Chapter 9 Techniques of Integration

9.1 Integration by Substitution

1. $\int 2x(x^2+4)^5 dx$

Let $u = x^2+4$; then $du = 2x\,dx$.

$$\int 2x(x^2+4)^5 dx = \int u^5\ du = \frac{1}{6}u^6 + C$$
$$= \frac{1}{6}(x^2+4)^6 + C$$

2. $\int 2(2x-1)^7 dx$

Let $u = 2x-1$; then $du = 2dx$.

$$\int 2(2x-1)^7 dx = \int u^7 du = \frac{1}{8}u^8 + C$$
$$= \frac{1}{8}(2x-1)^8 + C$$

3. $\int \frac{2x+1}{\sqrt{x^2+x+3}}\ dx$

Let $u = x^2+x+3$; then $du = (2x+1)\ dx$.

$$\int \frac{2x+1}{\sqrt{x^2+x+3}}\ dx = \int \frac{1}{\sqrt{u}} du = 2u^{1/2} + C$$
$$= 2\sqrt{x^2+x+3} + C$$

4. $\int \left(x^2+2x+3\right)^6 (x+1) dx$

Let $u = x^2+2x+3$; then

$du = (2x+2)dx \Rightarrow \frac{1}{2}du = (x+1)dx$.

$$\int \left(x^2+x+3\right)^6 (x+1) dx = \frac{1}{2}\int u^6 du$$
$$= \frac{1}{14}u^7 + C$$
$$= \frac{1}{14}\left(x^2+2x+3\right)^7 + C$$

5. $\int 3x^2 e^{(x^3-1)} dx$

Let $u = x^3-1$; then $du = 3x^2 dx$.

$$\int 3x^2 e^{(x^3-1)} dx = \int e^u du = e^u + C = e^{x^3-1} + C$$

6. $\int 2xe^{-x^2}\ dx$

Let $u = -x^2$; then $du = -2xdx \Rightarrow -du = 2xdx$

$$\int 2xe^{-x^2}\ dx = -\int e^u du = -e^u + C = -e^{-x^2} + C$$

7. $\int x\sqrt{4-x^2}\ dx$

Let $u = 4-x^2$; then

$du = -2xdx \Rightarrow -\frac{1}{2}du = xdx$.

$$\int x\sqrt{4-x^2}\ dx = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{3}u^{3/2} + C$$
$$= -\frac{1}{3}\left(4-x^2\right)^{3/2} + C$$

8. $\int \frac{(1+\ln x)^3}{x} dx$

Let $u = 1+\ln x$; then $du = \frac{1}{x}\ dx$.

$$\int \frac{(1+\ln x)^3}{x} dx = \int u^3 du = \frac{1}{4}u^4 + C$$
$$= \frac{1}{4}(1+\ln x)^4 + C$$

9. $\int \frac{1}{\sqrt{2x+1}}\ dx$

Let $u = 2x+1$; then $du = 2dx \Rightarrow \frac{1}{2}du = dx$.

$$\int \frac{1}{\sqrt{2x+1}}\ dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}du$$
$$= \left(\frac{1}{2}\right)(2u^{1/2}) + C = u^{1/2} + C$$
$$= \sqrt{2x+1} + C$$

10. $\int (x^3-6x)^7(x^2-2)\ dx$

Let $u = x^3-6x$; then

$du = (3x^2-6)\ dx \Rightarrow \frac{1}{3}du = (x^2-2)dx$.

$$\int (x^3-6x)^7(x^2-2)\ dx = \frac{1}{3}\int u^7 du = \frac{1}{24}u^8 + C$$
$$= \frac{1}{24}(x^3-6x)^8 + C$$

11. $\int xe^{x^2} dx$

Let $u = x^2$; then $du = 2xdx \Rightarrow \frac{1}{2}du = xdx$.

$$\int xe^{x^2} dx = \frac{1}{2}\int e^u du = \frac{e^u}{2} + C = \frac{1}{2}e^{x^2} + C$$

12. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$; then

$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2du = \frac{1}{\sqrt{x}}\,dx$.

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$

13. $\int \frac{\ln(2x)}{x}\,dx$

Let $u = \ln(2x)$; then $du = \frac{1}{2x}(2)\,dx = \frac{1}{x}\,dx$.

$\int \frac{\ln(2x)}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln(2x))^2}{2} + C$

14. $\int \frac{\sqrt{\ln x}}{x}\,dx$

Let $u = \ln x$; $du = \frac{1}{x}\,dx$.

$\int \frac{\sqrt{\ln x}}{x}\,dx = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C$
$= \frac{2}{3}(\ln x)^{3/2} + C$

15. $\int \frac{x^4}{x^5+1}\,dx$

Let $u = x^5 + 1$; then

$du = 5x^4dx \Rightarrow \frac{1}{5}du = x^4dx$

$\int \frac{x^4}{x^5+1}\,dx = \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|u| + C$
$= \frac{1}{5}\ln\left|x^5+1\right| + C$

16. $\int \frac{x}{\sqrt{x^2+1}}\,dx$

Let $u = x^2 + 1$; then $du = 2xdx \Rightarrow \frac{1}{2}du = xdx$.

$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int u^{-1/2}du = \left(\frac{1}{2}\right)2u^{1/2} + C$
$= u^{1/2} + C = (x^2+1)^{1/2} + C$

17. $\int \frac{x-3}{(1-6x+x^2)^2}\,dx$

Let $u = 1 - 6x + x^2$; then

$du = (-6+2x)dx \Rightarrow \frac{1}{2}du = (x-3)dx$.

$$\begin{aligned}\int \frac{x-3}{(1-6x+x^2)^2}\,dx &= \frac{1}{2}\int \frac{1}{u^2}\,du \\ &= \left(\frac{1}{2}\right)\left(-\frac{1}{u}\right) + C \\ &= -\frac{1}{2(1-6x+x^2)} + C \\ &= -\frac{1}{2-12x+2x^2} + C\end{aligned}$$

18. $\int x^{-2}\left(\frac{1}{x}+2\right)^5 dx$

Let $u = x^{-1} + 2$; then

$du = -x^{-2}dx \Rightarrow -du = x^{-2}dx$

$$\begin{aligned}\int x^{-2}\left(\frac{1}{x}+2\right)^5 dx &= -\int u^5du = -\frac{u^6}{6} + C \\ &= -\frac{1}{6}(x^{-1}+2)^6 + C\end{aligned}$$

19. $\int \frac{\ln\sqrt{x}}{x}\,dx$

Let $u = \ln x$; then $du = \frac{1}{x}\,dx$.

$$\begin{aligned}\int \frac{\ln\sqrt{x}}{x}\,dx &= \int \frac{\ln(x^{1/2})}{x}\,dx = \int \frac{\left(\frac{1}{2}\right)\ln x}{x}\,dx \\ &= \frac{1}{2}\int \frac{\ln x}{x}\,dx = \frac{1}{2}\int u\,du \\ &= \left(\frac{1}{2}\right)\frac{1}{2}u^2 + C = \frac{1}{4}u^2 + C \\ &= \frac{1}{4}(\ln x)^2 + C = (\ln\sqrt{x})^2 + C\end{aligned}$$

20. $\int \frac{x^2}{3-x^3}\,dx$

Let $u = 3 - x^3$; then

$du = -3x^2dx \Rightarrow -\frac{1}{3}du = x^2dx$

$$\begin{aligned}\int \frac{x^2}{3-x^3}\,dx &= -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + C \\ &= -\frac{1}{3}\ln\left|3-x^3\right| + C\end{aligned}$$

21. $\int \frac{x^2-2x}{x^3-3x^2+1}\,dx$

Let $u = x^3-3x^2+1$; then $du = (3x^2-6x)\,dx$

$\Rightarrow \frac{1}{3}du = (x^2-2x)dx$

$$\int \frac{x^2-2x}{x^3-3x^2+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3-3x^2+1\right| + C$$

22. $\int \frac{\ln(3x)}{3x}\,dx$

Let $u = \ln(3x)$; then $du = \frac{1}{3x}(3)dx = \frac{1}{x}dx \Rightarrow$

$\frac{1}{3}du = \frac{1}{3x}dx$

$$\int \frac{\ln(3x)}{3x}\,dx = \frac{1}{3}\int u\,du = \frac{1}{3}\left(\frac{1}{2}u^2\right) + C = \frac{1}{6}(\ln(3x))^2 + C$$

23. $\int \frac{8x}{e^{x^2}}\,dx$

Let $u = x^2$; then $du = 2x\,dx \Rightarrow 4du = 8xdx$

$$\int \frac{8x}{e^{x^2}}\,dx = 4\int e^{-u}\,du = -4e^{-u} + C = -4e^{-x^2} + C$$

24. $\int \frac{3}{(2x+1)^3}\,dx$

Let $u = 2x+1$; then $du = 2\,dx \Rightarrow \frac{3}{2}du = 3dx$

$$\int \frac{3}{(2x+1)^3}\,dx = \frac{3}{2}\int \frac{1}{u^3}du = \frac{3}{2}\left(-\frac{1}{2}u^{-2}\right) + C = -\frac{3}{4(2x+1)^2} + C$$

25. $\int \frac{1}{x\ln x^2}\,dx$

Let $u = \ln x^2 = 2\ln x$; then

$du = \frac{2}{x}dx \Rightarrow \frac{1}{2}du = \frac{1}{x}dx$

$$\int \frac{1}{x\ln x^2}\,dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|\ln x^2\right| + C$$

26. $\int \frac{2}{x(\ln x)^4}\,dx$

Let $u = \ln x$; then $du = \frac{1}{x}\,dx$.

$$\int \frac{2}{x(\ln x)^4}\,dx = 2\int \frac{1}{u^4}\,du = (2)\left(-\frac{1}{3}u^{-3}\right) + C = -\frac{2}{3u^3} + C = -\frac{2}{3(\ln x)^3} + C$$

27. $\int (3-x)(x^2-6x)^4dx$

Let $u = x^2-6x$; then $du = (2x-6)dx \Rightarrow$

$-\frac{1}{2}du = (3-x)dx$

$$\int (3-x)(x^2-6x)^4dx = -\frac{1}{2}\int u^4du = -\frac{1}{10}u^5 + C = -\frac{1}{10}(x^2-6x)^5 + C$$

28. $\int \frac{dx}{3-5x}$

Let $u = 3-5x$; then $du = -5dx \Rightarrow -\frac{1}{5}du = dx$

$$\int \frac{1}{3-5x}dx = -\frac{1}{5}\int \frac{1}{u}du = -\frac{1}{5}\ln|u| + C = -\frac{1}{5}\ln|3-5x| + C$$

29. $\int e^x(1+e^x)dx$

Let $u = 1+e^x$; then $du = e^x\,dx$.

$$\int e^x(1+e^x)dx = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}(1+e^x)^6 + C$$

30. $\int e^x\sqrt{1+e^x}\,dx$

Let $u = 1+e^x$; then $du = e^x dx$.

$$\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

31. $\int \frac{e^x}{1+2e^x}dx$

Let $u = 1+2e^x$; then $du = 2e^x dx \Rightarrow \frac{1}{2}du = e^x$

$$\int \frac{e^x}{1+2e^x}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln\left|1+2e^x\right| + C$$

32. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

Let $u = e^x - e^{-x}$; then $du = \left(e^x + e^{-x}\right)dx$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln\left|e^x - e^{-x}\right| + C$$

