z, \lambda) = xy + 3xz + 3yz + \lambda(9 - xyz)$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= y + 3z - \lambda yz = 0\\ \frac{\partial F}{\partial y} &= x + 3z - \lambda xz = 0\\ \frac{\partial F}{\partial z} &= 3x + 3y - \lambda xy = 0\\ \frac{\partial F}{\partial \lambda} &= 9 - xyz = 0\end{aligned}\right\} \left.\begin{aligned}\lambda &= \frac{1}{z} + \frac{3}{y}\\ \lambda &= \frac{1}{z} + \frac{3}{x}\\ \lambda &= \frac{3}{y} + \frac{3}{x}\\ xyz &= 9\end{aligned}\right\} \left.\begin{aligned}\frac{3}{y} &= \frac{3}{x}\\ \frac{1}{z} &= \frac{3}{y}\\ xyz &= 9\end{aligned}\right\} \left.\begin{aligned}x &= y\\ z &= \frac{y}{3}\\ yy\left(\frac{y}{3}\right) &= 9 \text{ or } y^3 = 27 \text{ or } y = 3\end{aligned}\right\} \begin{aligned}x &= 3\\ y &= 3\\ z &= 1\end{aligned}$$

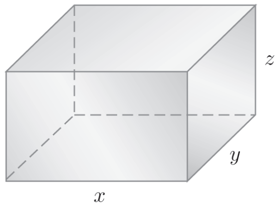
29. $F(x, y, z, \lambda) = 3x + 5y + z - x^2 - y^2 - z^2 + \lambda(6 - x - y - z)$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= 3 - 2x - \lambda = 0\\ \frac{\partial F}{\partial y} &= 5 - 2y - \lambda = 0\\ \frac{\partial F}{\partial z} &= 1 - 2z - \lambda = 0\\ \frac{\partial F}{\partial \lambda} &= 6 - x - y - z = 0\end{aligned}\right\} \left.\begin{aligned}\lambda &= 3 - 2x\\ \lambda &= 5 - 2y\\ \lambda &= 1 - 2z\\ x + y + z &= 6\end{aligned}\right\} \left.\begin{aligned}-2x + 2y &= 2\\ -2y + 2z &= -4\\ x + y + z &= 6\end{aligned}\right\} \left.\begin{aligned}x &= -1 + y\\ z &= -2 + y\\ (-1 + y) + y + (-2 + y) &= 6\end{aligned}\right\} \begin{aligned}x &= 2\\ y &= 3\\ z &= 1\end{aligned}$$

30. $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - 3x - 5y - z + \lambda(20 - 2x - y - z)$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= 2x - 3 - 2\lambda = 0\\ \frac{\partial F}{\partial y} &= 2y - 5 - \lambda = 0\\ \frac{\partial F}{\partial z} &= 2z - 1 - \lambda = 0\\ \frac{\partial F}{\partial \lambda} &= 20 - 2x - y - z = 0\end{aligned}\right\} \left.\begin{aligned}\lambda &= x - \frac{3}{2}\\ \lambda &= 2y - 5\\ \lambda &= 2z - 1\\ 2x + y + z &= 20\end{aligned}\right\} \left.\begin{aligned}x - 2y &= -\frac{7}{2}\\ 2y - 2z &= 4\\ 2x + y + z &= 20\end{aligned}\right\} \left.\begin{aligned}x &= 2y - \frac{7}{2}\\ z &= -2 + y\\ 2\left(2y - \frac{7}{2}\right) + y + y - 2 &= 20\end{aligned}\right\} \begin{aligned}x &= \frac{37}{6}\\ y &= \frac{29}{6}\\ z &= \frac{17}{6}\end{aligned}$$

31.



The problem is to minimize $3xy + 2xz + 2yz$ subject to the constraint $xyz = 12$ or $xyz - 12 = 0$.
$F(x, y, z, \lambda) = 3xy + 2xz + 2yz + \lambda(12 - xyz)$
(Note that $xyz = 12$ implies that $x \neq 0, y \neq 0, z \neq 0$.)

$$\begin{array}{l|l|l|l}
\dfrac{\partial F}{\partial x} = 3y + 2z - \lambda yz = 0 & \lambda = \dfrac{3}{z} + \dfrac{2}{y} & x = y & x = 2 \\
\dfrac{\partial F}{\partial y} = 3x + 2z - \lambda xz = 0 & \lambda = \dfrac{3}{z} + \dfrac{2}{x} & z = \dfrac{3}{2}y & y = 2 \\
\dfrac{\partial F}{\partial z} = 2x + 2y - \lambda xy = 0 & \lambda = \dfrac{2}{y} + \dfrac{2}{x} & & z = 3 \\
\dfrac{\partial F}{\partial \lambda} = 12 - xyz = 0 & xyz = 12 & y^2\left(\dfrac{3}{2}y\right) = 12 &
\end{array}$$

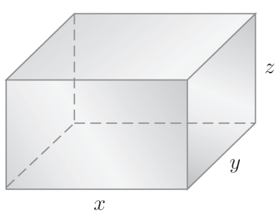
The box is 2 ft × 2 ft × 3 ft.

32. The problem is to maximize xyz subject to $x + y + z - 15 = 0$, $x > 0$, $y > 0$, $z > 0$.
$F(x, y, z, \lambda) = xyz + \lambda(x + y + z - 15)$

$$\begin{array}{l|l|l|l}
\dfrac{\partial F}{\partial x} = yz + \lambda = 0 & \lambda = -yz & x = y & x = 5 \\
\dfrac{\partial F}{\partial y} = xz + \lambda = 0 & \lambda = -xz & z = y & y = 5 \\
\dfrac{\partial F}{\partial z} = xy + \lambda = 0 & \lambda = -xy & 3y = 15 & z = 5 \\
\dfrac{\partial F}{\partial \lambda} = x + y + z - 15 = 0 & x + y + z = 15 & &
\end{array}$$

The numbers are 5, 5 and 5.

33.



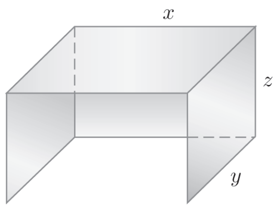
Let x, y, z be as shown in the figure. The problem is to minimize $xy + 2xz + 2yz$ subject to $xyz = 32$ or $32 - xyz = 0$.

