Chapter 3 Techniques of Differentiation

3.1 The Product and Quotient Rules

1. $\frac{d}{dx}\left[(x+1)\left(x^3+5x+2\right)\right]=(x+1)\left(3x^2+5\right)+\left(x^3+5x+2\right)=4x^3+3x^2+10x+7$

2. $\frac{d}{dx}\left[\left(-x^3+2\right)\left(\frac{x}{2}-1\right)\right]=\left(-x^3+2\right)\left(\frac{1}{2}\right)+\left(\frac{x}{2}-1\right)\left(-3x^2\right)=-2x^3+3x^2+1$

3. $\frac{d}{dx}\left[\left(2x^4-x+1\right)\left(-x^5+1\right)\right]=\left(2x^4-x+1\right)\left(-5x^4\right)+\left(8x^3-1\right)\left(-x^5+1\right)=-18x^8+6x^5-5x^4+8x^3-1$

4. $$\begin{aligned}\frac{d}{dx}\left[\left(x^2+x+1\right)^3(x-1)^4\right]&=3\left(x^2+x+1\right)^2(2x+1)(x-1)^4+4(x-1)^3\left(x^2+x+1\right)^3\\&=\left(x^2+x+1\right)^2(x-1)^3\left(10x^2+x+1\right)\end{aligned}$$

5. $\frac{d}{dx}\left[x\left(x^2+1\right)^4\right]=x(4)\left(x^2+1\right)^3(2x)+(1)\left(x^2+1\right)^4=8x^2\left(x^2+1\right)^3+\left(x^2+1\right)^4=\left(x^2+1\right)^3\left(9x^2+1\right)$

6. $\frac{d}{dx}\left[x\sqrt{x}\right]=x\left(\frac{1}{2\sqrt{x}}\right)+\sqrt{x}(1)=\frac{3\sqrt{x}}{2}$

7. $\frac{d}{dx}\left[\left(x^2+3\right)\left(x^2-3\right)^{10}\right]=\left(x^2+3\right)(10)\left(x^2-3\right)^9(2x)+\left(x^2-3\right)^{10}(2x)=2x\left(x^2-3\right)^9\left(11x^2+27\right)$

8. $$\begin{aligned}\frac{d}{dx}\left[\left(-2x^3+x\right)(6x-3)\right]^4&=4\left[\left(-2x^3+x\right)(6x-3)\right]^3\frac{d}{dx}\left[\left(-2x^3+x\right)(6x-3)\right]\\&=4\left[\left(-2x^3+x\right)(6x-3)\right]^3\left[\left(-2x^3+x\right)(6)+(6x-3)\left(-6x^2+1\right)\right]\\&=4\left[\left(-2x^3+x\right)(6x-3)\right]^3\left[-12x^3+6x-36x^3+18x^2+6x-3\right]\\&=4\left[\left(-2x^3+x\right)(6x-3)\right]^3\left[-48x^3+18x^2+12x-3\right]\end{aligned}$$

9. $\frac{d}{dx}\left[(5x+1)\left(x^2-1\right)+\frac{2x+1}{3}\right]=(5x+1)(2x)+5\left(x^2-1\right)+\frac{1}{3}(2)=15x^2+2x-\frac{13}{3}$

10. $$\begin{aligned}\frac{d}{dx}\left[x^7\left(3x^4+12x-1\right)^2\right]&=x^7(2)\left(3x^4+12x-1\right)\left(12x^3+12\right)+7x^6\left(3x^4+12x-1\right)^2\\&=\left(3x^4+12x-1\right)\left(45x^{10}+108x^7-7x^6\right)\end{aligned}$$

11. $\frac{d}{dx}\left[\frac{x-1}{x+1}\right]=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}=\frac{2}{(x+1)^2}$

12. $\frac{d}{dx}\left[\frac{1}{x^2+x+7}\right]=\frac{\left(x^2+x+7\right)(0)-(1)(2x+1)}{\left(x^2+x+7\right)^2}=\frac{-2x-1}{\left(x^2+x+7\right)^2}$

13. $\frac{d}{dx}\left[\frac{x^2-1}{x^2+1}\right]=\frac{\left(x^2+1\right)(2x)-\left(x^2-1\right)(2x)}{\left(x^2+1\right)^2}=\frac{4x}{\left(x^2+1\right)^2}$

14. $$\frac{d}{dx}\left[\frac{x}{x+\frac{1}{x}}\right]=\frac{d}{dx}\left[\frac{x^2}{x^2+1}\right]=\frac{\left(x^2+1\right)(2x)-x^2(2x)}{\left(x^2+1\right)^2}=\frac{2x}{\left(x^2+1\right)^2}$$

15. $$\frac{d}{dx}\left[\frac{x+3}{(2x+1)^2}\right]=\frac{(2x+1)^2(1)-(x+3)(2)(2x+1)(2)}{\left[(2x+1)^2\right]^2}=-\frac{4x^2+24x+11}{(2x+1)^4}=-\frac{(2x+1)(2x+11)}{(2x+1)^4}=-\frac{2x+11}{(2x+1)^3}$$

16. $$\frac{d}{dx}\left[x^4+4\sqrt[4]{x}\right]=4x^3+\frac{1}{4}\cdot 4x^{-3/4}=4x^3+x^{-3/4}=4x^3+\frac{1}{\sqrt[4]{x^3}}$$

17. $$\frac{d}{dx}\left[\frac{1}{\pi}+\frac{2}{x^2+1}\right]=0+\frac{\left(x^2+1\right)(0)-2(2x)}{\left(x^2+1\right)^2}=\frac{-4x}{\left(x^2+1\right)^2}$$

18. $$\frac{d}{dx}\left[\frac{ax+b}{cx+d}\right]=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}=\frac{ad-bc}{(cx+d)^2}$$

19. $$\frac{d}{dx}\left[\frac{x^2}{\left(x^2+1\right)^2}\right]=\frac{\left(x^2+1\right)^2(2x)-x^2(2)\left(x^2+1\right)(2x)}{\left[\left(x^2+1\right)^2\right]^2}=\frac{2x^5+4x^3+2x-4x^5-4x^3}{\left(x^2+1\right)^4}=\frac{2x-2x^5}{\left(x^2+1\right)^4}$$

Alternatively, we can factor $2x\left(x^2+1\right)$ in the numerator to obtain

$$\frac{\left(x^2+1\right)^2(2x)-x^2(2)\left(x^2+1\right)(2x)}{\left[\left(x^2+1\right)^2\right]^2}=\frac{2x\left(x^2+1\right)\left[\left(x^2+1\right)-2x^2\right]}{\left(x^2+1\right)^4}=\frac{2x\left(1-x^2\right)}{\left(x^2+1\right)^3}=\frac{2x-2x^3}{\left(x^2+1\right)^3}$$

20. $$\frac{d}{dx}\left[\frac{(x+1)^3}{(x-5)^2}\right]=\frac{(x-5)^2(3)(x+1)^2(1)-(x+1)^3(2)(x-5)(1)}{\left[(x-5)^2\right]^2}=\frac{3(x-5)^2(x+1)^2-2(x+1)^3(x-5)}{(x-5)^4}$$
$$=\frac{3(x-5)(x+1)^2-2(x+1)^3}{(x-5)^4}=\frac{(x+1)^2\left[3(x-5)-2(x+1)\right]}{(x-5)^3}=\frac{(x+1)^2(x-17)}{(x-5)^3}$$

21. $$\frac{d}{dx}\left(\left[\left(3x^2+2x+2\right)(x-2)\right]^2\right)=2\left[\left(3x^2+2x+2\right)(x-2)\right]\left[\left(3x^2+2x+2\right)(1)+(6x+2)(x-2)\right]$$
$$=2(x-2)\left(3x^2+2x+2\right)\left(9x^2-8x-2\right)$$

22. $$\frac{d}{dx}\left[\frac{1}{\sqrt{x}-2}\right]=\frac{d}{dx}\left[\left(\sqrt{x}-2\right)^{-1}\right]=-\left(\sqrt{x}-2\right)^{-2}\left(\frac{1}{2}x^{-1/2}\right)=-\frac{1}{2\sqrt{x}\left(\sqrt{x}-2\right)^2}$$

23. $$\frac{d}{dx}\left[\frac{1}{\sqrt{x}+1}\right]=\frac{d}{dx}\left(x^{1/2}+1\right)^{-1}=-\left(\sqrt{x}+1\right)^{-2}\left(\frac{1}{2}x^{-1/2}\right)=-\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$$

24. $$\frac{d}{dx}\left[\frac{3}{\sqrt[3]{x}+1}\right]=\frac{d}{dx}3\left(x^{1/3}+1\right)^{-1}=-3\left(\sqrt[3]{x}+1\right)^{-2}\left(\frac{1}{3}x^{-2/3}\right)=-\frac{1}{\sqrt[3]{x^2}\left(\sqrt[3]{x}+1\right)^2}$$

25. $\frac{d}{dx}\left(\frac{x+11}{x-3}\right)^3 = 3\left(\frac{x+11}{x-3}\right)^2 \frac{d}{dx}\left(\frac{x+11}{x-3}\right) = 3\left(\frac{x+11}{x-3}\right)^2\left(\frac{(x-3)(1)-(x+11)(1)}{(x-3)^2}\right)$

$= 3\left(\frac{x+11}{x-3}\right)^2\left(\frac{-14}{(x-3)^2}\right) = -42\left(\frac{(x+11)^2}{(x-3)^4}\right)$

26. $\frac{d}{dx}\sqrt{\frac{x+3}{x^2+1}} = \frac{d}{dx}\left(\frac{x+3}{x^2+1}\right)^{1/2} = \frac{1}{2}\left(\frac{x+3}{x^2+1}\right)^{-1/2}\frac{d}{dx}\left(\frac{x+3}{x^2+1}\right) = \frac{1}{2}\sqrt{\frac{x^2+1}{x+3}}\cdot\frac{(x^2+1)(1)-(x+3)\cdot 2x}{(x^2+1)^2}$

$= \frac{\sqrt{x^2+1}}{2\sqrt{x+3}}\cdot\frac{-x^2-6x+1}{(x^2+1)^2} = \frac{-x^2-6x+1}{2\sqrt{x+3}\,(x^2+1)^{3/2}}$

27. $\frac{d}{dx}\left[\sqrt{x+2}(2x+1)^2\right] = \frac{d}{dx}\left[(x+2)^{1/2}(2x+1)^2\right] = (x+2)^{1/2}(2)(2x+1)(2) + \frac{1}{2}(x+2)^{-1/2}(2x+1)^2$

$= \sqrt{x+2}\,(8x+4) + \frac{4x^2+4x+1}{2\sqrt{x+2}} = \frac{20x^2+44x+17}{2\sqrt{x+2}} = \frac{(2x+1)(10x+17)}{2\sqrt{x+2}}$

28. $\frac{d}{dx}\left[\frac{\sqrt{3x-1}}{x}\right] = \frac{d}{dx}\left[\frac{(3x-1)^{1/2}}{x}\right] = \frac{x\left(\frac{1}{2}(3x-1)^{-1/2}\right)(3)-(3x-1)^{1/2}(1)}{x^2} = \frac{2-3x}{2x^2\sqrt{3x-1}}$

29. $y = (x-2)^5(x+1)^2$

$\frac{dy}{dx} = (x-2)^5(2)(x+1) + (x+1)^2(5)(x-2)^4 = (x-2)^4(x+1)(7x+1)$

$\left.\frac{dy}{dx}\right|_{x=3} = 88$

Equation of tangent line: $y - 16 = 88(x-3)$ or $y = 88x - 248$

30. $y = \frac{x+1}{x-1}$

$\frac{dy}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$\left.\frac{dy}{dx}\right|_{x=2} = -2$

Equation of tangent line: $y - 3 = -2(x-2)$ or $y = -2x + 7$

31. $y = \frac{(x-2)^5}{(x-4)^3}$

$\frac{dy}{dx} = \frac{(x-4)^3(5)(x-2)^4-(x-2)^5(3)(x-4)^2}{(x-4)^6} = \frac{(x-2)^4(2x-14)}{(x-4)^4}$

The tangent line is horizontal when $\frac{dy}{dx} = 0.$

$\frac{(x-2)^4(2x-14)}{(x-4)^4} = 0 \Rightarrow (x-2)^4(2x-14) = 0 \Rightarrow x = 2$ or $x = 7$

32. $y = \dfrac{1}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)(0)-(1)(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2(-2)-(-2x)(2)(x^2+1)(2x)}{\left[(x^2+1)^2\right]^2} = \frac{-2(x^2+1)^2+8x^2(x^2+1)}{\left[(x^2+1)^2\right]^2}$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } -2(x^2+1)^2+8x^2(x^2+1)=0 \Rightarrow 2(x^2+1)^2 = 8x^2(x^2+1) \Rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

Inflection points are at $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$ and $\left(\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$.