33. $\int \frac{e^{-x}}{1-e^{-x}}dx$

Let $u = 1 - e^{-x}$; then $du = e^{-x}dx$

$$\int \frac{e^{-x}}{1-e^{-x}}dx = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln\left|1-e^{-x}\right| + C$$

34. $\int \frac{\left(1+e^{-x}\right)^3}{e^x}dx = \int \left(1+e^{-x}\right)^3 e^{-x}dx$

Let $u = 1 + e^{-x}$; then $du = -e^{-x}dx$

$\Rightarrow -du = e^{-x}dx$

$$\int \frac{\left(1+e^{-x}\right)^3}{e^x}dx = -\int u^3 du = -\frac{1}{4}u^4 + C$$
$$= -\frac{1}{4}\left(1+e^{-x}\right)^4 + C$$

35. $\int \frac{1}{1+e^x}dx$. Using hint,

$\int \frac{1}{1+e^x}dx = \int \frac{e^{-x}}{e^{-x}+1}dx$ Let $u = e^{-x} + 1$; then

$du = -e^{-x}dx \Rightarrow -du = e^{-x}dx$

$$\int \frac{1}{1+e^x}dx = -\int \frac{1}{u}du = -\ln|u| + C$$
$$= -\ln\left(e^{-x}+1\right) + C$$

36. $\int \frac{e^{2x}-1}{e^{2x}+1}dx$. Using hint,

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$$

Let $u = e^x + e^{-x}$ then $du = \left(e^x - e^{-x}\right)dx$

$$\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln\left(e^x + e^{-x}\right) + C$$
$$= \ln\left(\frac{e^{2x}+1}{e^x}\right) + C$$

37. $f'(x) = \frac{x}{\sqrt{x^2+9}}$, so $f(x) = \int \frac{x}{\sqrt{x^2+9}}dx$

Let $u = x^2 + 9$; then $du = 2xdx \Rightarrow \frac{1}{2}du = xdx$.

$$\int \frac{x}{\sqrt{x^2+9}}dx = \frac{1}{2}\int u^{-1/2}du = \left(\frac{1}{2}\right)2u^{1/2} + C$$
$$= u^{1/2} + C = (x^2+9)^{1/2} + C$$

But $f(4) = 8$, so $8 = (4^2+9)^{1/2} + C$, hence $C = 3$, and $f(x) = (x^2+9)^{1/2} + 3$.

38. $f'(x) = \frac{2\sqrt{x}+1}{\sqrt{x}}$, so

$$f(x) = \int \frac{2\sqrt{x}+1}{\sqrt{x}}dx = 2\int dx + \int \frac{dx}{\sqrt{x}}$$
$$= 2x + 2\sqrt{x} + C$$

But $f(4) = 15$, so $15 = 8 + 4 + C$, hence $C = 3$, and $f(x) = 2x + 2\sqrt{x} + 3$.

39. $\int (x+5)^{-1/2} e^{\sqrt{x+5}}dx$

Let $u = \sqrt{x+5}$; then $du = \frac{1}{2}(x+5)^{-1/2}dx \Rightarrow$

$2du = (x+5)^{-1/2}dx$

$$\int (x+5)^{-1/2} e^{(x+5)^{1/2}}dx = 2\int e^u du = 2e^u + C$$
$$= 2e^{(x+5)^{1/2}} + C$$

40. $\int \frac{x^4}{x^5-7}\ln(x^5-7)\ dx$

Let $u=\ln(x^5-7)$; then $du=\frac{1}{x^5-7}(5x^4)\ dx$

$\Rightarrow \frac{1}{5}du=\frac{x^4}{x^5-7}\ dx$

$$\int \frac{x^4}{x^5-7}\ln(x^5-7)\ dx=\frac{1}{5}\int u\ du=\left(\frac{1}{5}\right)\frac{1}{2}u^2+C=\frac{1}{10}\left[\ln(x^5-7)\right]^2+C$$

41. $\int x\sec^2(x^2)\ dx$

Let $u=x^2$; then $du=2x\ dx \Rightarrow \frac{1}{2}du=xdx$

$$\int x\sec^2(x^2)\ dx=\frac{1}{2}\int \sec^2 u\ du=\frac{1}{2}\tan u+C=\frac{1}{2}\tan x^2+C$$

42. $\int (1+\ln x)\sin(x\ln x)\ dx$

Let $u=x\ln x$; then

$du=\left(\ln x+x\left(\frac{1}{x}\right)\right)\ dx=(\ln x+1)\ dx$.

$$\int (1+\ln x)\sin(x\ln x)\ dx=\int \sin u\ du=-\cos u+C=-\cos(x\ln x)+C$$

43. $\int \sin x\cos x\ dx$

Let $u=\sin x$; then $du=\cos x\ dx$.

$$\int \sin x\cos x\ dx=\int u\ du=\frac{u^2}{2}+C=\frac{\sin^2 x}{2}+C$$

44. $\int 2x\cos(x^2)\ dx$

Let $u=x^2$; then $du=2x\ dx$.

$$\int 2x\cos(x^2)\ dx=\int \cos u\ du=\sin u+C=\sin(x^2)+C$$

45. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx$

Let $u=\sqrt{x} \Rightarrow du=\frac{1}{2}x^{-1/2}dx=\frac{1}{2\sqrt{x}}\ dx$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx=2\int \cos u\ du=2\sin u+C=2\sin\sqrt{x}+C$$

46. $\int \frac{\cos x}{(2+\sin x)^3}\ dx$

Let $u=2+\sin x$; $du=\cos x\ dx$.

$$\int \frac{\cos x}{(2+\sin x)^3}\ dx=\int u^{-3}du=-\frac{1}{2}u^{-2}+C=\frac{-1}{2(2+\sin x)^2}+C$$

47. $\int \cos^3 x\sin x\ dx$

Let $u=\cos x$; then $du=-\sin x\ dx \Rightarrow$ $-du=\sin xdx$

$$\int \cos^3 x\sin x\ dx=-\int u^3du=-\frac{1}{4}u^4+C=-\frac{1}{4}\cos^4 x+C$$

48. $\int (\sin 2x)e^{\cos 2x}\ dx$

Let $u=\cos 2x$; then $du=-2\sin 2x\ dx \Rightarrow$ $-\frac{1}{2}du=\sin 2xdx$

$$\int (\sin 2x)e^{\cos 2x}dx=-\frac{1}{2}\int e^u du=-\frac{1}{2}e^u+C=-\frac{1}{2}e^{\cos 2x}+C$$

49. $\int \frac{\cos 3x}{\sqrt{2-\sin 3x}}\ dx$

Let $u=2-\sin 3x$; then $du=-3\cos 3x\ dx \Rightarrow$ $-\frac{1}{3}du=\cos 3xdx$

$$\int \frac{\cos 3x}{\sqrt{2-\sin 3x}}\ dx=-\frac{1}{3}\int \frac{1}{\sqrt{u}}\ du=\left(-\frac{1}{3}\right)(2u^{1/2})+C=-\frac{2}{3}\sqrt{2-\sin 3x}+C$$

50. $\int \cot xdx=\int \frac{\cos x}{\sin x}dx$

Let $u=\sin x$; then $du=\cos x\ dx$

$$\int \cot xdx=\int \frac{1}{u}\ du=\ln|u|+C=\ln|\sin x|+C$$

51. $\int \frac{\sin x+\cos x}{\sin x-\cos x}dx$

Let $u=\sin x-\cos x$; then $du=(\cos x+\sin x)dx$

$$\int \frac{\sin x+\cos x}{\sin x-\cos x}dx=\int \frac{1}{u}\ du=\ln|u|+C=\ln|\sin x-\cos x|+C$$

52. $\int \tan x \sec^2 x dx$

Let $u = \tan x$; then $du = \sec^2 x dx$

$$\int \tan x \sec^2 x dx = \int u\, du = \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}\tan^2 x + C$$

53. $\int 2x(x^2+5)\ dx$

Let $u = x^2 + 5$; then $du = 2x\ dx$.

$$\int 2x(x^2+5)\ dx = \int u\ du = \frac{1}{2}u^2 + C$$
$$= \frac{1}{2}(x^2+5)^2 + C$$
$$= \frac{1}{2}x^4 + 5x^2 + \frac{25}{2} + C$$
$$\int 2x(x^2+5)\ dx = \int (2x^3 + 10x)\ dx$$
$$= \frac{1}{2}x^4 + 5x^2 + C_1.$$

The two methods differ by a constant.

9.2 Integration by Parts

1. $\int xe^{5x} dx$

Let $f(x) = x,\ g(x) = e^{5x}$.

Then $f'(x) = 1$ and $G(x) = \frac{1}{5}e^{5x}$.

$$\int xe^{5x}dx = x\left(\frac{1}{5}e^{5x}\right) - \int (1)\frac{1}{5}e^{5x}$$
$$= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C$$

2. $\int xe^{x/2}dx$

Let $f(x) = x,\ g(x) = e^{x/2}$.

Then $f'(x) = 1$ and $G(x) = 2e^{x/2}$.

$$\int xe^{x/2}dx = x\left(2e^{x/2}\right) - \int (1)\left(2e^{x/2}\right)\ dx$$
$$= 2xe^{x/2} - 4e^{x/2} + C$$

3. $\int x(x+7)^4 dx$

Let $f(x) = x,\ g(x) = (x+7)^4$.

Then $f'(x) = 1$ and $G(x) = \frac{1}{5}(x+7)^5$.

$$\int x(x+7)^4 dx$$
$$= x\left(\frac{1}{5}(x+7)^5\right) - \int (1)\frac{1}{5}(x+7)^5 dx$$
$$= \frac{1}{5}x(x+7)^5 - \frac{1}{30}(x+7)^6 + C$$

4. $\int x(2x-3)^2 dx$

Let $f(x) = x,\ \ g(x) = (2x-3)^2$.

Then $f'(x) = 1$ and $G(x) = \frac{1}{6}(2x-3)^3$.

$$\int x(2x-3)^3 dx$$
$$= x\left(\frac{1}{6}\right)(2x-3)^3 - \int (1)\left(\frac{1}{6}\right)(2x-3)^3 dx$$
$$= \frac{1}{6}x(2x-3)^3 - \frac{1}{6}\left(\frac{1}{8}\right)(2x-3)^4 + C$$
$$= \frac{1}{6}x(2x-3)^3 - \frac{1}{48}(2x-3)^4 + C$$

5. $\int \frac{x}{e^x}dx = \int xe^{-x}dx$

Let $f(x) = x,\ g(x) = e^{-x}$.

Then $f'(x) = 1$ and $G(x) = -e^{-x}$.

$$\int xe^{-x}dx = x(-e^{-x}) - \int (1)(-e^{-x})\ dx$$
$$= -xe^{-x} - e^{-x} + C$$

6. $\int x^2 e^x dx$

Let $f(x) = x^2,\ g(x) = e^x$.

Then $f'(x) = 2x$ and $G(x) = e^x$.

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$$

The integral $\int 2xe^x dx$ itself needs to be handled by integration by parts:

Let $f(x) = 2x,\ g(x) = e^x$. Then $f'(x) = 2$ and $G(x) = e^x$.

$$\int 2xe^x dx = 2xe^x - \int 2e^x dx = 2xe^x - 2e^x + C$$

Combining, we have

$$\int x^2 e^x dx = x^2 e^x - \left(2xe^x - 2e^x\right) + C$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

7. $\int \frac{x}{\sqrt{x+1}}\ dx$

Let $f(x) = x,\ g(x) = (x+1)^{-1/2}$.