$F(x, y, z, \lambda) = xy + 2xz + 2yz + \lambda(32 - xyz)$
(Note that $xyz = 32$ implies $x \neq 0, y \neq 0, z \neq 0$.)

$$\begin{array}{l|l|l|l|l}
\dfrac{\partial F}{\partial x} = y + 2z - \lambda yz = 0 & \lambda = \dfrac{1}{z} + \dfrac{2}{y} & \text{Equating} & x = y & x = 4 \\
\dfrac{\partial F}{\partial y} = x + 2z - \lambda xz = 0 & \lambda = \dfrac{1}{z} + \dfrac{2}{x} & \text{expressions for } \lambda \text{ gives} & z = \dfrac{1}{2}y & y = 4 \\
\dfrac{\partial F}{\partial z} = 2x + 2y - \lambda xy = 0 & \lambda = \dfrac{2}{y} + \dfrac{2}{x} & & \dfrac{1}{2}y^3 = 32 & z = 2 \\
\dfrac{\partial F}{\partial \lambda} = xyz - 32 = 0 & xyz = 32 & & &
\end{array}$$

The dimensions of the tank are 4 ft × 4 ft × 2 ft.

34.



Let x, y, z be as shown in the figure. The problem is to maximize xyz subject to $xy + xz + 2yz = 96$, $x > 0$, $y > 0$, $z > 0$.

$F(x, y, z, \lambda) = xyz + \lambda(xy + xz + 2yz - 96)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= yz + \lambda(y+z) = 0 \\ \frac{\partial F}{\partial y} &= xz + \lambda(x+2z) = 0 \\ \frac{\partial F}{\partial z} &= xy + \lambda(x+2y) = 0 \\ \frac{\partial F}{\partial \lambda} &= xy + xz + 2yz - 96 = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -\frac{yz}{y+z} \\ \lambda &= -\frac{xz}{x+2z} \\ \lambda &= -\frac{xy}{x+2y} \\ xy + xz &+ 2yz = 96 \end{aligned}\right\} \begin{array}{l}\text{Equating}\\ \text{expressions}\\ \text{for } \lambda \text{ gives}\end{array} \left.\begin{aligned} x &= 2y \\ z &= y \\ 2y^2 + 2y^2 + 2y^2 &= 96 \end{aligned}\right\} \begin{aligned} x &= 8 \\ y &= 4 \\ z &= 4 \end{aligned}$$

The dimensions of the shelter are 8 ft $\times$ 4 ft $\times$ 4 ft.

35. $F(x, y, \lambda) = f(x, y) + \lambda(c - ax - by)$

The values of x and y that minimize production subject to the cost constraint satisfy

$$\left.\begin{aligned} \frac{\partial F}{\partial x}(x, y) &= \frac{\partial f}{\partial x}(x, y) - \lambda a = 0 \\ \frac{\partial F}{\partial y}(x, y) &= \frac{\partial f}{\partial y}(x, y) - \lambda b = 0 \\ \frac{\partial F}{\partial \lambda}(x, y) &= c - ax - by = 0 \end{aligned}\right\} \left.\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \lambda a \\ \frac{\partial f}{\partial y}(x, y) &= \lambda b \end{aligned}\right\} \begin{array}{l}\text{Dividing the 1st}\\ \text{equation by the}\\ \text{second gives}\end{array} \quad \frac{\frac{\partial f}{\partial x}(x, y)}{\frac{\partial f}{\partial y}(x, y)} = \frac{a}{b}.$$

36. Given $f(x, y) = kx^{\alpha}y^{\beta}$, $\dfrac{\partial f}{\partial x} = \alpha k x^{\alpha-1}y^{\beta}$ and $\dfrac{\partial f}{\partial y} = \beta k x^{\alpha}y^{\beta-1}$.

Applying the result of Exercise 35, at the optimal values (x, y), $\dfrac{\alpha k x^{\alpha-1}y^{\beta}}{\beta k x^{\alpha}y^{\beta-1}} = \dfrac{a}{b}$; so $\dfrac{\alpha y}{\beta x} = \dfrac{a}{b}$ or $\dfrac{y}{x} = \dfrac{\beta a}{\alpha b}$.

7.5 The Method of Least Squares

1. The given points are (1, 3), (2, 6), (3, 8), and (4, 6) with straight line $y = 1.1x + 3$. When $x = 1, 2, 3, 4$ the corresponding y-coordinates are $1.1 + 3$, $2(1.1) + 3$, $3(1.1) + 3$, $4(1.1) + 3$ or 4.1, 5.2, 6.3, and 7.4, respectively. Then

$E_1^2 = (4.1-3)^2 = (1.1)^2 = 1.21$,

$E_2^2 = (5.2-6)^2 = (-.8)^2 = .64$,

$E_3^2 = (6.3-8)^2 = (-1.7)^2 = 2.89$,

$E_4^2 = (7.4-6)^2 = (1.4)^2 = 1.96$ so the least-squares error E is

$$\begin{aligned} E &= E_1^2 + E_2^2 + E_3^2 + E_4^2 \\ &= 1.21 + .64 + 2.89 + 1.96 = 6.7. \end{aligned}$$

2. The given points are (1, 8), (2, 5), (3, 3), (4, 4), and (5, 2) with straight line $y = -1.3x + 8.3$. When $x = 1, 2, 3, 4, 5$ the corresponding y-coordinates are $-1.3 + 8.3$, $2(-1.3) + 8.3$, $3(-1.3) + 8.3$, $4(-1.3) + 8.3$, $5(-1.3) + 8.3$ or 7, 5.7, 4.4, 3.1, 1.8, respectively. Then $E_1^2 = (7-8)^2 = 1$,

$E_2^2 = (5.7-5)^2 = (.7)^2 = .49$,

$E_3^2 = (4.4-3)^2 = (1.4)^2 = 1.96$,

$E_4^2 = (3.1-4)^2 = (-.9)^2 = .81$,

$E_5^2 = (1.8-2)^2 = (-.2)^2 = .04$.

Thus the least-squares error is

$$\begin{aligned} E &= E_1^2 + E_2^2 + E_3^2 + E_4^2 + E_5^2 \\ &= 1 + .49 + 1.96 + .81 + .04 = 4.3 \end{aligned}$$

3. $E = (2A + B - 6)^2 + (5A + B - 10)^2 + (9A + B - 15)^2$

4. $E = (8A + B - 4)^2 + (9A + B - 2)^2 + (10A + B - 3)^2$

5. Let the straight line be $y = Ax + B$.

$E_1^2 = (A + B - 2)^2$, $E_2^2 = (2A + B - 5)^2$,

$E_3^2 = (3A + B - 11)^2$

$$f(A, B) = E_1^2 + E_2^2 + E_3^2 = (A + B - 2)^2 + (2A + B - 5)^2 + (3A + B - 11)^2$$

$$\frac{\partial f}{\partial A} = 2(A + B - 2) + 2(2A + B - 5) \cdot 2 + 2(3A + B - 11) \cdot 3 = 28A + 12B - 90$$

$$\frac{\partial f}{\partial B} = 2(A + B - 2) + 2(2A + B - 5) + 2(3A + B - 11) = 12A + 6B - 36$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero, we obtain the system

$$\begin{cases} 28A + 12B = 90 \\ 12A + 6B = 36 \end{cases}.$$

Then $A = 4.5$ and $B = -3$, so the line with the best least-squares fit to the data points is $y = 4.5x - 3$.