33. $y = (x^2-4)^3(2x^2+5)^5$

$$\frac{dy}{dx} = (x^2-4)^3(5)(2x^2+5)^4(4x)+(2x^2+5)^5(3)(x^2-4)^2(2x) = (2x)(x^2-4)^2(2x^2+5)^4(16x^2-25)$$

$$\frac{dy}{dx} = 0 \text{ for } x = 0, \pm 2, \pm\frac{5}{4}$$

34. $y = (x^2-1)^4(x^2+1)^5$

$$\frac{dy}{dx} = 2x(x^2-1)^3(x^2+1)^4(9x^2-1)$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \pm 1, \pm\frac{1}{3}$$

Local maximums occur at $\left(-\dfrac{1}{3}, 1.0572\right)$ and $\left(\dfrac{1}{3}, 1.0572\right)$.

Local minimums occur at $(-1, 0)$, $(0, 1)$,and $(1, 0)$.

35. $y = \dfrac{x^2+3x-1}{x}$

$$\frac{dy}{dx} = \frac{x(2x+3)-(x^2+3x-1)(1)}{x^2} = \frac{x^2+1}{x^2}$$

$$\frac{dy}{dx} = 5 \Rightarrow \frac{x^2+1}{x^2} = 5 \Rightarrow 4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}$$

Points on the graph are $\left(\dfrac{1}{2}, \dfrac{3}{2}\right), \left(-\dfrac{1}{2}, \dfrac{9}{2}\right)$.

36. $y = (2x^4+1)(x-5)$

$$\frac{dy}{dx} = (2x^4+1)(1)+(x-5)(8x^3) = 10x^4-40x^3+1$$

$$\frac{dy}{dx} = 1 \text{ when } x^3(10x-40) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

The points on the graph are (0, –5) and (4, –513).

37. $y = \left(x^2+1\right)^4$

$$\frac{dy}{dx} = 4\left(x^2+1\right)^3(2x) = 8x\left(x^2+1\right)^3$$

$$\frac{d^2y}{dx^2} = 8x(3)\left(x^2+1\right)^2(2x) + \left(x^2+1\right)^3(8) = 8\left(x^2+1\right)^2\left(7x^2+1\right)$$

38. $y = \sqrt{x^2+1}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{x^2+1}(1) - x\cdot\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)}{\left(\sqrt{x^2+1}\right)^2} = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2} = \frac{1}{\left(x^2+1\right)^{3/2}}$$

39. $y = x\sqrt{x+1}$

$$\frac{dy}{dx} = x\cdot\frac{1}{2\sqrt{x+1}} + \sqrt{x+1}(1) = \frac{x}{2\sqrt{x+1}} + \sqrt{x+1} = \frac{3x+2}{2\sqrt{x+1}}$$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{x+1}(3) - (3x+2)\left[(2)\left(\frac{1}{2\sqrt{x+1}}\right) + \sqrt{x+1}(0)\right]}{\left(2\sqrt{x+1}\right)^2} = \frac{6\sqrt{x+1} - \frac{3x+2}{\sqrt{x+1}}}{\left(2\sqrt{x+1}\right)^2} = \frac{6(x+1)-(3x+2)}{4(x+1)^{3/2}} = \frac{3x+4}{4(x+1)^{3/2}}$$

40. $y = \dfrac{2}{2+x^2}$

$$\frac{dy}{dx} = \frac{\left(2+x^2\right)(0) - 2(2x)}{\left(2+x^2\right)^2} = -\frac{4x}{\left(2+x^2\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(2+x^2\right)^2(-4) - (-4x)(2)\left(2+x^2\right)(2x)}{\left(2+x^2\right)^4} = \frac{-4\left(2+x^2\right)\left(2-3x^2\right)}{\left(2+x^2\right)^4} = \frac{12x^2-8}{\left(2+x^2\right)^3}$$

41. $h(x) = xf(x)$

$$h'(x) = \frac{d}{dx}[xf(x)] = xf'(x) + (1)f(x) = xf'(x) + f(x)$$

42. $h(x) = \left(x^2+2x-1\right)f(x)$

$$h'(x) = \frac{d}{dx}\left[\left(x^2+2x-1\right)f(x)\right] = \left(x^2+2x-1\right)f'(x) + f(x)(2x+2)$$

43. $h(x) = \dfrac{f(x)}{x^2+1}$

$$h'(x) = \frac{d}{dx}\left[\frac{f(x)}{x^2+1}\right] = \frac{\left(x^2+1\right)f'(x) - f(x)(2x)}{\left(x^2+1\right)^2}$$

44. $h(x) = \left(\dfrac{f(x)}{x}\right)^2 = \dfrac{(f(x))^2}{x^2}$

$$h'(x) = \frac{d}{dx}\left[\frac{(f(x))^2}{x^2}\right] = \frac{x^2(2)f(x)f'(x) - (f(x))^2(2x)}{x^4} = \frac{2f(x)\left(xf'(x) - f(x)\right)}{x^3}$$

45. Let x be the width of the box and let h be its height. Constraint: $16 = 2xh + 2(3h) + 3x$

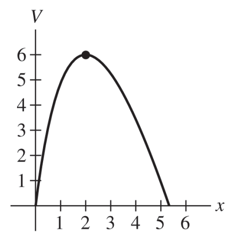
$\left(h = \dfrac{16-3x}{2x+6}\right)$

Objective: $V = 3xh = 3x\left(\dfrac{16-3x}{2x+6}\right) = \dfrac{48x-9x^2}{2x+6}$

$$\frac{dV}{dx} = \frac{(2x+6)(48-18x)-(48x-9x^2)(2)}{(2x+6)^2} = \frac{-18(x+8)(x-2)}{(2x+6)^2} \text{ or } \frac{-18(x^2+6x-16)}{(2x+6)^2}$$

$$\frac{d^2V}{dx^2} = -18\left[\frac{(2x+6)^2(2x+6)-(x^2+6x-16)(2)(2x+6)(2)}{(2x+6)^4}\right] = -\frac{225}{(x+3)^3}$$

The maximum value of V occurs at $x = 2$.
Answer: $x = 2$, $h = 1$, i.e. optimal dimensions are $2\text{ ft} \times 3\text{ ft} \times 1\text{ ft}$



46. Let x be the length of the other side of the box and let h be its height.
Constraint: $240 = (1)x(20) + (1)x(10) + 2xh(10) + (2)(1)h(10) = 30x + 20xh + 20h$

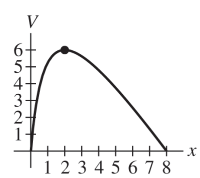
$\left(h = \dfrac{24-3x}{2x+2}\right)$

Objective: $V = xh = x\left(\dfrac{24-3x}{2x+2}\right) = \dfrac{24x-3x^2}{2x+2}$

$$\frac{dV}{dx} = \frac{(2x+2)(24-6x)-(24x-3x^2)(2)}{(2x+2)^2} = \frac{-6(x+4)(x-2)}{(2x+2)^2} \text{ or } \frac{-6(x^2+2x-8)}{(2x+2)^2}$$

$$\frac{d^2V}{dx^2} = -6\left[\frac{(2x+2)^2(2x+2)-(x^2+2x-8)(2)(2x+2)(2)}{(2x+2)^4}\right] = -\frac{27}{(x+1)^3}$$

The maximum value for V occurs at $x = 2$.
Answer: $x = 2$, $h = 3$, i.e. optimal dimensions are $1\text{ m} \times 2\text{ m} \times 3\text{ m}$.



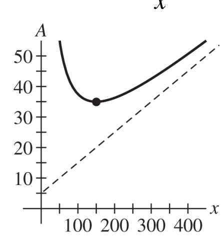
47. Let $AC(x)$ be the average cost of producing x units and let $C(x)$ be the total cost.

Then $AC(x) = \dfrac{C(x)}{x} = \dfrac{.1x^2+5x+2250}{x}$.

$$AC'(x) = \frac{x(.2x+5)-(.1x^2+5x+2250)(1)}{x^2} = \frac{.1x^2-2250}{x^2} = \frac{(.1)(x+150)(x-150)}{x^2}$$

$$AC''(x) = \frac{x^2(.2x)-(.1x^2-2250)(2x)}{x^4} = \frac{.2x^3-.2x^3+4500x}{x^4} = \frac{4500}{x^3}$$

The minimum value of $A(x)$ occurs at $x = 150$.
The marginal cost function is $C'(x) = .2x + 5$.
At the level of 150 units, we have
$AC(150) = 35 = .2(150) + 5 = C'(150)$.



48. The cost function is

$$C(x) = 50x\frac{(x+200)}{(x+100)} = \frac{50x^2+10,000x}{(x+100)}.$$

The average cost is $A(x) = \dfrac{C(x)}{x} = \dfrac{50x^2+10,000x}{x(x+100)} = \dfrac{50x+10,000}{x+100}$

$$C'(x) = \frac{(x+100)(100x+10,000)-(50x^2+10,000x)(1)}{(x+100)^2} = \frac{50x^2+10,000x+1,000,000}{(x+100)^2} > 0 \text{ for all } x > 0.$$

Thus, cost increases as output increases.

$$A'(x) = \frac{(x+100)(50)-(50x+10,000)(1)}{(x+100)^2} = \frac{-5000}{(x+100)^2} < 0 \text{ for all } x.$$

So, the average cost decreases as output increases.

49. Recall that the marginal revenue, MR, is defined by $MR = R'(x)$. The average revenue is maximized when

$$\frac{d}{dx}(AR) = 0.$$

$$\frac{d}{dx}(AR) = \frac{d}{dx}\left(\frac{R(x)}{x}\right) = \frac{xR'(x)-R(x)(1)}{x^2}$$

If $\dfrac{xR'(x)-R(x)}{x^2} = 0$, then $xR'(x)-R(x) = 0 \Rightarrow xR'(x) = R(x) \Rightarrow R'(x) = \dfrac{R(x)}{x} \Rightarrow MR = AR.$

50. If the average velocity, $\overline{v}(t) = \dfrac{s(t)}{t}$, is maximized at time t_0, then $\overline{v}'(t_0) = 0$. Using the quotient rule,

$$\overline{v}'(t) = \frac{d}{dt}\left[\frac{s(t)}{t}\right] = \frac{ts'(t)-s(t)(1)}{t^2}$$

Thus if $\overline{v}'(t_0) = 0$, we must have $t_0 s'(t_0) = s(t_0)$ or $s'(t_0) = \dfrac{s(t_0)}{t_0} = \overline{v}(t_0)$.

51. $A = W(t)L(t)$:

$$\frac{dA}{dt} = W(t)L'(t) + W'(t)L(t) = 5\cdot 4 + 3\cdot 6 = 38$$

The area of the rectangle is increasing by 38 square inches per second.

52. $f(x) = \dfrac{3x}{105-x}$, $0 \le x \le 100$

$$f'(x) = \frac{3(105-x)-3x(-1)}{(105-x)^2} = \frac{315}{(105-x)^2}$$

$$f'(x) = 1.4 \Rightarrow \frac{315}{(105-x)^2} = 1.4 \Rightarrow \frac{315}{1.4} = (105-x)^2 \Rightarrow 225 = (105-x)^2 \Rightarrow \pm 15 = 105 - x \Rightarrow x = 90 \text{ or } x = 120$$

Since $0 \le x \le 100$, $x = 90$ when the rate of increase of the cost-benefit function is \$1.4 million per unit.

53. Let x be the number of years since the beginning of 1998, $P(x)$ the population, $c(x)$ the annual per capita consumption in gallons, and $t(x)$ the total annual consumption in gallons.