Then $f'(x) = 1$ and $G(x) = 2(x+1)^{1/2}$.

$$\int \frac{x}{\sqrt{x+1}}\ dx = x\left[2(x+1)^{1/2}\right] - \int (1)2(x+1)^{1/2}dx$$
$$= 2x(x+1)^{1/2} - \frac{4}{3}(x+1)^{3/2} + C$$

8. $\int \frac{x}{\sqrt{3+2x}}\,dx$

Let $f(x) = x,\ g(x) = (3+2x)^{-1/2}$.

Then $f'(x) = 1$ and $G(x) = (3+2x)^{1/2}$.

$$\int \frac{x}{\sqrt{3+2x}}\,dx = x(3+2x)^{1/2} - \int (1)(3+2x)^{1/2}dx$$
$$= x(3+2x)^{1/2} - \frac{1}{3}(3+2x)^{3/2} + C$$

9. $\int e^{2x}(1-3x)\,dx$

Let $f(x) = 1 - 3x,\ g(x) = e^{2x}$.

Then $f'(x) = -3$ and $G(x) = \frac{1}{2}e^{2x}$.

$$\int e^{2x}(1-3x)\,dx$$
$$= (1-3x)\left(\frac{1}{2}e^{2x}\right) - \int (-3)\left(\frac{1}{2}e^{2x}\right)dx$$
$$= (1-3x)\left(\frac{1}{2}e^{2x}\right) + \left(\frac{3}{2}\right)\int e^{2x}dx$$
$$= (1-3x)\left(\frac{1}{2}e^{2x}\right) + \frac{3}{4}e^{2x} + C$$

10. $\int (1+x)^2 e^{2x}dx$

Let $f(x) = (1+x)^2,\ g(x) = e^{2x}$.
Then
$f'(x) = 2(1+x) = 2x+2$ and $G(x) = \frac{1}{2}e^{2x}$.

$$\int (1+x)^2 e^{2x}dx$$
$$= (1+x)^2\left(\frac{1}{2}\right)e^{2x} - \int (2x+2)\left(\frac{1}{2}e^{2x}\right)dx$$
$$= \frac{1}{2}(1+x)^2 e^{2x} - \int (x+1)e^{2x}dx$$

The integral $\int (x+1)e^{2x}dx$ itself needs to be handled by integration by parts:
Let $f(x) = x+1,\ g(x) = e^{2x}$. Then $f'(x) = 1$ and $G(x) = \frac{1}{2}e^{2x}$.

$$\int (x+1)e^{2x}dx$$
$$= (1+x)\left(\frac{1}{2}e^{2x}\right) - \int (1)\left(\frac{1}{2}e^{2x}\right)dx$$
$$= \frac{1}{2}(1+x)e^{2x} - \frac{1}{4}e^{2x}dx + C$$

Combining, we have

$$\int (1+x)^2 e^{2x}dx$$
$$= \frac{1}{2}(1+x)^2 e^{2x} - \left(\frac{1}{2}(1+x)e^{2x} - \frac{1}{4}e^{2x}dx\right) + C$$
$$= \frac{1}{2}(1+x)^2 e^{2x} - \frac{1}{2}(1+x)e^{2x} + \frac{1}{4}e^{2x} + C$$

11. $\int \frac{6x}{e^{3x}}\,dx$

Let $f(x) = 6x,\ g(x) = e^{-3x}$.

Then $f'(x) = 6$ and $G(x) = -\frac{1}{3}e^{-3x}$.

$$\int \frac{6x}{e^{3x}}\,dx = 6x\left(-\frac{1}{3}e^{-3x}\right) - \int 6\left(-\frac{1}{3}e^{-3x}\right)dx$$
$$= -2xe^{-3x} + (2)\int e^{-3x}dx$$
$$= -2xe^{-3x} - \frac{2}{3}e^{-3x} + C$$

12. $\int \frac{x+2}{e^{2x}}\,dx$

Let $f(x) = x + 2,\ g(x) = e^{-2x}$.

Then $f'(x) = 1$ and $G(x) = -\frac{1}{2}e^{-2x}$.

$$\int \frac{x+2}{e^{2x}}\,dx$$
$$= (x+2)\left(-\frac{1}{2}e^{-2x}\right) - \int (1)\left(-\frac{1}{2}e^{-2x}\right)dx$$
$$= (x+2)\left(-\frac{1}{2}e^{-2x}\right) + \left(\frac{1}{2}\right)\int e^{-2x}dx$$
$$= (x+2)\left(-\frac{1}{2}e^{-2x}\right) - \frac{1}{4}e^{-2x} + C$$

13. $\int x\sqrt{x+1}\,dx$

Let $f(x) = x,\ g(x) = (x+1)^{1/2}$.

Then $f'(x) = 1$ and $G(x) = \frac{2}{3}(x+1)^{3/2}$.

$$\int x\sqrt{x+1}dx$$
$$= x\left[\frac{2}{3}(x+1)^{3/2}\right] - \int (1)\left(\frac{2}{3}(x+1)^{3/2}\right)dx$$
$$= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C$$

14. $\int x\sqrt{2-x}dx$

Let $f(x)=x,\ g(x)=(2-x)^{1/2}$.

Then $f'(x)=1$ and $G(x)=-\frac{2}{3}(2-x)^{3/2}$.

$$\int x\sqrt{2-x}\ dx$$
$$=x\left[-\frac{2}{3}(2-x)^{3/2}\right]-\int(1)\left(-\frac{2}{3}(2-x)^{3/2}\right)dx$$
$$=-\frac{2}{3}x(2-x)^{3/2}-\frac{4}{15}(2-x)^{5/2}+C$$

15. $\int\sqrt{x}\ln\sqrt{x}\ dx$

Let $f(x)=\ln\sqrt{x},\ g(x)=\sqrt{x}$.

Then $f'(x)=\frac{1}{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right)=\frac{1}{2x}$ and

$G(x)=\frac{2}{3}x^{3/2}$.

$$\int\sqrt{x}\ln\sqrt{x}\ dx$$
$$=\ln\sqrt{x}\left[\frac{2}{3}x^{3/2}\right]-\int\left(\frac{1}{2x}\right)\frac{2}{3}x^{3/2}dx$$
$$=\frac{2}{3}x^{3/2}\ln\sqrt{x}-\int\frac{1}{3}x^{1/2}dx$$
$$=\frac{2}{3}x^{3/2}\ln\sqrt{x}-\frac{2}{9}x^{3/2}+C$$
$$=\frac{1}{3}x^{3/2}\ln x-\frac{2}{9}x^{3/2}+C$$

16. $\int x^5\ln x\ dx$

Let $f(x)=\ln x,\ g(x)=x^5$.

Then $f'(x)=\frac{1}{x}$ and $G(x)=\frac{1}{6}x^6$.

$$\int x^5\ln x\ dx=(\ln x)\left(\frac{1}{6}x^6\right)-\int\left(\frac{1}{x}\right)\frac{1}{6}x^6dx$$
$$=\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+C$$

17. $\int x\cos x\ dx$

Let $f(x)=x,\ g(x)=\cos x$.

Then $f'(x)=1$ and $G(x)=\sin x$.

$$\int x\cos x\ dx=x\sin x-\int(1)(\sin x)\ dx$$
$$=x\sin x+\cos x+C$$

18. $\int x\sin 8x\ dx$

Let $f(x)=x,\ g(x)=\sin 8x$.

Then $f'(x)=1$ and $G(x)=-\frac{1}{8}\cos 8x$.

$$\int x\sin 8x\ dx$$
$$=x\left(-\frac{1}{8}\cos 8x\right)-\int(1)\left(-\frac{1}{8}\cos 8x\right)dx$$
$$=-\frac{1}{8}x\cos 8x+\frac{1}{64}\sin 8x+C$$

19. $\int x\ln 5x\ dx$

Let $f(x)=\ln 5x,\ g(x)=x$.

Then $f'(x)=\frac{1}{5x}(5)=\frac{1}{x}$ and $G(x)=\frac{x^2}{2}$.

$$\int x\ln 5x\ dx=(\ln 5x)\left(\frac{x^2}{2}\right)-\int\frac{1}{x}\left(\frac{x^2}{2}\right)dx$$
$$=\frac{1}{2}x^2\ln 5x-\frac{1}{4}x^2+C$$

20. $\int x^{-3}\ln x\ dx$

Let $f(x)=\ln x,\ g(x)=x^{-3}$.

Then $f'(x)=\frac{1}{x}$ and $G(x)=-\frac{1}{2}x^{-2}$.

$$\int x^{-3}\ln x\ dx$$
$$=(\ln x)\left(-\frac{1}{2}x^{-2}\right)-\int\frac{1}{x}\left(-\frac{1}{2}x^{-2}\right)dx$$
$$=-\frac{1}{2}x^{-2}\ln x-\frac{1}{4}x^{-2}+C$$

21. $\int\ln x^4dx$

Let $f(x)=\ln x^4,\ g(x)=1$.

Then $f'(x)=\frac{1}{x^4}(4x^3)=\frac{4}{x}$ and $G(x)=x$.

$$\int\ln x^4dx=(\ln x^4)(x)-\int\frac{4}{x}(x)\ dx$$
$$=x\ln x^4-4x+C=4x\ln x-4x+C$$

22. $\int\frac{\ln(\ln x)}{x}dx$

Let $f(x)=\ln(\ln x),\ g(x)=x^{-1}$.

Then $f'(x)=\frac{1}{\ln x}\cdot\frac{1}{x}$ and $G(x)=\ln x$.

$$\int\frac{\ln(\ln x)}{x}dx=\ln(\ln x)\ln x-\int\frac{1}{\ln x}\cdot\frac{1}{x}\ln x dx$$
$$=\ln(\ln x)\ln x-\int\frac{1}{x}dx$$
$$=\ln(\ln x)\ln x-\ln x+C$$

23. $\int x^2e^{-x}dx$

Let $f(x) = x^2, g(x) = e^{-x}$.

Then $f'(x) = 2x$ and $G(x) = -e^{-x}$.

$$\int x^2e^{-x}dx = x^2(-e^{-x}) - \int 2x(-e^{-x})\ dx$$
$$= -x^2e^{-x} + \int 2xe^{-x}dx$$

To evaluate $\int 2xe^{-x}dx$, use parts again:

Let $f(x) = 2x,\ g(x) = e^{-x}$.

Then $f'(x) = 2$ and $G(x) = -e^{-x}$.

$$\int 2xe^{-x}dx = 2x(-e^{-x}) - \int 2(-e^{-x})\ dx$$
$$= -2xe^{-x} - 2e^{-x} + C$$

Therefore,

$$\int x^2e^{-x}dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$
$$= -e^{-x}\left(x^2 + 2x + 2\right) + C$$

24. $\int \ln\sqrt{x+1}\ dx$

Let $f(x) = \ln\sqrt{x+1}, g(x) = 1$.