6. Let the straight line be $y = Ax + B$.

$E_1^2 = (A + B - 8)^2$, $E_2^2 = (2A + B - 4)^2$,

$E_3^2 = (4A + B - 3)^2$

$$f(A, B) = E_1^2 + E_2^2 + E_3^2 = (A + B - 8)^2 + (2A + B - 4)^2 + (4A + B - 3)^2$$

$$\frac{\partial f}{\partial A} = 2(A + B - 8) + 2(2A + B - 4) \cdot 2 + 2(4A + B - 3) \cdot 4 = 42A + 14B - 56$$

$$\frac{\partial f}{\partial B} = 2(A + B - 8) + 2(2A + B - 4) + 2(4A + B - 3) = 14A + 6B - 30$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero, we obtain the system

$$\begin{cases} 42A + 14B = 56 \\ 14A + 6B = 30 \end{cases}.$$

Then $A = -1.5$ and $B = 8.5$, so the line with the best least-squares fit to the data points is $y = -1.5x + 8.5$.

7. Let the straight line be $y = Ax + B$.

$E_1^2 = (A + B - 9)^2$, $E_2^2 = (2A + B - 8)^2$,

$E_3^2 = (3A + B - 6)^2$, $E_4^2 = (4A + B - 3)^2$

$$f(A, B) = E_1^2 + E_2^2 + E_3^2 + E_4^2 = (A + B - 9)^2 + (2A + B - 8)^2 + (3A + B - 6)^2 + (4A + B - 3)^2.$$

$$\frac{\partial f}{\partial A} = 2(A + B - 9) + 2(2A + B - 8) \cdot 2 + 2(3A + B - 6) \cdot 3 + 2(4A + B - 3) \cdot 4 = 60A + 20B - 110$$

$$\frac{\partial f}{\partial B} = 2(A + B - 9) + 2(2A + B - 8) + 2(3A + B - 6) + 2(4A + B - 3) = 20A + 8B - 52$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero, we obtain the system

$$\begin{cases} 60A + 20B = 110 \\ 20A + 8B = 52 \end{cases}.$$

Then $A = -2$ and $B = 11.5$ so the line with the best least-squares fit to the data points is $y = -2x + 11.5$.

8. Let the straight line be $y = Ax + B$.

$E_1^2 = (A + B - 5)^2$, $E_2^2 = (2A + B - 7)^2$,

$E_2^3 = (3A + B - 6)^2$, $E_4^2 = (4A + B - 10)^2$

$$f(A, B) = E_1^2 + E_2^2 + E_3^2 + E_4^2 = (A + B - 5)^2 + (2A + B - 7)^2 + (3A + B - 6)^2 + (4A + B - 10)^2$$

$$\frac{\partial f}{\partial A} = 2(A + B - 5) + 2(2A + B - 7) \cdot 2 + 2(3A + B - 6) \cdot 3 + 2(4A + B - 10) \cdot 4 = 60A + 20B - 154$$

$$\frac{\partial f}{\partial B} = 2(A + B - 5) + 2(2A + B - 7) + 2(3A + B - 6) + 2(4A + B - 10) = 20A + 8B - 56$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero, we obtain the system

$$\begin{matrix} 60A + 20B = 154 \\ 20A + 8B = 56 \end{matrix}.$$

(*continued on next page*)

(*continued*)

Then $A = 1.4$ and $B = 3.5$ so the line with the best least-squares fit to the data points is $y = 1.4x + 3.5$.

9.

x	y	xy	x^2
1	7	7	1
2	6	12	4
3	4	12	9
4	3	12	16
$\Sigma x = 10$	$\Sigma y = 20$	$\Sigma xy = 43$	$\Sigma x^2 = 30$

Let $y = Ax + B$. Then using

$$A = \frac{N\cdot\Sigma xy - \Sigma x\cdot\Sigma y}{N\cdot\Sigma x^2 - (\Sigma x)^2} \text{ and } B = \frac{\Sigma y - A\cdot\Sigma x}{N}$$

we have: $A = \dfrac{4\cdot 43 - 10\cdot 20}{4\cdot 30 - 10^2} = -1.4$ and

$$B = \frac{20 - (-1.4)\cdot 10}{4} = 8.5\,.$$

The least-squares line is $y = -1.4x + 8.5$.

10.

x	y	xy	x^2
1	2	2	1
2	3	6	4
3	7	21	9
4	9	36	16
5	12	60	25
$\Sigma x = 15$	$\Sigma y = 33$	$\Sigma xy = 125$	$\Sigma x^2 = 55$

Let $y = Ax + B$. Then using

$$A = \frac{N\cdot\Sigma xy - \Sigma x\cdot\Sigma y}{N\cdot\Sigma x^2 - (\Sigma x)^2} \text{ and } B = \frac{\Sigma y - A\cdot\Sigma x}{N}$$

we have: $A = \dfrac{5\cdot 125 - 15\cdot 33}{5\cdot 55 - 15^2} = 2.6$ and

$$B = \frac{33 - 2.6\cdot 15}{5} = -1.2.$$

The least-squares line is $y = 2.6x - 1.2$.

11. a.

x	y	xy	x^2
9	8175	73,575	81
10	8428	84,280	100
12	8996	107,952	144
13	9255	120,315	169
$\Sigma x = 44$	$\Sigma y = 34{,}854$	$\Sigma xy = 386{,}122$	$\Sigma x^2 = 494$

Let $y = Ax + B$, where x = years after 2000 and y = dollars in thousands.

Then using $A = \dfrac{N\cdot\Sigma xy - \Sigma x\cdot\Sigma y}{N\cdot\Sigma x^2 - (\Sigma x)^2}$ and

$$B = \frac{\Sigma y - A\cdot\Sigma x}{N}$$

we have

$$A = \frac{4\cdot 386{,}122 - 44\cdot 34{,}854}{4\cdot 494 - (44)^2} = 272.8$$

$$B = \frac{34{,}854 - 272.8\cdot 44}{4} = 5712.7$$

The least-squares line is $y = 272.8x + 5712.7$.

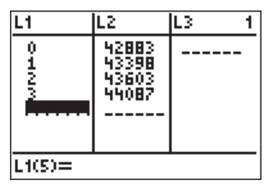
b. For $x = 16$,

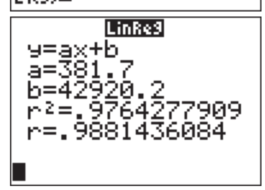
$$y = 272.8(16) + 5712.7 = 10{,}077.5$$

c. $12{,}000 = 272.8x + 5712.7 \Rightarrow x \approx 23$

Per capita health care expenditures will reach \$12,000 sometime during 2023.

12. a.





Using a graphing utility, we obtain $y = 381.7x + 42{,}920.2$. (Note that the year 2012 is represented by $x = 0$).