$P(x) = 1,856,000x + 268,924,000$

$c(x) = .2x + 52.3$

$t(x) = P(x)c(x)$

$t'(x) = P(x)c'(x) + P'(x)c(x) = (1,856,000x + 268,924,000)(.2) + (1,856,000)(.2x + 52.3)$

$t'(0) = (268,924,000)(.2) + (1,856,000)(52.3) = 150,853,600$

Total annual consumption was increasing at the beginning of 1998 at a rate of 150,853,600 gallons per year.

54. Let x be the number of years since the beginning of 1998, $P(x)$ the population, $a(x)$ the annual consumption in millions of pints, and $c(x)$ the annual per capita consumption in millions of pints.
$P(x) = 1.856x + 268.924$
$a(x) = 212x + 12.582$

$$c(x) = \frac{a(x)}{P(x)}$$

$$c'(x) = \frac{P(x)a'(x) - a(x)P'(x)}{P(x)^2} = \frac{(1.856x + 268.924)(212) - (212x + 12.582)(1.856)}{(1.856x + 268.924)^2}$$

$$c'(0) = \frac{(268.924)(212) - (12.582)(1.856)}{268.924^2} \approx .788$$

The annual per capita consumption of ice cream at the beginning of 1998 was increasing at a rate of approximately 0.788 pint per capita per year.

55. $$\frac{dy}{dx} = \frac{\left(1 + .25x^2\right)10 - 10x(.5x)}{\left(1 + .25x^2\right)^2} = \frac{10 - 2.5x^2}{\left(1 + .25x^2\right)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{10 - 2.5x^2}{\left(1 + .25x^2\right)^2} = 0 \Rightarrow 10 = 2.5x^2 \Rightarrow x = \pm 2$$

When $x = 2$, $y = 10$, so the maximum point is at (2, 10).

56. $$\frac{dy}{dx} = \frac{\left(x^2 - 2x + 2\right)(2x - 2) - \left(x^2 - 2x + 1\right)(2x - 2)}{\left(x^2 - 2x + 2\right)^2} = \frac{2x - 2}{\left(x^2 - 2x + 2\right)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{2x - 2}{\left(x^2 - 2x + 2\right)^2} = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$$

When $x = 1$, $y = \frac{1}{2}$, so the minimum point is at $\left(1, \frac{1}{2}\right)$.

57. $$\frac{d}{dx}\left[f(x)g(x)\right]\Big|_{x=2} = g(2)f'(2) + f(2)g'(2) = 3(3) + 3\left(\frac{1}{3}\right) = 10$$

58. $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]\Bigg|_{x=2} = \frac{g(2)f'(2) - f(2)g'(2)}{\left(g(2)\right)^2} = \frac{3(3) - 3\left(\frac{1}{3}\right)}{3^2} = \frac{8}{9}$$

59. $$\frac{d}{dx}\left[\left(f(x)\right)^2\right]\Big|_{x=2} = 2f(2)f'(2) = 2(3)(3) = 18$$

60. $$\frac{d}{dx}\left[\left(g(x)\right)^2\right]\Big|_{x=2} = 2g(2)g'(2) = 2(3)\left(\frac{1}{3}\right) = 2$$

61. $$\frac{d}{dx}\left[xf(x)\right]\Big|_{x=2} = xf'(x) + f(x)(1) = 2(3) + 3 = 9$$

62. $$\begin{aligned}\frac{d}{dx}\left[x\left(g(x) - f(x)\right)\right]\Big|_{x=2} &= \left(g(x) - f(x)\right)\frac{d}{dx}x + x\frac{d}{dx}\left(g(x) - f(x)\right)\\ &= \left(g(2) - f(2)\right)(1) + x\left(g'(2)\right) - x\left(f'(2)\right)\\ &= 3 - 3 + 2\left(\frac{1}{3}\right) - 2(3) = -\frac{16}{3}\end{aligned}$$

63. $f(x)=\frac{1}{x},\ g(x)=x^3$

a. Using the product rule,

$$\frac{d}{dx}\left[\left(\frac{1}{x}\right)x^3\right]=\left(\frac{1}{x}\right)(3x^2)+x^3(-1)x^{-2}=3x-x=2x=\frac{d}{dx}\left[x^2\right]$$

b. $f'(x)=-\frac{1}{x^2}$ and $g'(x)=3x^2$, so $f'(x)g'(x)=-3$.

Now $f(x)g(x)=\left(\frac{1}{x}\right)x^3=x^2$ which has derivative $2x$. Thus, $f'(x)g'(x)\neq(f(x)g(x))'$.

64. Using the quotient rule, $\frac{d}{dx}\left[\frac{x^3-4x}{x}\right]=\frac{x(3x^2-4)-(x^3-4x)}{x^2}=\frac{2x^3}{x^2}=2x$

65.
$$\begin{aligned}\frac{d}{dx}[f(x)g(x)h(x)]&=\frac{d}{dx}\left[\left(f(x)g(x)\right)\cdot h(x)\right]=h(x)\frac{d}{dx}[f(x)g(x)]+[g(x)h(x)]h'(x)\\&=h(x)[f'(x)g(x)+f(x)g'(x)]+g(x)h(x)h'(x)\\&=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)\end{aligned}$$

66. Let f and g be differentiable functions. From the identity in the statement of the problem,

$$\frac{d}{dx}[fg]=\frac{d}{dx}\left[\frac{1}{4}\left((f+g)^2-(f-g)^2\right)\right]=\frac{1}{4}\left[\frac{d}{dx}(f+g)^2-\frac{d}{dx}(f-g)^2\right]$$

and by the special case

$$=\frac{1}{4}[2(f+g)(f'+g')-2(f-g)(f'-g')]=\frac{1}{2}[ff'+fg'+f'g+gg'-ff'+fg'+f'g-gg']=fg'+f'g$$

67. $b(t)=\frac{w(t)}{[h(t)]^2}$

$$b'(t)=\frac{d}{dt}\left[\frac{w(t)}{[h(t)]^2}\right]=\frac{[h(t)]^2w'(t)-w(t)(2)h(t)h'(t)}{[h(t)]^4}=\frac{h(t)w'(t)-2h'(t)w(t)}{[h(t)]^3}$$

68. a. $b(t)=\frac{w(t)}{[h(t)]^2}$

$b(12)=\frac{50}{(1.55)^2}\approx 20.81$

Since $20.81 < 24$, he is not overweight. Since $20.81 < 21$, he is not at risk of being overweight.

b. In Exercise 67, we found

$$b'(t)=\frac{h(t)w'(t)-2h'(t)w(t)}{[h(t)]^3}$$

so that

$$b'(12)=\frac{h(12)w'(12)-2h'(12)w(12)}{[h(12)]^3}.$$

Substituting $h(12)=1.55$, $w(12)=50$, $h'(12)=.05$, and $w'(12)=7$ yields the result.

$$b'(12)=\frac{1.55(7)-2(.05)(50)}{[1.55]^3}\approx 1.57$$

c. Formula (2), Section 1.8 gives the approximation

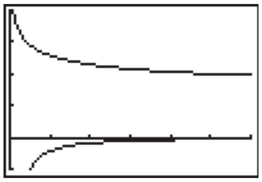
$$f(a+h)-f(a)\approx f'(a)\cdot h\Rightarrow$$
$$f(a+h)\approx f(a)+f'(a)\cdot h$$

Applying this formula gives us

$$\begin{aligned}b(13)&\approx b(12)+b'(12)\cdot 1\\&\approx 20.81+1.57\\&\approx 22.38\end{aligned}$$

Since $22.38 > 22$, the child is at risk of being overweight when he turns 13.

69. **a.**



[0, 6] by [−5, 20]

b. $f(3) \approx 10.8$ square millimeters

c. Using a graphing calculator to find when $f(x) = 11$, $x \approx 2.61$ units of light.

d. $f'(3) \approx -.55$ square millimeters per unit of light.

3.2 The Chain Rule and the General Power Rule

1. $f(x) = \dfrac{x}{x+1},\ g(x) = x^3;\ f(g(x)) = \dfrac{x^3}{x^3+1}$

2. $f(x) = x - 1,\ g(x) = \dfrac{1}{x+1}$

$f(g(x)) = \dfrac{1}{x+1} - 1 = -\dfrac{x}{x+1}$

3. $f(x) = x(x^2+1),\ g(x) = \sqrt{x}$

$f(g(x)) = \sqrt{x}\left(\left(\sqrt{x}\right)^2 + 1\right) = \sqrt{x}\,(x+1)$

4. $f(x) = \dfrac{x+1}{x-3},\ g(x) = x + 3;$

$f(g(x)) = \dfrac{x+3+1}{x+3-3} = \dfrac{x+4}{x}$

5. $f(g(x)) = (x^3 + 8x - 2)^5$

$f(x) = x^5,\ g(x) = x^3 + 8x - 2$

6. $f(g(x)) = (9x^2 + 2x - 5)^7$

$f(x) = x^7,\ g(x) = 9x^2 + 2x - 5$

7. $f(g(x)) = \sqrt{4 - x^2}$

$f(x) = \sqrt{x},\ g(x) = 4 - x^2$

8. $f(g(x)) = (5x^2 + 1)^{-1/2}$

$f(x) = x^{-1/2} = \dfrac{1}{\sqrt{x}},\ g(x) = 5x^2 + 1$

9. $f(g(x)) = \dfrac{1}{x^3 - 5x^2 + 1}$

$f(x) = \dfrac{1}{x},\ g(x) = x^3 - 5x^2 + 1$

10. $f(g(x)) = (4x-3)^3 + \dfrac{1}{4x-3}$

$f(x) = x^3 + \dfrac{1}{x},\ g(x) = 4x - 3$

11. $\dfrac{d}{dx}(x^2+5)^{15} = 15(x^2+5)^{14}(2x)$

$= 30x(x^2+5)^{14}$

12. $\dfrac{d}{dx}\left(x^4 + x^2\right)^{10} = 10\left(x^4 + x^2\right)^9\left(4x^3 + 2x\right)$

13. $\dfrac{d}{dx}6x^2(x-1)^3 = 6x^2(3)(x-1)^2(1) + 12x(x-1)^3$

$= 18x^2(x-1)^2 + 12x(x-1)^3$

$= 6x(x-1)^2(5x-2)$

14. $\dfrac{d}{dx}5x^3(2-x)^4$

$= 5x^3(4)(2-x)^3(-1) + 15x^2(2-x)^4$

$= -20x^3(2-x)^3 + 15x^2(2-x)^4$

$= 5x^2(2-x)^3(6-7x)$

15. $\frac{d}{dx}\left[2(x^3-1)(3x^2+1)^4\right] = 2\left[4(3x^2+1)^3(6x)(x^3-1)+3x^2(3x^2+1)^4\right] = 6x(3x^2+1)^3(11x^3+x-8)$

16. $\frac{d}{dx}\left[2(2x-1)^{5/4}(2x+1)^{3/4}\right] = 2\left[\frac{5}{4}(2x-1)^{1/4}(2)(2x+1)^{3/4}+\frac{3}{4}(2x+1)^{-1/4}(2)(2x-1)^{5/4}\right]$
$= 2(2x-1)^{1/4}(2x+1)^{-1/4}(8x+1)$

17. $\frac{d}{dx}\left[f(g(x))\right]\Big|_{x=1} = f'(g(1))g'(1)$
$= (f'(3))(4)$
$= 2(4) = 8$

18. $\frac{d}{dx}\left[g(f(x))\right]\Big|_{x=1} = g'(f(1))f'(1)$
$= (g'(1))(5)$
$= 4(5) = 20$

19. $\frac{d}{dx}\left[f(f(x))\right]\Big|_{x=1} = f'(f(1))f'(1)$
$= (f'(1))(5)$
$= 5(5) = 25$

20. $\frac{d}{dx}\left[g(g(x))\right]\Big|_{x=1} = g'(g(1))g'(1)$
$= (g'(3))(4)$
$= 6(4) = 24$

21. $h(x) = f(x^2)$
$h'(x) = \frac{d}{dx}\left[f(x^2)\right]$
$= f'(x^2)(2x)$
$= 2xf'(x^2)$

22. $h(x) = 2f(2x+1)$
$h'(x) = \frac{d}{dx}[2f(2x+1)] = 2\frac{d}{dx}[f(2x+1)]$
$= 2f'(2x+1)(2) = 4f'(2x+1)$

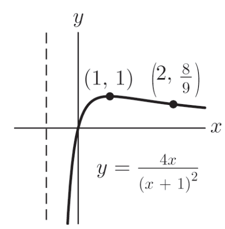
23. $h(x) = -f(-x)$
$h'(x) = \frac{d}{dx}[-f(-x)] = -\frac{d}{dx}[f(-x)]$
$= -f'(-x)(-1) = f'(-x)$

24. $h(x) = f(f(x))$
$h'(x) = \frac{d}{dx}[f(f(x))] = f'(f(x))f'(x)$

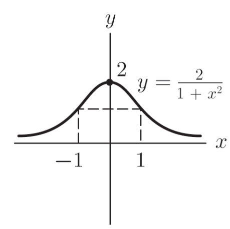
25. $h(x) = \frac{f(x^2)}{x}$
$h'(x) = \frac{d}{dx}\left[\frac{f(x^2)}{x}\right] = \frac{xf'(x^2)(2x)-f(x^2)(1)}{x^2}$
$= \frac{2x^2f'(x^2)-f(x^2)}{x^2}$

26. $h(x) = \sqrt{f(x^2)} = \left[f(x^2)\right]^{1/2}$
$h'(x) = \frac{d}{dx}\left[f(x^2)\right]^{1/2}$
$= \frac{1}{2}\left[f(x^2)\right]^{-1/2} f'(x^2)(2x)$
$= \frac{xf'(x^2)}{\sqrt{f(x^2)}}$

27.



