Chapter 10 Differential Equations

10.1 Solutions of Differential Equations

1. The differential equation says that $y' - 2ty = t$ for all values of *t*. We must show that this result holds if *y* is replaced by $\frac{3}{2}e^{t^2} - \frac{1}{2}$ and y' is replaced by $\left(\frac{3}{2}e^{t^2} - \frac{1}{2}\right)'$.

$$y = f(t) = \frac{3}{2}e^{t^2} - \frac{1}{2} \Rightarrow y' = 3te^{t^2}$$

$$y' - 2ty = 3te^{t^2} - 2t\left[\frac{3}{2}e^{t^2} - \frac{1}{2}\right] = t$$

Thus, $f(t) = \frac{3}{2}e^{t^2} - \frac{1}{2}$ is a solution of the differential equation.

2. The differential equation says that $(y')^2 - 4y = 2$ for all values of *t*. We must show that this result holds if *y* is replaced by $t^2 - \frac{1}{2}$ and y' is replaced by $\left(t^2 - \frac{1}{2}\right)'$.

$$y = f(t) = t^2 - \frac{1}{2} \Rightarrow y' = 2t$$

$$(y')^2 - 4y = (2t)^2 - 4\left[t^2 - \frac{1}{2}\right] = 2$$

Thus, $f(t) = t^2 - \frac{1}{2}$ is a solution of the differential equation.

3. The differential equation says that $y'' - 3y' + 2y = 0,\ y(0) = 5,\ y'(0) = 10.$ We must show that this result holds if *y* is replaced by $5e^{2t}$, y' is replaced by $\left(5e^{2t}\right)'$, and y'' is replaced by $\left(5e^{2t}\right)''$.

$$y = f(t) = 5e^{2t} \Rightarrow y' = 10e^{2t} \Rightarrow y'' = 20e^{2t}$$

$$y'' - 3y' + 2y = 20e^{2t} - 3(10e^{2t}) + 2(5e^{2t}) = 0$$

$$y(0) = 5e^0 = 5,\ y'(0) = 10e^0 = 10$$

Thus, $f(t) = 5e^{2t}$ is a solution of the differential equation.

4. The differential equation says that $y' + y^2 = y,\ y(0) = \frac{1}{2}$. We must show that this result holds if *y* is replaced by $\left(e^{-t} + 1\right)^{-1}$ and y' is replaced by $\left(\left(e^{-t} + 1\right)^{-1}\right)'$.

$$y = f(t) = (e^{-t} + 1)^{-1} \Rightarrow y' = \frac{e^{-t}}{(e^{-t} + 1)^2}$$

$$y' + y^2 = \frac{e^{-t}}{(e^{-t} + 1)^2} + \frac{1}{(e^{-t} + 1)^2} = \frac{1}{e^{-t} + 1} = y$$

$$y(0) = \frac{1}{e^0 + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

Thus, $f(t) = \left(e^{-t} + 1\right)^{-1}$ is a solution of the differential equation.

5. The highest derivative is the second derivative y'', so the differential equation is of second order.

$$y(t) = t \Rightarrow y'(t) = 1 \Rightarrow y''(t) = 0$$

$$(1 - t^2)y'' - 2ty' + 2y = (1 - t^2)(0) - 2t(1) + 2(t) = 0$$

6. The highest derivative is the second derivative y'', so the differential equation is of second order.

$$y(t) = \frac{1}{2}(3t^2 - 1) \Rightarrow y'(t) = 3t \Rightarrow y''(t) = 3$$

$$(1 - t^2)y'' - 2ty' + 6y = (1 - t^2)(3) - 2t(3t) + 6\left[\frac{1}{2}(3t^2 - 1)\right] = 0$$

7. Yes. Given $y = f(t) = 3$, $y' = 0 = 6 - 2(3)$.

8. Yes. Given $y = f(t) = -4$, $y' = 0 = t^2(-4 + 4)$ for all *t*.

9. $y' = t^2 y - 5t^2 = t^2(y - 5)$; $y = 5$ is a constant solution. $\frac{dy}{dt} = 0$, and $t^2(y - 5) = 0$ at $y = 5$.

10. $y' = 4y(y - 7)$; $y = 0$ or $y = 7$ are constant solutions. $\frac{dy}{dt} = 0$, and $4y(y - 7) = 0$ at $y = 0$ and $y = 7$.

11. $f(0) = y(0) = 4$. Since $y' = 2y - 3$, we have in particular $y'(0) = 2y(0) - 3 = 2(4) - 3 = 5$.

12. $f(0) = y(0) = 0$. Since $y' = e^t + y$, we have in particular $y'(0) = e^0 + y(0) = 1 + 0 = 1$.

13. $y' = .2(160 - y)$. When $y = 60$,
$y' = .2(160 - 60) = .2(100) = 20$ ft/sec/sec

14. $y' = 0.0004y(1000-y)$.

When $y = \frac{1}{2}(1000) = 500$,

$y' = .0004(500)(1000-500) = 100$. So the fish population is growing at a rate of 100 fish per month at the time there are 500 fish in the lake.

15. $y' = 0.05y - 10{,}000$.

a. When $t = 1$, $y = 150{,}000$.
$y' = .05(150{,}000) - 10{,}000 = -2500$
So the balance is decreasing at a rate of \$2500 per year after 1 year.

b. $y' = .05(y - 200{,}000)$

c. The rate of change of the savings account balance is proportional to the difference between the balance at the end of t years and \$200,000.

16. $y' = .04y + 2000$

a. When $t = 1$, $y = 10{,}000$.
$y' = .04(10{,}000) + 2000 = 2400$
The balance is increasing at a rate of \$2400 per year after 1 year.

b. $y' = .04(y + 50{,}000)$

c. The rate of change of the savings account balance is proportional to the sum of the balance at the end of t years and \$50,000.

17. The number of people who have heard the news broadcast after t hours is increasing at a rate that is proportional to the difference between that number and 200,000. At the beginning of the broadcast there are 10 people tuned in.

18. The size of the paramecium population after t days is increasing at a rate that is proportional to the difference between that number and 500. There are 20 paramecia at the beginning of the count.

19. $y' = k(C-y)$, $k > 0$

20. $y' = k(20-y)$, $k > 0$

21. $y' = k(P_b - y)$, $y(0) = P_0$, $k > 0$

22. **a.** $\frac{dy}{dt} = t - y;\ \left.\frac{dy}{dt}\right|_{\substack{t=0\\y=-1}} = 0-(-1) = 1.$

The line through $(0, -1)$ with slope 1 is
$y-(-1) = 1(t-0) \Rightarrow y = t-1$

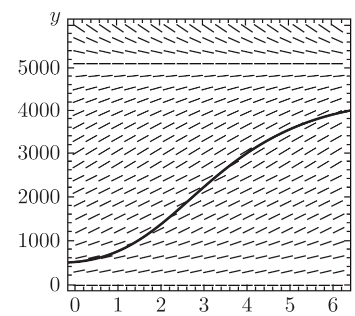
b. $y = t - 1 \Rightarrow y' = 1$
$t - y = t - (t-1) = 1 = y'$, so $y' = t - y$

23. $y = f(t) = 2e^{-t} + t - 1 \Rightarrow y' = -2e^{-t} + 1$

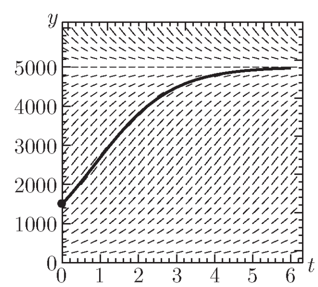
$t - y = t - (2e^{-t} + t - 1) = -2e^{-t} + 1 = y'$, so $f(t)$ is a solution of the differential equation.

$y(0) = f(0) = 2e^0 + 0 - 1 = 1$, so $f(t)$ satisfies the initial value condition.

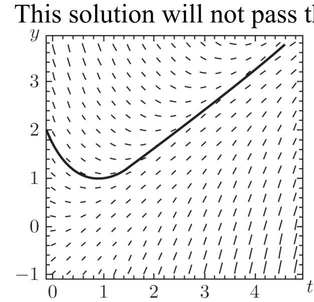
24.



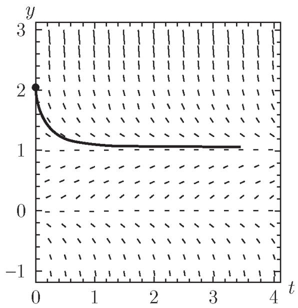
25. The solution to $y' = .0002y(5000-y)$, $y(0) = 1500$ approaches $y = 5000$ asymptotically from below. So $f(t)$ will never exceed 5000.



26. This solution will not pass through (0.5, 2.2).



27. The slope field indicates that $y = 0$ and $y = 1$ may be the constant solutions.



Check: $y = 0,\ y' = 0.$

So $2y(1-y) = 2(0)(1-0) = 0 = y'.$

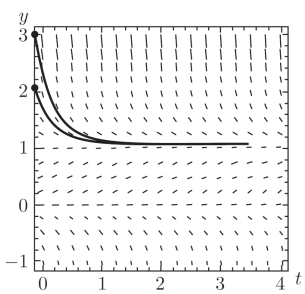
Thus $y = 0$ is a solution.

Check: $y = 1,\ y' = 0.$

So $2y(1-y) = 2(1)(1-1) = 0 = y'.$

Thus $y = 1$ is also a solution.

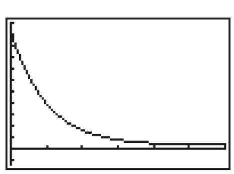
28.



29. Yes. The solution will approach $y = 1$ asymptotically from above, and is decreasing for all $t > 0$.

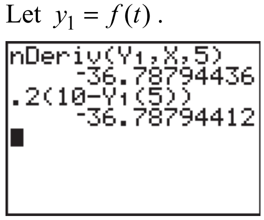
30. Since $y_0 > 1, 1 - y_0 < 0,$ so $2y_0(1-y_0) < 0.$ Thus $y'(0)$ is negative. The solution of the initial value problem will approach $y = 1$ asymptotically from above, so $y(t) > 1$ for all $t \ge 0$. This implies $1 - y < 0$, so $y' = 2y(1-y) < 0$ for all $t \ge 0$. $f'(t)$ is negative for all $t \ge 0$.

31. a.

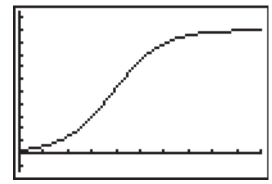


$[0, 30]$ *by* $[-75, 550]$

b. Let $y_1 = f(t)$.

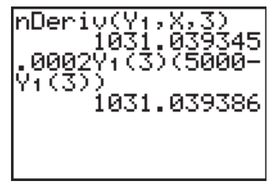


32. a.



$[0, 10]$ by $[-750, 5750]$

b. Let $y_1 = f(t)$.



10.2 Separation of Variables

1.
$$\frac{dy}{dt} = \frac{5-t}{y^2} \Rightarrow y^2\frac{dy}{dt} = 5 - t$$
$$\int y^2 dy = \int (5-t)\ dt$$
$$\frac{y^3}{3} + C_1 = 5t - \frac{t^2}{2} + C_2$$
$$y = \sqrt[3]{15t - \frac{3}{2}t^2 + C}$$

2.
$$y' = te^{2y} \Rightarrow \frac{dy}{dt}e^{-2y} = t$$
$$\int e^{-2y} dy = \int t\ dt$$
$$-\frac{1}{2}e^{-2y} + C_1 = \frac{t^2}{2} + C_2$$
$$e^{-2y} = -t^2 + C \Rightarrow y = -\frac{1}{2}\ln(-t^2 + C)$$

3.
$$\frac{dy}{dt} = \frac{e^t}{e^y} \Rightarrow \frac{dy}{dt}e^y = e^t \Rightarrow \int e^y dy = \int e^t dt \Rightarrow$$
$$e^y + C_1 = e^t + C_2 \Rightarrow y = \ln\left|e^t + C\right|$$

4.
$$\frac{dy}{dt} = \frac{-1}{y^2t^2} \Rightarrow y^2\frac{dy}{dt} = \frac{-1}{t^2}$$
$$\int y^2 dy = \int \frac{-1}{t^2} dt$$
$$\frac{y^3}{3} + C_1 = \frac{1}{t} + C_2 \Rightarrow y = \sqrt[3]{\frac{3}{t} + C}$$

5. First check for constant solutions; if $y = 0$, then $\frac{dy}{dt} = 0$ and $t^{1/2}y^2 = 0$; otherwise assume $y \neq 0$.

$y' = t^{1/2}y^2 \Rightarrow y'y^{-2} = t^{1/2}$ (Assuming $y \neq 0$)

$$\int y^{-2}dy = \int t^{1/2}dt \Rightarrow -\frac{1}{y} + C_1 = \frac{2}{3}t^{3/2} + C_2 \Rightarrow$$

$$y = \frac{1}{C - \left(\frac{2}{3}\right)t^{3/2}}$$

The solutions are $y = \frac{1}{C - \left(\frac{2}{3}\right)t^{3/2}}$ or $y = 0$.

6. First check for constant solutions; if $y = 0$, then $\frac{dy}{dt} = 0$ and $\frac{t^2y^2}{t^3+8} = 0$. If $y \neq 0$,

$$y' = \frac{t^2y^2}{t^3+8} \Rightarrow \frac{dy}{dt}y^{-2} = \frac{t^2}{t^3+8}$$

$$\int y^{-2}dy = \int \frac{t^2}{t^3+8}dt$$

$$-y^{-1} + C_1 = \frac{1}{3}\ln\left|t^3+8\right| + C_2$$

$$y = -\frac{3}{\ln\left|t^3+8\right| + C}$$

The solutions are $y = -\frac{3}{\ln\left|t^3+8\right| + C}$ and $y = 0$.

7.
$$y' = \frac{e^{t^3}t^2}{y^2} \Rightarrow y^2\frac{dy}{dt} = t^2e^{t^3}$$

$$\int y^2dy = \int t^2e^{t^3}dt$$

$$\frac{y^3}{3} + C_1 = \frac{e^{t^3}}{3} + C_2$$

$$y = \sqrt[3]{e^{t^3} + C}$$

8.
$$y' = e^{4y}t^3 - e^{4y} = e^{4y}(t^3-1)$$

$$\frac{dy}{dt}e^{-4y} = t^3 - 1$$

$$\int e^{-4y}dy = \int (t^3-1)\ dt$$

$$-\frac{1}{4}e^{-4y} + C_1 = \frac{t^4}{4} - t + C_2$$

$$e^{-4y} = -t^4 + 4t + C$$

$$y = -\frac{1}{4}\ln(-t^4 + 4t + C)$$

9. $\frac{dy}{dt} = \sqrt{\frac{y}{t}} = \frac{\sqrt{y}}{\sqrt{t}} \Rightarrow y^{-1/2}\frac{dy}{dt} = t^{-1/2}$

(Assuming $y \neq 0$)

$$\int y^{-1/2}dy = \int t^{-1/2}dt$$

$$2y^{1/2} + C_1 = 2t^{1/2} + C_2 \Rightarrow y = (\sqrt{t} + C)^2$$

$y = 0$ is also a solution.

10.
$$\frac{dy}{dt} = \left(\frac{e^t}{y}\right)^2 \Rightarrow \frac{dy}{dt}y^2 = e^{2t}$$

$$\int y^2dy = \int e^{2t}dt$$

$$\frac{y^3}{3} + C_1 = \frac{1}{2}e^{2t} + C_2$$

$$y = \sqrt[3]{\frac{3}{2}e^{2t} + C}$$

11. $y' = 3t^2y^2 \Rightarrow y^{-2}\frac{dy}{dt} = 3t^2$ (assuming $y \neq 0$)

$$\int \frac{1}{y^2}dy = \int 3t^2dt \Rightarrow -\frac{1}{y} + C_1 = t^3 + C_2 \Rightarrow$$

$$y = -\frac{1}{t^3 + C}$$

$y = 0$ is also a solution.