By hand, we have $N = 4$, $\Sigma x = 6$, $\Sigma y = 173{,}971$, $\Sigma xy = 262{,}865$, and $\Sigma x^2 = 14$.

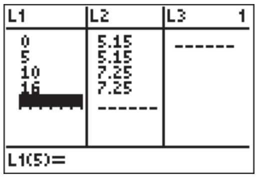
The regression equation is the same.

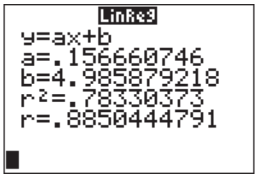
b. $46{,}000 = 381.7x + 42{,}920.2 \Rightarrow x \approx 8$

The university should build more housing 8 years after 2012, or in 2020.

13. Given the table, convert *year* to *years after 2000*. Then we have the data points (0, 5.15), (5, 5.15), (10, 7.25), (16, 7.25).

a.





Using a graphing utility, we obtain $y = .157x + 4.986$

By hand, we have
$N = 4,\ \Sigma x = 31,\ \Sigma y = 24.8,$

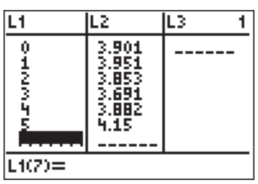
$\Sigma xy = 214.25,$ and $\Sigma x^2 = 381.$ The regression equation is the same.

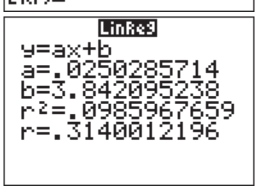
b. $y = .157(8) + 4.986 = \$6.24$ per hour

c. $10 = .157x + 4.986 \Rightarrow x \approx 31.9$
The minimum wage will reach \$10 per hour in about 31.9 years, or in the year 2032.

14. Given the table, convert *year* to *years after 2010*. Then we have the data points(0, 3.901), 1, 3.951), (2, 3.853), (3, 3.691), (4, 3.882), (5, 4.150).

a.





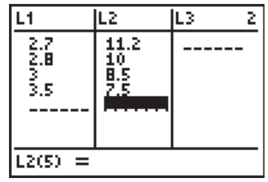
Using a graphing utility, we obtain $y = .025x + 3.842.$

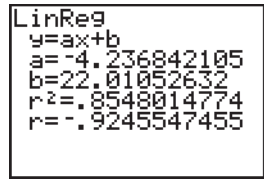
By hand, we have
$N = 6,\ \Sigma x = 15,\ \Sigma y = 23.428,$

$\Sigma xy = 59.008,$ and $\Sigma x^2 = 55.$
The regression equation is the same.

b. $y = .025(7) + 3.842$
$= 4.017$ million visitors

15. a.





Using a graphing utility, we obtain $y = -4.24x + 22.01$. By hand, we have $N = 4,\ \Sigma x = 12,\ \Sigma y = 37.2,$

$\Sigma xy = 109.99,$ and $\Sigma x^2 = 36.38.$
The regression equation is the same.

b. $y = -4.24(3.2) + 22.01 = 8.442°\text{C}.$

7.6 Double Integrals

1.
$$\begin{aligned}\int_0^1\left(\int_0^1 e^{x+y}dy\right)dx &= \int_0^1\left(e^{x+y}\Big|_{y=0}^{1}\right)dx\\ &= \int_0^1 (e^{x+1} - e^x)\ dx\\ &= e^{x+1} - e^x\Big|_0^1\\ &= e^2 - e - e + 1\\ &= e^2 - 2e + 1\end{aligned}$$

2.
$$\begin{aligned}\int_{-1}^1\left(\int_{-1}^1 xy\ dx\right)dy &= \int_{-1}^1\left(\left[\frac{1}{2}x^2y\right]\Bigg|_{x=-1}^{1}\right)dy\\ &= \int_{-1}^1\left(\frac{1}{2}y - \frac{1}{2}y\right)dy\\ &= \int_{-1}^1 0\ dy = 0\end{aligned}$$

3.
$$\begin{aligned}\int_{-1}^1\left(\int_{-2}^0 xe^{xy}dy\right)dx &= \int_{-1}^1\left(e^{xy}\Big|_{y=-2}^{0}\right)dx\\ &= \int_{-1}^1\left(1 - e^{-2x}\right)dx\\ &= x + \frac{e^{-2x}}{2}\Bigg|_{-1}^{1}\\ &= 1 + \frac{e^{-2}}{2} - \left(-1 + \frac{e^2}{2}\right)\\ &= 2 + \frac{e^{-2}}{2} - \frac{e^2}{2}\end{aligned}$$

4. $$\int_0^1\left(\int_{-1}^1 \frac{1}{3}y^3x\ dy\right)dx = \int_0^1\left(\left(\frac{1}{12}y^4x\right)\Bigg|_{y=-1}^{1}\right)dx$$
$$= \int_0^1\left(\frac{1}{12}x - \frac{1}{12}x\right)dx$$
$$= \int_0^1 0\ dx = 0$$

5. $$\int_1^4\left(\int_x^{x^2} xy\ dy\right)dx = \int_1^4\left(\frac{1}{2}xy^2\Bigg|_{y=x}^{x^2}\right)dx$$
$$= \int_1^4\left(\frac{1}{2}x^5 - \frac{1}{2}x^3\right)dx$$
$$= \frac{1}{12}x^6 - \frac{1}{8}x^4\Bigg|_1^4$$
$$= \frac{1}{12}4^6 - \frac{1}{8}4^4 - \frac{1}{12} + \frac{1}{8}$$
$$= 309\frac{3}{8}$$

6. $$\int_0^3\left(\int_x^{2x} y\ dy\right)dx = \int_0^3\left(\frac{1}{2}y^2\Bigg|_{y=x}^{2x}\right)dx$$
$$= \int_0^3\left(2x^2 - \frac{1}{2}x^2\right)dx$$
$$= \int_0^3 \frac{3}{2}x^2 dx = \frac{1}{2}x^3\Bigg|_0^3 = \frac{27}{2}$$

7. $$\int_{-1}^1\left(\int_x^{2x}(x+y)\ dy\right)dx$$
$$= \int_{-1}^1\left(\left[xy + \frac{1}{2}y^2\right]\Bigg|_{y=x}^{2x}\right)dx$$
$$= \int_{-1}^1\left(2x^2 + 2x^2 - x^2 - \frac{1}{2}x^2\right)dx$$
$$= \int_{-1}^1 \frac{5}{2}x^2 dx = \frac{5}{6}x^3\Bigg|_{-1}^1 = \frac{5}{6} + \frac{5}{6} = \frac{5}{3}$$