28.



29. $f(x) = x^5,\ g(x) = 6x-1$
$\frac{d}{dx}f(g(x)) = \frac{d}{dx}(6x-1)^5 = 5(6x-1)^4(6)$
$= 30(6x-1)^4$

30. $f(x) = \sqrt{x},\ g(x) = x^2+1$
$\frac{d}{dx}f(g(x)) = \frac{d}{dx}\sqrt{x^2+1} = (x^2+1)^{1/2}$
$= \frac{1}{2}(x^2+1)^{-1/2}(2x)$
$= \frac{x}{\sqrt{x^2+1}}$

31. $f(x) = \dfrac{1}{x},\ g(x) = 1 - x^2$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}\left[\frac{1}{1-x^2}\right] = \frac{0(1-x^2) - (-2x)(1)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

32. $f(x) = \dfrac{1}{1+\sqrt{x}},\ g(x) = \dfrac{1}{x}$

$$\frac{d}{dx} f\left(g(x)\right) = \frac{d}{dx}\left[\frac{1}{1+\sqrt{\frac{1}{x}}}\right] = \frac{d}{dx}\left[\frac{\sqrt{x}}{\sqrt{x}+1}\right] = \frac{\left(\sqrt{x}+1\right)\frac{1}{2\sqrt{x}} - \sqrt{x}\,\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}+1\right)^2} = \frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$$

33. $f(x) = x^4 - x^2,\ g(x) = x^2 - 4$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}\left[(x^2-4)^4 - (x^2-4)^2\right] = 4(x^2-4)^3(2x) - 2(x^2-4)(2x) = 8x(x^2-4)^3 - 4x(x^2-4)$$

34. $f(x) = \dfrac{4}{x} + x^2,\ g(x) = 1 - x^4$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}\left[\frac{4}{1-x^4} + (1-x^4)^2\right] = \frac{-4(-4x^3)}{(1-x^4)^2} + 2(1-x^4)(-4x^3) = \frac{16x^3}{(1-x^4)^2} - 8x^3(1-x^4)$$

35. $f(x) = (x^3+1)^2,\ g(x) = x^2 + 5$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}\left[(x^2+5)^3+1\right]^2 = 2\left[(x^2+5)^3+1\right]3(x^2+5)^2(2x) = 12x\left[(x^2+5)^3+1\right](x^2+5)^2$$

36. $f(x) = x(x-2)^4,\ g(x) = x^3$

$$\frac{d}{dx} f\left(g(x)\right) = \frac{d}{dx}\left[x^3\left(x^3-2\right)^4\right] = x^3(4)\left(x^3-2\right)^3\left(3x^2\right) + 3x^2\left(x^3-2\right)^4 = 3x^2\left(x^3-2\right)^3\left(5x^3-2\right)$$

37. $y = u^{3/2}$, $u = 4x + 1$

$$\frac{dy}{du} = \frac{3}{2}u^{1/2};\ \frac{du}{dx} = 4;\ \ \frac{dy}{dx} = 6u^{1/2} = 6(4x+1)^{1/2}$$

38. $y = \sqrt{u+1},\ u = 2x^2$

$$\frac{dy}{du} = \frac{1}{2}(u+1)^{-1/2} = \frac{1}{2\sqrt{u+1}};\ \frac{du}{dx} = 4x;\ \ \frac{dy}{dx} = \frac{4x}{2\sqrt{u+1}} = \frac{2x}{\sqrt{2x^2+1}}$$

39. $y = \dfrac{u}{2} + \dfrac{2}{u},\ u = x - x^2,\ \ \dfrac{dy}{du} = \dfrac{1}{2} - \dfrac{2}{u^2};\ \dfrac{du}{dx} = 1 - 2x$

$$\frac{dy}{dx} = \left[\frac{1}{2} - \frac{2}{(x-x^2)^2}\right](1-2x) = \frac{\left(\left(x-x^2\right)^2 - 4\right)(1-2x)}{2(x-x^2)^2}$$

40. $y = \dfrac{u^2 + 2u}{u+1}$, $u = x(x+1) = x^2 + x$

$$\frac{dy}{du} = \frac{(u+1)(2u+2) - (u^2+2u)}{(u+1)^2} = \frac{u^2+2u+2}{(u+1)^2};\ \frac{du}{dx} = 2x+1$$

$$\frac{dy}{dx} = \left(\frac{u^2+2u+2}{(u+1)^2}\right)(2x+1) = \frac{\left[\left(x^2+x\right)^2 + 2\left(x^2+x\right)+2\right]}{\left(x^2+x+1\right)^2}(2x+1) = \frac{(2x+1)\left(x^4+2x^3+3x^2+2x+2\right)}{\left(x^2+x+1\right)^2}$$

41. $y = x^2 - 3x,\ x = t^2 + 3,\ t_0 = 0$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (2x-3)(2t) = \left(2\left(t^2+3\right)-3\right)(2t) = (2t^2+6-3)(2t) = 4t^3 + 6t \Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 0$$

42. $y = (x^2 - 2x + 4)^2,\ x = \dfrac{1}{t+1},\ t_0 = 1$

$$\frac{dy}{dt} = 2(x^2-2x+4)(2x-2)\left(\frac{-1}{(t+1)^2}\right) = \frac{-2\left(\left(\frac{1}{t+1}\right)^2 - 2\left(\frac{1}{t+1}\right)+4\right)\left(2\left(\frac{1}{t+1}\right)-2\right)}{(t+1)^2}$$

$$\left.\frac{dy}{dt}\right|_{t=1} = \frac{-2\left(\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)+4\right)\left(2\left(\frac{1}{2}\right)-2\right)}{(2)^2} = \frac{13}{8}$$

43. $y = \dfrac{x+1}{x-1},\ x = \dfrac{t^2}{4} = \dfrac{1}{4}t^2,\ t_0 = 3$

$$\frac{dy}{dt} = \left(\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right)\left(\frac{1}{2}t\right)$$

$$= \frac{-t}{(x-1)^2} = \frac{-t}{\left(\frac{1}{4}t^2-1\right)^2}$$

$$\left.\frac{dy}{dt}\right|_{t=3} = \frac{-3}{\left(\frac{9}{4}-1\right)^2} = -\frac{48}{25}$$

44. $y = \sqrt{x+1},\ x = \sqrt{t+1},\ t_0 = 0$

$$\frac{dy}{dt} = \frac{1}{2}(x+1)^{-1/2}\left(\frac{1}{2}\right)(t+1)^{-1/2}$$

$$= \frac{1}{4\sqrt{\sqrt{t+1}+1}\sqrt{t+1}}$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{1}{4\sqrt{\sqrt{1}+1}\sqrt{1}} = \frac{1}{4\sqrt{2}}$$

45. $\dfrac{dy}{dx} = 2(x-4)^6 + 12x(x-4)^5;\ \left.\dfrac{dy}{dx}\right|_{x=5} = 62$

The tangent line is $y - 10 = 62(x-5)$ or $y = 62x - 300$.

46. $$\frac{dy}{dx} = \frac{\sqrt{2-x^2}(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2}$$

$\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{1+1}{1} = 2$; the tangent line is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

47. A horizontal tangent line has $\dfrac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 3(-x^2+4x-3)^2(-2x+4) = 0 \Rightarrow$$

$-2x + 4 = 0 \Rightarrow x = 2$ or $-x^2 + 4x - 3 = 0 \Rightarrow$
$(x-3)(x-1) = 0 \Rightarrow x = 1, x = 3$
The x-coordinates are 1, 2, and 3.

48. $f'(x) = \dfrac{1}{2}(x^2 - 6x + 10)^{-1/2}(2x-6)$

$$f'(x) = 0 \Rightarrow \frac{1}{2\sqrt{x^2-6x+10}}(2x-6) = 0 \Rightarrow$$

$2x - 6 = 0 \Rightarrow x = 3;\ y = \sqrt{9-18+10} = 1$

The relative minimum point is (3, 1).

49. a. $\dfrac{dV}{dt} = \dfrac{dV}{dx} \cdot \dfrac{dx}{dt}$

b. By the chain rule $\dfrac{dV}{dt} = 3x^2 \dfrac{dx}{dt}$.

We want $\dfrac{dV}{dt} = 12\dfrac{dx}{dt}$; so $3x^2 = 12$ or $x = 2$.

50. $\dfrac{dw}{dt} = 3(446)x^2 \dfrac{dx}{dt}$, $x = .4$ and $\dfrac{dx}{dt} = .2$, so

$\dfrac{dw}{dt} = 3(446)(.4)^2(.2) = 42.816$ grams/year.

51. a. $\dfrac{dy}{dt}, \dfrac{dP}{dy}, \dfrac{dP}{dt}$ **b.** $\dfrac{dP}{dt} = \dfrac{dP}{dy} \cdot \dfrac{dy}{dt}$

52. a. $\dfrac{dx}{dy}, \dfrac{dQ}{dy}, \dfrac{dQ}{dx}$ **b.** $\dfrac{dQ}{dy} = \dfrac{dQ}{dx} \cdot \dfrac{dx}{dy}$

53. $P = \dfrac{200x}{100 + x^2}$, $x = 4 + 2t$

a.
$$\begin{aligned}\frac{dP}{dx} &= \frac{(100+x^2)(200) - (200x)(2x)}{(100+x^2)^2} \\ &= \frac{20,000 + 200x^2 - 400x^2}{(100+x^2)^2} \\ &= \frac{200(100 - x^2)}{(100+x^2)^2}\end{aligned}$$

b. $\dfrac{dx}{dt} = 2$
$$\begin{aligned}\frac{dP}{dt} &= \left[\frac{200(100-(4+2t)^2)}{(100+(4+2t)^2)^2}\right]2 \\ &= \frac{400\left[100-(4+2t)^2\right]}{\left[100+(4+2t)^2\right]^2}\end{aligned}$$

c. When $t = 8$,
$$\frac{dP}{dt} = \frac{400(100-(4+16)^2)}{(100+(4+16)^2)^2} = -.48 .$$
Profits are falling at \$480/week

54. $C = 3x + 4\sqrt{x} + 2$, $x = 6200 + 100t$

a. $\dfrac{dC}{dx} = 3 + \dfrac{2}{\sqrt{x}}$

b. $\dfrac{dx}{dt} = 100$
$$\begin{aligned}\frac{dC}{dt} &= 100\left[3 + \frac{2}{\sqrt{6200+100t}}\right] \\ &= 100\left[3 + \frac{2}{10\sqrt{62+t}}\right] = 300 + \frac{20}{\sqrt{62+t}}\end{aligned}$$

c. When $t = 2$,
$$\begin{aligned}\left.\frac{dC}{dt}\right|_{t=2} &= \left[3 + \frac{2}{\sqrt{6200+100(2)}}\right]\frac{dx}{dt} \\ &= \left[3 + \frac{2}{\sqrt{6200+200}}\right](100) \\ &= 302.5\end{aligned}$$
Costs are rising at \$302.50 per week.