Then $f'(x) = \dfrac{1}{2(x+1)}$ and $G(x) = x$.

$$\int \ln\sqrt{x+1}\ dx$$
$$= x\ln\sqrt{x+1} - \frac{1}{2}\int \frac{x}{x+1}dx$$
$$= x\ln\sqrt{x+1} - \frac{1}{2}\left(x\ln(x+1) - \int \ln(x+1)dx\right)$$
$$= x\ln\sqrt{x+1} - \frac{1}{2}x\ln(x+1)$$
$$+\frac{1}{2}\left(\ln(x+1)(x+1) - (x+1)\right) + C$$
$$= x\ln\sqrt{x+1} - x\ln\sqrt{x+1} + \ln\sqrt{x+1}(x+1)$$
$$-\frac{1}{2}x - \frac{1}{2} + C$$
$$= \ln\sqrt{x+1}(x+1) - \frac{1}{2}x - \frac{1}{2} + C$$
$$= \ln\sqrt{x+1}(x+1) - \frac{1}{2}x + C$$

25. $\int x(x+5)^4dx$

Use integration by parts: Let $f(x) = x$, $g(x) = (x+5)^4$. Then $f'(x) = 1$ and $G(x) = \frac{1}{5}(x+5)^5$.

$$\int x(x+5)^4dx = x\left[\frac{1}{5}(x+5)^5\right] - \int (1)\frac{1}{5}(x+5)^5dx$$
$$= \frac{1}{5}x(x+5)^5 - \frac{1}{30}(x+5)^6 + C$$

26. $\int 4x\cos(x^2+1)\ dx$

Use the substitution, $u = x^2 + 1$; then $du = 2x\ dx \Rightarrow 2du = 4xdx$

$$\int 4x\cos(x^2+1)\ dx = 2\int \cos u\ du = 2\sin u + C$$
$$= 2\sin(x^2+1) + C$$

27. $\int x(x^2+5)^4dx$

Use the substitution, $u = x^2 + 5$; then $du = 2x\ dx \Rightarrow \frac{1}{2}du = xdx$

$$\int x(x^2+5)^4dx = \frac{1}{2}\int u^4du = \left(\frac{1}{2}\right)\frac{1}{5}u^5 + C$$
$$= \frac{1}{10}u^5 + C = \frac{1}{10}(x^2+5)^5 + C$$

28. $\int 4x\cos(x+1)\ dx$

Use integration by parts with $f(x) = 4x$, $g(x) = \cos(x+1)$.
Then $f'(x) = 4$ and $G(x) = \sin(x+1)$.

$$\int 4x\cos(x+1)\ dx$$
$$= 4x\sin(x+1) - \int 4\sin(x+1)\ dx$$
$$= 4x\sin(x+1) + 4\cos(x+1) + C$$

29. $\int (3x+1)e^{x/3}dx$

Use integration by parts with $f(x) = 3x + 1$, $g(x) = e^{x/3}$. Then $f'(x) = 3$ and $G(x) = 3e^{x/3}$.

$$\int (3x+1)e^{x/3}dx$$
$$= (3x+1)(3e^{x/3}) - \int (3)3e^{x/3}dx$$
$$= (9x+3)e^{x/3} - 27e^{x/3} + C$$

30. $\int \frac{(\ln x)^5}{x}\ dx$

Use the substitution, $u = \ln x;\ du = \frac{1}{x}dx$.

$$\int \frac{(\ln x)^5}{x}\ dx = \int u^5du = \frac{1}{6}u^6 + C$$
$$= \frac{1}{6}(\ln x)^6 + C$$

31. $\int x\sec^2(x^2+1)\ dx$

Use the substitution, $u = x^2+1$; then $du = 2x\ dx \Rightarrow \frac{1}{2}du = xdx$

$$\int x\sec^2(x^2+1)\ dx = \frac{1}{2}\int \sec^2 u\ du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan\left(x^2+1\right)+C$$

32. $\int \frac{\ln x}{x^5}\ dx$

Let $f(x) = \ln x,\ g(x) = x^{-5}$.

Then $f'(x) = \frac{1}{x}$ and $G(x) = -\frac{1}{4}x^{-4}$.

$$\int \frac{\ln x}{x^5}\ dx = (\ln x)\left(-\frac{1}{4}x^{-4}\right) - \int \frac{1}{x}\left(-\frac{1}{4}x^{-4}\right)dx$$
$$= -\frac{1}{4}x^{-4}\ln x + \frac{1}{4}\int x^{-5}dx$$
$$= -\frac{1}{4}x^{-4}\ln x + \left(\frac{1}{4}\right)\left(-\frac{1}{4}x^{-4}\right)+C$$
$$= -\frac{1}{4}x^{-4}\ln x - \frac{1}{16}x^{-4}+C$$

33. $\int (xe^{2x}+x^2)\ dx = \int xe^{2x}dx + \int x^2 dx$

$\int x^2 dx = \frac{1}{3}x^3 + C_1$

To evaluate $\int xe^{2x}dx$, use integration by parts with $f(x) = x,\ g(x) = e^{2x}$.

Then $f'(x) = 1$ and $G(x) = \frac{1}{2}e^{2x}$.

$$\int xe^{2x}dx = x\left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C_2$$

Therefore,

$$\int (xe^{2x}+x^2)\ dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + \frac{1}{3}x^3 + C$$

34. $\int (x^{3/2}+\ln 2x)\ dx = \int x^{3/2}dx + \int \ln 2x\ dx$

$\int x^{3/2}dx = \frac{2}{5}x^{5/2} + C_1$

To evaluate $\int \ln 2x\ dx$, use integration by parts with $f(x) = \ln 2x$, $g(x) = 1$.

Then $f'(x) = \frac{1}{2x}(2) = \frac{1}{x}$ and $G(x) = x$.

$$\int \ln 2x\ dx = (\ln 2x)(x) - \int \frac{1}{x}(x)\ dx = x\ln 2x - x + C_2$$

Therefore,

$$\int (x^{3/2}+\ln 2x)\ dx = \frac{2}{5}x^{5/2} + x\ln 2x - x + C.$$

35. $\int (xe^{x^2} - 2x)\ dx = \int xe^{x^2}dx - \int 2x\ dx$

$\int 2x\ dx = x^2 + C_1$

To evaluate $\int xe^{x^2}dx$, use the substitution,

$u = x^2$; $du = 2x\ dx \Rightarrow \frac{1}{2}du = xdx$

$$\int xe^{x^2}dx = \frac{1}{2}\int e^{x^2}2x\ dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C_2 = \frac{1}{2}e^{x^2} + C_2$$

Therefore, $\int (xe^{x^2} - 2x)\ dx = \frac{1}{2}e^{x^2} - x^2 + C$.

36. $\int (x^2 - x\sin 2x)\ dx = \int x^2 dx - \int x\sin 2x\ dx$

$\int x^2 dx = \frac{1}{3}x^3 + C_1$

To evaluate $\int x\sin 2x\ dx$, use integration by parts with $f(x) = x$, $g(x) = \sin 2x$.

Then $f'(x) = 1$ and $G(x) = -\frac{1}{2}\cos 2x$.

$$\int x\sin 2x\ dx = x\left[-\frac{1}{2}\cos 2x\right] - \int -\frac{1}{2}\cos 2x\ dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C_2$$

Therefore,

$$\int (x^2 - x\sin 2x)\ dx = \frac{1}{3}x^3 + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$$

37. The slope is $\frac{x}{\sqrt{x+9}}$, so $f'(x)=\frac{x}{\sqrt{x+9}}$ and thus $f(x)=\int \frac{x}{\sqrt{x+9}}\,dx$. Let $f(x)=x$ and $g(x)=(x+9)^{-1/2}$.

Then $f'(x)=1$ and $G(x)=2(x+9)^{1/2}$.

$$\int \frac{x}{\sqrt{x+9}}\,dx = 2x(x+9)^{1/2} - \int 2(x+9)^{1/2}\,dx$$
$$= 2x(x+9)^{1/2} - \frac{4}{3}(x+9)^{3/2} + C \Rightarrow$$
$$f(x) = 2x(x+9)^{1/2} - \frac{4}{3}(x+9)^{3/2} + C$$
$$f(0)=2 \Rightarrow$$
$$2(0)(0+9)^{1/2} - \frac{4}{3}(0+9)^{3/2} + C = 2 \Rightarrow$$
$$-36 + C = 2 \Rightarrow C = 38$$

Therefore,

$$f(x) = 2x(x+9)^{1/2} - \frac{4}{3}(x+9)^{3/2} + 38$$

38. Slope is $\frac{x}{e^{x/3}}$, so $\frac{x}{e^{x/3}}$, $f(x)=\int xe^{-x/3}dx$.

By parts:

$h(x)=x, g(x)=e^{-x/3}$

$h'(x)=1, G(x)=-3e^{-x/3}$

$$f(x)=\int xe^{-x/3}dx = -3xe^{-x/3} + 3\int e^{-x/3}dx$$
$$= -3xe^{-x/3} - 9e^{-x/3} + C$$

$f(0) = 6$, $6 = -9 + C \Rightarrow C = 15$, so

$f(x) = -3xe^{-x/3} - 9e^{-x/3} + 15$

39. $\int \frac{xe^x}{(x+1)^2}\,dx$

Use integration by parts with $f(x)=xe^x$ and $g(x)=\frac{1}{(x+1)^2}$.

Then $f'(x)=e^x(x+1)$ and $G(x)=-\frac{1}{x+1}$.

$$\int \frac{xe^x}{(x+1)^2}dx$$
$$= xe^x\left(-\frac{1}{x+1}\right) - \int\left(-\frac{1}{x+1}\right)e^x(x+1)\,dx$$
$$= -\frac{xe^x}{x+1} - \int(-e^x)\,dx = -\frac{xe^x}{x+1} + e^x + C$$
$$= \frac{e^x}{x+1} + C$$

40. $\int x^7 e^{x^4}\,dx$

First, let $u = x^4$. Then

$du = 4x^3dx \Rightarrow \frac{1}{4}du = x^3dx$. So, we have

$$\int x^7 e^{x^4}dx = \frac{1}{4}\int x^4 e^u du = \frac{1}{4}\int u\,e^u du.$$

Now use integration by parts with $f(u)=u$ and $g(u)=e^u$. Then $f'(u)=du$ and $G(u)=e^u$.

$$\frac{1}{4}\int u\,e^u du = \frac{1}{4}ue^u - \frac{1}{4}\int e^u\,du$$
$$= \frac{1}{4}ue^u - \frac{1}{4}e^u + C$$

Substitute back:

$$\frac{1}{4}ue^u - \frac{1}{4}e^u + C = \frac{1}{4}x^4e^{x^4} - \frac{1}{4}e^{x^4} + C$$

9.3 Evaluation of Definite Integrals

1. $\int_{5/2}^{3} 2(2x-5)^{14}\,dx$

Let $u = 2x - 5$; $du = 2\,dx$. When $x=\frac{5}{2}$, $u = 2\left(\frac{5}{2}\right)-5=0$.

When $x = 3$, $u = 2(3) - 5 = 1$.

$$\int_{5/2}^{3} 2(2x-5)^{14}\,dx = \int_0^1 u^{14}du = \frac{1}{15}u^{15}\Big|_0^1$$
$$= \frac{1}{15}(1)^{15} - 0 = \frac{1}{15}$$

2. $\int_2^6 \frac{1}{\sqrt{4x+1}}\,dx$

Let $u = 4x+1; du = 4\ dx \Rightarrow \frac{1}{4}du = dx$.