8. $$\int_0^1\left(\int_0^x e^{x+y}\ dy\right)dx = \int_0^1\left(e^{x+y}\Big|_{y=0}^{x}\right)dx$$
$$= \int_0^1 (e^{2x} - e^x)\ dx$$
$$= \frac{1}{2}e^{2x} - e^x\Bigg|_0^1$$
$$= \frac{1}{2}e^2 - e - \frac{1}{2} + 1$$
$$= \frac{1}{2}e^2 - e + \frac{1}{2}$$

9. $$\int_0^2\left(\int_2^3 xy^2 dy\right)dx = \int_0^2\left(\frac{1}{3}xy^3\Bigg|_{y=2}^{3}\right)dx$$
$$= \int_0^2\left(9x - \frac{8}{3}x\right)dx$$
$$= \int_0^2 \frac{19}{3}x\ dx = \frac{19}{6}x^2\Bigg|_0^2 = \frac{38}{3}$$

10. $$\int_0^2\left(\int_2^3 (xy + y^2)\ dy\right)dx$$
$$= \int_0^2\left(\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]\Bigg|_{y=2}^{3}\right)dx$$
$$= \int_0^2\left(\frac{9}{2}x + 9 - 2x - \frac{8}{3}\right)dx$$
$$= \int_0^2\left(\frac{5}{2}x + \frac{19}{3}\right)dx = \frac{5}{4}x^2 + \frac{19}{3}x\Bigg|_0^2$$
$$= 5 + \frac{38}{3} = \frac{53}{3}$$

11. $$\int_0^2\left(\int_2^3 e^{-x-y}dy\right)dx = \int_0^2\left(-e^{-x-y}\Big|_{y=2}^{3}\right)dx$$
$$= \int_0^2 (e^{-x-2} - e^{-x-3})\ dx$$
$$= -e^{-x-2} + e^{-x-3}\Big|_0^2$$
$$= -e^{-4} + e^{-5} + e^{-2} - e^{-3}$$

12. $$\int_0^2\left(\int_2^3 e^{y-x}dy\right)dx = \int_0^2\left(e^{y-x}\Big|_{y=2}^{3}\right)dx$$
$$= \int_0^2 (e^{3-x} - e^{2-x})\ dx$$
$$= -e^{3-x} + e^{2-x}\Big|_0^2$$
$$= -e + 1 + e^3 - e^2$$
$$= e^3 - e^2 - e + 1$$

13. $$\int_1^3\left[\int_0^1 (x^2 + y^2)\ dy\right]dx$$
$$= \int_1^3\left[\left[x^2y + \frac{1}{3}y^3\right]\Bigg|_{y=0}^{1}\right]dx$$
$$= \int_1^3\left(x^2 + \frac{1}{3}\right)dx = \frac{1}{3}x^3 + \frac{1}{3}x\Bigg|_1^3$$
$$= 9 + 1 - \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{28}{3}$$

14. $\int_0^1 \left[\int_0^{\sqrt[3]{x}} (x^2 + y^2)\, dy \right] dx$

$= \int_0^1 \left[\left[x^2 y + \frac{1}{3} y^3 \right] \Big|_0^{\sqrt[3]{x}} \right] dx$

$= \int_0^1 \left(x^{7/3} + \frac{1}{3} x \right) dx = \frac{3}{10} x^{10/3} + \frac{1}{6} x^2 \Big|_0^1$

$= \frac{3}{10} + \frac{1}{6} = \frac{7}{15}$

Chapter 7 Fundamental Concept Check Exercises

1. Answers will vary. Sample answer: For $f(x, y) = x^2 y + x,$ the level curve at height 10 is the graph of $f(x, y) = 10,$ or $x^2 y + x = 10.$ Solving for *y*, we have $y = \frac{10 - x}{x^2}.$

2. To find the first partial derivative with respect to *x* of $f(x, y),$ treat *y* as a constant and differentiate the formula for $f(x, y)$ with respect to *x*. To find the first partial derivative with respect to *y* of $f(x, y),$ treat *x* as a constant and differentiate the formula for $f(x, y)$ with respect to *y*.

3. To find a second partial derivative of $f(x, y),$ for example, the second partial derivative with respect to *x*, $\frac{\partial^2 f}{\partial x^2},$ treat *y* as a constant and differentiate the formula for $\frac{\partial f}{\partial x}$ with respect to *x*. Use a similar method to find $\frac{\partial^2 f}{\partial y^2}.$ To find $\frac{\partial^2 f}{\partial x \partial y},$ differentiate the formula for $\frac{\partial f}{\partial y}$ with respect to *x*. To find $\frac{\partial^2 f}{\partial y \partial x},$ differentiate the formula for $\frac{\partial f}{\partial x}$ with respect to *y*.

4. $f(a + h, b) - f(a, b) = \frac{\partial f}{\partial x}(a, b) \cdot h$

5. If *x* is kept constant at 2 and *y* is allowed to vary near 3, then $f(x, y)$ changes at a rate that is $\frac{\partial f}{\partial x}(2, 3)$ times the change in *y*.

6. Answers will vary. Sample answer: An example of a Cobb-Douglas production function is $f(x, y) = 600x^{3/5} y^{2/5}.$ The marginal productivity of labor is $\frac{\partial f}{\partial x}.$ The marginal productivity of capital is $\frac{\partial f}{\partial y}.$

7. To find possible relative extreme point for a function of several variables, look for points where all the first partial derivatives are zero. Then apply the Second Derivative Test to determine if the point is a relative maximum or a relative minimum.

8. Suppose that $f(x, y)$ is a function and (a, b) is a point at which $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0,$ and let

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

- If $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0,$ then $f(x, y)$ has a relative minimum at $(a, b).$
- If $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0,$ then $f(x, y)$ has a relative maximum at $(a, b).$
- If $D(a, b) < 0,$ then $f(x, y)$ has neither a relative maximum nor a relative minimum at $(a, b).$
- If $D(a, b) = 0,$ no conclusion can be drawn from this test.

9. To find a relative maximum or minimum of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$, use the Lagrange multiplier method as follows: Form the function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. Compute the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.
Solve the system of equations $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $g(x, y) = 0$. If you find more than one value for x, evaluate f for each value. The largest of these values is the maximum value for f, and the smallest is the minimum value of f.

10. The least-squares line approximation to a set of data points is the line that best fits the data in the sense that it minimizes the least-squares error, the sum of the squares of the distances from the given data points to the line. If the lien is $y = Ax + b$, then the coefficients are computed as follows:
$$A = \frac{N \cdot \Sigma xy - \Sigma x \cdot \Sigma y}{N \cdot \Sigma x^2 - (\Sigma x)^2} \text{ and } B = \frac{\Sigma y - A \cdot \Sigma x}{N}$$

11. $\iint_R f(x, y)\,dx\,dy$ is the volume of the solid bounded above by $f(x, y) \geq 0$ and lying over the region R in the xy-plane.