55. $L = 10 + .4x + .0001x^2$, $x = 752 + 23t + .5t^2$

a. $\dfrac{dL}{dx} = .4 + .0002x$

b. $\dfrac{dx}{dt} = 23 + t \Rightarrow \left.\dfrac{dx}{dt}\right|_{t=2} = 25$

The population is increasing at the rate of 25 thousand persons per year.

c. When $t = 2$, $x = 800$ and $\dfrac{dL}{dx} = .56$.

$(.56)(25) = 14$ ppm per year

56. When $t = 5$, $x = .05(5^2) + 2(5) + 5 = 16.25$.

$\dfrac{dP}{dx} = .002x + .1$, so

$\left.\dfrac{dP}{dx}\right|_{x=16.25} = (.002)16.25 + 1 = 1.0325$.

$\dfrac{dx}{dt} = .1t + 2$, so $\left.\dfrac{dx}{dt}\right|_{t=5} = 2.5$.

Therefore, $\left.\dfrac{dP}{dt}\right|_{t=5} = 2.5(.1325) = .33125$.

Profit will be increasing at \$.33125 million per month or \$331,250 per month.

57.
$$\begin{aligned}\frac{d}{dx}f(g(x)) &= f'(g(x)) \cdot g'(x) \\ &= 3x^2 \cdot f'(x^3+1)\end{aligned}$$
So $g(x) = x^3 + 1$.

58. $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) = \frac{g'(x)}{g(x)}$

$= \frac{2x+5}{x^2+5x-4}$

So $g(x) = x^2 + 5x - 4$.

59. $\left. \frac{d}{dx} f(g(x)) \right|_{x=1} = f'(g(1)) \cdot g'(1)$

$= f'(5) \cdot 6 = 4 \cdot 6 = 24$

60. $\left. \frac{d}{dx} g(f(x)) \right|_{x=1} = g'(f(1)) \cdot f'(1) = g'(2) \cdot 3$

$= 7 \cdot 3 = 21$

61. a. From Fig 1(a), $x = 40$ when $t = 1.5$ and $x = 30$ when $t = 3.5$.

$W(40) = 10\left(\frac{12+8(40)}{3+40}\right) \approx 77.209$

million dollars

$W(30) = 10\left(\frac{12+8(30)}{3+30}\right) \approx 76.364$ million

dollars.

b. The slope of $x(t)$ at $t = 1.5$ is given by

slope$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{50-10}{2-0} = 20.$

So, $\left. \frac{dx}{dt} \right|_{t=1.5} = 20.$

A month and a half after the company went public, the share price of \$40 was increasing at a rate of \$20 per month. The slope of $x(t)$ at $t = 3.5$ is zero, since the graph is horizontal.

So, $\left. \frac{dx}{dt} \right|_{t=3.5} = 0.$

Three and a half months after the company went public, the share price was holding steady at \$30.

62. $W = \frac{120+80x}{3+x}$

$\frac{dW}{dx} = \frac{(3+x)(80) - (120+80x)(1)}{(3+x)^2}$

$= \frac{120}{(3+x)^2}$

$\frac{dW}{dt} = \frac{dW}{dx}\frac{dx}{dt}$

$\left. \frac{dW}{dt} \right|_{t=1.5} = \left. \frac{dW}{dx}\frac{dx}{dt} \right|_{t=1.5}$

$= \frac{120}{(3+40)^2} \cdot 20 \approx 1.297999$

At $t = 1.5$, the value of the company is increasing at the rate of about \$1,297,999 per month.

$\left. \frac{dW}{dt} \right|_{t=3.5} = \left. \frac{dW}{dx}\frac{dx}{dt} \right|_{t=3.5}$

$= \frac{120}{(3+30)^2} \cdot 0 = 0$

At $t = 3.5$, the value of the company is not changing.

63. a. $\left. \frac{dx}{dt} \right|_{t=2.5} =$ slope of $x(t)$ at $t = 2.5$.

slope$= \frac{30-50}{3-2} = -20 = \left. \frac{dx}{dt} \right|_{t=2.5}$

The share price is dropping at a rate of \$20 per month.

$\left. \frac{dx}{dt} \right|_{t=4} =$ slope of $x(t)$ at $t = 4$. Since $x(t)$

is horizontal at $t = 4$, $\left. \frac{dx}{dt} \right|_{t=4} = 0.$

The share price is steady at \$30 per share.

b. $W = \frac{120+80x}{3+x}$

$\frac{dW}{dx} = \frac{(3+x)(80) - (120+80x)(1)}{(3+x)^2}$

$= \frac{120}{(3+x)^2}$

From Fig 1(a), $x = 40$ when $t = 2.5$ and $x = 30$ when $t = 4$.

$\left. \frac{dW}{dt} \right|_{t=2.5} = \left. \frac{dW}{dx}\frac{dx}{dt} \right|_{t=2.5}$

$= \frac{120}{(3+40)^2} \cdot (-20) \approx -1.3$

At $t = 2.5$, the value of the company is decreasing at the rate of 1.3 million dollars per month.

$\left. \frac{dW}{dt} \right|_{t=4} = \left. \frac{dW}{dx}\frac{dx}{dt} \right|_{t=4} = \frac{120}{(3+30)^2} \cdot 0 = 0$

At $t = 4$, the value of +.

64. a. The maximum value of the company corresponds to the maximum share price. The maximum share price of \$50 occurred at $t = 2$. The maximum value of the company was

$$W(50) = 10\frac{12+8(50)}{3+50} \approx 77.736 \text{ million}$$

dollars. This value was obtained two months after the company went public.

b. If the share price is increasing, the figure shows the total value of the company approaches 80 million dollars asymptotically. The limit on the total value is 80 million dollars.

65. The derivative of the composite function $f(g(x))$ is the derivative of the outer function evaluated at the inner function and then multiplied by the derivative of the inner function.

3.3 Implicit Differentiation and Related Rates

1. $x^2 - y^2 = 1$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

2. $x^3 + y^3 - 6 = 0$

$$3x^2 + 3y^2\frac{dy}{dx} - 0 = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}$$

3. $y^5 - 3x^2 = x$

$$5y^4\frac{dy}{dx} - 6x = 1 \Rightarrow \frac{dy}{dx} = \frac{1+6x}{5y^4}$$

4. $x^4 + (y+3)^4 = x^2$

$$4x^3 + 4(y+3)^3\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x-4x^3}{4(y+3)^3} = \frac{x-2x^3}{2(y+3)^3}$$

5. $y^4 - x^4 = y^2 - x^2$

$$4y^3\frac{dy}{dx} - 4x^3 = 2y\frac{dy}{dx} - 2x$$

$$\frac{dy}{dx}(4y^3 - 2y) = 4x^3 - 2x$$

$$\frac{dy}{dx} = \frac{2x^3 - x}{2y^3 - y}$$

6. $x^3 + y^3 = x^2 + y^2$

$$3x^2 + 3y^2\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x-3x^2}{3y^2-2y} \text{ or } \frac{3x^2-2x}{2y-3y^2}$$

7. $2x^3 + y = 2y^3 + x$

$$6x^2 + \frac{dy}{dx} = 6y^2\frac{dy}{dx} + 1$$

$$\frac{dy}{dx}(1-6y^2) = 1 - 6x^2$$

$$\frac{dy}{dx} = \frac{1-6x^2}{1-6y^2}$$

8. $x^4 + 4y = x - 4y^3$

$$4x^3 + 4\frac{dy}{dx} = 1 - 12y^2\frac{dy}{dx}$$

$$\frac{dy}{dx}(4+12y^2) = 1 - 4x^3$$

$$\frac{dy}{dx} = \frac{1-4x^3}{4+12y^2} \text{ or } \frac{4x^3-1}{-12y^2-4}$$

9. $xy = 5$

$$1(y) + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

10. $xy^3 = 2$

$$(1)y^3 + 3y^2\frac{dy}{dx}(x) = 0$$

$$\frac{dy}{dx} = -\frac{y^3}{3y^2x} = -\frac{y}{3x}$$

11. $x(y+2)^5 = 8$

$$(1)(y+2)^5 + 5(y+2)^4(x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(y+2)^5}{5x(y+2)^4} = -\frac{y+2}{5x}$$

12. $x^2y^3 = 6$

$$2xy^3 + 3y^2(x^2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2xy^3}{3y^2x^2} = -\frac{2y}{3x}$$

13. $x^3y^2 - 4x^2 = 1$

$$3x^2y^2 + 2yx^3\frac{dy}{dx} - 8x = 0$$

$$\frac{dy}{dx} = \frac{8x - 3x^2y^2}{2yx^3} = \frac{8 - 3xy^2}{2x^2y}$$

14. $(x+1)^2(y-1)^2 = 1$

$$2(x+1)(y-1)^2 + 2(y-1)(x+1)^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2(x+1)(y-1)^2}{2(x+1)^2(y-1)} = -\frac{y-1}{x+1}$$

15. $x^3 + y^3 = x^3y^3$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x^2y^3 + 3y^2x^3\frac{dy}{dx}$$

$$\frac{dy}{dx}(3y^2 - 3y^2x^3) = 3x^2y^3 - 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2y^3 - 3x^2}{3y^2 - 3y^2x^3}$$

$$\frac{dy}{dx} = \frac{x^2(y^3-1)}{y^2(1-x^3)}$$

16. $x^2 + 4xy + 4y = 1$

$$2x + 4\left(y + x\frac{dy}{dx}\right) + 4\frac{dy}{dx} = 0$$

$$4\frac{dy}{dx}(x+1) = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2(x+2y)}{4(x+1)} = \frac{-x-2y}{2(x+1)} = -\frac{x+2y}{2x+2}$$

17. $x^2y + y^2x = 3$

$$2xy + x^2\frac{dy}{dx} + 2yx\frac{dy}{dx} + y^2 = 0$$

$$x\frac{dy}{dx}(2y + x) = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x(2y+x)} = \frac{-2xy - y^2}{2xy + x^2} = -\frac{2xy + y^2}{2xy + x^2}$$

18. $x^3y + xy^3 = 4$

$$3x^2y + x^3\frac{dy}{dx} + y^3 + 3y^2x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx}(x^2 + 3y^2) = -3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x(x^2+3y^2)} = \frac{-3x^2y - y^3}{x^3 + 3xy^2} = -\frac{3x^2y + y^3}{x^3 + 3xy^2}$$

19. $4y^3 - x^2 = -5$

$$12y^2\frac{dy}{dx} - 2x = 0$$

$$\text{slope} = \frac{dy}{dx} = \frac{2x}{12y^2} = \frac{x}{6y^2}$$

When $x = 3$, $y = 1$, $\text{slope} = \frac{dy}{dx} = \frac{3}{6(1)^2} = \frac{1}{2}$.

20. $y^2 = x^3 + 1$

$$2y\frac{dy}{dx} = 3x^2 + 0 \Rightarrow \text{slope} = \frac{dy}{dx} = \frac{3x^2}{2y}$$

When $x = 2$, $y = -3$, $\text{slope} = \frac{dy}{dx} = \frac{3(2)^2}{2(-3)} = -2$.