12. $(1+t^2)y' = ty^2 \Rightarrow y^{-2}\frac{dy}{dt} = \frac{t}{1+t^2}$

(assuming $y \neq 0$)

$$\int \frac{1}{y^2}dy = \int \frac{t}{1+t^2}dt$$

$$-\frac{1}{y} + C_1 = \frac{1}{2}\ln\left(1+t^2\right) + C_2$$

$$y = -\frac{2}{\ln\left(1+t^2\right) + C}$$

$y = 0$ is also a solution.

13. $y'e^y = te^{t^2} \Rightarrow e^y\frac{dy}{dt} = te^{t^2}$

$$\int e^ydy = \int te^{t^2}dt \Rightarrow e^y + C_1 = \frac{1}{2}e^{t^2} + C_2 \Rightarrow$$

$$y = \ln\left(\frac{1}{2}e^{t^2} + C\right)$$

14. $y' = \frac{1}{ty+y} = \frac{1}{y(t+1)} \Rightarrow \frac{dy}{dt}y = \frac{1}{t+1}$

$$\int y\ dy = \int \frac{1}{t+1}\ dt$$

$$\frac{y^2}{2} + C_1 = \ln|t+1| + C_2$$

$$y = \pm\sqrt{2\ln|t+1| + C}$$

15. $y' = \frac{\ln t}{ty} \Rightarrow y\frac{dy}{dt} = \frac{\ln t}{t} \Rightarrow \int y\ dy = \int \frac{\ln t}{t}\ dt \Rightarrow$

$\frac{y^2}{2} + C_1 = \frac{(\ln t)^2}{2} + C_2 \Rightarrow y = \pm\sqrt{(\ln t)^2 + C}$

16. $y^2 y' = \tan t \Rightarrow y^2 \frac{dy}{dt} = \tan t \Rightarrow$

$\int y^2 dy = \int \tan t\, dt$

$\frac{1}{3}y^3 + C_1 = -\ln|\cos t| + C_2$

$y = \sqrt[3]{-3\ln|\cos t| + C}$

17. $y' = (y-3)^2 \ln t \Rightarrow (y-3)^{-2}\frac{dy}{dt} = \ln t$

(Assuming $y \neq 3$)

$\int (y-3)^{-2} dy = \int \ln t\ dt$

$-(y-3)^{-1} + C_1 = t\ln t - t + C_2$

$y = 3 + \frac{1}{t - t\ln t + C}$

$y = 3$ is also a solution.

18. $yy' = t\sin(t^2+1)$

$\int y\ dy = \int t\sin(t^2+1)\ dt$

$\frac{y^2}{2} + C_1 = -\frac{1}{2}\cos(t^2+1) + C_2$

$y = \pm\sqrt{-\cos(t^2+1) + C}$

19. $y' = 2te^{-2y} - e^{-2y}$, $y(0) = 3$

$e^{2y}\frac{dy}{dt} = 2t - 1$

$\int e^{2y} dy = \int (2t-1)\ dt$

$\frac{1}{2}e^{2y} + C_1 = t^2 - t + C_2$

$e^{2y} = 2t^2 - 2t + C$

$y = \frac{1}{2}\ln(2t^2 - 2t + C)$

$y(0) = 3 = \frac{1}{2}\ln C$, so $C = e^6$ and the solution is $y = \frac{1}{2}\ln(2t^2 - 2t + e^6)$.

20. $y' = y^2 - e^{3t}y^2 = y^2(1 - e^{3t}),\ y(0) = 1$

Assuming $y \neq 0$

$y'y^{-2} = (1 - e^{3t})$

$\int y^{-2} dy = \int (1 - e^{3t})\ dt$

$-\frac{1}{y} + C_1 = t - \frac{1}{3}e^{3t} + C_2$

$y = \frac{3}{e^{3t} - 3t + C}$

Given that $y(0) = 1 = \frac{3}{1+C} \Rightarrow C = 2.$

$y = \frac{3}{e^{3t} - 3t + 2}$

21. $y^2 y' = t\cos t$, $y(0) = 2$

$\int y^2 dy = \int t\cos t\ dt$

$\frac{y^3}{3} + C_1 = t\sin t + \cos t + C_2$

$y = \sqrt[3]{3t\sin t + 3\cos t + C}$

$y(0) = 2 = \sqrt[3]{3 + C}$ so $C = 5$ and the solution is $y = \sqrt[3]{3t\sin t + 3\cos t + 5}$.

22. $y' = t^2 e^{-3y}$, $y(0) = 2$

$e^{3y}\frac{dy}{dt} = t^2 \Rightarrow \int e^{3y}\ dy = \int t^2 dt$

$\frac{1}{3}e^{3y} + C_1 = \frac{t^3}{3} + C_2$

$e^{3y} = t^3 + C$

$y = \frac{\ln(t^3 + C)}{3}$

$y(0) = 2 = \frac{\ln(0^3 + C)}{3}$

So $C = e^6$ and the solution is $y = \frac{\ln(t^3 + e^6)}{3}$.

23. $3y^2 y' = -\sin t,\ y\left(\frac{\pi}{2}\right) = 1$

$3y^2\frac{dy}{dt} = -\sin t \Rightarrow \int 3y^2 dy = \int (-\sin t)\, dt$

$y^3 + C_1 = \cos t + C_2 \Rightarrow y = \sqrt[3]{\cos t + C}$

$y\left(\frac{\pi}{2}\right) = 1 = \sqrt[3]{\cos\frac{\pi}{2} + C}$

so $C = 1$ and the solution is $y = \sqrt[3]{\cos t + 1}$.

24. $y' = -y^2 \sin t,\ y\left(\frac{\pi}{2}\right) = 1$

$\frac{1}{y^2}\frac{dy}{dt} = -\sin t$ (assuming $y \neq 0$)

$\int \frac{1}{y^2}\,dy = \int (-\sin t)\,dt$

$-\frac{1}{y} + C_1 = \cos t + C_2 \Rightarrow y = -\frac{1}{\cos t + C}$

$y\left(\frac{\pi}{2}\right) = 1 = -\frac{1}{\cos\frac{\pi}{2} + C}$, so $C = -1$ and the solution is $y = \frac{1}{1-\cos t}$.

25. $\frac{dy}{dt} = \frac{t+1}{ty},\ t > 0, y(1) = -3.$

$y\frac{dy}{dt} = \frac{t+1}{t} \Rightarrow \int y\ dy = \int\left(1+\frac{1}{t}\right)dt$

$\frac{y^2}{2} = t + \ln|t| + C_1$

$y = \pm\sqrt{2t + 2\ln|t| + C}$

Since $y(1) = -3$, the negative solution applies and $-3 = -\sqrt{2+C}$, so $C = 7$ and the solution is $y = -\sqrt{2t + 2\ln t + 7}$.

26. $y' = \left(\frac{1+t}{1+y}\right)^2,\ y(0) = 2$

$(1+y)^2\frac{dy}{dt} = (1+t)^2$

$\int (1+y)^2 dy = \int (1+t)^2 dt$

$\frac{(1+y)^3}{3} = \frac{(1+t)^3}{3} + C_1$

$y = \sqrt[3]{(1+t)^3 + C} - 1$

$y(0) = 2 = \sqrt[3]{1+C} - 1$, so $C = 26$ and the solution is $y = \sqrt[3]{(1+t)^3 + 26} - 1$.

27. $y' = 5ty - 2t,\ y(0) = 1$

$y' = 5t\left(y - \frac{2}{5}\right)$

$\left(y - \frac{2}{5}\right)^{-1}\frac{dy}{dt} = 5t$

$\int\left(y-\frac{2}{5}\right)^{-1}dy = \int 5t\ dt$

$\ln\left|y - \frac{2}{5}\right| = \frac{5}{2}t^2 + C$

$y = \frac{2}{5} + Ae^{(5/2)t^2}$

$y(0) = 1 = \frac{2}{5} + A$, so $A = \frac{3}{5}$ and the solution is $y = \frac{2}{5} + \frac{3}{5}e^{(5/2)t^2}$.

28. $y' = \frac{t^2}{y},\ y(0) = -5$

$y\frac{dy}{dt} = t^2 \Rightarrow \int y\ dy = \int t^2 dt \Rightarrow$

$\frac{y^2}{2} + C_1 = \frac{t^3}{3} + C_2 \Rightarrow y = \pm\sqrt{\frac{2}{3}t^3 + C}$

Since $y(0) = -5$, the negative solution applies and $-5 = -\sqrt{C}$, so $C = 25$ and the solution is $y = -\sqrt{\frac{2}{3}t^3 + 25}$.

29. $\frac{dy}{dx} = \frac{\ln x}{\sqrt{xy}},\ y(1) = 4$

$\int \sqrt{y}\ dy = \int \frac{\ln x}{\sqrt{x}}\,dx = \int \frac{2\ln\sqrt{x}}{\sqrt{x}}\,dx$

$\frac{2}{3}y^{3/2} = 2\sqrt{x}\ln\sqrt{x} - 4\sqrt{x} + C_1$

$y = \left(3\sqrt{x}\ln\sqrt{x} - 6\sqrt{x} + C\right)^{2/3}$

$y(1) = 4 = (-6+C)^{2/3} \Rightarrow C = 14$ and the solution is $y = \left(3\sqrt{x}\ln x - 6\sqrt{x} + 14\right)^{2/3}$.

30. $\frac{dN}{dt} = 2tN^2,\ N(0) = 5$

$\frac{dN}{dt}N^{-2} = 2t \Rightarrow \int N^{-2}dN = \int 2t\ dt \Rightarrow$

$-\frac{1}{N} = t^2 + C_1 \Rightarrow N = \frac{1}{-t^2 + C}$

$N(0) = 5 = \frac{1}{C}$, so $C = \frac{1}{5}$ and the solution is

$N = \frac{1}{-t^2 + \left(\frac{1}{5}\right)} = \frac{5}{1-5t^2}$.

31. $\frac{dy}{dp} = -\frac{1}{2}\left(\frac{y}{p+3}\right) \Rightarrow y^{-1}\frac{dy}{dp} = -\frac{1}{2(p+3)}$

(Assuming $y \neq 0$)

$\int y^{-1}dy = -\frac{1}{2}\int \frac{1}{p+3}\,dp \Rightarrow$

$\ln|y| = -\frac{1}{2}\ln|p+3| + C \Rightarrow y = \frac{A}{\sqrt{p+3}}$

$A > 0$ because $y > 0$.

32. $\frac{dy}{ds} = k\frac{y}{s} \Rightarrow y^{-1}dy = \frac{k}{x}ds \Rightarrow$

$\int y^{-1}dy = k\int s^{-1}ds$

(Assuming $y \neq 0$)

$\ln|y| = k\ln|s| + C \Rightarrow y = As^k \ (A \geq 0, s > 0)$

33. $\frac{dp}{dt} = k(1-p);\ \ 0 \leq p \leq 1, p(0) = 0.$

$\int (1-p)^{-1}dp = \int k\ dt$ (Assuming $p \neq 1$)

$-\ln|1-p| = kt + C_1 \Rightarrow 1 - p = e^{-kt+C} \Rightarrow$

$p = 1 - e^{-kt+C} = 1 - Ae^{-kt}$

$p(0) = 0 = 1 - A$, so $A = 1$ and the solution is

$y = 1 - e^{-kt}$.

34. $\frac{dy}{dt} = k(M-y);\ \ 0 \leq y \leq M, f(0) = 0$

$\int (M-y)^{-1}dy = \int k\,dt$ (Assuming $y < M$)

$-\ln|M-y| = kt + C_1 \Rightarrow M - y = e^{-kt+C} \Rightarrow$

$y = M - e^{-kt+C} = M - Ae^{-kt}$

$f(0) = 0 = M - A$, so $A = M$, and the solution is

$y = f(t) = M - Me^{-kt} = M\left(1 - e^{-kt}\right).$

35. $\frac{dV}{dt} = kV^{2/3}$, $k < 0$ (since V decreases), $V(0) = 27$, $V(4) = 15.625$. Solving the differential equation gives

$\int V^{-2/3}dV = \int kdt \Rightarrow 3V^{1/3} = kt + C_1$, or

$V = \left(\frac{kt}{3} + C\right)^3$. Using the conditions

$V(0) = 27 \Rightarrow 27 = C^3$ or $C = 3$.

$V(4) = 15.625 \Rightarrow 15.625 = \left(\frac{4k}{3} + 3\right)^3$, or

$k = -\frac{3}{8}$. Hence $V = \left(3 - \frac{t}{8}\right)^3$, $V = 0$ when $t = 24$ weeks.

36. $y' = ky$; $k > 0$, $y(0) = 100$, $y(2) = 115$

$\frac{dy}{dt}y^{-1} = k$ (Assuming $y \neq 0$)

$\int \frac{dy}{y} = \int k\ dt \Rightarrow \ln|y| + C_1 = kt + C_2 \Rightarrow$

$y = Ae^{kt}$

$f(0) = 100 \Rightarrow 100 = Ae^0 \Rightarrow A = 100$

$f(2) = 115 \Rightarrow 115 = 100e^{2k} \Rightarrow k = 0.07$

Hence $y = 100e^{0.07t}$.

$y(t) = 200 = 100e^{0.07t}$, or $2 = e^{0.07t}$.

Thus $.07t = \ln 2 \Rightarrow t = 9.9$. The CCI will reach 200 when $t = 9.9$, in approximately November or December of 1999.

37. $\frac{dy}{dt} = -ay\ln\left(\frac{y}{b}\right) \Rightarrow \int \frac{1}{y\ln\left(\frac{y}{b}\right)}dy = \int -a\ dt$

$(y \neq 0, b)$

$\ln\left|\ln\left(\frac{y}{b}\right)\right| = -at + C \Rightarrow \ln\left(\frac{y}{b}\right) = Ce^{-at} \Rightarrow$

$y = be^{Ce^{-at}}$

($y = 0$ is not a solution and the solution $y = b$ corresponds to $C = 0$.)

38. $y' = ky^2, k < 0$

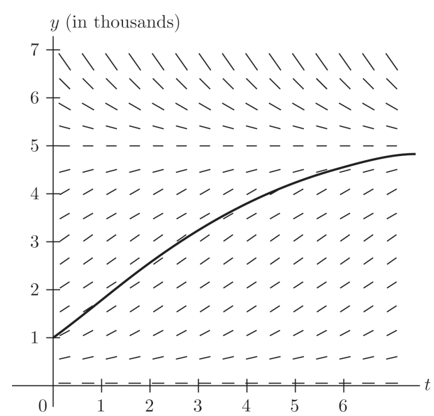
$\int y^{-2}dy = \int k\ dt\ \ (y \neq 0)$

$-\frac{1}{y} = kt + C_1 \Rightarrow y = -\frac{1}{kt + C_1} = \frac{1}{C - kt}$

39. a. The slope lines are pointing downward at (0, 6000), so an initial population of 6000 will decrease and approach 5000.

b. The slope lines are pointing upward at (0, 1000), so an initial population of 1000 will increase and approach 5000.

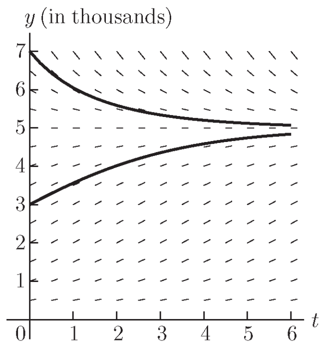
c.



The solution shows that an initial population of 1000 fish will increase over time, asymptotically approaching a maximum of 5000 fish.