12. Let R be the region in the xy-plane bounded by the graphs of $y = g(x)$, $y = h(x)$, and the vertical lines $x = a$ and $x = b$. Then,
$$\iint_R f(x, y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)\,dy \right) dx.$$

Chapter 7 Review Exercises

1. $f(x, y) = \frac{x\sqrt{y}}{1+x}$

$f(2, 9) = \frac{2\sqrt{9}}{1+2} = \frac{6}{3} = 2$

$f(5, 1) = \frac{5\sqrt{1}}{1+5} = \frac{5}{6}$; $f(0, 0) = \frac{0\sqrt{0}}{1+0} = 0$

2. $f(x, y, z) = x^2 e^{y/z}$

$f(-1, 0, 1) = (-1)^2 e^{0/1} = 1e^0 = 1$

$f(1, 3, 3) = 1^2 e^{3/3} = e$

$f(5, -2, 2) = 5^2 e^{-2/2} = \frac{25}{e}$

3. $f(A, t) = Ae^{0.06t}$

$f(10, 11.5) = 10e^{(0.06)(11.5)} \approx 19.94$

Ten dollars increases to approximately 20 dollars in 11.5 years.

4. $f(x, y, \lambda) = xy + \lambda(5 - x - y)$
$f(1, 2, 3) = (1)(2) + 3(5 - 1 - 2) = 2 + 3(2) = 8$

5. $f(x, y) = 3x^2 + xy + 5y^2$

$\frac{\partial f}{\partial x} = 6x + y$; $\frac{\partial f}{\partial y} = x + 10y$

6. $f(x, y) = 3x - \frac{1}{2}y^4 + 1$

$\frac{\partial f}{\partial x} = 3$; $\frac{\partial f}{\partial y} = -2y^3$

7. $f(x, y) = e^{x/y}$
$\frac{\partial f}{\partial x} = \frac{1}{y}e^{x/y}$; $\frac{\partial f}{\partial y} = -\frac{x}{y^2}e^{x/y}$

8. $f(x, y) = \frac{x}{x - 2y}$

$$\frac{\partial f}{\partial x} = \frac{(1)(x-2y) - (x)(1)}{(x-2y)^2} = \frac{-2y}{(x-2y)^2}$$
$$\frac{\partial f}{\partial y} = \frac{(0)(x-2y) - (-2)(x)}{(x-2y)^2} = \frac{2x}{(x-2y)^2}$$

9. $f(x, y, z) = x^3 - yz^2$

$\frac{\partial f}{\partial x} = 3x^2$; $\frac{\partial f}{\partial y} = -z^2$; $\frac{\partial f}{\partial z} = -2yz$

10. $f(x, y, \lambda) = xy + \lambda(5 - x - y)$
$= xy + 5\lambda - x\lambda - y\lambda$

$\frac{\partial f}{\partial x} = y - \lambda$; $\frac{\partial f}{\partial y} = x - \lambda$; $\frac{\partial f}{\partial \lambda} = 5 - x - y$

11. $f(x, y) = x^3 y + 8$

$\frac{\partial f}{\partial x} = 3x^2 y$; $\frac{\partial f}{\partial x}(1, 2) = 3(1)^2(2) = 6$

$\frac{\partial f}{\partial y} = x^3$; $\frac{\partial f}{\partial y}(1, 2) = (1)^3 = 1$

12. $f(x, y, z) = (x + y)z = xz + yz$

$\frac{\partial f}{\partial y} = z$

$\frac{\partial f}{\partial y}(2, 3, 4) = 4$

13. $f(x,\ y) = x^5 - 2x^3y + \dfrac{y^4}{2}$

$\dfrac{\partial f}{\partial x} = 5x^4 - 6x^2y \Rightarrow \dfrac{\partial^2 f}{\partial x^2} = 20x^3 - 12xy$

$\dfrac{\partial f}{\partial y} = -2x^3 + 2y^3 \Rightarrow \dfrac{\partial^2 f}{\partial y^2} = 6y^2$

$\dfrac{\partial^2 f}{\partial x \partial y} = -6x^2; \dfrac{\partial^2 f}{\partial y \partial x} = -6x^2$

14. $f(x,\ y) = 2x^3 + x^2y - y^2$

$\dfrac{\partial f}{\partial x} = 6x^2 + 2xy \Rightarrow \dfrac{\partial^2 f}{\partial x^2} = 12x + 2y$

$\dfrac{\partial f}{\partial y} = x^2 - 2y \Rightarrow \dfrac{\partial^2 f}{\partial y^2} = -2$

$\dfrac{\partial^2 f}{\partial x^2}(1,\ 2) = 16;\ \dfrac{\partial^2 f}{\partial y^2}(1,\ 2) = -2$

$\dfrac{\partial^2 f}{\partial x \partial y} = 2x;\ \dfrac{\partial^2 f}{\partial x \partial y}(1,\ 2) = 2$

15. $f(p, t) = -p + 6t - 0.02pt$

$\dfrac{\partial f}{\partial p} = -1 - 0.02t;\ \dfrac{\partial f}{\partial p}(25,\ 10{,}000) = -201$

$\dfrac{\partial f}{\partial t} = 6 - 0.02p; \dfrac{\partial f}{\partial t}(25,\ 10{,}000) = 5.5$

At the level $p = 25$, $t = 10{,}000$, an increase in price of \$1 will result in a loss in sales of approximately 201 calculators, and an increase in advertising of \$1 will result in the sale of approximately 5.5 additional calculators.

16. The crime rate increases with increased unemployment and decreases with increased social services and police force size.

17. $f(x,\ y) = -x^2 + 2y^2 + 6x - 8y + 5$

$\dfrac{\partial f}{\partial x} = -2x + 6;\ \dfrac{\partial f}{\partial y} = 4y - 8$

$-2x + 6 = 0 \Rightarrow x = 3;\ 4y - 8 = 0 \Rightarrow y = 2$

The only possibility is $(x, y) = (3, 2)$.

18. $f(x,\ y) = x^2 + 3xy - y^2 - x - 8y + 4$

$\dfrac{\partial f}{\partial x} = 2x + 3y - 1; \dfrac{\partial f}{\partial y} = 3x - 2y - 8$

$\left.\begin{array}{l} 2x + 3y = 1 \\ 3x - 2y = 8 \end{array}\right\} \begin{array}{l} x = 2 \\ y = -1 \end{array}$

The only possibility is $(x, y) = (2, -1)$.