21. $xy^3 = 2$

$$(1)y^3 + 3y^2x\frac{dy}{dx} = 0$$

$$\text{slope} = \frac{dy}{dx} = \frac{-y^3}{3y^2x} = \frac{-y}{3x}$$

When $x = -\frac{1}{4}$, $y = -2$,

$$\text{slope} = \frac{dy}{dx} = \frac{-(-2)}{3\left(-\frac{1}{4}\right)} = -\frac{8}{3}.$$

22. $\sqrt{x} + \sqrt{y} = 7$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\text{slope} = \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

When $x = 9$, $y = 16$,

$$\text{slope} = \frac{dy}{dx} = -\frac{\sqrt{16}}{\sqrt{9}} = -\frac{4}{3}.$$

23. $xy + y^3 = 14$

$$(1)y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\text{slope} = \frac{dy}{dx} = \frac{-y}{x + 3y^2}$$

When $x = 3$, $y = 2$,

$$\text{slope} = \frac{dy}{dx} = \frac{-2}{3 + 3(2)^2} = -\frac{2}{15}.$$

24. $y^2 = 3xy - 5$

$$2y\frac{dy}{dx} = 3(1)y + 3x\frac{dy}{dx} - 0$$

$$\text{slope} = \frac{dy}{dx} = \frac{3y}{2y - 3x}$$

When $x = 2, y = 1,$

$$\text{slope} = \frac{dy}{dx} = \frac{3(1)}{2(1) - 3(2)} = -\frac{3}{4}.$$

25. $x^2y^4 = 1$

$$2xy^4 + 4y^3x^2\frac{dy}{dx} = 0$$

$$\text{slope} = \frac{dy}{dx} = \frac{-2xy^4}{4y^3x^2} = -\frac{y}{2x}$$

When $x = 4,\ y = \frac{1}{2}$, slope $= -\frac{1/2}{2(4)} = -\frac{1}{16}.$

Let $(x_1, y_1) = \left(4, \frac{1}{2}\right).$

$$y - \frac{1}{2} = -\frac{1}{16}(x - 4) \Rightarrow y = -\frac{1}{16}x + \frac{3}{4}$$

When $x = 4,\ y = -\frac{1}{2}$, slope $= -\frac{-1/2}{2(4)} = \frac{1}{16}.$

Let $(x_1, y_1) = \left(4, -\frac{1}{2}\right).$

$$y + \frac{1}{2} = \frac{1}{16}(x - 4) \Rightarrow y = \frac{1}{16}x + \frac{1}{4}$$

26. $x^4y^2 = 144$

$$4x^3y^2 + 2yx^4\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^3y^2}{2yx^4} = -\frac{2y}{x}$$

$$\text{slope} = \frac{dy}{dx} = -\frac{2y}{x}$$

When $x = 2, y = 3,$ slope $= \frac{-2(3)}{2} = -3.$

Let $(x_1, y_1) = (2, 3).$

$y - 3 = -3(x - 2) \Rightarrow y = -3x + 9$

When $x = 2, y = -3,$ slope $= \frac{-2(-3)}{2} = 3.$

Let $(x_1, y_1) = (2, -3).$

$y + 3 = 3(x - 2) \Rightarrow y = 3x - 9$

27. a. $x^4 + 2x^2y^2 + y^4 = 4x^2 - 4y^2$

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 8x - 8y\frac{dy}{dx}$$

$$\frac{dy}{dx}(4x^2y + 4y^3 + 8y) = 8x - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{4(2x - x^3 - xy^2)}{4(x^2y + y^3 + 2y)} = \frac{2x - x^3 - xy^2}{x^2y + y^3 + 2y}$$

b. $\text{slope} = \frac{dy}{dx} = \frac{2x - x^3 - xy^2}{x^2y + y^3 + 2y}$

When

$x = \frac{\sqrt{6}}{2},\ y = \frac{\sqrt{2}}{2},$

$$\text{slope} = \frac{\sqrt{6} - \frac{\sqrt{6^3}}{8} - \frac{\sqrt{6}}{2}\cdot\frac{2}{4}}{\frac{6}{4}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2^3}}{8} + \sqrt{2}} = 0.$$

28. a. $x^4 + 2x^2y^2 + y^4 = 9x^2 - 9y^2$

$$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 18x - 18y\frac{dy}{dx}$$

$$\frac{dy}{dx}(4x^2y + 4y^3 + 18y) = 18x - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{18x - 4x^3 - 4xy^2}{4x^2y + 4y^3 + 18y} = \frac{9x - 2x^3 - 2xy^2}{2x^2y + 2y^3 + 9y}$$

b. $\text{slope} = \frac{dy}{dx} = \frac{9x - 2x^3 - 2xy^2}{2x^2y + 2y^3 + 9y}$

When $x = \sqrt{5},\ y = -1,$

$$\text{slope} = \frac{9\sqrt{5} - 10\sqrt{5} - 2\sqrt{5}}{-10 - 2 - 9} = \frac{3\sqrt{5}}{21} = \frac{\sqrt{5}}{7}.$$

29. a. $30x^{1/3}y^{2/3} = 1080$

$$10x^{-2/3}y^{2/3} + 20x^{1/3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-10y^{2/3}x^{-2/3}}{20x^{1/3}y^{-1/3}} = -\frac{y}{2x}$$

b. marginal rate of substitution of x for y is

$$\left|\frac{dy}{dx}\right|_{\substack{x=16\\y=54}} = \left|-\frac{27}{16}\right| = \frac{27}{16}.$$

30. $10x^{1/2}y^{1/2} = 600$

$$5x^{-1/2}y^{1/2} + 5y^{-1/2}x^{1/2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^{-1/2}y^{1/2}}{y^{-1/2}x^{1/2}} = -\frac{y}{x}$$

When $x = 50$, $y = 72$, $\frac{dy}{dx} = -\frac{72}{50} = -\frac{36}{25}$.

31. $x^4 + y^4 = 1$

$$4x^3\frac{dx}{dt} + 4y^3\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{4x^3}{4y^3}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x^3}{y^3}\frac{dx}{dt}$$

32. $y^4 - x^2 = 1$

$$4y^3\frac{dy}{dt} - 2x\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \frac{x}{2y^3}\frac{dx}{dt}$$

33. $3xy - 3x^2 = 4$

$$3\frac{dx}{dt}y + 3x\frac{dy}{dt} - 6x\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \frac{6x - 3y}{3x}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2x - y}{x}\frac{dx}{dt}$$

34. $y^2 = 8 + xy$

$$2y\frac{dy}{dt} = 0 + y\frac{dx}{dt} + x\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{y}{2y - x}\frac{dx}{dt}$$

35. $x^2 + 2xy = y^3$

$$2x\frac{dx}{dt} + 2y\frac{dx}{dt} + 2x\frac{dy}{dt} = 3y^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{2x + 2y}{3y^2 - 2x}\frac{dx}{dt}$$

36. $x^2y^2 = 2y^3 + 1$

$$2x\frac{dx}{dt}y^2 + 2y\frac{dy}{dt}x^2 = 6y^2\frac{dy}{dt} + 0$$

$$\frac{dy}{dt} = \frac{-2xy^2}{2yx^2 - 6y^2}\frac{dx}{dt} = \frac{xy^2}{3y^2 - x^2}\frac{dx}{dt}$$

37. $x^2 - 4y^2 = 9$

$$2x\frac{dx}{dt} - 8y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{2x}{8y}\frac{dx}{dt} = \frac{x}{4y}\frac{dx}{dt}$$

When $x = 5$, $y = -2$, $\frac{dx}{dt} = 3$ per sec,

$$\frac{dy}{dt} = \frac{5}{-8}(3) = -\frac{15}{8}.$$

The y-coordinate is decreasing at $\frac{15}{8}$ units per second.

38. $x^3y^2 = 200$

$$3x^2\frac{dx}{dt}y^2 + 2y\frac{dy}{dt}x^3 = 0$$

$$\frac{dy}{dt} = -\frac{3x^2y^2}{2yx^3}\frac{dx}{dt} = \frac{-3y}{2x}\frac{dx}{dt}$$

When $x = 2$, $y = 5$, $\frac{dx}{dt} = -4$ units per min,

$$\frac{dy}{dt} = \frac{-3(5)}{2(2)}(-4) = 15.$$

The y-coordinate is increasing at 15 units per minute.

39. $2p^3 + x^2 = 4500$

$$6p^2\frac{dp}{dt} + 2x\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{6p^2}{2x}\frac{dp}{dt} = -\frac{3p^2}{x}\frac{dp}{dt}$$

When $p = 10$, $x = 50$, $\frac{dp}{dt} = -.5$ per week,

$$\frac{dx}{dt} = -\frac{3(100)}{50}(-.5) = 3.$$

The sales are rising at 3 thousand units per week.

40. $6p + x + xp = 94$

$$6\frac{dp}{dt} + \frac{dx}{dt} + p\frac{dx}{dt} + x\frac{dp}{dt} = 0$$

$$\frac{dx}{dt} = \frac{-x - 6}{p + 1}\frac{dp}{dt}$$

When $x = 4$, $p = 9$, $\frac{dp}{dt} = 2$,

$$\frac{dx}{dt} = \frac{-4 - 6}{9 + 1}(2) = -2.$$

The demand is decreasing at the rate of 2 thousand units per week.

41. $A = 6\sqrt{x^2 - 400}$, $x \ge 20$, $A = 6(x^2 - 400)^{1/2}$

$$\frac{dA}{dx} = 3(x^2 - 400)^{-1/2}(2x) = \frac{6x}{\sqrt{x^2 - 400}}$$

$$\frac{dA}{dt} = \frac{dA}{dx}\cdot\frac{dx}{dt} = \frac{6x}{\sqrt{x^2 - 400}}\cdot\frac{dx}{dt}$$

When $x = 25$, $\frac{dx}{dt} = 2$,

$$\frac{dA}{dt} = \frac{6(25)}{\sqrt{25^2 - 400}}(2) = 20.$$

The revenue is increasing at the rate of 20 thousand dollars per month.

42. $px + 7x + 8p = 328$

$$x\frac{dp}{dt} + p\frac{dx}{dt} + 7\frac{dx}{dt} + 8\frac{dp}{dt} = 0$$

$$\frac{dp}{dt} = \frac{-p-7}{x+8}\frac{dx}{dt}$$

When $p = 25$, $x = 4$, $\frac{dx}{dt} = -.3$,

$$\frac{dp}{dt} = \frac{-25-7}{4+8}(-.3) = .8.$$

The price is increasing at the rate of \$ .80 per day.

43. a. $x^2 + y^2 = (10)^2 \Rightarrow x^2 + y^2 = 100$

b. $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$

When $x = 8$, $y = 6$,

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -\frac{8}{6}(3) = -4.$$

The top end is sliding down at 4 feet per second.

44. a. $x^2 + (5000)^2 = y^2$

b. When $y = 13{,}000$,

$$x^2 + 25{,}000{,}000 = (13{,}000)^2$$
$$x = 12{,}000$$

c. $2y\frac{dy}{dx} = 2x + 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{x}{y}\frac{dx}{dt}$$

When $x = 12{,}000$, $y = 13{,}000$, $\frac{dx}{dt} = 390$.

$$\frac{dy}{dt} = \frac{12{,}000}{13{,}000}(390) = 360$$

The distance is increasing at 360 feet per second.

45. $x^2 + 90^2 = y^2$; differentiate with respect to t:

$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ and $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$.

If $x = 45$ and $\frac{dx}{dt} = 22$ and by the equation above $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, then

$$\frac{dy}{dt} = \frac{45\cdot 22}{45\sqrt{5}} = \frac{22}{\sqrt{5}} \text{ ft/sec.}$$

46. Using similar triangles, we have

$$\frac{h}{x} = \frac{100}{1000} \Rightarrow h = \frac{x}{10}$$

Differentiate with respect to t:

$$\frac{dh}{dt} = \frac{1}{10}\frac{dx}{dt}$$

Since $\frac{dx}{dt} = 80, \frac{dh}{dt} = \frac{1}{10}\cdot 80 = 8.$ The motorcyclist is rising at 8 miles per hour.

Chapter 3 Fundamental Concept Check Exercises

1. Product rule: $(f\cdot g)' = f\cdot g' + g\cdot f'$

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$

2. Chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ or

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Example: If $y = u^3 + 2u - 1$ and $u = x^2 + 1$, then $\frac{dy}{du} = 3u^2 + 2$ and $\frac{du}{dx} = 2x$. Then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (3u^2 + 2)(2x)$. Now substitute $x^2 + 1$ for u to obtain

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(3(x^2+1)^2 + 2\right)(2x).$$

3. The chain rule implies the general power rule.

4. An implicit relation is an equation that relates x and y. An example is $xy = 1$.

5. $\frac{d}{dx}y^r = ry^{r-1}\frac{dy}{dx}$

6. To solve a related rate problem, use the following procedure.
1. Draw a picture if possible.
2. Assign letters to quantities that vary, and identify one variable, for example t, on which the other variables depend.
3. Find an equation that relates the variables to each other.
4. Differentiate the equation with respect to the independent variable t. Use the chain rule whenever appropriate.
5. Substitute all specified values for the variables and their derivatives.
6. Solve for the derivative that gives the unknown rate.