40. a. The slope lines at $y = 0$ are all horizontal, so an initial population of 0 will remain 0 for all t. The slope lines at $y = 5$ are also horizontal, so $y = 5$ is also a constant solution.

b.



For an initial population greater than 5000, the solution will start above $y = 5$ and decrease steadily. The solution will approach $y = 5$ asymptotically from above. For an initial population less than 5000, the solution will start below 5 and increase steadily, approaching $y = 5$ asymptotically from below.

10.3 First-Order Linear Differential Equations

1. $y' - 2y = t \Rightarrow a(t) = -2$

$A(t) = \int (-2)dt = -2t; e^{A(t)} = e^{-2t}$

The integrating factor is e^{-2t}.

2. $y' + ty = 6t \Rightarrow a(t) = t$

$A(t) = \int t\,dt = \frac{1}{2}t^2; e^{A(t)} = e^{t^2/2}$.

The integrating factor is $e^{t^2/2}$.

3. $t^3 y' + y = 0 \Rightarrow y' + \frac{1}{t^3}y = 0$ (since $t > 0$)

$a(t) = \frac{1}{t^3}; A(t) = \int \frac{1}{t^3}dt = -\frac{1}{2}t^{-2}$

$e^{A(t)} = e^{-t^{-2}/2}$

The integrating factor is $e^{-1/\left(2t^2\right)}$.

4. $y' + \sqrt{t}\,y = 2(t+1) \Rightarrow a(t) = \sqrt{t} = t^{1/2}$

$A(t) = \int t^{1/2}dt = \frac{2}{3}t^{3/2}; e^{A(t)} = e^{2t^{3/2}/2}$

The integrating factor is $e^{2t^{3/2}/2}$.

5. $y' - \frac{y}{10+t} = 2 \Rightarrow a(t) = -\frac{1}{10+t}$

$$\begin{aligned} A(t) &= \int \left(-\frac{1}{10+t}\right)dt = -\int \frac{1}{10+t}dt \\ &= -\ln|10+t| \\ &= -\ln(10+t) \\ &\qquad \left[|10+t| = 10+t, \text{ since } t > 0\right] \end{aligned}$$

$e^{A(t)} = e^{-\ln(10+t)} = \frac{1}{10+t}$

The integrating factor is $\frac{1}{10+t}$.

6. $y' = t^2(y+1) \Rightarrow y' = t^2 y + t^2 \Rightarrow y' - t^2 y = t^2$

$a(t) = -t^2; A(t) = \int (-t^2)dt = -\frac{1}{3}t^3$

$e^{A(t)} = e^{-t^3/3}$.

The integrating factor is $e^{-t^3/3}$.

7. $y' + y = 1 \Rightarrow a(t) = 1; b(t) = 1$

$A(t) = \int 1dt = t \Rightarrow e^{A(t)} = e^t; e^{-A(t)} = e^{-t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-t}\left[\int (1)e^t dt + C\right] \\ &= e^{-t}\left[\int e^t dt + C\right] = e^{-t}[e^t + C] \\ &= 1 + Ce^{-t} \end{aligned}$$

8. $y' + 2ty = 0 \Rightarrow a(t) = 2t; b(t) = 0$

$A(t) = \int 2t\,dt = t^2; e^{A(t)} = e^{t^2}; e^{-A(t)} = e^{-t^2}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-t^2}\left[\int e^{t^2}(0)dt + C\right] \\ &= e^{-t^2}[0 + C] = Ce^{-t^2} \end{aligned}$$

9. $y' - 2ty = -4t \Rightarrow a(t) = -2t; b(t) = -4t$

$A(t) = \int (-2t)dt = -t^2$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-(-t^2)}\left[\int e^{-t^2}(-4t)dt + C\right] \\ &= e^{t^2}\left[-\int 4te^{-t^2}dt + C\right] = e^{t^2}\left[2e^{-t^2} + C\right] \\ &= 2 + Ce^{t^2} \end{aligned}$$

10. $y' = 2(20-y) \Rightarrow y' = 40-2y \Rightarrow y'+2y = 40$

$a(t)=2;\ b(t)=40;\ A(t)=\int 2\,dt = 2t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-2t}\left[\int 40e^{2t}dt + C\right] \\ &= e^{-2t}[20e^{2t}+C] \\ &= 20+Ce^{-2t} \end{aligned}$$

11. $y' = 0.5(35-y) \Rightarrow y' = 17.5-0.5y$
$y'+0.5y = 17.5$

$a(t)=0.5;\ b(t)=17.5;\ A(t)=\int 0.5\,dt = 0.5t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-0.5t}\left[\int 17.5e^{0.5t}dt + C\right] \\ &= e^{-0.5t}[35e^{0.5t}+C] = 35+Ce^{-0.5t} \end{aligned}$$

12. $y'+y = e^{-t}+1 \Rightarrow a(t)=1; b(t)=e^{-t}+1$

$A(t)=\int 1\,dt = t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-t}\left[\int e^{t}(e^{-t}+1)dt + C\right] \\ &= e^{-t}\left[\int (1+e^{t})dt + C\right] = e^{-t}[t+e^{t}+C] \\ &= 1+te^{-t}+Ce^{-t} \end{aligned}$$

13. $y'+\dfrac{y}{10+t} = 0 \Rightarrow a(t)=\dfrac{1}{10+t};\ b(t)=0$

$A(t)=\int \dfrac{1}{10+t}dt = \ln|10+t| = \ln(10+t)$, since $t>0$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-\ln(10+t)}\left[\int e^{\ln(10+t)}(0)dt + C\right] \\ &= \frac{1}{10+t}[0+C] = \frac{C}{10+t} \end{aligned}$$

14. $y'-2y = e^{2t};\ a(t)=-2;\ b(t)=e^{2t}$

$A(t)=\int (-2)dt = -2t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-(-2t)}\left[\int e^{-2t}e^{2t}dt + C\right] \\ &= e^{2t}\left[\int dt + C\right] = e^{2t}[t+C] \\ &= te^{2t}+Ce^{2t} \end{aligned}$$

15. $(1+t)y'+y=-1 \Rightarrow y'+\dfrac{1}{1+t}y = -\dfrac{1}{1+t}$, since $t>0$

$a(t)=\dfrac{1}{1+t};\ b(t)=-\dfrac{1}{1+t}$

$A(t)=\int \dfrac{1}{1+t}dt = \ln|1+t| = \ln(1+t)$ since $t>0$

$e^{A(t)} = e^{\ln(1+t)} = 1+t;\ e^{-A(t)} = e^{-\ln(1+t)} = \dfrac{1}{1+t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= \frac{1}{1+t}\left[\int (1+t)\left(\frac{-1}{1+t}\right)dt + C\right] \\ &= \frac{1}{1+t}\left[\int (-1)dt + C\right] = \frac{1}{1+t}[-t+C] \\ &= \frac{C-t}{1+t} \end{aligned}$$

16. $y' = e^{-t}(y+1) \Rightarrow y' = ye^{-t}+e^{-t} \Rightarrow$
$y'-ye^{-t} = e^{-t}$

$a(t)=-e^{-t};\ b(t)=e^{-t};\ A(t)=\int(-e^{-t})dt = e^{-t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-e^{-t}}\left[\int e^{e^{-t}}e^{-t}dt + C\right] \\ &= e^{-e^{-t}}\left[-e^{e^{-t}}+C\right] = -1+Ce^{-e^{-t}} \end{aligned}$$

17. $6y'+ty = t \Rightarrow y'+\dfrac{1}{6}ty = \dfrac{1}{6}t$

$a(t)=\dfrac{1}{6}t;\ b(t)=\dfrac{1}{6}t;\ A(t)=\int \dfrac{1}{6}t\,dt = \dfrac{1}{12}t^2$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-t^2/12}\left[\int e^{t^2/12}\left(\frac{1}{6}t\right)dt + C\right] \\ &= e^{-t^2/12}\left[e^{t^2/12}+C\right] \\ &= 1+Ce^{-t^2/12} \end{aligned}$$

18. $e^{t}y'+y = 1 \Rightarrow y'+e^{-t}y = e^{-t}$

$a(t)=e^{-t};\ b(t)=e^{-t};\ A(t)=\int e^{-t}dt = -e^{-t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-(-e^{-t})}\left[\int e^{-e^{-t}}e^{t}dt + C\right] \\ &= e^{e^{-t}}\left[e^{-e^{-t}}+C\right] = 1+Ce^{e^{-t}} \end{aligned}$$

19. $y' + y = 2 - e^t \Rightarrow a(t) = 1; b(t) = 2 - e^t$

$A(t) = \int 1\,dt = t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-t}\left[\int e^t(2-e^t)dt + C\right] \\ &= e^{-t}\left[\int (2e^t - e^{2t})dt + C\right] \\ &= e^{-t}\left[2e^t - \frac{1}{2}e^{2t} + C\right] \\ &= 2 - \frac{1}{2}e^t + Ce^{-t} \end{aligned}$$

20. $\frac{1}{\sqrt{t+1}}y' + y = 1 \Rightarrow y' + y\sqrt{t+1} = \sqrt{t+1}$

$a(t) = (t+1)^{1/2}; b(t) = (t+1)^{1/2}$

$A(t) = \int (t+1)^{1/2}dt = \frac{2}{3}(t+1)^{3/2}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t) + dt + C\right] \\ &= e^{-\frac{2}{3}(t+1)^{3/2}}\left[\int (t+1)^{1/2}e^{\frac{2}{3}(t+1)^{3/2}}dt + C\right] \\ &= e^{-\frac{2}{3}(t+1)^{3/2}}\left[e^{\frac{2}{3}(t+1)^{3/2}} + C\right] \\ &= 1 + Ce^{-\frac{2}{3}(t+1)^{3/2}} \end{aligned}$$

21. $y' + 2y = 1, y(0) = 1; a(t) = 2; b(t) = 1$

$A(t) = \int 2\,dt = 2t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-2t}\left[\int e^{2t}(1)dt + C\right] \\ &= e^{-2t}\left[\frac{1}{2}e^{2t} + C\right] = \frac{1}{2} + Ce^{-2t} \end{aligned}$$

$y(0) = 1 = \frac{1}{2} + Ce^0$; so $C = \frac{1}{2}$ and the solution is $y = \frac{1}{2}\left(1 + e^{-2t}\right)$.

22. $ty' + y = \ln t, y(e) = 0$

$y' + \frac{1}{t}y = \frac{1}{t}\ln t \Rightarrow a(t) = \frac{1}{t}; b(t) = \frac{1}{t}\ln t$

$A(t) = \int \frac{1}{t}dt = \ln t$

$e^{A(t)} = e^{\ln t} = t \Rightarrow e^{-A(t)} = e^{-\ln t} = \frac{1}{t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= \frac{1}{t}\left[\int t\left(\frac{1}{t}\ln t\right)dt + C\right] \\ &= \frac{1}{t}\left[t\ln t - t + C\right] = \ln t - 1 + \frac{1}{t}C \end{aligned}$$

$y(e) = 0 = \ln e - 1 + \frac{1}{e}C$; so $C = 0$ and the solution is $y = \ln t - 1$.

23. $y' + \frac{y}{1+t} = 20, y(0) = 10 \Rightarrow$

$a(t) = \frac{1}{1+t}; b(t) = 20$

$A(t) = \int \frac{1}{1+t}dt = \ln|1+t| = \ln(1+t)$, assuming $t \geq 0$.

$e^{A(t)} = e^{\ln(1+t)} = 1+t; e^{-A(t)} = e^{-\ln(1+t)} = \frac{1}{1+t}$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= \frac{1}{1+t}\left[\int (1+t)20\,dt + C\right] \\ &= \frac{1}{1+t}\left[\int (20t+20)dt + C\right] \\ &= \frac{1}{1+t}[10t^2 + 20t + C] \end{aligned}$$

$y(0) = 10 = \frac{1}{1+0}[10(0)^2 + 20(0) + C]$, so $C = 10$ and the solution is

$$\begin{aligned} y &= \frac{1}{1+t}[10t^2 + 20t + 10] \\ &= \frac{1}{1+t}\left[10(t+1)^2\right] = 10t + 10. \end{aligned}$$

24. $y' = 2(10 - y), y(0) = 1$

$y' = 20 - 2y \Rightarrow y' + 2y = 20$

$a(t) = 2; b(t) = 20$

$A(t) = \int 2\,dt = 2t$

$$\begin{aligned} y &= e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right] \\ &= e^{-2t}\left[\int e^{2t}(20)dt + C\right] \\ &= e^{-2t}[10e^{2t} + C] = 10 + Ce^{-2t} \end{aligned}$$

$y(0) = 1 = 10 + Ce^0$, so $C = -9$ and the solution is $y = 10 - 9e^{-2t}$.

25. $y' + y = e^{2t}$, $y(0) = -1$; $a(t) = 1$; $b(t) = e^{2t}$

$A(t) = \int (1)dt = t$

$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$

$= e^{-t}\left[\int e^{t}e^{2t}dt + C\right]$

$= e^{-t}\left[\int e^{3t}dt + C\right] = e^{-t}\left[\frac{1}{3}e^{3t} + C\right]$

$= \frac{1}{3}e^{2t} + Ce^{-t}$

$y(0) = -1 = \frac{1}{3}e^{0} + Ce^{0}$, so $C = -\frac{4}{3}$ and the solution is $y = \frac{1}{3}e^{2t} - \frac{4}{3}e^{-t}$.

26. $ty' - y = -1$, $y(1) = 1$

$y' - \frac{1}{t}y = -\frac{1}{t} \Rightarrow a(t) = -\frac{1}{t}; b(t) = -\frac{1}{t}$

$A(t) = \int\left(-\frac{1}{t}\right)dt = -\ln|t| = -\ln t$, assuming $t > 0$.

$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$

$= e^{-(-\ln t)}\left[\int e^{-\ln t}\left(-\frac{1}{t}\right)dt + C\right]$

$= t\left[-\int \frac{1}{t^2}dt + C\right] = t\left[\frac{1}{t} + C\right] = 1 + Ct$

$y(1) = 1 = 1 + C(1)$, so $C = 0$ and the solution is $y = 1$.

27. $y' + 2y\cos(2t) = 2\cos(2t)$, $y\left(\frac{\pi}{2}\right) = 0$

$a(t) = 2\cos(2t); b(t) = 2\cos(2t)$

$A(t) = \int 2\cos(2t)dt = \sin(2t)$

$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$

$= e^{-\sin(2t)}\left[\int e^{\sin(2t)}(2\cos(2t))dt + C\right]$

$= e^{-\sin(2t)}\left[e^{\sin(2t)} + C\right] = 1 + Ce^{-\sin(2t)}$

$y\left(\frac{\pi}{2}\right) = 0 = 1 + Ce^{-\sin(0)} = 1 + Ce^{0}$, so $C = -1$

and the solution is $y = 1 - e^{-\sin(2t)}$.

28. $ty' + y = \sin t$, $y\left(\frac{\pi}{2}\right) = 0$

$y' + \frac{1}{t}y = \frac{1}{t}\sin t \Rightarrow a(t) = \frac{1}{t}; b(t) = \frac{1}{t}\sin t$

$A(t) = \int \frac{1}{t}dt = \ln|t| = \ln t$, assuming $t > 0$.