19. $f(x,\ y) = x^3 + 3x^2 + 3y^2 - 6y + 7$

$\dfrac{\partial f}{\partial x} = 3x^2 + 6x; \dfrac{\partial f}{\partial y} = 6y - 6$

$3x^2 + 6x = 0 \Rightarrow 3x(x + 2) = 0 \Rightarrow x = 0,\ -2$

$6y - 6 = 0 \Rightarrow y = 1$

$(x, y) = (0, 1), (-2, 1)$

20. $f(x,\ y) = \dfrac{1}{2}x^2 + 4xy + y^3 + 8y^2 + 3x + 2$

$\dfrac{\partial f}{\partial x} = x + 4y + 3$

$\dfrac{\partial f}{\partial y} = 4x + 3y^2 + 16y$

$x + 4y + 3 = 0 \Rightarrow x = -4y - 3$

$4x + 3y^2 + 16y = 0 \Rightarrow$

$4(-4y - 3) + 3y^2 + 16y = 0 \Rightarrow$

$-16y - 12 + 3y^2 + 16y = 0 \Rightarrow y = \pm 2$

$x = -4(-2) - 3 = 5;\ x = -4(2) - 3 = -11;$

$(x, y) = (-11, 2), (5, -2)$

21. $f(x,\ y) = x^2 + 3xy + 4y^2 - 13x - 30y + 12$

$\dfrac{\partial f}{\partial x} = 2x + 3y - 13 \Rightarrow \dfrac{\partial^2 f}{\partial x^2} = 2$

$\dfrac{\partial f}{\partial y} = 3x + 8y - 30 \Rightarrow \dfrac{\partial^2 f}{\partial y^2} = 8;\ \dfrac{\partial^2 f}{\partial x \partial y} = 3$

$\left.\begin{array}{l} 2x + 3y = 13 \\ 3x + 8y = 30 \end{array}\right\} \begin{array}{l} x = 2 \\ y = 3 \end{array}$

$D(x, y) = \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 \Rightarrow$

$D(2, 3) = 2 \cdot 8 - 3^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$,

so $f(x, y)$ has a relative minimum at $(2, 3)$.

22. $f(x, y) = 7x^2 - 5xy + y^2 + x - y + 6$

$$\frac{\partial f}{\partial x} = 14x - 5y + 1; \frac{\partial^2 f}{\partial x^2} = 14; \frac{\partial f}{\partial y} = -5x + 2y - 1; \frac{\partial^2 f}{\partial y^2} = 2; \frac{\partial^2 f}{\partial x \partial y} = -5$$

$$\left.\begin{array}{l} 14x - 5y = -1 \\ -5x + 2y = 1 \end{array}\right\} \begin{array}{l} x = 1 \\ y = 3 \end{array}$$

$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \Rightarrow D(1, 3) = 14 \cdot 2 - (-5)^2 > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0,$ so $f(x, y)$ has a relative minimum at (1, 3).

23. $f(x, y) = x^3 + y^2 - 3x - 8y + 12$

$$\frac{\partial f}{\partial x} = 3x^2 - 3; \quad \frac{\partial^2 f}{\partial x^2} = 6x; \quad \frac{\partial f}{\partial y} = 2y - 8; \quad \frac{\partial^2 f}{\partial y^2} = 2; \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$3x^2 = 3 \Rightarrow x = \pm 1$
$2y = 8 \Rightarrow y = 4$

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \Rightarrow$$

$D(1, 4) = 6(1)(2) - 0^2 > 0$ and $\frac{\partial^2 f}{\partial x^2}(1, 4) > 0,$ so $f(x, y)$ has a relative minimum at (1, 4).

$D(-1, 4) = 6(-1)(2) - 0^2 < 0$, so $f(x, y)$ has neither a maximum nor a minimum at (–1, 4).

24. $f(x, y, z) = x^2 + 4y^2 + 5z^2 - 6x + 8y + 3$

$\frac{\partial f}{\partial x} = 2x - 6 = 0 \Rightarrow x = 3$

$\frac{\partial f}{\partial y} = 8y + 8 = 0 \Rightarrow y = -1$

$\frac{\partial f}{\partial z} = 10z = 0 \Rightarrow z = 0$

$f(x, y, z)$ must assume its minimum value at (3, –1, 0).

25. $F(x, y, \lambda) = 3x^2 + 2xy - y^2 + \lambda(5 - 2x - y)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 6x + 2y - 2\lambda = 0 \\ \frac{\partial F}{\partial y} = 2x - 2y - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 5 - 2x - y = 0 \end{array}\right\} \left.\begin{array}{l} \lambda = 3x + y \\ \lambda = 2x - 2y \\ 2x + y = 5 \end{array}\right\} \left.\begin{array}{l} x + 3y = 0 \\ 2x + y = 5 \end{array}\right\} \begin{array}{l} x = 3 \\ y = -1 \end{array}$$

The maximum value of $f(x, y)$ subject to the constraint is $3(3)^2 + 2(3)(-1) - (-1)^2 = 20$ occurs at $(x, y) = (3, -1)$.

26. $F(x, y, \lambda) = -x^2 - 3xy - \frac{1}{2}y^2 + y + 10 + \lambda(10 - x - y)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= -2x - 3y - \lambda = 0 \\ \frac{\partial F}{\partial y} &= -3x - y + 1 - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 10 - x - y = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= -2x - 3y \\ \lambda &= -3x - y + 1 \\ x + y &= 10 \end{aligned}\right\} \left.\begin{aligned} x - 2y &= 1 \\ x + y &= 10 \end{aligned}\right\} \begin{aligned} x &= 7 \\ y &= 3 \end{aligned}$$

27. $F(x, y, z, \lambda) = 3x^2 + 2y^2 + z^2 + 4x + y + 3z + \lambda(4 - x - y - z)$

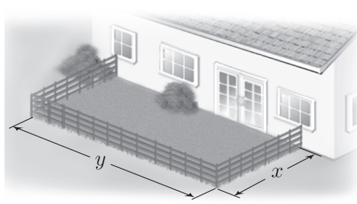
$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 6x + 4 - \lambda = 0 \\ \frac{\partial F}{\partial y} &= 4y + 1 - \lambda = 0 \\ \frac{\partial F}{\partial z} &= 2z + 3 - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 4 - x - y - z = 0 \end{aligned}\right\} \left.\begin{aligned} 6x + 4 &= 4y + 1 \\ 4y + 1 &= 2z + 3 \\ x + y + z &= 4 \end{aligned}\right\} \left.\begin{aligned} x &= \frac{2}{3}y - \frac{1}{2} \\ z &= 2y - 1 \\ \frac{2}{3}y - \frac{3}{2} &+ y + 2y = 4 \end{aligned}\right\} \begin{aligned} x &= \frac{1}{2} \\ y &= \frac{3}{2} \\ z &= 2 \end{aligned}$$

28. The problem is to minimize $x + y + z$ subject to $xyz = 1000$.
$(x > 0, y > 0, z > 0)$
$F(x, y, z, \lambda) = x + y + z + \lambda(1000 - xyz)$
(Assuming $x \neq 0, y \neq 0, z \neq 0$)

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 1 - \lambda yz \\ \frac{\partial F}{\partial y} &= 1 - \lambda xz \\ \frac{\partial F}{\partial z} &= 1 - \lambda xy \\ \frac{\partial F}{\partial \lambda} &= 1000 - xyz \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{1}{yz} \\ \lambda &= \frac{1}{xz} \\ \lambda &= \frac{1}{xy} \\ xyz &= 1000 \end{aligned}\right\} \left.\begin{aligned} yz &= xz \\ xz &= xy \\ xyz &= 1000 \end{aligned}\right\} \left.\begin{aligned} x = y &= z \\ xyz &= 1000 \end{aligned}\right\} \begin{aligned} x &= 10 \\ y &= 10 \\ z &= 10 \end{aligned}$$

The optimal dimensions are $10 \text{ in.} \times 10 \text{ in.} \times 10 \text{ in.}$

29.