When $x = 2$, $u = 4(2)+1 = 9$. When $x = 6, u = 4(6)+1 = 25$.

$$\int_2^6 \frac{1}{\sqrt{4x+1}}\,dx = \frac{1}{4}\int_9^{25}\frac{1}{\sqrt{u}}\,du = \frac{1}{4}\cdot 2u^{1/2}\Big|_9^{25}$$
$$= \frac{5}{2} - \frac{3}{2} = 1$$

3. $\int_0^2 4x(1+x^2)^3\ dx$

Let $u = 1+x^2$; $du = 2x\ dx \Rightarrow 2du = 4xdx$

When $x = 0$, $u = 1+0^2 = 1$. When $x = 2$, $u = 1+2^2 = 5$.

$$\int_0^2 4x\left(1+x^2\right)^3 dx = 2\int_1^5 u^3 du = \frac{1}{2}u^4\Big|_1^5 = \frac{625}{2} - \frac{1}{2} = 312$$

4. $\int_0^1 \frac{2x}{\sqrt{x^2+1}}\ dx$

Let $u = x^2+1$; $du = 2x\ dx$

When $x = 0$, $u = 0^2+1 = 1$. When $x = 1$, $u = 1^2+1 = 2$.

$$\int_0^1 \frac{2x}{\sqrt{x^2+1}}\ dx = \int_1^2 \frac{1}{\sqrt{u}}du = 2u^{1/2}\Big|_1^2 = 2\sqrt{2}-2$$

5. $\int_0^3 \frac{x}{\sqrt{x+1}}\ dx$

Use integration by parts with $f(x) = x,\ g(x) = \frac{1}{\sqrt{x+1}}$. Then $f'(x) = 1,\ G(x) = 2\sqrt{x+1}$

$$\int_0^3 \frac{x}{\sqrt{x+1}}\ dx = x\left(2\sqrt{x+1}\right)\Big|_0^3 - \int_0^3 2\sqrt{x+1}\ dx$$
$$= 12 - \frac{4}{3}(x+1)^{3/2}\Big|_0^3$$
$$= 12 - \left(\frac{32}{3} - \frac{4}{3}\right) = 12 - \frac{28}{3} = \frac{8}{3}$$

6. $\int_0^1 x(3+x)^5\ dx$

Use integration by parts with $f(x) = x,\ g(x) = (3+x)^5$. Then $f'(x) = 1,\ G(x) = \frac{1}{6}(3+x)^5$

$$\int_0^1 x(3+x)^5\ dx$$
$$= \frac{1}{6}x(3+x)^6\Big|_0^1 - \int_0^1 \frac{1}{6}(3+x)^6\ dx$$
$$= \frac{4096}{6} - \frac{1}{42}(3+x)^7\Big|_0^1$$
$$= \frac{4096}{6} - \left(\frac{16384}{42} - \frac{2187}{42}\right) = \frac{14475}{42} = \frac{4825}{14}$$

7. $\int_3^5 x\sqrt{x^2-9}\ dx$

Let $u = x^2-9$, $du = 2x\ dx \Rightarrow \frac{1}{2}du = xdx$

When $x = 3$, $u = 3^2-9 = 0$. When $x = 5$, $u = 5^2-9 = 16$.

$$\int_2^5 x\sqrt{x^2-9}\ dx = \frac{1}{2}\int_2^5 \sqrt{x^2-9}\ 2x\,dx$$
$$= \frac{1}{2}\int_0^{16} u^{1/2}du = \left(\frac{1}{2}\right)\frac{2}{3}u^{3/2}\Big|_0^{16}$$
$$= \frac{1}{3}16^{3/2} - 0 = \frac{64}{3}$$

8. $\int_0^1 \frac{1}{(1+2x)^4}dx$

Let $u = 1+2x$, $du = 2dx \Rightarrow \frac{1}{2}du = dx$

When $x = 0$, $u = 1+2(0) = 1$. When $x = 1$, $u = 1+2(1) = 3$

$$\int_0^1 \frac{1}{(1+2x)^4}dx = \frac{1}{2}\int_1^3 \frac{1}{u^4}du = -\frac{1}{6}u^{-3}\Big|_1^3$$
$$= -\frac{1}{6}\left(\frac{1}{27}-1\right) = \frac{13}{81}$$

9. $\int_{-1}^2 \left(x^2-1\right)\left(x^3-3x\right)^4 dx$

Let $u = x^3-3x;\ du = \left(3x^2-3\right)dx \Rightarrow$

$\frac{1}{3}du = \left(x^2-1\right)dx$

When $x = -1$, $u = (-1)^3 - 3(-1) = 2.$

When $x = 2$, $u = 2^3 - 3(2) = 2.$

$$\int_{-1}^2 \left(x^2-1\right)\left(x^3-3x\right)^4 dx = \frac{1}{3}\int_2^2 u^4\ du$$
$$= \left(\frac{1}{15}u^5\right)\Big|_2^2$$
$$= \frac{32}{15} - \frac{32}{15} = 0$$

10. $\int_0^1 (2x-1)(x^2-x)^{10}\,dx$

Let $u = x^2 - x;\ du = (2x-1)\,dx$,

When $x = 0,\ u = 0^2 - 0 = 0$. When $x = 1$,
$u = 1^2 - 1 = 0$.

$$\int_0^1 (2x-1)(x^2-x)^{10}\,dx = \int_0^0 u^{10}\,du = \left(\frac{1}{11}u^{11}\right)\Big|_0^0 = 0 - 0 = 0$$

11. $\int_0^1 \frac{x}{x^2+3}\,dx$

Let $u = x^2 + 3$. Then $du = 2xdx \Rightarrow \frac{1}{2}du = xdx$.

When $x = 0$, $u = 3$. When $x = 1$, $u = 4$.

$$\int_0^1 \frac{x}{x^2+3}\,dx = \frac{1}{2}\int_3^4 \frac{1}{u}\,du = \frac{1}{2}\ln u\Big|_3^4$$
$$= \frac{1}{2}(\ln 4 - \ln 3) = \frac{1}{2}\ln\frac{4}{3}$$

12. $\int_0^4 8x(x+4)^{-3}\,dx$

Use integration by parts with
$f(x) = 8x,\ g(x) = (x+4)^{-3}$. Then
$f'(x) = 8,\ G(x) = -\frac{1}{2}(x+4)^{-2}$

$$\int_0^4 8x(x+4)^{-3}\,dx$$
$$= -4x(x+4)^{-2}\Big|_0^4 - \int_0^4 (-4)(x+4)^{-2}\,dx$$
$$= -\frac{16}{64} - 4(x+4)^{-1}\Big|_0^4 = -\frac{1}{4} - \left(\frac{4}{8} - 1\right) = \frac{1}{4}$$

13. $\int_1^3 x^2 e^{x^3}\,dx$

Let $u = x^3;\ du = 3x^2dx \Rightarrow \frac{1}{3}du = x^2dx$.

When $x = 1,\ u = 1^3 = 1$. When $x = 3$,
$u = 3^3 = 27$.

$$\int_1^3 x^2 e^{x^3}\,dx = \frac{1}{3}\int_1^3 e^{x^3}\,3x^2dx$$
$$= \frac{1}{3}\int_1^{27} e^u\,du = \frac{1}{3}e^u\Big|_1^{27} = \frac{1}{3}e^{27} - \frac{1}{3}e$$

14. $\int_{-1}^1 2xe^x\,dx$

Use integration by parts with $f(x) = 2x$,
$g(x) = e^x$. Then $f'(x) = 2$ and $G(x) = e^x$.

$$\int_{-1}^1 2xe^x\,dx = 2xe^x\Big|_{-1}^1 - \int_{-1}^1 2e^x\,dx$$
$$= 2e + 2e^{-1} - 2e^x\Big|_{-1}^1$$
$$= 2e + 2e^{-1} - 2e + 2e^{-1} = 4e^{-1}$$

15. $\int_1^e \frac{\ln x}{x}\,dx$

Let $u = \ln x;\ du = \frac{1}{x}dx$.

When $x = 1,\ u = \ln 1 = 0$. When $x = e$,
$u = \ln e = 1$.

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}$$

16. $\int_1^e \ln x\,dx$

Use integration by parts with $f(x) = \ln x$,
$g(x) = 1$. Then $f'(x) = \frac{1}{x}$ and $G(x) = x$.

$$\int_1^e \ln x\,dx = x\ln x\Big|_1^e - \int_1^e \frac{1}{x}x\,dx$$
$$= e\ln e - \ln 1 - x\Big|_1^e = e - e + 1 = 1$$

17. $\int_0^\pi e^{\sin x}\cos x\,dx$

Let $u = \sin x;\ du = \cos x\,dx$.

When $x = 0,\ u = \sin 0 = 0$. When $x = \pi$,
$u = \sin\pi = 0$.

$$\int_0^\pi e^{\sin x}\cos x\,dx = \int_0^0 e^u\,du = e^u\Big|_0^0 = 1 - 1 = 0$$

18. $\int_0^{\pi/4} \tan x\,dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x;\ du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$.

When $x = 0,\ u = \cos 0 = 1$. When $x = \frac{\pi}{4}$,

$u = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$$\int_0^{\pi/4} \tan x\,dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx = -\int_1^{\sqrt{2}/2} \frac{1}{u}\,du$$
$$= -\ln u\Big|_1^{\sqrt{2}/2}$$
$$= -\ln\left(\frac{\sqrt{2}}{2}\right) + \ln(1) = \ln\sqrt{2}$$

19. $\int_0^1 x\sin \pi x\,dx$

Use integration by parts, with $f(x) = x$, $g(x) = \sin \pi x$. Then $f'(x) = 1$ and

$G(x) = -\frac{1}{\pi}\cos \pi x.$

$$\int_0^1 x\sin \pi x\,dx$$
$$= x\left[-\frac{1}{\pi}\cos \pi x\right]_0^1 - \int_0^1\left(-\frac{1}{\pi}\cos \pi x\right)dx$$
$$= \left[-\frac{1}{\pi}\cos \pi - 0\right] + \frac{1}{\pi^2}\sin \pi x\Big|_0^1$$
$$= \frac{1}{\pi} + \frac{1}{\pi^2}\sin \pi - \frac{1}{\pi^2}\sin 0 = \frac{1}{\pi}$$

20. $\int_0^{\pi/2} \sin\left(2x - \frac{\pi}{2}\right)dx$

Let $u = 2x - \frac{\pi}{2};\ du = 2dx \Rightarrow \frac{1}{2}du = dx$.

When $x = 0$, $u = 2(0) - \frac{\pi}{2} = -\frac{\pi}{2}$. When $x = \frac{\pi}{2}$, $u = 2\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = \frac{\pi}{2}$

$$\int_0^{\pi/2} \sin\left(2x - \frac{\pi}{2}\right)dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin u\,du$$
$$= -\frac{1}{2}\cos u\Big|_{-\pi/2}^{\pi/2}$$
$$= -\frac{1}{2}(0) - \left(-\frac{1}{2}\right)(0) = 0$$

21. $\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 x}\cos x\,dx$

Let $u = \sin x$; $du = \cos x\,dx$.