$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$

$= e^{-\ln t}\left[\int e^{\ln t}\left(\frac{1}{t}\sin t\right)dt + C\right]$

$= \frac{1}{t}\left[\int \sin t + C\right] = \frac{1}{t}[-\cos t + C]$

$y\left(\frac{\pi}{2}\right) = 0 = \frac{1}{\frac{\pi}{2}}\left[-\cos\frac{\pi}{2} + C\right] = \frac{2}{\pi}C$ so $C = 0$

and the solution is $y = -\frac{1}{t}\cos t$

29. a. $y'(0) = -\frac{y(0)}{1+0} + 10 = -\frac{50}{1} + 10 = -40$

Since $y'(0)$ is negative, the solution $y(t)$ is decreasing when $t = 0$.

b. $y' = -\frac{y}{1+t} + 10$; $y(0) = 50$

$y' + \frac{1}{1+t}y = 10; a(t) = \frac{1}{1+t}; b(t) = 10$

$A(t) = \int \frac{1}{1+t}dt = \ln|1+t| = \ln(1+t)$, assuming $t > 0$.

$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$

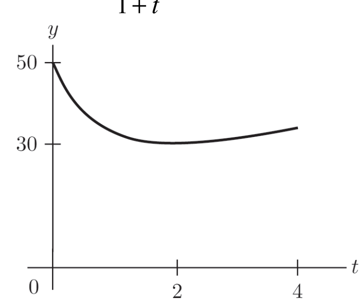
$= e^{-\ln(1+t)}\left[\int e^{\ln(1+t)}(10)dt + C\right]$

$= \frac{1}{1+t}\left[\int (10t + 10)dt + C\right]$

$= \frac{1}{1+t}[5t^2 + 10t + C]$

$y(0) = 50 = \frac{1}{1+0}[5(0)^2 + 10(0) + C] = C$, so $C = 50$ and the solution is

$y = \frac{5t^2 + 10t + 50}{1+t}$.



10.4 Applications of First-Order Linear Differential Equations

1. a. $y' = 0.06y + 2400$, so

$$y'|_{y=30,000} = .06(30,000) + 2400 = 4200$$

The account was growing at \$4200 per year.

b. Set $y' = 2(2400) = 4800$, and solve for y.

$.06y + 2400 = 4800 \Rightarrow y = 40,000$

The account contained \$40,000.

c. Solve $P(t) = 40,000$ for t.

$$P(t) = -40,000 + 41,000e^{0.06t} = 40,000$$

$$.06t = \ln\left(\frac{80}{41}\right); t = \frac{1}{.06}\ln\left(\frac{80}{41}\right) \approx 11.14$$

About 11 years and 2 months.

2. $y' - 0.07y = -4800; y(0) = 25,000$

$a(t) = -0.07; b(t) = -4800$

$A(t) = \int(-0.07)dt = -0.07t$

$$y = e^{-(-0.07t)}\left[\int e^{-0.07t}(-4800)dt + C\right]$$

$$= e^{0.07t}\left[\frac{4800}{.07}e^{-0.07t} + C\right]$$

$$y(0) = 25,000 = e^{0}\left[\frac{4800}{.07}e^{0} + C\right], \text{ so}$$

$C = 25,000 - \dfrac{4800}{.07}$ and the solution is

$$y = \frac{4800}{0.07} + \left(25,000 - \frac{4800}{0.07}\right)e^{0.07t}$$

To find the length of time needed to pay off the loan, set $y = 0$ and solve for t.

$$y = 0 = \frac{4800}{.07} + \left(25,000 - \frac{4800}{.07}\right)e^{0.07t}$$

$$0 \approx 68,571.43 - 43,571.43e^{0.07t}$$

$.07t \approx .4535$; $t \approx 6.48$; about $6\frac{1}{2}$ years.

3. a. $y' = .05y + 3600$

b. $y' - .05y = 3600$, $y(0) = 0$; $a(t) = -.05$; $b(t) = 3600$

$A(t) = \int(-.05)dt = -.05t$

$$y = e^{-(-0.05t)}\left[\int e^{-0.05t}(3600)dt + C\right]$$

$$= e^{0.05t}\left[-72,000e^{-0.05t} + C\right]$$

$$= -72,000 + Ce^{0.05t}$$

$f(0) = 0 = -72,000 + Ce^{0}$, so $C = 72,000$ and the solution is

$$f(t) = -72,000 + 72,000e^{0.05t}$$

After 25 years, the account will contain

$$f(25) = -72,000 + 72,000e^{0.05(25)} \approx \$179,305$$

4. a. $y' = .06y + A$, $y(0) = 10,000$

b. $y' - .06y = A$; $a(t) = -.06$; $b(t) = A$

$A(t) = \int(-.06)dt = -.06t$

$$y = e^{-(-0.06t)}\left[\int e^{-0.06t}(A)dt + C\right]$$

$$= e^{.06t}\left[-\frac{A}{.06}e^{-0.06t} + C\right]$$

$$= -\frac{A}{.06} + Ce^{0.06t} \Rightarrow$$

$$f(t) = -\frac{A}{.06} + Ce^{0.06t}$$

$$f(0) = 10,000 = -\frac{A}{.06} + Ce^{0} \Rightarrow$$

$$C = 10,000 + \frac{A}{.06}$$

The solution is

$$f(t) = \left(10,000 + \frac{A}{0.06}\right)e^{0.06t} - \frac{A}{0.06}$$

c. Set $f(5) = 20,000$ and solve for A.

$$f(5) = 20,000 = \left(10,000 + \frac{A}{.06}\right)e^{0.06(5)} - \frac{A}{.06}$$

$$A = \frac{.06(20,000 - 10,000e^{.3})}{e^{0.3} - 1} \approx 1114.98$$

So \$1114.98 should be deposited each year.

5. Both accounts satisfy the differential equation $y' = 0.05y + A$, $y(0) = 0$, where A is the yearly contribution. The solution is

$$y = \frac{A}{0.05}(e^{0.05t} - 1).$$

At her retirement, Kelly will have

$$y(20) = \frac{1200}{0.05}(e^{0.05(20)} - 1) \approx \$41,238.76$$

John will have

$$y(10) = \frac{2400}{0.05}(e^{0.05(10)} - 1) \approx \$31,138.62$$

Kelly has more than John.

6. Both accounts satisfy the differential equation $y' = .05y + A,\ y(0) = 0,$ where A is the yearly contribution. The solution is

$$y = \frac{A}{.05}(e^{0.05t} - 1).$$

At her retirement, Kelly will have

$$y(20) = \frac{1200}{.05}(e^{0.05(20)} - 1) \approx \$41,238.76$$

John will have

$$y(10) = \frac{3000}{.05}\left(e^{0.05(10)} - 1\right) \approx \$38,923.28$$

Kelly has more than John.

7. Let $f(t)$ be the amount owed at time t. $f(t)$ satisfies the differential equation $y' = .075 - A,\ y(0) = 100,000,$ where A is the annual rate of payment. The solution is

$$f(t) = 100,000e^{0.075t} + \frac{A}{.075}\left(1 - e^{0.075t}\right)$$

For the loan to be paid in 10 years, $f(10) = 0$. Set $f(10) = 0$ and solve for A.

$$\begin{aligned} f(10) &= 100,000e^{0.075(10)} + \frac{A}{.075}\left(1 - e^{0.075(10)}\right) \\ &= 0 \end{aligned}$$

$$A = \frac{7500e^{0.75}}{e^{0.75} - 1} \approx \$14,214.41$$

The person should pay approximately \$14,214.41 per year.

8. a. $y' = .035y - 6000,\ \ y(0) = 30,303$

b. The solution is

$$f(t) = \frac{6000}{.035} + \left(30,303 - \frac{6000}{.035}\right)e^{0.035t}$$

To find how long it takes to pay off the loan, set $f(t) = 0$ and solve for t.

$$\begin{aligned} f(t) = 0 &= \frac{6000}{.035} + \left(30,303 - \frac{6000}{.035}\right)e^{0.035t} \\ e^{0.035t} &= \frac{6000}{6000 - .035(30,303)} \approx 1.2147 \\ t &\approx \frac{1}{.035}\ln(1.2147) \approx 5.6 \end{aligned}$$

It will take about 5.6 years to pay off the loan.

9. a. Since the person paid 10% down, the amount of the mortgage is

$$.9\left(278,900\right) = \$251,010.$$

$$y' = .031y - A,\ y(0) = 251,010$$

b. The solution is

$$f(t) = 251,010e^{0.031t} + \frac{A}{.031}\left(1 - e^{0.031t}\right)$$

To find A, set $f(30) = 0$ and solve for A.

$$\begin{aligned} f(30) &= 251,010e^{0.031(30)} + \frac{A}{0.031}\left(1 - e^{0.031(30)}\right) \\ &= 0 \end{aligned}$$

$$A = \frac{0.031(251,010)e^{0.93}}{e^{0.93} - 1} \approx 12,852.19$$

The monthly payment will be \$12,852.19/12 or about \$1071.

c. Subtract the amount borrowed from the total amount paid.
30(12,852.19) – 251,010 ≈ \$134,556

10. a. $y' = .06y - A,\ y(0) = 251,010$

b. The solution is

$$f(t) = 251,010e^{0.06t} + \frac{A}{.06}\left(1 - e^{0.06t}\right)$$

To find A, set $f(15) = 0$ and solve for A.

$$\begin{aligned} f(15) &= 251,010e^{0.06(15)} + \frac{A}{.06}\left(1 - e^{0.06(15)}\right) \\ &= 0 \end{aligned}$$

$$A = \frac{.06(251,010)e^{0.9}}{e^{0.9} - 1} \approx 25,378.88$$

The monthly payments will be \$2114.91.

c. Subtract the amount borrowed from the total amount paid.
15(25,378.88) – 251,010 ≈ \$129,673

11. a. Substitute $E(p) = p + 1$ and $q = f(p)$ in $E(p) = \dfrac{-pf'(p)}{f(p)},$ which yields the result.

$-pq' = (p+1)q$
(or $-py' = (p+1)y$ with $y = q$)

b. $-p\dfrac{dq}{dp} = (p+1)q;\ \dfrac{1}{q}\dfrac{dq}{dp} = -1 - \dfrac{1}{p}$

$$\begin{aligned} \int \frac{1}{q}\,dq &= -\int\left(1 + \frac{1}{p}\right)dp \\ \ln|q| &= -\left(p + \ln|p|\right) + C \end{aligned}$$

q and p are positive, so we have

$\ln q = -(p + \ln p) + C;\ q = Ae^{-(p+\ln p)}$

So $f(p) = Ae^{-(p+\ln p)} = \dfrac{Ae^{-p}}{p}.$

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$f(1) = 100 = Ae = Ae^{-1}$
so $A = 100e$, and the solution is
$f(p) = 100\dfrac{e^{1-p}}{p}.$

12. Substitute $E(p) = ap + b$ and $q = f(p)$ in $E(p) = \dfrac{-pf'(p)}{f(p)}$, which yields the differential equation $-pq' = (ap+b)q$. Solve by separation of variables.

$\dfrac{1}{q}\dfrac{dq}{dp} = -a - \dfrac{b}{p} \Rightarrow \int \dfrac{1}{q}dq = \int\left(-a - \dfrac{b}{p}\right)dp \Rightarrow$

$\ln|q| = -ap - b\ln|p| + C$; p and q are positive,

so $\ln q = -ap - b\ln p + C \Rightarrow q = Ae^{-(ap+b\ln p)}$

The demand function is

$f(p) = Ae^{-(ap+b\ln p)} = A\dfrac{e^{-ap}}{p^b}.$

13. $y' = .1(10 - y), y(0) = 350$
$y' + .1y = 1 \Rightarrow a(t) = .1; b(t) = 1$
$A(t) = \int (.1)dt = .1t$
$y = e^{-0.1t}\left[\int (1)e^{0.1t}dt + C\right] = 10 + Ce^{-0.1t}$
$y(0) = 350 = 10 + Ce^0$; so $C = 340$ and the solution is $f(t) = 10 + 340e^{-0.1t}$.

14. $y' = .2(10 - y), y(0) = 350$
$y' + .2y = 2 \Rightarrow a(t) = .2; b(t) = 2$
$A(t) = \int (.2)dt = .2t$
$y = e^{-0.2t}\left[\int 2e^{0.2t}dt + C\right] = 10 + Ce^{-0.2t}$
$y(0) = 350 = 10 + Ce^0$, so $C = 340$ and the solution is $f(t) = 10 + 340e^{-0.2t}$.

Since $y' = k(10 - y)$, the rate of change of the temperature will be greater for $k = .2$ than for $k = 0.1$. So the rod with $k = .2$ will cool faster than the rod with $k = 0.1$. In general, larger values of k will give faster cooling.

15. **a.** $T = 70$

b. Let $y' = -5$, $T = 70$, and $y = 80$ in $y' = k(T - y)$, and solve for k.
$-5 = k(70 - 80) \Rightarrow k = .5$

c. The initial value problem is
$y' = .5(70 - y),\ y(0) = 98.$
$y' + .5y = 35 \Rightarrow a(t) = .5; b(t) = 35$
$A(t) = \int .5\,dt = .5t$
$y = e^{-0.5t}\left[\int 35e^{0.5t}dt + C\right] = 70 + Ce^{-0.5t}$
$y(0) = 98 = 70 + Ce^0$; so $C = 28$ and the solution is $f(t) = 70 + 28e^{-0.5t}$.

d. Set $f(t) = 85$ and solve for t.
$f(t) = 85 = 70 + 28e^{-0.5t}$
$t = -2\ln\left(\dfrac{15}{28}\right) \approx 1.25$
The person died about 1 hour and 15 minutes before being discovered.

16. $y' - \dfrac{3}{125}y = -0.004e^{0.04t} - 0.04,\ y(0) = 2$

$a(t) = -\dfrac{3}{125}; b(t) = -0.004e^{0.04t} - 0.04$

$A(t) = \int\left(-\dfrac{3}{125}\right)dt = -\dfrac{3}{125}t$

$$y = e^{-\left(-\frac{3t}{125}\right)}\left[\int\left(-0.004e^{0.04t} - .04\right)e^{-\frac{3t}{125}}dt + C\right]$$
$$= e^{\frac{3}{125}t}\left[\int\left(-0.004e^{\frac{2}{125}t} - 0.04e^{-\frac{3}{125}t}\right)dt + C\right]$$
$$= e^{\frac{3}{125}t}\left[-\frac{1}{4}e^{\frac{2}{125}t} + \frac{5}{3}e^{-\frac{3}{125}t} + C\right]$$
$$= -\frac{1}{4}e^{\frac{t}{25}} + \frac{5}{3} + Ce^{\frac{3}{125}t}$$

$y(0) = 2 = -\dfrac{1}{4}e^0 + \dfrac{5}{3} + Ce^0$, so $C = \dfrac{7}{12}$ and the solution is

$P(t) = \dfrac{7}{12}e^{\frac{3}{125}t} - \dfrac{1}{4}e^{\frac{t}{25}} + \dfrac{5}{3}$

17. $y' = .45y + e^{0.03t} + 2$

18. The initial value problem is
$y' - .45y = e^{0.03t} + 2, y(0) = 10$
$a(t) = -.45; b(t) = e^{0.03t} + 2$
$A(t) = \int (-.45)dt = -.45t$

$$y = e^{-(-0.45t)}\left[\int (e^{0.03t} + 2)e^{-0.45t}dt + C\right]$$
$$= e^{0.45t}\left[\int\left(e^{-0.42t} + 2e^{-0.45t}\right)dt + C\right]$$
$$= e^{0.45t}\left[-\frac{1}{0.42}e^{-0.42t} - \frac{2}{0.45}e^{-0.45t} + C\right]$$
$$= -2.381e^{0.03t} - 4.444 + Ce^{0.45t}$$

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$y(0) = 10 = -2.381e^0 - 4.444 + Ce^0$, so $C = 16.825$ and the solution is

$f(t) = 16.825e^{0.45t} - 2.381e^{0.03t} - 4.444.$

19. a. Substitute $y' = 10$ and $y = 75$ into $y' = k(110 - y)$ and solve for k.

$$10 = k\,(110 - 75) \Rightarrow k = \frac{10}{35} = \frac{2}{7}$$

b. $y'(0) = \frac{2}{7}\left(110 - y(0)\right) = \frac{2}{7}(110) = \frac{220}{7} \approx 31.43$ grams/liter/hour.