Chapter 3 Review Exercises

1. $$\frac{d}{dx}\left[(4x-1)(3x+1)^4\right] = (4x-1)(4)(3x+1)^3(3) + (3x+1)^4(4) = 4(3x+1)^3\left[(3x+1)+3(4x-1)\right]$$
$$= 4(3x+1)^3(15x-2)$$

2. $$\frac{d}{dx}\left[2(5-x)^3(6x-1)\right] = 2\left[(5-x)^3(6)+(6x-1)3(5-x)^2(-1)\right] = 2\left[-3(5-x)^2(6x-1)+6(5-x)^3\right]$$
$$= 6\left[2(5-x)^3-(5-x)^2(6x-1)\right] = 6(5-x)^2\left[10-2x-6x+1\right] = 6(5-x)^2(11-8x)$$

3. $$\frac{d}{dx}\left[x(x^5-1)^3\right] = (x)(3)(x^5-1)^2(5x^4)+(x^5-1)^3(1) = (x^5-1)^2(16x^5-1)$$

4. $$\frac{d}{dx}\left[(2x+1)^{5/2}(4x-1)^{3/2}\right] = (2x+1)^{5/2}\left(\frac{3}{2}\right)(4x-1)^{1/2}(4)+(4x-1)^{3/2}\left(\frac{5}{2}\right)(2x+1)^{3/2}(2)$$
$$= 5(2x+1)^{3/2}(4x-1)^{3/2}+6(4x-1)^{1/2}(2x+1)^{5/2}$$
$$= (2x+1)^{3/2}(4x-1)^{1/2}\left[5(4x-1)+6(2x+1)\right]$$
$$= (2x+1)^{3/2}(4x-1)^{1/2}(32x+1)$$

5. $$\frac{d}{dx}\left[5\left(\sqrt{x}-1\right)^4\left(\sqrt{x}-2\right)^2\right] = 5\left[\left(\sqrt{x}-1\right)^4(2)\left(\sqrt{x}-2\right)\left(\frac{1}{2}x^{-1/2}\right)+\left(\sqrt{x}-2\right)^2(4)\left(\sqrt{x}-1\right)^3\left(\frac{1}{2}x^{-1/2}\right)\right]$$
$$= 5\left[2x^{-1/2}\left(\sqrt{x}-1\right)^3\left(\sqrt{x}-2\right)^2+x^{-1/2}\left(\sqrt{x}-2\right)\left(\sqrt{x}-1\right)^4\right]$$
$$= 5x^{-1/2}\left(\sqrt{x}-1\right)^3\left(\sqrt{x}-2\right)\left[2\left(\sqrt{x}-2\right)+\left(\sqrt{x}-1\right)\right]$$
$$= \frac{5\left(\sqrt{x}-1\right)^3\left(\sqrt{x}-2\right)\left(3\sqrt{x}-5\right)}{\sqrt{x}}$$

6. $$\frac{d}{dx}\left[\frac{\sqrt{x}}{\sqrt{x}+4}\right] = \frac{\left(\sqrt{x}+4\right)\left(\frac{1}{2\sqrt{x}}\right)-\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}+4\right)^2} = \frac{\frac{1}{2\sqrt{x}}\left(\sqrt{x}+4-\sqrt{x}\right)}{\left(\sqrt{x}+4\right)^2} = \frac{2}{\sqrt{x}\left(\sqrt{x}+4\right)^2}$$

7. $$\frac{d}{dx}\left[3(x^2-1)^3(x^2+1)^5\right] = 3\left[(x^2-1)^3(5)(x^2+1)^4(2x)+(x^2+1)^5(3)(x^2-1)^2(2x)\right]$$
$$= 3(x^2-1)^2(2x)(x^2+1)^4\left[3(x^2+1)+5(x^2-1)\right]$$
$$= 12x(x^2-1)^2(x^2+1)^4(4x^2-1)$$

8. $$\frac{d}{dx}\left[\frac{1}{(x^2+5x+1)^6}\right] = \frac{(x^2+5x+1)^6(0)-(1)(6)(x^2+5x+1)^5(2x+5)}{(x^2+5x+1)^{12}} = \frac{-12x-30}{(x^2+5x+1)^7}$$

9. $\frac{d}{dx}\left[\frac{x^2-6x}{x-2}\right]=\frac{(x-2)(2x-6)-(x^2-6x)(1)}{(x-2)^2}=\frac{2x^2-10x+12-x^2+6x}{(x-2)^2}=\frac{x^2-4x+12}{(x-2)^2}$

10. $\frac{d}{dx}\left[\frac{2x}{2-3x}\right]=\frac{(2-3x)(2)-(2x)(-3)}{(2-3x)^2}=\frac{4-6x+6x}{(2-3x)^2}=\frac{4}{(2-3x)^2}$

11. $\frac{d}{dx}\left[\left(\frac{3-x^2}{x^3}\right)^2\right]=2\left(\frac{3-x^2}{x^3}\right)\left(\frac{(x^3)(-2x)-(3-x^2)(3x^2)}{x^6}\right)=2\left(\frac{3-x^2}{x^3}\right)\left(\frac{-2x^4-9x^2+3x^4}{x^6}\right)$

$=2\left(\frac{3-x^2}{x^3}\right)\left(\frac{x^4-9x^2}{x^6}\right)=2\left(\frac{3-x^2}{x^3}\right)\left(\frac{x^2-9}{x^4}\right)=\frac{2(3-x^2)(x^2-9)}{x^7}$

12. $\frac{d}{dx}\left[\frac{x^3+x}{x^2-x}\right]=\frac{(x^2-x)(3x^2+1)-(x^3+x)(2x-1)}{(x^2-x)^2}=\frac{3x^4-3x^3+x^2-x-2x^4-2x^2+x^3+x}{(x^2-x)^2}$

$=\frac{x^4-2x^3-x^2}{x^4-2x^3+x^2}=\frac{x^2-2x-1}{x^2-2x+1}=\frac{x^2-2x-1}{(x-1)^2}$

13. $f(x)=(3x+1)^4(3-x)^5$

$f'(x)=(3x+1)^4(5)(3-x)^4(-1)+(3-x)^5(4)(3x+1)^3(3)=12(3x+1)^3(3-x)^5-5(3-x)^4(3x+1)^4$

$=(3x+1)^3(3-x)^4\left[12(3-x)-5(3x+1)\right]=(3x+1)^3(3-x)^4(-27x+31)$

Let $f'(x)=0$ and solve for x.

$(3x+1)^3(3-x)^4(31-27x)=0\Rightarrow x=-\frac{1}{3}$ or $x=3$ or $x=\frac{31}{27}$

14. $f(x)=\frac{x^2+1}{x^2+5}$

$f'(x)=\frac{(x^2+5)(2x)-(x^2+1)(2x)}{(x^2+5)^2}=\frac{2x^3+10x-2x^3-2x}{(x^2+5)^2}=\frac{8x}{(x^2+5)^2}$

Let $f'(x)=0$ and solve for x.

$\frac{8x}{(x^2+5)^2}=0\Rightarrow x=0$

15. $y=(x^3-1)(x^2+1)^4$

slope = y'

$y'=(x^3-1)(4)(x^2+1)^3(2x)+(x^2+1)^4(3x^2)=3x^2(x^2+1)^4+8x(x^2+1)^3(x^3-1)$

When $x=-1$,

slope $=3(-1)^2(1+1)^4+8(-1)(1+1)^3(-1-1)=48+128=176$

When $x=-1$, $y=(-1-1)(1+1)^4=-32$.

Let $(x_1,\ y_1)=(-1,\ -32)$. Then $y+32=176(x+1)$ or $y=176x+144$.

16. $y = \dfrac{x-3}{\sqrt{4+x^2}}$

slope = y'

$$y' = \frac{(4+x^2)^{1/2}(1) - (x-3)(\frac{1}{2})(4+x^2)^{-1/2}(2x)}{4+x^2}$$
$$= \frac{(4+x^2)^{1/2}}{4+x^2} - \frac{(x^2-3x)}{(4+x^2)^{3/2}}$$
$$= \frac{4+x^2-x^2+3x}{(4+x^2)^{3/2}} = \frac{3x+4}{(4+x^2)^{3/2}}$$

When $x = 0$, $y = -\dfrac{3}{2}$. When $x = 0$, slope $= \dfrac{4}{8} = \dfrac{1}{2}$. Let $(x_1, y_1) = \left(0, -\dfrac{3}{2}\right)$. Then $y + \dfrac{3}{2} = \dfrac{1}{2}(x-0)$ or $y = \dfrac{1}{2}x - \dfrac{3}{2}$.

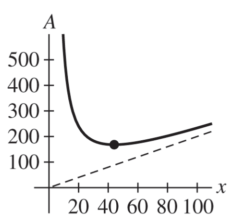
17. The objective is $A = 2y(2) + (x-4)(2) = 4y + 2x - 8$. The constraint is $(x-4)(y-2) = 800$ or

$$y = \frac{800}{x-4} + 2.$$

Thus,

$$A(x) = 4\left(\frac{800}{x-4} + 2\right) + 2x - 8 = \frac{3200}{x-4} + 2x$$
$$A'(x) = -\frac{3200}{(x-4)^2} + 2 \Rightarrow A'(44) = 0$$

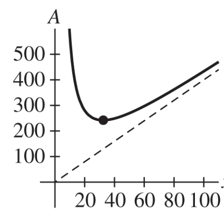
The minimum value of $A(x)$ for $x > 0$ occurs at $x = 44$. Thus, the optimal values of x and y are $x = 44$ m, $y = \dfrac{800}{44-4} + 2 = 22$ m.



18. The objective is $A = 2(x-4)(2) + 2(y)(2) = 4x - 16 + 4y$ and the constraint is $(x-4)(y-4) = 800$ or

$$y = \frac{800}{x-4} + 4.$$
$$A(x) = 4x + 4\left(\frac{800}{x-4} + 4\right) - 16 = 4x + \frac{3200}{x-4}$$
$$A'(x) = 4 - \frac{3200}{(x-4)^2} \Rightarrow A'\left(4 + 20\sqrt{2}\right) = 0$$

The minimum value of $A(x)$ occurs at $x = 4 + 20\sqrt{2}$. The optimal dimensions are $x = y = 4 + 20\sqrt{2} \approx 32.3$ meters.



19. We are given that $C(x) = 40x + 30$ and $\dfrac{dx}{dt} = 3.$

By the chain rule, $\dfrac{dC}{dt} = \dfrac{dC}{dx}\dfrac{dx}{dt} = 40 \cdot 3 = 120$

Costs are rising at \$120 per day.