When $x = -\frac{\pi}{2}$, $u = \sin\left(-\frac{\pi}{2}\right) = -1.$

When $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1.$

$$\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 x}\cos x\,dx = \int_{-1}^{1}\sqrt{1-u^2}\,du$$

= the area of the top half of the circle with radius $1 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

22. $\int_0^{\sqrt{2}} \sqrt{4-x^4}\,2x\,dx$

Let $u = x^2;\ du = 2x\,dx$. When $x = 0$, $u = 0^2 = 0.$ When $x = \sqrt{2}$, $u = \left(\sqrt{2}\right)^2 = 2.$

$$\int_0^{\sqrt{2}} \sqrt{4-x^4}\,2x\,dx = \int_0^2 \sqrt{4-u^2}\,du$$

= the area of top right-hand quarter of the circle with radius 2 $= \frac{1}{4}\pi(2)^2 = \pi$

23. $\int_{-6}^{0} \sqrt{-x^2-6x}\,dx = \int_{-6}^{0}\sqrt{9-(x+3)^2}\,dx$

Let $u = (x + 3)$; $du = dx$. When $x = -6$, $u = -6 + 3 = -3$. When $x = 0$, $u = 0 + 3 = 3$.

$$\int_{-6}^{0}\sqrt{9-(x+3)^2}\,dx = \int_{-3}^{3}\sqrt{9-u^2}\,du$$

= the area of the top half of the circle with radius 3 $= \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$

24. $y = -x\sqrt{9-x^2}$

If $y = 0$, then $9 - x^2 = 0$ or $x = 0$. Thus, $x = -3$, 0, 3 are the intersection points with the x-axis. Area of the portion from $x = -3$ to $x = 0$ is given by $\int_{-3}^{0} -x\sqrt{9-x^2}\,dx$.

Let $u = 9 - x^2$, $du = -2x\,dx$.

$$\int -x\sqrt{9-x^2}\,dx = \frac{1}{2}\int u^{1/2}du$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(9-x^2)^{3/2} + C$$
$$\int_{-3}^{0} -x\sqrt{9-x^2}\,dx = \frac{1}{3}(9-x^2)^{3/2}\Big|_{-3}^{0} = 9$$

By symmetry, area from $x = 0$ to $x = 3$ is also 9, so the total area is 18.

25. $y = x\sqrt{4-x^2}$

If $y = 0$, then $4 - x^2 = 0$ or $x = 0$. Thus, $x = -2$, 0, 2 are the intersection points with the x-axis. Area of the portion from $x = 0$ to $x = 2$ is given by $\int_0^2 x\sqrt{4-x^2}\,dx.$ Let $u = 4 - x^2 \Rightarrow du = -2x\,dx$.

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{3}u^{3/2} + C$$
$$= -\frac{1}{3}(4-x^2)^{3/2} + C$$
$$\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{3}(4-x^2)^{3/2}\Big|_0^2 = \frac{8}{3}$$

By symmetry, area from $x = -2$ to $x = 0$ is also $\frac{8}{3}$, so the total area is $\frac{16}{3}$.

9.4 Approximation of Definite Integrals

1. $\Delta x = \dfrac{(5-3)}{5} = \dfrac{2}{5} = .4$

$a_0 = 3,\ a_1 = 3.4,\ a_2 = 3.8,\ a_3 = 4.2,\ a_4 = 4.6,$
$a_5 = 5$

2. $\Delta x = \dfrac{(2-(-1))}{5} = \dfrac{3}{5} = .6$

$a_0 = -1,\ a_1 = -.4,\ a_2 = .2,\ a_3 = .8,$
$a_4 = 1.4,\ a_5 = 2$

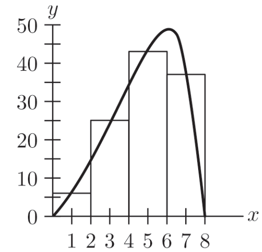
3. $\Delta x = \dfrac{(1-(-1))}{4} = .5$

$x_1 = -1 + \dfrac{.5}{2} = -.75,\ x_2 = -.25,\ x_3 = .25,$
$x_4 = .75$

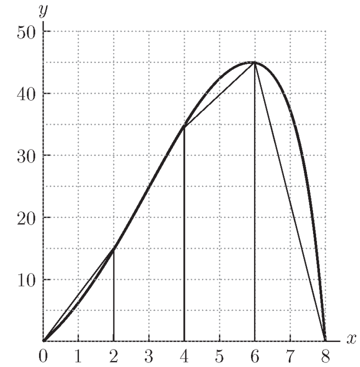
4. $\Delta x = \dfrac{(3-0)}{6} = .5$

$x_1 = 0 + \dfrac{.5}{2} = .25,\ x_2 = .75,\ x_3 = 1.25,$
$x_4 = 1.75,\ x_5 = 2.25,\ x_6 = 2.75$

5.



6.



$$\text{Area} = [0 + 2(15) + 2(35) + 2(45) + 0]\frac{2}{2} = 190$$

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7. If $n = 2$, then $\Delta x = \dfrac{(4-0)}{2} = 2$. The first midpoint is $x_1 = 0 + \dfrac{2}{2} = 1$. The second midpoint is $x_2 = 1 + 2 = 3$.

Using the midpoint rule, we have $\int_0^4 (x^2+5)dx \approx [f(1) + f(3)] \cdot 2 = (6+14) \cdot 2 = 40.$

If $n = 4$, then $\Delta x = \dfrac{(4-0)}{4} = 1,\ \ x_1 = 0 + \dfrac{1}{2} = .5,\ \ x_2 = 1.5,\ x_3 = 2.5,\ x_4 = 3.5.$ Hence,

$$\int_0^4 (x^2+5)dx \approx [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\Delta x$$
$$= [[(.5)^2+5] + [(1.5)^2+5] + [(2.5)^2+5] + [(3.5)^2+5]](1) = 41$$

Evaluating the integral directly, we have

$$\int_0^4 (x^2+5)dx = \left(\frac{1}{3}x^3 + 5x\right)\Bigg|_0^4 = \frac{64}{3} + 20 = 41\frac{1}{3}.$$

8. $\int_1^5 (x-1)^2\,dx$

Using the midpoint rule with $n = 2$, we have $\Delta x = \dfrac{(5-1)}{2} = 2$. The midpoints are $x_1 = 1 + \dfrac{2}{2} = 2$ and $x_2 = 4$.

$$\int_1^5 (x-1)^2\,dx \approx 2[(2-1)^2 + (4-1)^2] = 20$$

With $n = 4$, $\Delta x = \dfrac{(5-1)}{4} = 1$. The midpoints are $x_1 = 1 + \dfrac{1}{2} = \dfrac{3}{2},\ x_2 = \dfrac{5}{2},\ x_3 = \dfrac{7}{2},\ x_4 = \dfrac{9}{2}.$

$$\int_1^5 (x-1)^2\,dx \approx (1)\left[\left(\frac{3}{2}-1\right)^2 + \left(\frac{5}{2}-1\right)^2 + \left(\frac{7}{2}-1\right)^2 + \left(\frac{9}{2}-1\right)^2\right] = 21$$

The exact value is $\displaystyle\int_1^5 (x-1)^2\,dx = \frac{1}{3}(x-1)^3\Bigg|_1^5 = \frac{1}{3}(4^3-0) = \frac{64}{3}.$

9. $\int_0^1 e^{-x}dx$

Using the midpoint rule with $n = 5$, we have $\Delta x = \frac{(1-0)}{5} = .2$ and the midpoints are

$x_1 = 0 + \frac{.2}{2} = .1,\ x_2 = .3,\ x_3 = .5,\ x_4 = .7,\ x_5 = .9$.

$\int_0^1 e^{-x}dx \approx .2[e^{-0.1} + e^{-0.3} + e^{-0.5} + e^{-0.7} + e^{-0.9}] \approx 0 = .63107$

The exact value is $\int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1 = -\frac{1}{e} + 1 \approx .63212.$

10. $\int_1^2 \frac{1}{x+1}dx$

Using the midpoint rule with $n = 5$, we have $\Delta x = \frac{(2-1)}{5} = .2$ and the midpoints are

$x_1 = 1 + \frac{.2}{2} = 1.1,\ x_2 = 1.3,\ x_3 = 1.5,\ x_4 = 1.7,$ and $x_5 = 1.9.$

$\int_1^2 \frac{1}{x+1}dx \approx .2\left[\frac{1}{1.1+1} + \frac{1}{1.3+1} + \frac{1}{1.5+1} + \frac{1}{1.7+1} + \frac{1}{1.9+1}\right] \approx .40523$

The exact value is $\int_1^2 \frac{1}{x+1}dx = \ln|x+1|\Big|_1^2 = \ln 3 - \ln 2 \approx .40547.$

11. $\int_0^1 \left(x - \frac{1}{2}\right)^2 dx$

Using the trapezoidal rule with $n = 4$, we have $\Delta x = \frac{(1-0)}{4} = .25$ and the endpoints are

$a_0 = 0,\ a_1 = 0.25,\ a_2 = 0.5,\ a_3 = 0.75$ and $a_4 = 1.$

$\int_0^1 \left(x - \frac{1}{2}\right)^2 dx \approx \left[\left(0 - \frac{1}{2}\right)^2 + 2\left(.25 - \frac{1}{2}\right)^2 + 2\left(.5 - \frac{1}{2}\right)^2 + 2\left(.75 - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2\right]\frac{.25}{2} = .09375$

The exact value is $\int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{3}\left(x - \frac{1}{2}\right)^3\Bigg|_0^1 = \frac{1}{3}[.5^3 - (-.5)^3] \approx .08333.$

12. $\int_4^9 \frac{1}{x-3}dx$

Using the trapezoidal rule with $n = 5$, we have $\Delta x = \frac{(9-4)}{5} = 1.$ The endpoints are

$a_0 = 4,\ a_1 = 5,\ a_2 = 6,\ a_3 = 7,\ a_4 = 8,$ and $a_5 = 9.$

$\int_4^9 \frac{1}{x-3}dx \approx \left[\frac{1}{4-3} + (2)\frac{1}{5-3} + (2)\frac{1}{6-3} + (2)\frac{1}{7-3} + (2)\frac{1}{8-3} + \frac{1}{9-3}\right]\frac{1}{2} \approx 1.86667$

The exact value is $\int_4^9 \frac{1}{x-3}dx = \ln|x-3|\Big|_4^9 = \ln 6 - \ln 1 \approx 1.79176.$

13. $\int_1^5 \frac{1}{x^2}\,dx$

Using the trapezoidal rule with $n = 3$, we have $\Delta x = \frac{(5-1)}{3} \approx 1.33333$. The endpoints are $a_0 = 1,\ a_1 = 2.33333,\ a_2 = 3.66667,$ and $a_3 = 5.$

$$\int_1^5 \frac{1}{x^2}\,dx \approx \left[\frac{1}{1^2} + (2)\frac{1}{2.33333^2} + (2)\frac{1}{3.66667^2} + \frac{1}{5^2}\right]\left(\frac{1.33333}{2}\right) \approx 1.03740$$

The exact value is $\int_1^5 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^5 = -\frac{1}{5} + 1 = .8$.