This is more than 3 times the rate at the end of a four-hour session. Draining and replacing the solution after 4 hours will greatly shorten the amount of time spent in dialysis.

20. a. $y' + 0.00043y = 0.003;\ y(0) = 0$

b. $a(t) = 0.00043;\ b(t) = 0.003$

$A(t) = \int 0.00043dt = 0.00043t$

$$y = e^{-0.00043t}\left[\int 0.003e^{0.00043t}dt + C\right] = e^{-0.00043t}\left[\frac{300}{43}e^{0.00043t} + C\right] = \frac{300}{43} + Ce^{-0.00043t}$$

$y(0) = 0 = \frac{300}{43} + Ce^0$, so $C = -\frac{300}{43}$ and the solution is $P(t) = \frac{300}{43}\left(1 - e^{-0.00043t}\right)$.

c. $\lim_{t\to\infty} P(t) = \frac{300}{43}(1-0) = \frac{300}{43} \approx 6.977$ grams

21. a. $y' - .04y = -2000 - 500t,\ y(0) = 100{,}000$

b. $a(t) = -0.04, b(t) = -2000 - 500t;\ A(t) = \int(-.04)\,dt = -.04t$

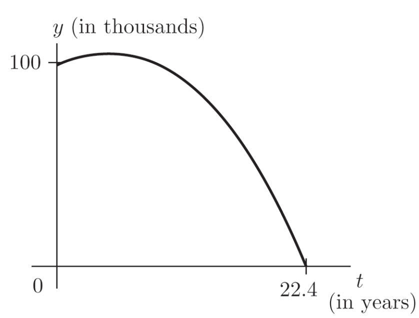
$$y = e^{-(-0.04t)}\left[\int(-2000 - 500t)e^{-0.04t}dt + C\right]$$

$$= e^{0.04t}\left[\frac{1}{(-0.04)^2}e^{-0.04t}\left(0.04(500)t + 0.04(2000) + 500\right) + C\right] = 12{,}500t + 362{,}500 + Ce^{0.04t}$$

$y(0) = 100{,}000 = 12{,}500(0) + 362{,}500 + Ce^0$, so $C = -262{,}500$ and the solution is

$f(t) = -262{,}500e^{0.04t} + 12{,}500t + 362{,}500.$

c.



The account will be empty in about 22 years and 4 months.

22. a. $y' - .06y = 90t + 810,\ y(0) = 500$

b. $a(t) = -.06; b(t) = 90t + 810; A(t) = \int(-.06)\,dt = -.06t$

$$y = e^{-(-0.06t)}\left[\int(90t+810)e^{-0.06t}dt + C\right] = e^{0.06t}\left[\frac{1}{(-0.06)^2}e^{-0.06t}\left(-0.06(90)t - 0.06(810) - 90\right) + C\right]$$
$$= -1500t - 38{,}500 + Ce^{0.06t}$$

$y(0) = 500 = -1500(0) - 38{,}500 + Ce^0$, so $C = 39{,}000$ and the solution is

$P(t) = -1500t - 38{,}500 + 39{,}000e^{0.06t}$.

23. a. $3000 - 500t = 0 \Rightarrow t = 6$
The person contributed for 6 years before starting to withdraw.

b. $y' - 0.04y = 3000 - 500t, y(0) = 10{,}000$

24. a. The graph crosses the t-axis at about 17.
The account will be empty in about 17 years.

b. $y' = .04y + 3000 - 500t$

$a(t) = -.04$; $b(t) = -500t + 3000$; $A(t) = \int(-.04)\,dt = -.04t$

$$y = e^{-(-0.04t)}\left[\int(-500t+3000)e^{-0.04t}dt + C\right]$$
$$= e^{0.04t}\left[\frac{1}{(-0.04)^2}e^{-0.04t}\left(.04(500)t - .04(3000) + 500\right) + C\right] = 12{,}500t + 237{,}500 + Ce^{0.04t}$$

$y(0) = 10{,}000 = 12{,}500(0) + 237{,}500 + Ce^0$, so $C = -227{,}500$ and the solution is

$P(t) = 12{,}500t - 227{,}500e^{0.04t} + 237{,}500$.

c. A calculator gives the zero of $P(t)$ at $t \approx 17.17$.

25. a. $y' + .35y = t$

b. Solve the initial value problem $y' + 0.35y = t, y(0) = 0$.

$a(t) = .35$; $b(t) = t$; $A(t) = \int .35\,dt = .35t$

$$y = e^{-0.35t}\left[\int te^{0.35t}dt + C\right] = e^{-0.35t}\left[\frac{1}{(0.35)^2}e^{0.35t}\left(.35(1)t + .35(0) - 1\right) + C\right] = \frac{1}{.35}t - \frac{1}{.1225} + Ce^{-0.35t}$$

$y(0) = 0 = \frac{1}{0.35}(0) - \frac{1}{0.1225} + Ce^0$, so $C = \frac{1}{0.1225}$ and the solution is

$f(t) = \frac{1}{0.35}t + \frac{1}{0.1225}(e^{-0.35t} - 1), t \le 8.$
$f(8) \approx 15.2$ milligrams

26. a. $y' + .5y = r$

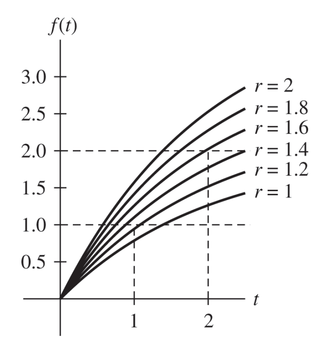
b. Solve the initial value problem
$y' + .5y = r, y(0) = 0.$

$a(t) = 0.5$; $b(t) = r$; $A(t) = \int .5\,dt = .5t$

$$y = e^{-0.5t}\left[\int re^{0.5t}dt + C\right]$$
$$= e^{-0.5t}[2re^{0.5t} + C] = 2r + Ce^{-0.5t}$$

$y(0) = 0 = 2r + Ce^0$, so $C = -2r$ and the solution is $f(t) = 2r\left(1 - e^{-0.5t}\right)$.

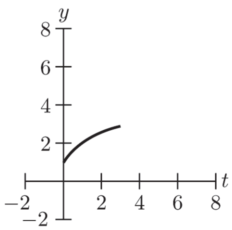
c.



The value of r should be at least 1.3 but no more than 1.5.

10.5 Graphing Solutions of Differential Equations

1.



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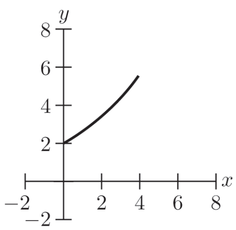

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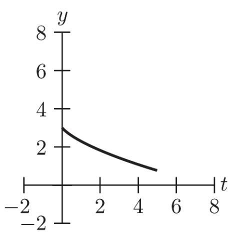

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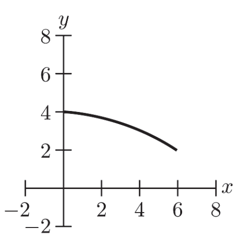

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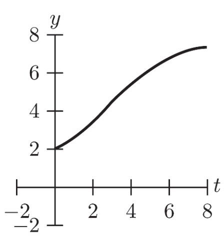

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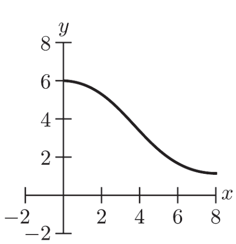

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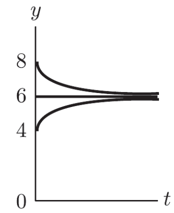

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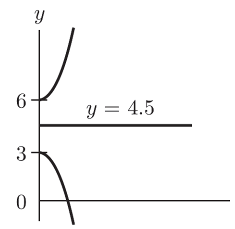

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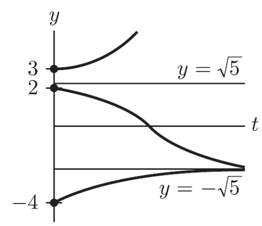

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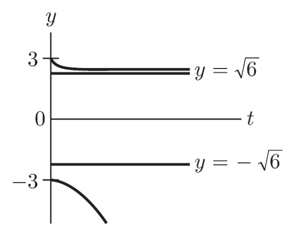

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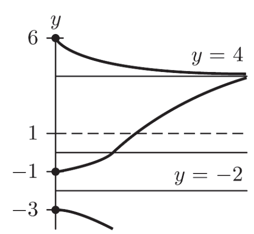

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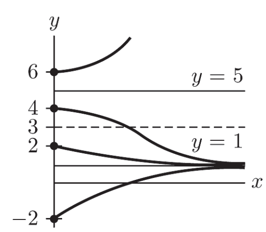

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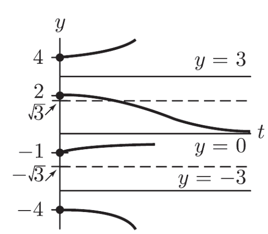

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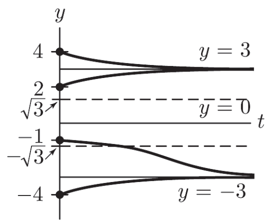

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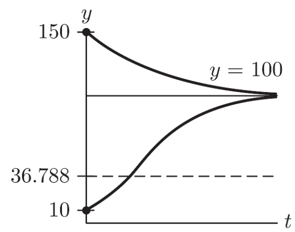

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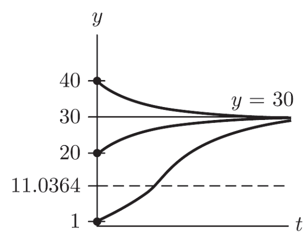

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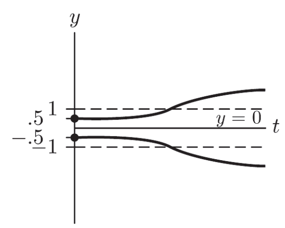

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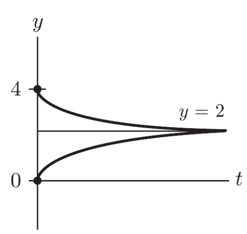

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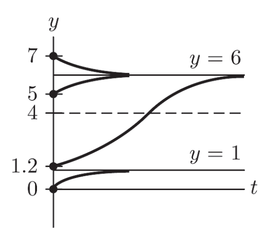

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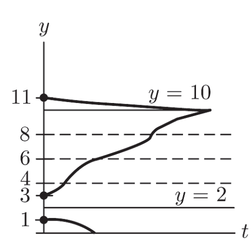

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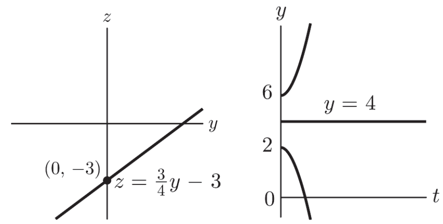

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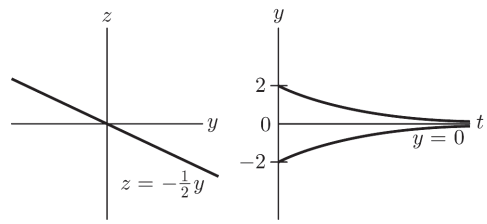

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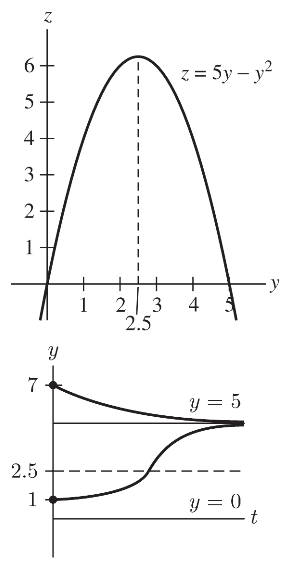

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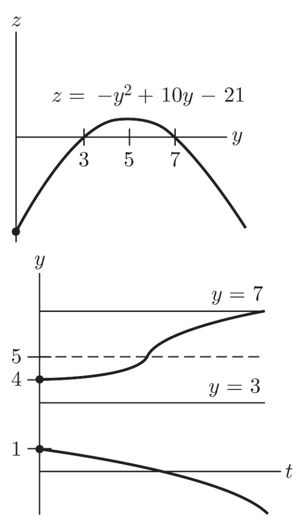

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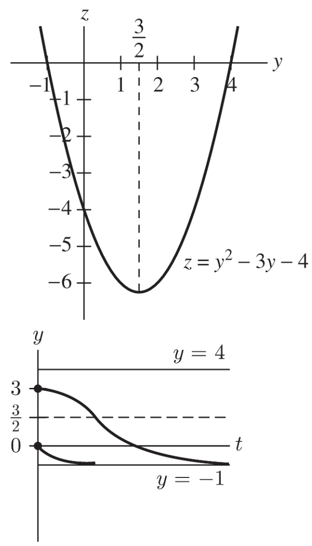

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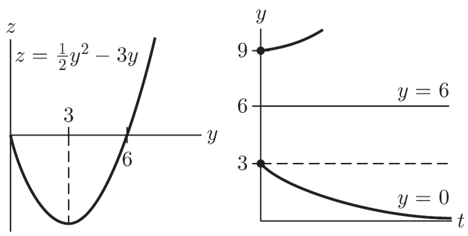

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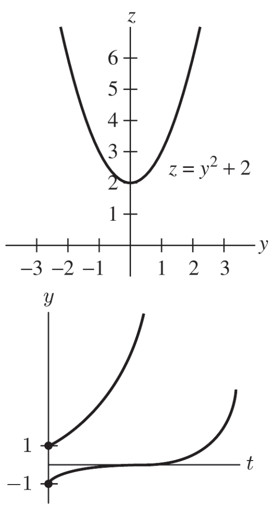

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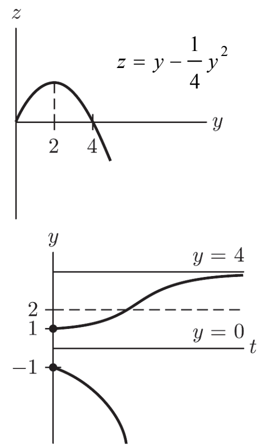

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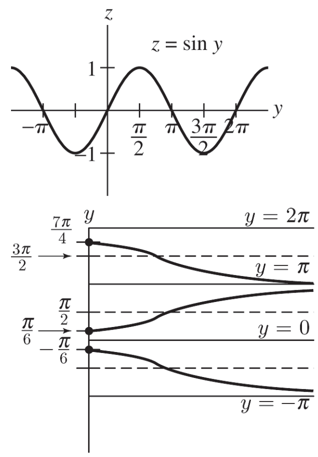

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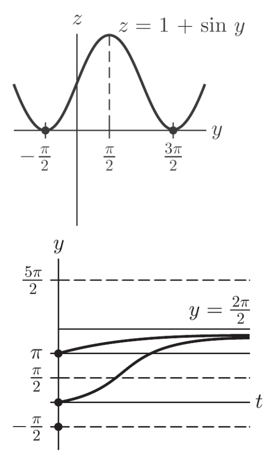

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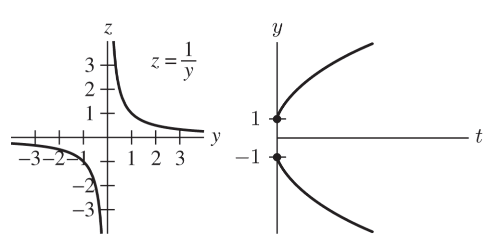