20. $\dfrac{dy}{dt} = \dfrac{dy}{dP} \cdot \dfrac{dP}{dt}$

21. $f'(x) = \dfrac{1}{(x^2+1)}$; $g(x) = x^3$; $g'(x) = 3x^2$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$
$$= f'(x^3)g'(x)$$
$$= \frac{1}{(x^3)^2 + 1}(3x^2) = \frac{3x^2}{x^6+1}$$

22. $f'(x) = \dfrac{1}{(x^2+1)}$; $g(x) = \dfrac{1}{x}$; $g'(x) = -\dfrac{1}{x^2}$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$
$$= f'\left(\frac{1}{x}\right)g'(x) = \frac{1}{\left(\frac{1}{x}\right)^2 + 1}\left(-\frac{1}{x^2}\right)$$
$$= \frac{1}{\frac{1+x^2}{x^2}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}$$

23. $f'(x) = \dfrac{1}{(x^2+1)}$; $g(x) = x^2 + 1$; $g'(x) = 2x$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$
$$= f'(x^2+1)g'(x)$$
$$= \frac{1}{(x^2+1)^2 + 1}(2x) = \frac{2x}{(x^2+1)^2 + 1}$$

24. $f'(x) = x\sqrt{1-x^2};\ g(x) = x^2;\ g'(x) = 2x$

$$\begin{aligned}\frac{d}{dx} f(g(x)) &= f'(g(x))g'(x)\\ &= f'(x^2)g'(x)\\ &= x^2\sqrt{1-(x^2)^2}\,(2x) = 2x^3\sqrt{1-x^4}\end{aligned}$$

25. $f'(x) = x\sqrt{1-x^2};\ g(x) = \sqrt{x};\ g'(x) = \dfrac{1}{2\sqrt{x}}$

$$\begin{aligned}\frac{d}{dx} f(g(x)) &= f'(g(x))g'(x)\\ &= f'(\sqrt{x})g'(x)\\ &= \sqrt{x}\sqrt{1-(\sqrt{x})^2}\,\frac{1}{2\sqrt{x}}\\ &= \frac{1}{2}\sqrt{1-x}\end{aligned}$$

26. $f'(x) = x\sqrt{1-x^2};\ g(x) = x^{3/2};\ g'(x) = \dfrac{3}{2}\sqrt{x}$

$$\begin{aligned}\frac{d}{dx} f(g(x)) &= f'(g(x))g'(x)\\ &= f'(x^{3/2})g'(x)\\ &= x^{3/2}\sqrt{1-(x^{3/2})^2}\left(\frac{3}{2}\sqrt{x}\right)\\ &= \frac{3}{2}x^2\sqrt{1-x^3}\end{aligned}$$

27. $\dfrac{dy}{du} = \dfrac{u}{u^2+1},\ u = x^{3/2},\ \dfrac{du}{dx} = \dfrac{3}{2}x^{1/2},$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{u}{u^2+1}\right)\frac{3}{2}x^{1/2}$$

Substitute $x^{3/2}$ for u.

$$\frac{dy}{dx} = \frac{x^{3/2}}{x^3+1}\left(\frac{3}{2}x^{1/2}\right) = \frac{3x^2}{2(x^3+1)}$$

28. $\dfrac{dy}{du} = \dfrac{u}{u^2+1},\ u = x^2+1,\ \dfrac{du}{dx} = 2x,$

$$\frac{dy}{dx} = \left(\frac{u}{u^2+1}\right)(2x)$$

Substitute (x^2+1) for u.

$$\frac{dy}{dx} = \frac{(x^2+1)}{(x^2+1)^2+1}(2x) = \frac{2x(x^2+1)}{x^4+2x+2}$$

29. $\dfrac{dy}{du} = \dfrac{u}{u^2+1},\ u = \dfrac{5}{x},\ \dfrac{du}{dx} = -\dfrac{5}{x^2}$

$$\frac{dy}{dx} = \frac{\frac{5}{x}}{\frac{25}{x^2}+1}\left(-\frac{5}{x^2}\right) = -\frac{25}{x(25+x^2)}$$

30. $\dfrac{dy}{du} = \dfrac{u}{\sqrt{1+u^4}},\ u = x^2,\ \dfrac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{1+x^8}}(2x) = \frac{2x^3}{\sqrt{1+x^8}}$$

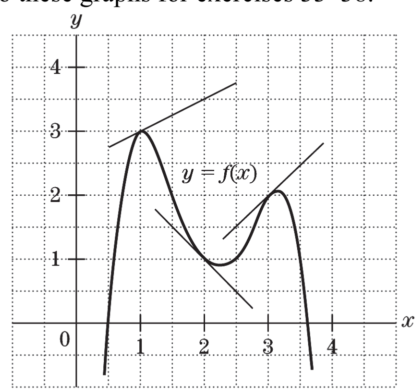
31. $\dfrac{dy}{du} = \dfrac{u}{\sqrt{1+u^4}},\ u = \sqrt{x},\ \dfrac{du}{dx} = \dfrac{1}{2}x^{-1/2}$

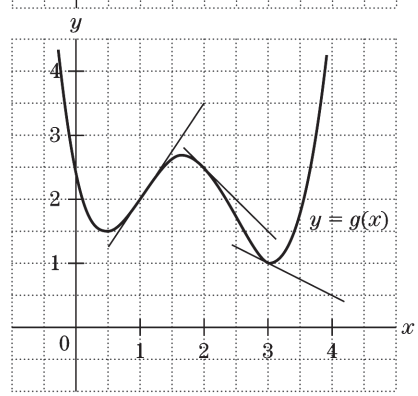
$$\frac{dy}{dx} = \frac{x^{1/2}}{\sqrt{1+x^2}}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{1+x^2}}$$

32. $\dfrac{dy}{du} = \dfrac{u}{\sqrt{1+u^4}},\ u = \dfrac{2}{x},\ \dfrac{du}{dx} = \dfrac{-2}{x^2}$

$$\frac{dy}{dx} = \frac{\frac{2}{x}}{\sqrt{1+\frac{16}{x^4}}}\left(-\frac{2}{x^2}\right) = -\frac{4}{x^3\sqrt{1+\frac{16}{x^4}}}$$

Refer to these graphs for exercises 33–38.





33. $f(1) = 3,\ f'(1) = \dfrac{1}{2},\ g(1) = 2,\ g'(1) = \dfrac{3}{2}$

$$h(x) = 2f(x) - 3g(x)$$

$$h(1) = 2f(1) - 3g(1) = 2(3) - 3(2) = 0$$

$$h'(1) = 2f'(1) - 3g'(1) = 2\left(\frac{1}{2}\right) - 3\left(\frac{3}{2}\right) = -\frac{7}{2}$$

34. $h(x) = f(x) \cdot g(x)$

$h'(x) = f(x)g'(x) + g(x)f'(x)$

$h(1) = 3(2) = 6;\ h'(1) = 3\left(\frac{3}{2}\right) + (2)\left(\frac{1}{2}\right) = \frac{11}{2}$

35. $h(x) = \frac{f(x)}{g(x)},\ h'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{[g(x)]^2}$,

$h(1) = \frac{3}{2},\ \ h'(x) = \frac{2\left(\frac{1}{2}\right) - 3\left(\frac{3}{2}\right)}{2^2} = -\frac{7}{8}$

36. $h(x) = [f(x)]^2,\ h'(x) = 2f(x)f'(x),$

$h(1) = 3^2 = 9,\ h'(1) = 2(3)\left(\frac{1}{2}\right) = 3$

37. $h(x) = f(g(x)),\ h'(x) = f'(g(x))g'(x)$

$g(1) = 2, f(2) = 1,\ f'(2) = -1,\ \ g'(1) = \frac{3}{2}$

$h(1) = 1,\ h'(1) = f'(2)g'(1) = -1\left(\frac{3}{2}\right) = -\frac{3}{2}$

38. $h(x) = g(f(x)),\ h'(x) = g'(f(x))f'(x)$

$f(1) = 3,\ f'(1) = \frac{1}{2},\ g(3) = 1,\ g'(3) = -\frac{1}{2}$

$h(1) = 1,\ h'(1) = g'(3)f'(1) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$

39. **a.** $\frac{dR}{dA}, \frac{dA}{dt}, \frac{dR}{dx}, \frac{dx}{dA}$

b. $\frac{dR}{dt} = \frac{dR}{dx}\frac{dx}{dA}\frac{dA}{dt}$

40. **a.** $\frac{dP}{dt}, \frac{dA}{dP}, \frac{dS}{dP}, \frac{dA}{dS}$

b. $\frac{dA}{dt} = \frac{dA}{dS} \cdot \frac{dS}{dP} \cdot \frac{dP}{dt}$

41. $x^{2/3} + y^{2/3} = 8$

a. $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2}{3}x^{-1/3} \cdot \frac{3}{2}y^{1/3} = -\frac{y^{1/3}}{x^{1/3}}$

b. slope $= \frac{dy}{dx}$

When $x = 8$, $y = -8$, slope = 1.

42. $x^3 + y^3 = 9xy$

a. $3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y$

$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$ or $\frac{x^2 - 3y}{3x - y^2}$

b. slope $= \frac{dy}{dx}$

When $x = 2$, $y = 4$, slope $= \frac{3(4) - 4}{16 - 6} = \frac{4}{5}$.

43. $x^2y^2 = 9$

$2xy^2 + 2yx^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2xy^2}{2yx^2} = -\frac{y}{x}$

When $x = 1$, $y = 3$, $\frac{dy}{dx} = -3$.

44. $xy^4 = 48$

$y^4 + 4y^3x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^4}{4y^3x} = -\frac{y}{4x}$

When $x = 3$, $y = 2$, $\frac{dy}{dx} = \frac{-2}{4(3)} = -\frac{1}{6}$.

45. $x^2 - xy^3 = 20$

$2x - \left(y^3 + 3y^2x\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{2x - y^3}{3y^2x}$

When $x = 5$, $y = 1$, $\frac{dy}{dx} = \frac{10 - 1}{3(1)(5)} = \frac{3}{5}$.

46. $xy^2 - x^3 = 10$

$\left(y^2 + 2yx\frac{dy}{dx}\right) - 3x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{3x^2 - y^2}{2yx}$

When $x = 2$, $y = 3$, $\frac{dy}{dx} = \frac{3(4) - 9}{2(3)(2)} = \frac{1}{4}$.

47. $y^2 - 5x^3 = 4$

a. $2y\frac{dy}{dx} - 15x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{15x^2}{2y}$

b. When $x = 4$ and $y = 18$,
$\frac{dy}{dx} = \frac{15(16)}{2(18)} = \frac{20}{3}$ (thousand dollars per thousand-unit increase in production).

c. $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{15x^2}{2y}\frac{dx}{dt}$

d. When $x = 4$, $y = 18$, and $\frac{dx}{dt} = .3$,

$\frac{dy}{dt} = \frac{15(16)}{2(18)}(.3) = 2$ (thousand dollars per week).

48. $y^3 - 8000x^2 = 0$

a. $3y^2\frac{dy}{dx} - 16,000x = 0 \Rightarrow \frac{dy}{dx} = \frac{16,000x}{3y^2}$

b. When $x = 27$, $y = 180$,

$\frac{dy}{dx} = \frac{16,000(27)}{3(32,400)} = \frac{40}{9} \approx 4.44$ (thousand books per person).

c. $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{16,000x}{3y^2}\frac{dx}{dt}$

d. When $x = 27$, $y = 180$, and $\frac{dx}{dt} = 1.8$,

$\frac{dy}{dt} = \frac{16,000(27)}{3(32,400)}(1.8) = 8$ (thousand books per year).

49. $6p + 5x + xp = 50$

$6 + 5\frac{dx}{dp} + p\frac{dx}{dp} + x = 0 \Rightarrow \frac{dx}{dp} = \frac{-x-6}{5+p}$

$\frac{dx}{dt} = \frac{dx}{dp}\frac{dp}{dt} = \left(\frac{-x-6}{5+p}\right)\frac{dp}{dt}$

When $x = 4$, $p = 3$, and $\frac{dp}{dt} = -2$,

$\frac{dx}{dt} = \left(\frac{-4-6}{5+3}\right)(-2) = 2.5$.

The quantity x is increasing at the rate of 2.5 units per unit time.

50. $V = .005\pi r^2$, so

$\frac{dV}{dt} = .005\pi(2)r\frac{dr}{dt} = .01\pi r\frac{dr}{dt}$.

Now, $\frac{dV}{dt} = 20$, so $20 = .01\pi r\frac{dr}{dt} \Rightarrow$

$\frac{dr}{dt} = \frac{2000}{\pi r}$.

When $r = 50$, $\frac{dr}{dt} = \frac{2000}{50\pi} = \frac{40}{\pi}$ m/hr, or approximately 12.73 meters per hour.

51. $S = .1W^{2/3}$, so

$\frac{dS}{dt} = \frac{2}{3}(.1)W^{-1/3}\frac{dW}{dt} = \frac{.2}{3}W^{-1/3}\frac{dW}{dt}$.

When $W = 350$ and $\frac{dW}{dt} = 200$,

$\frac{dS}{dt} = \frac{.2}{3\sqrt[3]{350}}(200) = \frac{40}{3\sqrt[3]{350}} \approx 1.89\ \text{m}^2/\text{yr}$.

52. $xy - 6x + 20y = 0$,

$\frac{dx}{dt}y + \frac{dy}{dt}x - 6\frac{dx}{dt} + 20\frac{dy}{dt} = 0$

Currently, $x = 10$, $y = 2$, $\frac{dx}{dt} = 1.5$. Thus,

$1.5(2) + \frac{dy}{dt}(10) - 6(1.5) + 20\frac{dy}{dt} = 0$, so

$30\frac{dy}{dt} = 6 \Rightarrow \frac{dy}{dt} = .2$ or 200 dishwashers/month.