14. $\int_{-1}^1 e^{2x}dx$

Using the trapezoidal rule with $n = 2$, we have $\Delta x = \frac{(1-(-1))}{2} = 1$ and the endpoints are $a_0 = -1,\ a_1 = 0$, and $a_2 = 1$.

$$\int_{-1}^1 e^{2x}dx \approx \left[e^{2(-1)} + 2e^{2(0)} + e^{2(1)}\right]\frac{1}{4} \approx 4.76220$$

With $n = 4$, $\Delta x = \frac{(1-(-1))}{4} = .5,\ a_0 = -1,\ a_1 = -.5,\ a_2 = 0,\ a_3 = .5,$ and $a_4 = 1.$

$$\int_{-1}^1 e^{2x}dx \approx \left[e^{2(-1)} + 2e^{2(-0.5)} + 2e^{2(0)} + 2e^{2(0.5)} + e^{2(1)}\right]\frac{.5}{2} \approx 3.92418$$

The exact value is $\int_{-1}^1 e^{2x}dx = \frac{1}{2}e^{2x}\Big|_{-1}^1 = \frac{1}{2}\left[e^2 - e^{-2}\right] \approx 3.62686.$

15. $\int_1^4 (2x-3)^3\,dx;\ n = 3$

$$\Delta x = \frac{(4-1)}{3} = 1$$

Midpoint rule: $x_1 = 1 + \frac{1}{2} = 1.5,\ x_2 = 2.5,\ x_3 = 3.5$

$$\int_1^4 (2x-3)^3\,dx \approx \left[(2(1.5)-3)^3 + (2(2.5)-3)^3 + (2(3.5)-3)^3\right] = 72$$

Trapezoidal rule: $a_0 = 1,\ a_1 = 2,\ a_2 = 3,\ a_3 = 4$

$$\int_1^4 (2x-3)^3\,dx \approx \left[(2(1)-3)^3 + 2(2(2)-3)^3 + 2(2(3)-3)^3 + (2(4)-3)^3\right]\frac{1}{2} = 90$$

Simpson's rule: $\int_1^4 (2x-3)^3\,dx \approx S = \frac{2M+T}{3} = \frac{2(72)+90}{3} = 78$

Exact value: $\int_1^4 (2x-3)^3\,dx = \frac{1}{8}(2x-3)^4\Big|_1^4 = \frac{1}{8}[(2(4)-3)^4 - (2(1)-3)^4] = 78$

16. $\int_{10}^{20} \frac{\ln x}{x}\,dx;\ n = 5$

$\Delta x = \frac{(20-10)}{5} = 2$

Midpoint rule: $x_1 = 10 + \frac{2}{2} = 11,\ x_2 = 13,\ x_3 = 15,\ x_4 = 17,\ x_5 = 19$

$$\int_{10}^{20} \frac{\ln x}{x}\,dx \approx 2\left[\frac{\ln 11}{11} + \frac{\ln 13}{13} + \frac{\ln 15}{15} + \frac{\ln 17}{17} + \frac{\ln 19}{19}\right] \approx 1.83492$$

Trapezoidal rule: $a_0 = 10,\ a_1 = 12,\ a_2 = 14,\ a_3 = 16,\ a_4 = 18,\ a_5 = 20$

$$\int_{10}^{20} \frac{\ln x}{x}\,dx \approx \left[\frac{\ln 10}{10} + 2\frac{\ln 12}{12} + 2\frac{\ln 14}{14} + 2\frac{\ln 16}{16} + 2\frac{\ln 18}{18} + \frac{\ln 20}{20}\right]\frac{2}{2} \approx 1.83893$$

Simpson's rule:

$$\int_{10}^{20} \frac{\ln x}{x}\,dx \approx S = \frac{2M+T}{3} = \frac{2(1.83492) + 1.83893}{3} = 1.83626$$

Exact value: $\int_{10}^{20} \frac{\ln x}{x}\,dx$

Let $u = \ln x,\ du = \frac{1}{x}dx.$ When $x = 10$, $u = \ln 10$, and when $x = 20$, $u = \ln 20$.

$$\int_{10}^{20} \frac{\ln x}{x}\,dx = \int_{\ln 10}^{\ln 20} u\,du = \frac{1}{2}u^2\Big|_{\ln 10}^{\ln 20} = \frac{1}{2}[(\ln 20)^2 - \ln(10)^2] \approx 1.83626$$

17. $\int_0^2 2xe^{x^2}\,dx;\ n = 4$

$\Delta x = \frac{(2-0)}{4} = .5$

Midpoint rule: $x_1 = 0 + \frac{.5}{2} = .25,\ x_2 = .75,\ x_3 = 1.25,\ x_4 = 1.75$

$$\int_0^2 2xe^{x^2}\,dx \approx 0.5[2(.25)e^{(0.25)^2} + 2(.75)e^{(0.75)^2} + 2(1.25)e^{(1.25)^2} + 2(1.75)e^{(1.75)^2}] \approx 44.96248$$

Trapezoidal rule: $a_0 = 0,\ a_1 = .5,\ a_2 = 1,\ a_3 = 1.5,\ a_4 = 2$

$$\int_0^2 2xe^{x^2}\,dx \approx \left[2(0)e^{0^2} + 2(2)(.5)e^{(0.5)^2} + 2(2)(1)e^{1^2} + 2(2)(1.5)e^{(1.5)^2} + 2(2)e^{2^2}\right]\frac{.5}{2} \approx 72.19005$$

Simpson's rule: $\int_0^2 2xe^{x^2}\,dx \approx S = \frac{2M+T}{3} = \frac{2(44.96248) + 72.19005}{3} = 54.03834$

Exact value: $\int_0^2 2xe^{x^2}\,dx = e^{x^2}\Big|_0^2 = e^4 - 1 \approx 53.59815$

18. $\int_0^3 x\sqrt{4-x}\,dx; n = 5$

$\Delta x = \dfrac{(3-0)}{5} = .6$

Midpoint rule: $x_1 = 0 + \dfrac{.6}{2} = .3,\ x_2 = .9,\ x_3 = 1.5,\ x_4 = 2.1,\ x_5 = 2.7$

$\int_0^3 x\sqrt{4-x}\ dx \approx .6\left[(.3)\sqrt{4-.3} + (.9)\sqrt{4-.9} + (1.5)\sqrt{4-1.5} + (2.1)\sqrt{4-2.1} + (2.7)\sqrt{4-2.7}\right] \approx 6.30390$

Trapezoidal rule: $a_0 = 0,\ a_1 = 0.6,\ a_2 = 1.2,\ a_3 = 1.8,\ a_4 = 2.4,\ a_5 = 3.0$

$\int_0^3 x\sqrt{4-x}\,dx$

$\approx \left[(0)\sqrt{4-0} + 2(.6)\sqrt{4-.6} + 2(1.2)\sqrt{4-1.2} + 2(1.8)\sqrt{4-1.8} + 2(2.4)\sqrt{4-2.4} + (3.0)\sqrt{4-3}\right]\dfrac{.6}{2}$

≈ 6.19197

Simpson's rule: $\int_0^3 x\sqrt{4-x}\,dx \approx \dfrac{2(6.3039) + 6.19197}{3} = 6.26659$

Exact value: $\int_0^3 x\sqrt{4-x}\,dx$

Use integration by parts with $f(x) = x,\ g(x) = \sqrt{4-x}$. Then $f'(x) = 1$ and $G(x) = -\dfrac{2}{3}(4-x)^{3/2}$ and

$\int_0^3 x\sqrt{4-x}\,dx = -\dfrac{2}{3}x(4-x)^{3/2}\Big|_0^3 - \int_0^3 -\dfrac{2}{3}(4-x)^{3/2}dx = \left[-\dfrac{2}{3}(3)1^{3/2} - 0\right] - \dfrac{4}{15}(4-x)^{5/2}\Big|_0^3 \approx 6.26667$.

19. $\int_2^5 xe^x dx;\ n = 5$

$\Delta x = \dfrac{(5-2)}{5} = 0.6$

Midpoint rule: $x_1 = 2 + \dfrac{.6}{2} = 2.3,\ x_2 = 2.9,\ x_3 = 3.5,\ x_4 = 4.1,\ x_5 = 4.7$

$\int_2^5 xe^x dx \approx .6[(2.3)e^{2.3} + (2.9)e^{2.9} + (3.5)e^{3.5} + (4.1)e^{4.1} + (4.7)e^{4.7}] \approx 573.41797$

Trapezoidal rule: $a_0 = 2,\ a_1 = 2.6,\ a_2 = 3.2,\ a_3 = 3.8,\ a_4 = 4.4,\ a_5 = 5$

$\int_2^5 xe^x dx \approx [2e^2 + 2(2.6)e^{2.6} + 2(3.2)e^{3.2} + 2(3.8)e^{3.8} + 2(4.4)e^{4.4} + 5e^5]\dfrac{.6}{2} \approx 612.10806$

Simpson's rule: $\int_2^5 xe^x dx \approx \dfrac{2(573.41797) + 612.10806}{3} \approx 586.31466$

Exact value: $\int_2^5 xe^x dx$

Use integration by parts with $f(x) = x,\ g(x) = e^x$.

Then $f'(x) = 1$ and $G(x) = e^x$.

$\int_2^5 xe^x dx = xe^x\Big|_2^5 - \int_2^5 e^x dx = (xe^x - e^x)\Big|_2^5 = e^x(x-1)\Big|_2^5 = 4e^5 - e^2 \approx 586.26358$

20. $\int_1^5 (4x^3 - 3x^2)dx;\ n = 2$

$\Delta x = \dfrac{(5-1)}{2} = 2$

Midpoint rule: $x_1 = 1 + \dfrac{2}{2} = 2,\ x_2 = 4$

$\int_1^5 (4x^3 - 3x^2)dx \approx 2[(4(2)^3 - 3(2)^2) + (4(4)^3 - 3(4)^2)] = 456$

Trapezoidal rule: $a_0 = 1,\ a_1 = 3,\ a_3 = 5$

$\int_1^5 (4x^3 - 3x^2)dx \approx \left[(4(1^3) - 3(1^2)) + 2(4(3^3) - 3(3^2)) + (4(5^3) - 3(5^2))\right]\dfrac{2}{2} = 588$

Simpson's rule: $\int_1^5 (4x^3 - 3x^2)dx \approx \dfrac{2(456) + 588}{3} = 500$

Exact value: $\int_1^5 (4x^3 - 3x^2)dx = (x^4 - x^3)\Big|_1^5 = 5^4 - 5^3 = 500$

21. $\int_0^2 \sqrt{1+x^3}\,dx;\ n = 4$

$\Delta x = \dfrac{(2-0)}{4} = .5$

$a_0 = 0,\ a_1 = .5,\ a_2 = 1,\ a_3 = 1.5,\ a_4 = 2,\ x_1 = .25,\ x_2 = .75,\ x_3 = 1.25,\ x_4 = 1.75$

$$\int_0^2 \sqrt{1+x^3}\,dx$$
$$\approx \left[\sqrt{1+0^3} + 4\sqrt{1+.25^3} + 2\sqrt{1+.5^3} + 4\sqrt{1+.75^3} + 2\sqrt{1+1^3} + 4\sqrt{1+1.25^3}\right.$$
$$\left. + 2\sqrt{1+1.5^3} + 4\sqrt{1+1.75^3} + \sqrt{1+2^3}\right]\frac{.5}{6}$$
$$\approx 3.24124$$

22. $\int_0^1 \dfrac{1}{x^3+1}dx;\ n = 2$

$\Delta x = \dfrac{(1-0)}{2} = .5$

$a_0 = 0,\ a_1 = .5,\ a_2 = 1,\ x_1 = .25,\ x_2 = .75$

$$\int_0^1 \frac{1}{x^3+1}dx \approx \left[\frac{1}{0^3+1} + (4)\frac{1}{.25^3+1} + (2)\frac{1}{.5^3+1} + (4)\frac{1}{.75^3+1} + \frac{1}{1+1^2}\right]\frac{.5}{6} \approx .83579$$