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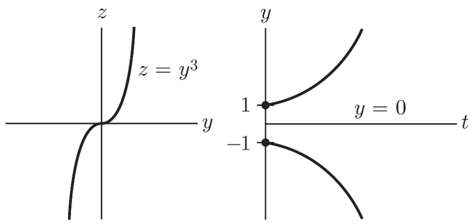

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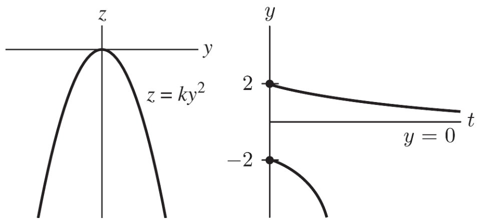

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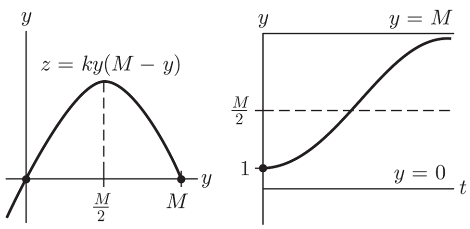

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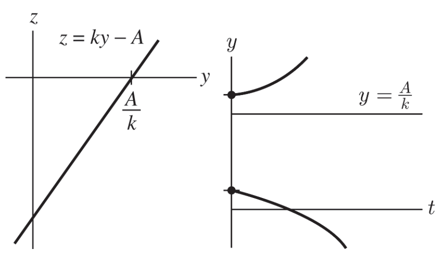

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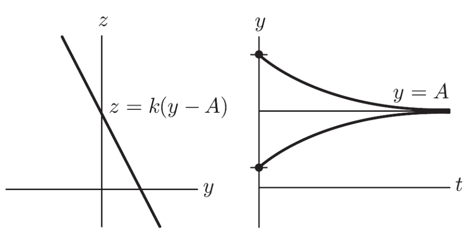
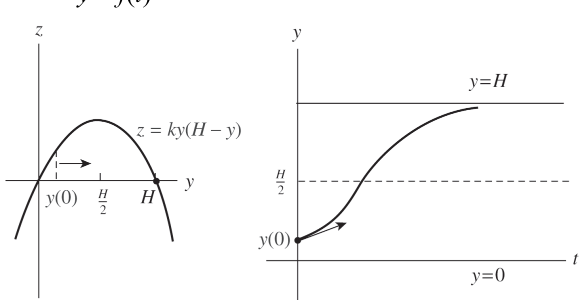


37. $y' = ky(H-y), k>0$

H = height at maturity

$y = f(t)$



38. $\dfrac{dv}{dt} = 32 - kv, \int (32-kv)^{-1}dy = \int dt$

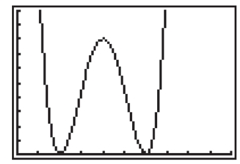
$(32 \neq kv)$

$-\dfrac{1}{k}\ln|32-kv| = t + C;\ y = \dfrac{32}{k} + Ae^{-kt}$

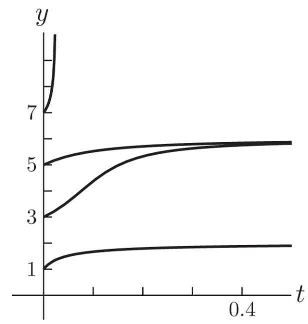
As $t \to \infty$, $v = 176$, so $176 = \dfrac{32}{k} + A \cdot 0$, or

$k = \dfrac{32}{176} = \dfrac{2}{11}$.

39.



[0, 10] *by* [0, 20]



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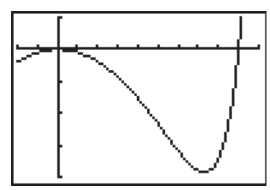
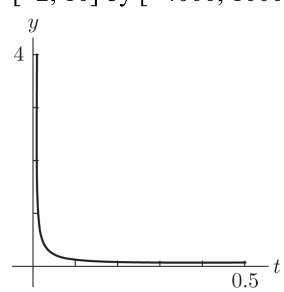


[−2, 10] by [−4000, 1000]

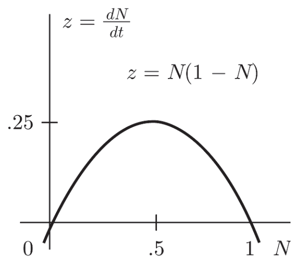


10.6 Applications of Differential Equations

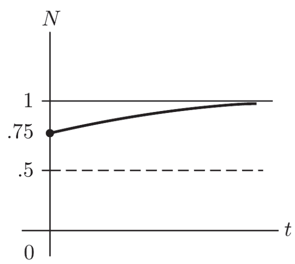
1. a. $\dfrac{dN}{dt} = N(1-N) = \dfrac{r}{K}N(K-N)$

Carrying capacity $K = 1$, intrinsic rate $r = 1$

b.



c., d.



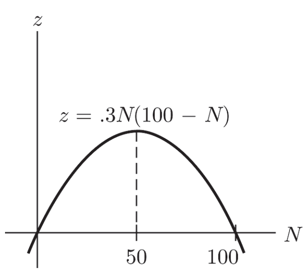
2. a. $\dfrac{dN}{dt} = .3N(100-N)$

$= \dfrac{30}{100}N(100-N)$

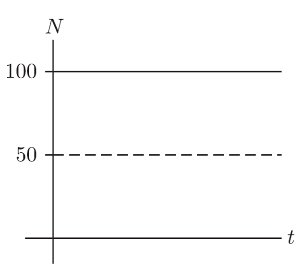
$= \dfrac{r}{K}N(K-N)$

Carrying capacity $K = 100$, intrinsic rate $r = 30$

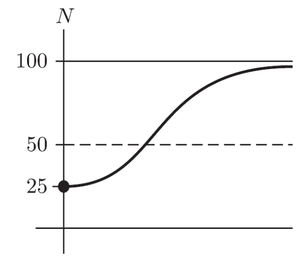
b.



c.



d.



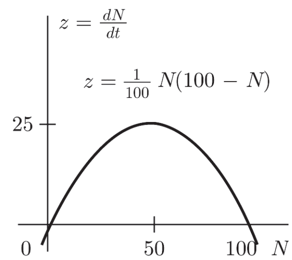
3. a. $\dfrac{dN}{dt} = -.01N^2 + N$

$= \dfrac{1}{100}N(100-N)$

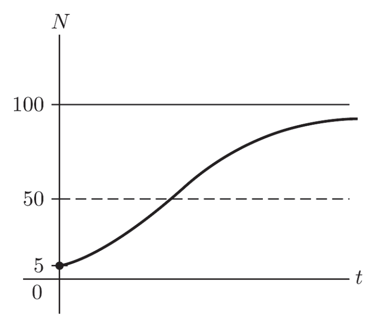
$= \dfrac{r}{K}N(K-N)$

Carrying capacity $K = 100$, intrinsic rate $r = 1$.

b.



c., d.

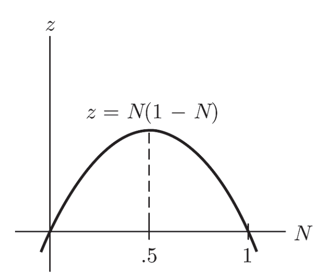


4. a. $\dfrac{dN}{dt} = -N^2 + N = N(1-N)$

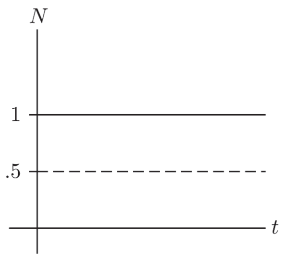
$= \dfrac{r}{K}N(K-N)$

carrying capacity = 1, intrinsic rate = 1

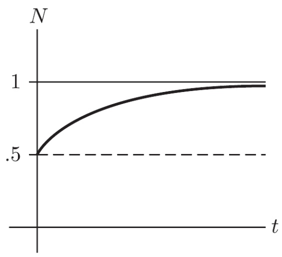
b.



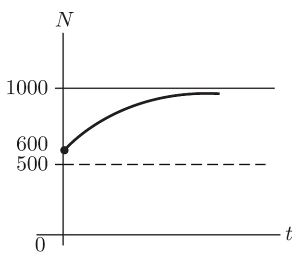
c.



d.

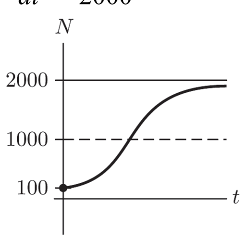


5. $\dfrac{dN}{dt} = \dfrac{r}{1000} N(1000 - N)$



This graph starts out at 600, which is much higher than the graph in Example 2. Both graphs, however, are concave down and approach $N = 1000$ asymptotically from below.

6. a. $\dfrac{dN}{dt} = \dfrac{r}{2000} N(2000 - N)$



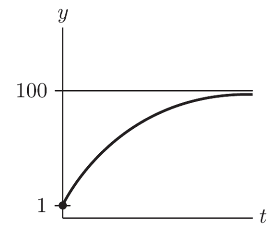
b. The rate of growth

$$\frac{dN}{dt} = \frac{r}{2000} N(2000 - N)$$

has its maximum at $N = 1000$. The size of the population with the highest growth rate is 1000. The rate is

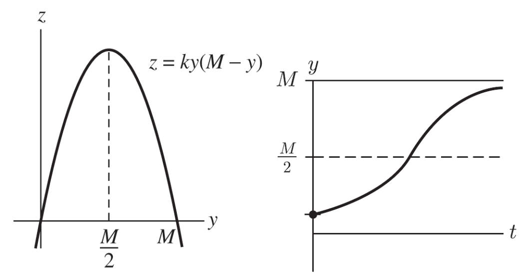
$$\left.\frac{dN}{dt}\right|_{N=1000} = \frac{0.3}{2000}(1000)(2000 - 1000) = 150 \text{ fish per month}$$

7. $y' = k(100 - y),\ k > 0$



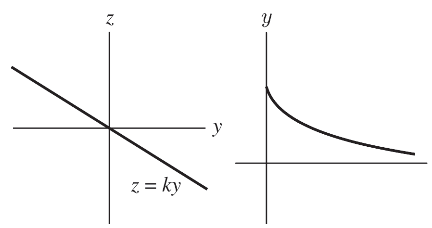
8. $y(0) = 0$. But this implies $y = Ce^{kt} \Rightarrow C = y(0) = 0$; so $y = 0$ is the only solution. This means that the object will not move, which is absurd.

9. $y' = ky(M - y),\ k > 0$

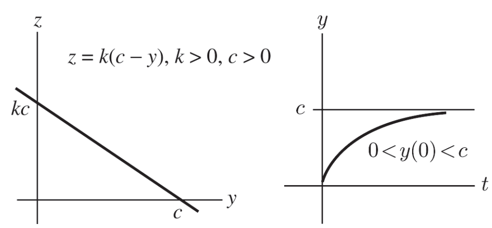


The reaction is fastest when $y = \dfrac{M}{2}$.

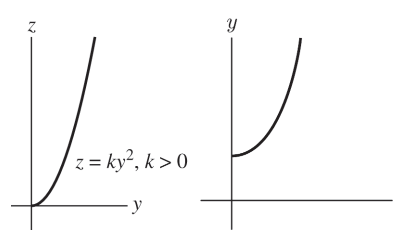
10. $y' = ky,\ k < 0$; so $y = Ce^{kt}$ where $C = y(0)$.



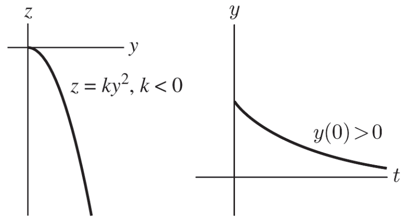
11. $y' = k(c - y),\ k > 0,\ c > 0$



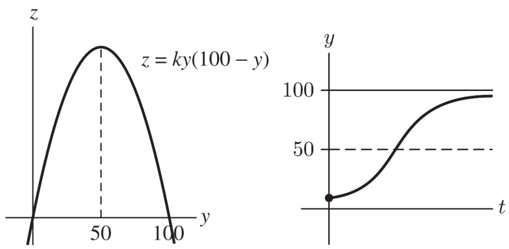
12. $y' = ky^2,\ k > 0$



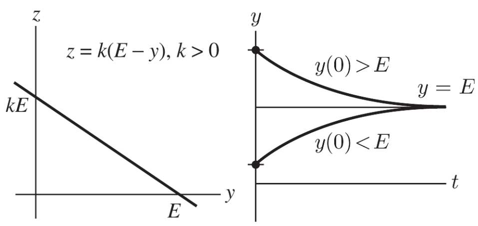
13. $y' = ky^2, k < 0$



14. $y' = ky(100 - y), k > 0$



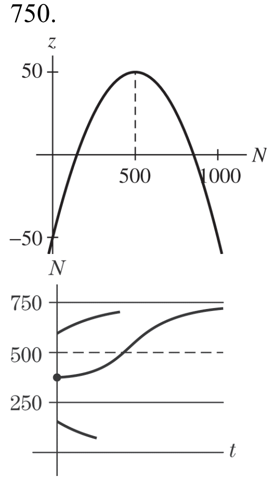
15. $y' = k(E - y), k > 0$



16. $p = k(p_0 - p), k > 0$

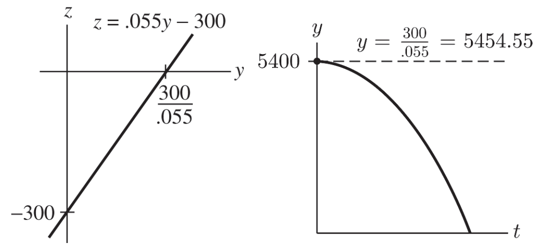
17. a. $\dfrac{dN}{dt} = \dfrac{.4}{1000}N(1000 - N) - 75$

b. $\dfrac{dN}{dt} = 0$ is a quadratic equation with solutions $N = 250$ and $N = 750$. So the new upper limit on the fish population is 750.



c. Catching 75 fish per year is sustainable. The fish population will continue to grow but will never come close to the pond's carrying capacity.

18. $y' = .055y - 300, y(0) = 5400$



19. a. $y' = .05y + 10{,}000, y(0) = 0$

b. $\dfrac{dy}{dt} = .05y + 10{,}000 \Rightarrow 20\dfrac{dt}{dy} = y + 200{,}000$

$$\int \frac{20}{y + 200{,}000}\,dy = \int dt$$

$$20\ln(y + 200{,}000) = t + C$$

$$y + 200{,}000 = Ae^{.05t};\ y = Ae^{.05t} - 200{,}000$$

Since $y(0) = 0$, $A = 200{,}000$; so the solution is $y = 200{,}000(e^{.05t} - 1)$.

$$y(5) = 200{,}000(e^{.05(5)} - 1) \approx \$56{,}805$$

20. The balance of the account after t years satisfies the differential equation $y' = .05y + P, y(0) = 0$.

$$20\frac{dy}{dt} = y + 20P \Rightarrow \int \frac{20}{y + 20P}\,dy = \int dt \Rightarrow$$

$$20\ln(y + 20P) = t \Rightarrow y + 20P = e^{.05t} + C \Rightarrow$$

$$y = Ae^{.05t} - 20P$$

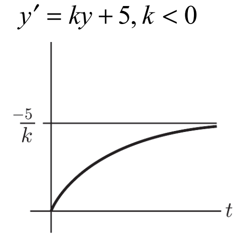
Since $y(0) = 0$, $A = 20P$; so $y = 20P(e^{.05t} - 1)$.

If we require $y(4) = 20P(e^{.2} - 1) = 50{,}000$, we must have $P \approx \$11{,}292$.