The problem is to maximize xy subject to $2x + y = 40$.
$F(x, y, \lambda) = xy + \lambda(40 - 2x - y)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y - 2\lambda = 0 \\ \frac{\partial F}{\partial y} &= x - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 40 - 2x - y = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{1}{2}y \\ \lambda &= x \\ 2x + y &= 40 \end{aligned}\right\} \left.\begin{aligned} x &= \frac{1}{2}y \\ 2y &= 40 \end{aligned}\right\} \begin{aligned} x &= 10 \text{ ft} \\ y &= 20 \text{ ft} \end{aligned}$$

The dimensions of the garden should be 10 ft × 20 ft.

30. Maximize xy subject to $2x + y = 41$.
$F(x, y, \lambda) = xy + \lambda(41 - 2x - y)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= y - 2\lambda = 0 \\ \frac{\partial F}{\partial y} &= x - \lambda = 0 \\ \frac{\partial F}{\partial \lambda} &= 41 - 2x - y = 0 \end{aligned}\right\} \left.\begin{aligned} \lambda &= \frac{1}{2}y \\ \lambda &= x \\ 2x + y &= 41 \end{aligned}\right\} \left.\begin{aligned} x &= \frac{1}{2}y \\ 2y &= 41 \end{aligned}\right\} \left.\begin{aligned} x &= 10.25 \text{ ft} \\ y &= 20.5 \text{ ft} \end{aligned}\right\}$$

The new area is $xy = (10.25)(20.5) = 210.125$ sq ft.
The increase in area (compared with the area in Exercise 29) is $210.125 - (10)(20) = 10.125$, which is approximately the value of λ.

31. Let the straight line be $y = Ax + B$.
$E_1^2 = (A + B - 1)^2$; $E_2^2 = (2A + B - 3)^2$; $E_3^2 = (3A + B - 6)^2$
Let $f(A, B) = E_1^2 + E_2^2 + E_3^2 = (A + B - 1)^2 + (2A + B - 3)^2 + (3A + B - 6)^2$.

$$\frac{\partial f}{\partial A} = 2(A + B - 1) + 2(2A + B - 3)(2) + 2(3A + B - 6)(3) = 28A + 12B - 50$$

$$\frac{\partial f}{\partial B} = 2(A + B - 1) + 2(2A + B - 3) + 2(3A + B - 6) = 12A + 6B - 20$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero we obtain the system $\begin{cases} 28A + 12B = 50 \\ 12A + 6B = 20 \end{cases}$.

Then $A = \frac{5}{2}$ and $B = -\frac{5}{3}$ so the line with the best least-squares fit to the data points is $y = \frac{5}{2}x - \frac{5}{3}$.

32. Let the straight line be $y = Ax + B$.
$E_1^2 = (A + B - 1)^2$; $E_2^2 = (3A + B - 4)^2$; $E_3^2 = (5A + B - 7)^2$
Let $f(A, B) = E_1^2 + E_2^2 + E_3^2 = (A + B - 1)^2 + (3A + B - 4)^2 + (5A + B - 7)^2$.

$$\frac{\partial f}{\partial A} = 2(A + B - 1) + 2(3A + B - 4)(3) + 2(5A + B - 7)(5) = 70A + 18B - 96$$

$$\frac{\partial f}{\partial B} = 2(A + B - 1) + 2(3A + B - 4) + 2(5A + B - 7) = 18A + 6B - 24$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero we obtain the system $\begin{cases} 70A + 18B = 96 \\ 18A + 6B = 24 \end{cases}$.

Then $A = \frac{3}{2}$ and $B = -\frac{1}{2}$ so the line with the best least-squares fit to the data points is $y = \frac{3}{2}x - \frac{1}{2}$.

33. Let the straight line be $y = Ax + B$.
$E_1^2 = (0A + B - 1)^2$; $E_2^2 = (A + B + 1)^2$; $E_3^2 = (2A + B + 3)^2$; $E_4^2 = (3A + B + 5)^2$
Let $f(A, B) = E_1^2 + E_2^2 + E_3^2 + E_4^2 = (0A + B - 1)^2 + (A + B + 1)^2 + (2A + B + 3)^2 + (3A + B + 5)^2$.

$$\frac{\partial f}{\partial A} = 2(A + B + 1) + 2(2A + B + 3)(2) + 2(3A + B + 5)(3) = 28A + 12B + 44$$

$$\frac{\partial f}{\partial B} = 2(B - 1) + 2(A + B + 1) + 2(2A + B + 3) + 2(3A + B + 5) = 12A + 8B + 16$$

Setting $\frac{\partial f}{\partial A}$ and $\frac{\partial f}{\partial B}$ equal to zero we obtain the system: $\begin{cases} 28A + 12B = -44 \\ 12A + 8B = -16 \end{cases}$.

Then $A = -2$ and $B = 1$ so the line with the best least-squares fit to the data points is $y = -2x + 1$.

34. $\int_0^1 \left(\int_0^4 (x\sqrt{y} + y)\ dy \right) dx = \int_0^1 \left(\frac{2}{3}xy^{3/2} + \frac{1}{2}y^2 \Big|_{y=0}^{4} \right) dx = \int_0^1 \left(\frac{16}{3}x + 8 \right) dx = \frac{8}{3}x^2 + 8x \Big|_0^1 = \frac{8}{3} + 8 = \frac{32}{3}$

35. $\int_0^5 \left(\int_1^4 (2xy^4 + 3)\ dy \right) dx = \int_0^5 \left(\frac{2}{5}xy^5 + 3y \Big|_{y=1}^{4} \right) dx = \int_0^5 \left(\frac{2046}{5}x + 9 \right) dx = \frac{1023}{5}x^2 + 9x \Big|_0^5 = 5115 + 45 = 5160$

36. $\int_1^3 \left(\int_0^4 (2x + 3y) dx \right) dy = \int_1^3 \left(x^2 + 3xy \Big|_{x=0}^{4} \right) dy = \int_1^3 (16 + 12y)\ dy = 16y + 6y^2 \Big|_1^3 = 80$

37. $\iint\limits_R 5\ dx\ dy$ represents the volume of the box with dimensions $(4-0)\times(3-1)\times 5$.

So $\iint\limits_R 5\ dx\ dy = 4 \cdot 2 \cdot 5 = 40.$

38.