21. a. $y' = .05 - .2y, y(0) = 6.25$

b. $y' = .13 - .2y, y(0) = 6.25$

22. $y' = ky + 5, k < 0$

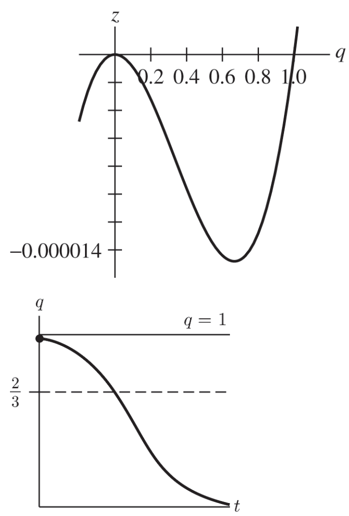


23. $y' = -.14y$

24. $y' = 10 - .8y$

The equilibrium point will be the value of y that makes $y' = 10 - .8y = 0$, i.e. $y = 12.5$.
If $y(0) < 12.5$, the solution $y(t)$ will increase approaching $y = 12.5$ asymptotically.
If $y(0) > 12.5$, $y(t)$ will be decreasing approaching $y = 12.5$ from above.

25.



10.7 Numerical Solution of Differential Equations

1. The slope of the graph = $y' = ty - 5$
$= 2 \cdot 4 - 5 = 3$.

2. $y' = t^2 - y^2 = 2^2 - 3^2 = -5$

3. $y' = y^2 + ty - 7, y(0) = 3$

$y'(0) = 3^2 + 0(3) - 7 = 2$

Thus $f(t)$ is increasing at $t = 0$.

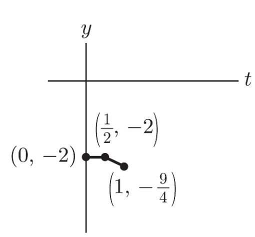
4. $y' = y^2 + ty - 7, y(0) = 2$

$y'(0) = 2^2 + 0(2) - 7 = -3$

Thus $f(t)$ is decreasing at $t = 0$.

5. $g(t, y) = y' = t^2 y$; $y(0) = -2$; $h = .5$.
The iterates are

t_1	y_1
0	–2
.5	–2
1	–2.25



So $y(1) \approx -\frac{9}{4}$.

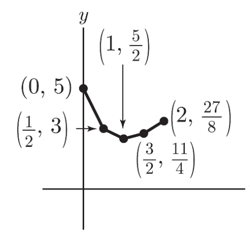
6. $g(t, y) = t - 2y$; $y(2) = 3$, $h = .5$. The iterates are

t_1	y_1
2	3
2.5	1
3	1.25

So $f(3) \approx \frac{5}{4}$.

7. $g(t, y) = 2t - y + 1$; $y(0) = 5$; $h = .5$.
The iterates are

t_1	y_1
0	5
.5	3
1	2.5
1.5	2.75
2	3.375



So $f(2) \approx \frac{27}{8}$.

8. $g(t, y) = y(2t - 1)$; $y(0) = 8$, $h = 0.25$.
The iterates are

t_1	y_1
0	8
.25	6
.5	5.25
.75	5.25
1	5.90625

So $f(1) \approx \frac{189}{32}$.

9. *Euler's Method*: $g(t, y) = -(t+1)y^2$; $y(0) = 1$, $h = .2$. The iterates are

t_1	y_1
0	1
.2	.8
.4	.6464
.6	.52941
.8	.43972
1	.37011

So $f(1) \approx 3.7011$.

(*continued on next page*)

(*continued*)

Exact Solution:

$$\int y^{-2}dy = \int -(t+1)\ dt$$

$$-\frac{1}{y} = \frac{-t^2}{2} - t + C_1$$

$$y = \frac{1}{\left(\frac{t^2}{2}\right) + t + C_2} = \frac{2}{t^2 + 2t + C}$$

$y(0) = 1 = \frac{2}{C}$, so $C = 2$ and the exact solution is $f(t) = \frac{2}{t^2 + 2t + 2}$.

$f(1) = \frac{2}{5} = .4$.

Therefore, the error in the estimate is $|.4 - .37011| = .02989$.

10. *Euler's Method*: $g(t, y) = 10 - y$; $y(0) = 1$, $h = .2$.

The iterates are

t_1	y_1
0	1
0.2	2.8
0.4	4.24
0.6	5.392
0.8	6.3136
1	7.05088

So $y(1) \approx 7.05088$.

Exact Solution:

$$\int (y-10)^{-1}dy = -\int dt$$

$$\ln|y - 10| = -t + C \Rightarrow y = 10 + Ae^{-t}$$

$y(0) = 1 = 10 + A$, so $A = -9$ and the exact solution is $y = 10 - 9e^{-t}$.

$$y(1) = 10 - \frac{9}{e} \approx 6.68909$$

11. a. $y' = .1(1 - y),\ y(0) = 0$

b. Using Euler's method with $n = 3$, $h = 1$ gives the sequence of iterates

t_1	y_1
0	0
1	0 + .1(1 – 0) = .1
2	0.1 + .1(1 – .1) = .19
3	0.19 + .1(1 – .19) = .271

c.

$$\int (y-1)^{-1}dy = -0.1\int dt$$

$$\ln|y - 1| = -0.1t + C$$

$$1 - y = Ae^{-0.1t}$$

$$y = 1 - Ae^{-0.1t}$$

$y(0) = 0 = 1 - A$, so $A = 1$.

So $y = 1 - e^{-0.1t}$ and

$y(3) = 1 - e^{-0.3} \approx .25918$.

d. Euler's method gives $y(3) = .271$.
The exact value is

$y(3) = 1 - e^{-0.3} \approx .25918$.

The error in Euler's method is about $|.271 - .25918| = .01182$.

12. a. $y' = -.3y,\ y(0) = 2$

b. Euler's method with $n = 2$, $h = \frac{1}{2}$ gives the sequence of iterates

t_1	y_1
0	2
$\frac{1}{2}$	$2 - .3(2)\left(\frac{1}{2}\right) = 1.7$
1	$1.7 - .3(1.7)\left(\frac{1}{2}\right) = 1.445$

c. $\int \frac{1}{y}dy = -.3\int dt;\ \ln|y| = -.3t + C$

$y = Ae^{-0.3t}$; $y(0) = 2 = Ae^{0}$, so $A = 2$.

The solution is

$y = 2e^{-0.3t}$, and $y(1) = 2e^{-0.3}$.

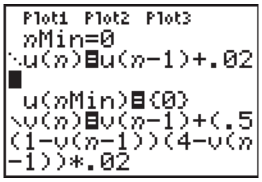
d. Euler's method gives 1.445.
The exact value is

$y(1) = 2e^{-0.3} \approx 1.48164$.

The error in Euler's method is about $|1.445 - 1.48164| \approx .03664$.

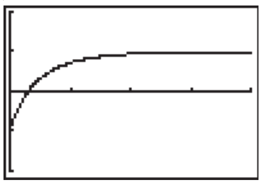
13. $y' = 0.5(1 - y)(4 - y)$

To generate the following graphs, set the TI83/TI84 in sequence mode and then use the following in the sequence **Y=** editor.



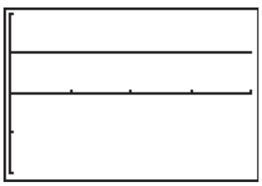
Set ***v*Min** equal to the value of $y(0)$.

a. $y(0) = -1$; solution is type C: increasing, concave down, and asymptotic to the line $y = 1$.



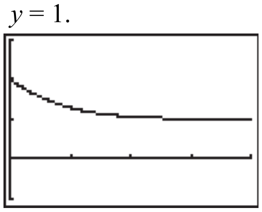
[0, 4] by [−2, 2]

b. $y(0) = 1$; solution is type A: constant solution.



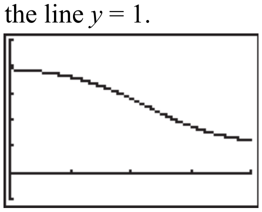
[0, 4] by [−2, 2]

c. $y(0) = 2$; solution is type E: decreasing, concave up, and asymptotic to the line $y = 1$.



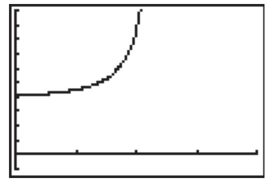
[0, 4] by [−1, 3]

d. $y(0) = 3.9$; solution is type B: decreasing, has an inflection point, and asymptotic to the line $y = 1$.



[0, 4] by [−1, 5]

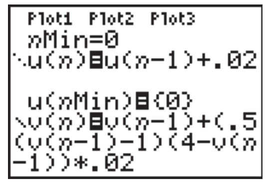
e. $y(0) = 4.1$; solution is type D: concave up and increasing indefinitely.



[0, 4] by [−1, 10]

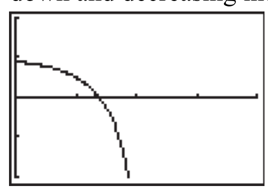
14. $y' = 0.5(y - 1)(4 - y)$

To generate the following graphs, set the TI83/TI84 in sequence mode and then use the following in the sequence **Y=** editor.



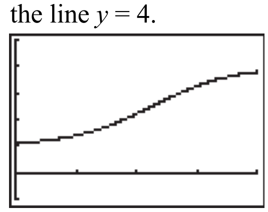
Set ***v*Min** equal to the value of $y(0)$.

a. $y(0) = 0.9$; solution is type E: concave down and decreasing indefinitely.



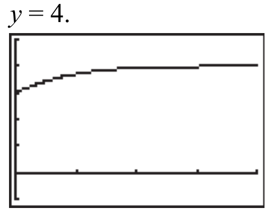
[0, 4] by [−2, 2]

b. $y(0) = 1.1$; solution is type C: increasing, has an inflection point, and asymptotic to the line $y = 4$.



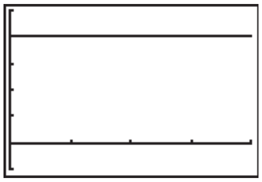
[0, 4] by [−1, 5]

c. $y(0) = 3$; solution is type D: increasing, concave down, and asymptotic to the line $y = 4$.



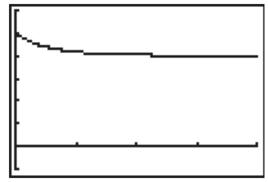
[0, 4] by [−1, 5]

d. $y(0) = 4$; solution is type A: constant solution.



[0, 4] by [−1, 5]

e. $y(0) = 5$; solution is type B: decreasing, concave up, and asymptotic to the line $y = 4$.



[0, 4] by [−1, 6]

15. $y' = e^t - 2y, y(0) = 1, y = \frac{1}{3}(2e^{-2t} + e^t)$

t_i	0	.25	.5	.75	1	1.25	1.5	1.75	2
y_i	1	.75	.6960	.7602	.9093	1.1342	1.4397	1.8403	2.3588
y	1	.8324	.7948	.8544	.9963	1.2182	1.5271	1.9383	2.4752

The greatest difference between y and y_i is 2.4752 – 2.3588 = .1164.

16. $y' = 2ty + e^{t^2}, y(0) = 5, y = (t+5)e^{t^2}$

t_i	0	.2	.4	.6	.8	1	1.2	1.4	1.6	1.8	2
y_i	5	5.2	5.824	6.991	8.955	12.2	17.624	26.927	43.427	73.807	132.054
y	5	5.412	6.337	8.027	11	16.31	26.17	45.44	85.38	173.6	382.2

The greatest difference between y and y_i is 382.2 – 132.054 = 250.146.

Chapter 10 Fundamental Concept Check Exercises

1. A differential equation is an equation involving an unknown function y and one or more of the derivatives $y', y'', y''',$ and so on.

2. A solution of a differential equation is any function $f(t)$ such that the differential equation becomes a true statement when y is replaced by $f(t)$, y' by $f'(t)$, y'' by $f'(t)$, and so forth.

3. A solution curve is the graph of a solution of a differential equation.

4. A constant solution to a differential equation is a constant function that satisfies the differential equation.

5. A slope field is a collection of small line segments. Each line segment is a small portion of a tangent line to a solution curve.

6. The separation of variables technique is a technique for solving differential equations of the form $y' = p(t)q(y,)$ where $p(t)$ is a function of t only and $q(y)$ is a function of y only.

7. The equation $y' + a(t)y = b(t)$ is a first-order linear differential equation in standard form.

8. An integrating factor is $e^{A(t)}$, where $A(t)$ is an antiderivative of $a(t)$. After we multiply both sides of a first-order linear differential equation by an integrating factor, we can combine the terms in y and y' by appealing to the product rule for differentiation.

9. An autonomous differential equation is of the form $y' = g(y)$, where the right side depends only on y and not on t.

10. The slopes of the line segments do not depend on t. All line segments at the same y-level are parallel.

11. See Properties I−IV in Section 10.5.

12. The logistic differential equation is of the form $y' = ky(a-y)$, where a and k are constants.

13. Euler's method is used to approximate the solution of an initial value problem of the form $y' = g(t, y)$, $y(a) = y_0$. Assume that $f(t)$ is a solution of (1) for $a \le t \le b$. If the graph of $y = f(t)$ passes through some given point (t, y), the slope of the graph (that is, the value of y') at that point is just $g(t, y)$, because $y' = g(t, y)$. See Section 10.7 for additional information.

Chapter 10 Review Exercises

1. $$y^2y' = 4t^3 - 3t^2 + 2$$
$$\int y^2 dy = \int (4t^3 - 3t^2 + 2)\, dt$$
$$\frac{y^3}{3} = t^4 - t^3 + 2t + C_1$$
$$y = (3t^4 - 3t^3 + 6t + C)^{1/3}$$

2. $$\frac{y'}{t+1} = y+1$$
$$\int (y+1)^{-1} dy = \int (t+1)\, dt \quad (y \ne -1)$$
$$\ln|y+1| = \frac{t^2}{2} + t + C$$
$$y = -1 + Ae^{t^2/2+t}$$

3. $$y' = \frac{y}{t} - 3y, t > 0$$
$$\int y^{-1} dy = \int \left(\frac{1}{t} - 3\right) dt \quad (y \ne 0)$$
$$\ln|y| = \ln|t| - 3t + C \Rightarrow y = Ate^{-3t}$$

4. $$(y')^2 = t \Rightarrow y' = \pm\sqrt{t}$$
$$\int dy = \pm\int t^{1/2} dt \Rightarrow y = \pm\frac{2}{3}t^{3/2} + C$$

5. $$y = 7y' + ty', y(0) = 3$$
$$\int y^{-1} dy = \int (7+t)^{-1} dt \quad (y \ne 0)$$
$$\ln|y| = \ln|7+t| + C$$
$$y = A(7+t)$$

$y(0) = 3 = 7A$ so $A = \frac{3}{7}$ and the solution is

$y = \frac{3}{7}(7+t) = 3 + \frac{3}{7}t$.

6. $$y' = te^{t+y}, y(0) = 0$$
$$\int e^{-y} dy = \int te^t dt$$
$$-e^{-y} = te^t - e^t + C_1$$
$$e^{-y} = -te^t + e^t + C$$
$$y = -\ln(-te^t + e^t + C)$$

$y(0) = 0 = -\ln(1 + C)$, so $C = 0$ and the solution is $y = -\ln(-te^t + e^t)$.

7. $$yy' + t = 6t^2, y(0) = 7$$
$$\int y\, dy = \int (6t^2 - t)\, dt$$
$$\frac{y^2}{2} = 2t^3 - \frac{t^2}{2} + C_1$$
$$y = \pm\sqrt{4t^3 - t^2 + C}$$

$y(0) = 7 = \sqrt{C}$, so $C = 49$ and the solution is $y = \sqrt{4t^3 - t^2 + 49}$.

8. $$y' = 5 - 8y, y(0) = 1$$
$$y' = -8\left(-\frac{5}{8} + y\right)$$
$$\int \left(y - \frac{5}{8}\right)^{-1} dy = \int -8\, dt$$
$$\ln\left|y - \frac{5}{8}\right| = -8t + C \Rightarrow y = \frac{5}{8} + Ae^{-8t}$$

$y(0) = 1 = \frac{5}{8} + A$, so $A = \frac{3}{8}$ and the solution is

$y = \frac{5}{8} + \frac{3}{8}e^{-8t}$.

9. $$y' - \frac{2}{1-t}y = (1-t)^4 \Rightarrow$$
$$a(t) = -\frac{2}{1-t},\ b(t) = (1-t)^4$$
$$A(t) = -\int \frac{2}{1-t} dt = 2\ln|1-t| = \ln(1-t)^2$$
$$e^{A(t)} = e^{\ln(1-t)^2} = (1-t)^2$$
$$e^{-A(t)} = e^{-\ln(1-t)^2} = \frac{1}{(1-t)^2}$$

(*continued on next page*)

(*continued*)

$$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$$
$$= \frac{1}{(1-t)^2}\left[\int (1-t)^2(1-t)^4 dt + C\right]$$
$$= \frac{1}{(1-t)^2}\left[\int (1-t)^6 dt + C\right]$$
$$= \frac{1}{(1-t)^2}\left[-\frac{1}{7}(1-t)^7 + C\right]$$
$$= -\frac{1}{7}(1-t)^5 + \frac{C}{(1-t)^2}$$
$$\text{or } y = \frac{1}{7}(t-1)^5 + \frac{C}{(t-1)^2}$$

10. $y' - \frac{1}{2(1+t)}y = 1+t \Rightarrow$

$$a(t) = -\frac{1}{2(1+t)}, b(t) = 1+t$$
$$A(t) = -\frac{1}{2}\int \frac{1}{1+t}dt = -\frac{1}{2}\ln|1+t|$$
$$= -\frac{1}{2}\ln(1+t), \ (t \ge 0)$$
$$= \ln\left[(1+t)^{-1/2}\right]$$
$$e^{A(t)} = e^{\ln\left[(1+t)^{-1/2}\right]} = (1+t)^{-1/2}$$
$$e^{-A(t)} = e^{-\ln\left[(1+t)^{-1/2}\right]} = (1+t)^{1/2}$$
$$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + C\right]$$
$$= (1+t)^{1/2}\left[\int (1+t)^{-1/2}(1+t)dt + C\right]$$
$$= (1+t)^{1/2}\left[\int (1+t)^{1/2}dt + C\right]$$
$$= (1+t)^{1/2}\left[\frac{2}{3}(1+t)^{3/2} + C\right]$$
$$= \frac{2}{3}(1+t)^2 + C\sqrt{1+t}$$

11. slope $= \frac{dy}{dx} = x + y; y' = x + y; y' - y = x$

$$a(x) = -1;\ b(x) = x$$
$$A(x) = \int(-1)dx = -x; e^{A(x)} = e^{-x}; e^{-A(x)} = e^x$$
$$y = e^{-A(x)}\left[\int e^{A(x)}b(x)dx + C\right]$$
$$= e^x\left[\int e^{-x}(x)dx + C\right]$$
$$= e^x\left[(-x-1)e^{-x} + C\right] = -x - 1 + Ce^x$$
$$y(0) = 0 = 0 - 1 + Ce^0; C = 1$$

The solution is then $y = -x - 1 + e^x$.

12. a. $\frac{dP}{dt} = k(D-S) = k(12 - 3.3P)$

$$\left.\frac{dP}{dt}\right|_{D=10,\,S=20} = -1 = k(10-20); k = .1$$
$$\frac{dP}{dt} = .1(12 - 3.3P) = 1.2 - .33P$$

b. $\frac{1}{1.2 - .33P}\frac{dP}{dt} = 1$

$$\int \frac{1}{1.2 - .33P}dP = \int dt$$
$$-\frac{1}{.33}\ln|1.2 - .33P| = t + C$$
$$1.2 - .33P = A_1 e^{-0.33t}\ (A_1 \ne 0)$$
$$P = \frac{1.2}{.33} - Ae^{-0.33t}$$
$$P(0) = 1 = \frac{1.2}{.33} - Ae^0$$
$$A = \frac{.87}{.33}$$

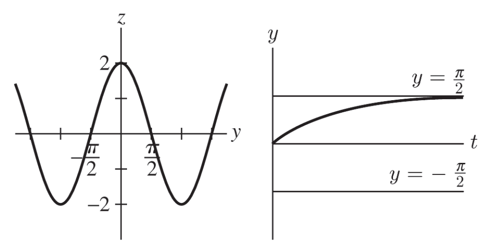
The solution is

$$P = \frac{1}{.33}\left(1.2 - 0.87e^{-0.33t}\right), \text{ or}$$
$$P = \frac{1}{11}\left(40 - 29e^{-0.33t}\right).$$

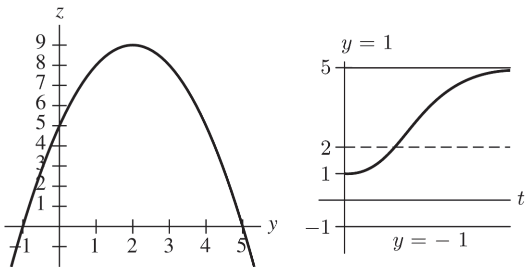
13. If $f(t) = y(t) = 3$, then $y'(t) = (2-3)e^{-3} = -e^{-3} < 0$, so $f(t)$ is decreasing at this point.

14. Note that the constant solution $y = 1$ is a solution to the given equation. Since this solution satisfies the initial condition $y(0) = 1$, it must be the desired particular solution.

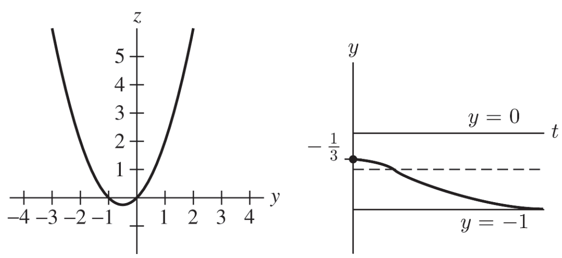
15. $z = 2\cos y$



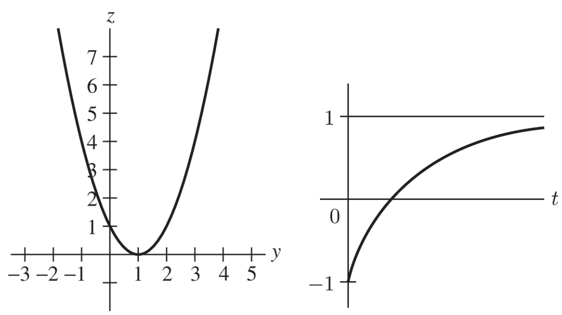
16. $z = 5 + 4y - y^2$



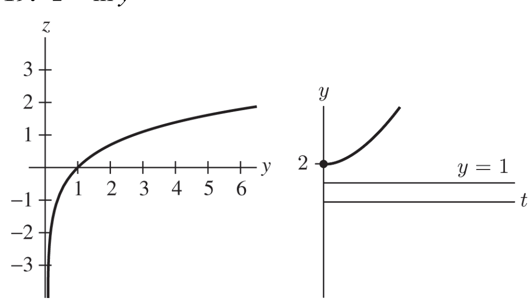
17. $z = y^2 + y$



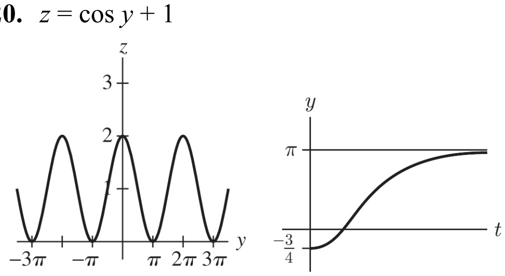
18. $z = y^2 - 2y + 1$



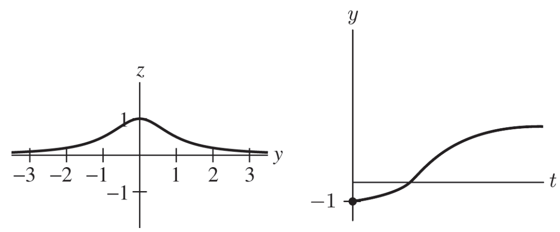
19. $z = \ln y$



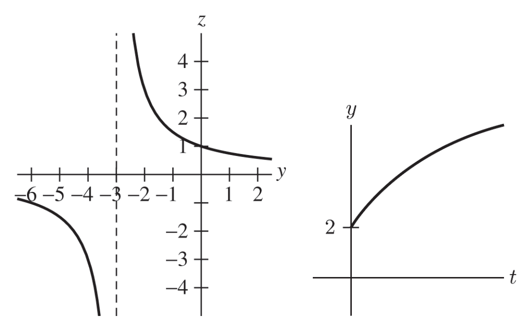
20. $z = \cos y + 1$



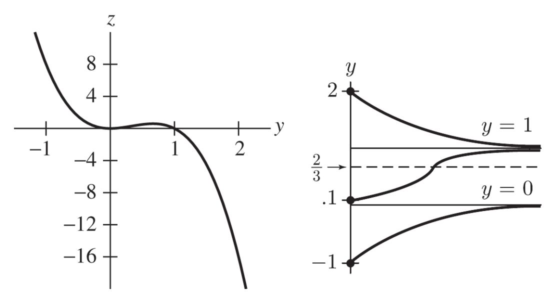
21. $z = \dfrac{1}{y^2 + 1}$



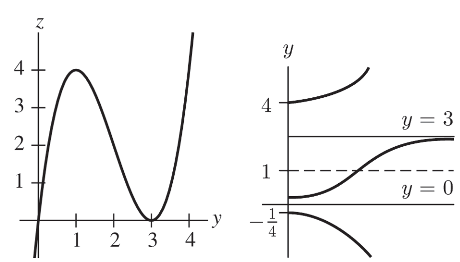
22. $z = \dfrac{3}{y + 3}$



23. $z = .4y^2(1 - y)$

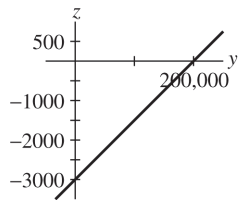


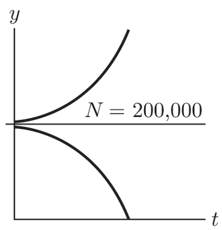
24. $z = y^3 - 6y^2 + 9y$



25. a. $N' = .015N - 3000$

b. There is a constant solution $N = 200{,}000$, but it is unstable. It is unlikely a city would have such a constant population.

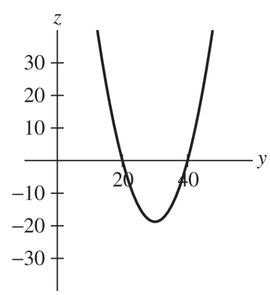


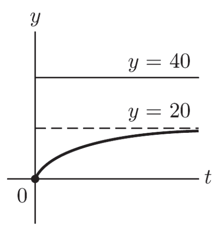


26. a. $\left(10 - \frac{1}{4}y\right)$ represents the amount of unreacted substance A present and $\left(15 - \frac{3}{4}y\right)$ that of B.

b. $k > 0$ since the amount of C is increasing.

c.





27. Let $f(t)$ be the balance in the account after t years. Then $f(t)$ satisfies the differential equation $y' = .05y - 2000$, $y(0) = 20{,}000$.

Thus $y' = .05(y - 40{,}000)$.

$$\int (y - 40{,}000)^{-1}\, dy = \int .05\ dt$$
$$\ln|y - 40{,}000| = .05t + C$$
$$y = 40{,}000 + Ae^{0.05t}$$

$y(0) = 20{,}000 = 40{,}000 + A$, so $A = -20{,}000$ and $f(t) = 40{,}000 - 20{,}000e^{0.05t}$.

We want to find t so that $f(t) = 0$, i.e.

$40{,}000 = 20{,}000e^{0.05t}$,

$2 = e^{0.05t} \Rightarrow \ln 2 = 0.05t \Rightarrow$

$t = 20 \ln 2 \approx 13.86294$ years.

28. Let $f(t)$ be the balance in the savings account after t years and let $M = f(0)$ be the initial amount. Then $f(t)$ satisfies the differential equation

$y' = .06y - 12{,}000 = .06(y - 200{,}000)$,

$y(0) = M$. Thus,

$$\int (y - 200{,}000)^{-1} dy = \int .06\ dt$$
$$\ln|y - 200{,}000| = .06t + C$$
$$y = 200{,}000 + Ae^{0.06t}$$

Now $y(0) = M = 200{,}000 + A$, so $A = M - 200{,}000$ and

$f(t) = 200{,}000 + (M - 200{,}000)e^{0.06t}$.

a. If the initial amount M is to fund the endowment forever, we must have $f(t) > 0$ for all t. This happens first in case $M - 200{,}000 \geq 0 \Rightarrow M \geq \$200{,}000$. (A \$200,000 endowment would give the constant solution $f(t) = 200{,}000$.)

b. Using the expression for $f(t)$ above, setting $f(20) = 0$ gives

$$-200{,}000 = (M - 200{,}000)e^{1.2} \Rightarrow$$
$$-\frac{200{,}000}{e^{1.2}} + 200{,}000 = M \Rightarrow$$

$M \approx \$139{,}761.16$.

29. *Euler's Method*: $y' = 2e^{2t-y}$, $y(0) = 0$, $n = 4$; $h = .5$. The iterates are

t_1	y_1
0	0
.5	1
1	2
1.5	3
2	4

So the estimate is $f(2) \approx 4$.

Exact Solution:

$$\int e^y dy = \int 2e^{2t} dt \Rightarrow e^y = e^{2t} + C \Rightarrow$$

$y = \ln(e^{2t} + C)$

$y(0) = 0 = \ln(1 + C)$, so $C = 0$ and the exact solution is $y = \ln(e^{2t}) = 2t$. Then $f(2) = 4$ and the estimate above is exact.

30. $y' = \frac{t+1}{y},\ y(0) = 1$

Euler's Method: $n = 3,\ h = \frac{1}{3}$. The iterates are

t_1	y_1
0	1
$\frac{1}{3}$	$\frac{4}{3}$
$\frac{2}{3}$	$\frac{5}{3}$
1	2

So $y(1) \approx 2$.

Exact Solution:

$$\int y\ dy = \int (t+1)\ dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + t + C_1$$

$$y = \pm\sqrt{t^2 + 2t + C}$$

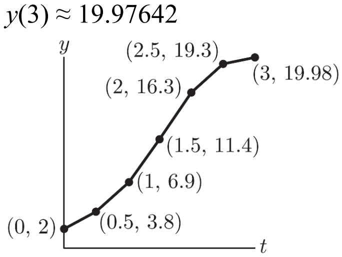
$y(0) = 1 = \sqrt{C}$, so $C = 1$ and the exact solution is $y = \sqrt{t^2 + 2t + 1} = \sqrt{(t+1)^2} = |t+1|$.

Thus $y(1) = 2$ and the above estimate is exact.

31. $y' = .1y(20 - y),\ y(0) = 2;\ n = 6,\ h = .5.$

t_1	y_1
0	2
.5	3.8
1	6.878
1.5	11.39066
2	16.29396
2.5	19.31326
3	19.97642

$y(3) \approx 19.97642$



32. $y' = \frac{1}{2}y(y - 10);\ y(0) = 9;\ n = 5,\ h = 0.2$

t_1	y_1
0	9
.2	8.1
.4	6.561
.6	4.30467
.8	1.85302
1	.34337

$y(1) \approx .34337$