23. $\int_0^2 \sqrt{\sin x}\,dx;\ n = 5$

$\Delta x = \dfrac{(2-0)}{5} = .4$

$a_0 = 0,\ a_1 = .4,\ a_2 = .8,\ a_3 = 1.2,\ a_4 = 1.6,\ a_5 = 2,\ x_1 = .2,\ x_2 = .6,\ x_3 = 1,\ x_4 = 1.4,\ x_5 = 1.8$

$$\int_0^2 \sqrt{\sin x}\,dx$$
$$\approx \left[\sqrt{\sin 0} + 4\sqrt{\sin(.2)} + 2\sqrt{\sin(.4)} + 4\sqrt{\sin(.6)} + 2\sqrt{\sin(.8)} + 4\sqrt{\sin 1} + 2\sqrt{\sin(1.2)} + 4\sqrt{\sin(1.4)}\right.$$
$$\left. + \quad 2\sqrt{\sin(1.6)} + 4\sqrt{\sin(1.8)} + \sqrt{\sin(2)}\right]\frac{.4}{6}$$
$$\approx 1.61347$$

24. $\int_{-1}^{1} \sqrt{1+x^4}\,dx$; $n = 4$

$\Delta x = \dfrac{(1-(-1))}{4} = .5$

$a_0 = -1, a_1 = -.5, a_2 = 0, a_3 = .5, a_4 = 1, x_1 = -.75, x_2 = -.25, x_3 = .25, x_4 = .75$

$$\int_{-1}^{1} \sqrt{1+x^4}\,dx \approx \Big[\sqrt{1+(-1)^4} + 4\sqrt{1+(-.75)^4} + 2\sqrt{1+(-.5)^4} + 4\sqrt{1+(-.25)^4} + 2\sqrt{1+0^4} + 4\sqrt{1+.25^4} + 2\sqrt{1+.5^4} + 4\sqrt{1+.75^4} + \sqrt{1+1^4}\Big]\frac{.5}{6}$$

≈ 2.17883

25. View the distance to the water as a function *f* of the position *x* of a point on the side. If *a* corresponds to the top of the diagram and *b* to the bottom, the area we wish to compute is $\int_a^b f(x)dx.$ Because the measurements are taken 50 feet apart, $\Delta x = 50$. Thus the trapezoidal rule with $n = 4$ gives

$\int_a^b f(x)dx \approx [100 + 2(90) + 2(125) + 2(150) + 200]\dfrac{50}{2} = 25{,}750$ sq. ft .

26. View the depth of the river as a function *f* of the position *x* of a point between the banks. If *a* corresponds to the bank on which the first reading was taken and *b* corresponds to the last reading, then the cross-sectional area we wish to compute is $\int_a^b f(x)dx.$ Using the trapezoidal rule with $n = 5$ and $\Delta x = 20$ (distance between readings), we have, after converting readings to yards,

$\int_a^b f(x)dx \approx [0 + 2(2) + 2(4) + 2(6) + 2(2) + 0]\dfrac{20}{2} = 280$ sq. yd .

27. (See hint.) using the trapezoidal rule with $\Delta t = 10$, $n = 10$,

$$s(10) = \int_0^{10} v(t)dt \approx [0 + 2(30) + 2(75) + 2(115) + 2(155) + 2(200) + 2(250) + 2(300) + 2(360) + 2(420) + 490]\frac{1}{2}$$

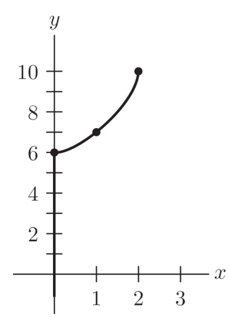
$= 2150$ ft

28. (See hint for Exercise 27.)

$s(5) = \int_0^5 v(t)dt$ with $v(t)$ velocity at time *t* measured in miles per minute.

$\int_0^5 v(t)dt \approx [33 + 2(32) + 2(28) + 2(30) + 2(32) + 35]\left(\dfrac{1}{2}\right)\dfrac{1}{60} = 2.6$ miles

29. a. $f(x) = \dfrac{1}{12}x^4 + 3x^2;\ \ f'(x) = \dfrac{1}{3}x^3 + 6x;\ \ f''(x) = x^2 + 6$



b. $f''(x) \le 10$ on [0, 2], so $A = 10$

c. $a = 0, b = 2, n = 10, A = 10$

$\dfrac{A(b-a)^3}{24n^2} = \dfrac{10(2-0)^3}{24(10)^2} = \dfrac{8}{240} \approx .0333$

d. 8.5089 – 8.5333 = –.0244, which satisfies the bound in part (c).

e. $a = 0, b = 2, n = 20, A = 10$

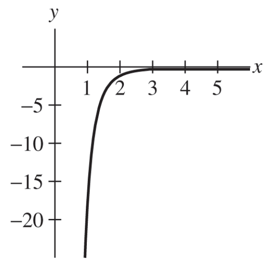
$$\frac{A(b-a)^3}{24n^2} = \frac{10(2-0)^3}{24(20)^2} = \frac{10(8)}{24(400)} = \frac{1}{4}\left(\frac{8}{240}\right)$$

The bound is quartered.

30. a. $f(x) = 3\ln x$

$$f'(x) = 3\left(\frac{1}{x}\right) = 3x^{-1};\ f''(x) = -3x^{-2}$$

$$f'''(x) = 6x^{-3};\ f^{(4)}(x) = -18x^{-4}$$



b. $f''''(1) = -18$ and $f''''(2) = -\frac{9}{8}$, so $f''''(x) \le |-18| = 18$

c. $a = 1, b = 2, n = 2,\ A = 18$

$$\frac{A(b-a)^5}{2880n^4} = \frac{18(2-1)^5}{2880(2)^4} = 18\left(\frac{1}{2880(16)}\right) = \frac{18}{46{,}080} \approx 3.9063\times 10^{-4}$$

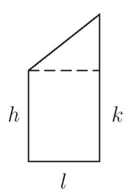
d. 1.1588 – 1.1589 = –.0001, which satisfies the bound in part (c).

e. $a = 1, b = 2, n = 6, A = 18$

$$\frac{A(b-a)^5}{2880n^4} = \frac{18(2-1)^5}{2880(6)^4} = \frac{18}{2880(2\cdot 3)^4} = \frac{18}{2880(2)^4(3)^4}$$

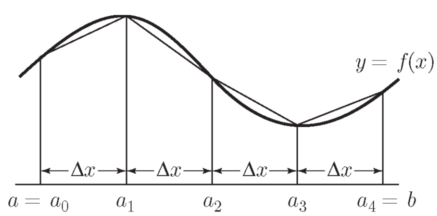
The bound is divided by 81.

31. a.



The area of the triangle on top is $\frac{1}{2}(k-h)l$ and the area of the rectangle on the bottom is lh. Thus the area of the trapezoid is $\frac{1}{2}(k-h)l + lh = \frac{1}{2}(h+k)l.$

b.



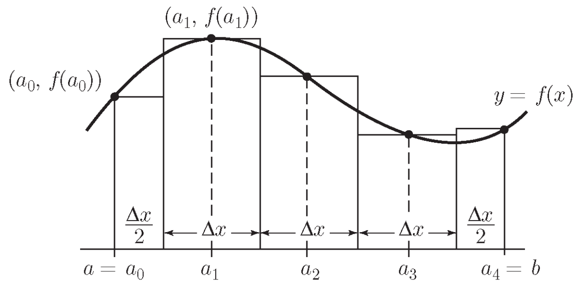
From part (a) with $f(a_0) = h,\ \Delta x = l$ and $f(a_1) = k,$ we have area $= \frac{1}{2}[f(a_0) + f(a_1)]\Delta x.$

c. Applying the result in (a) as in part (b), we see that the sum of the areas of the 4 trapezoids in the figure is

$$\frac{1}{2}\left[f(a_0)+f(a_1)\right]\Delta x + \frac{1}{2}\left[f(a_1)+f(a_2)\right]\Delta x + \frac{1}{2}\left[f(a_2)+f(a_3)\right]\Delta x + \frac{1}{2}\left[f(a_3)+f(a_4)\right]\Delta x$$

$$= \left[f(a_0) + 2f(a_1) + 2f(a_2) + 2f(a_3) + f(a_4)\right]\frac{\Delta x}{2}$$

32.



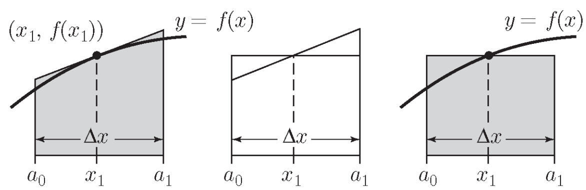
Using the approximation described in the problem,

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2} f(a_0) + \Delta x f(a_1) + \Delta x f(a_2) + \Delta x f(a_3) + \frac{\Delta x}{2} f(a_4)$$

$$= \left[f(a_0) + 2f(a_1) + 2f(a_2) + 2f(a_3) + f(a_4)\right]\frac{\Delta x}{2}$$

= estimate using the trapezoidal rule with $n = 4$

33. a.



(a) (b) (c)

Let T_1 be the top triangle in Fig. 15(b) and let T_2 be the bottom one. Both are right triangles, their bases have the same length $\left(\frac{\Delta x}{2}\right)$, and the angle between the base and hypotenuse is the same in each. Therefore the area of T_1, $A(T_1) = A(T_2)$, the area of T_2. The shaded area in Fig. 15(a) = area shaded in 15(c) + $A(T_1) - A(T_2)$.

b. $\int_a^b f(x)dx \le M$ follows from (a), using the fact that

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx \ldots + \int_{a_{n-1}}^b f(x)dx$$

Decomposing $\int_a^b f(x)dx$ as above, $T \le \int_a^b f(x)dx$ follows from the fact that if *f* is concave downward, then each of the trapezoids whose area is summed to obtain *T* is entirely *below* the graph of *f*. (See, for example, Fig. 13(b) in exercise 31, where the first two trapezoids are drawn over a region on which *f* is concave downward.) Thus, $T \le \int_a^b f(x)dx \le M$.

34. a. *R* is defined to be the rate at which the heart pumps blood, measured in liters per minute. Thus $\frac{R}{60}$ measures the same quantity in liters per sec. So in Δt seconds (the length of the *i*th time interval) approximately $\left(\frac{R}{60}\right)\Delta t$ liters of blood will flow past the monitoring point.

b. If $\left(\frac{R}{60}\right)\Delta t$ liters of blood flow past the monitoring point during the ith interval and the concentration of dye in the blood is approximately $c(t_i)$ mg/liter (here we are using $c(t_i)$ to approximate $c(t)$ for all t in the ith interval), then approximately $c(t_i)\left(\frac{R}{60}\right)\Delta t$ mg of dye will flow past the monitoring point during this time.

c. If essentially all of the dye flows past the monitoring point during the first 22 seconds, then $D \approx (R/60)[c(t_1)+\cdots+c(t_n)]\Delta t$ follows directly from (b), since D is the total quantity of dye. The approximation improves as n gets large, since as Δt gets smaller, $c(t_i)$ becomes a better approximation for $c(t)$ on the ith interval.

d. $[c(t_1)+c(t_2)+\cdots+c(t_n)]\Delta t$ is a Riemann sum of $c(t)$ on [0, 22]. The result in (c) says that

$$\lim_{\Delta t\to 0}[c(t_1)+c(t_2)+\cdots+c(t_n)]\Delta t = \frac{60}{R}D.$$

Thus $\int_0^{22} c(t)dt = \frac{60}{R}D$ or $D = \int_0^{22}\frac{R}{60}c(t)dt.$

Therefore, $R = \dfrac{60D}{\int_0^{22} c(t)dt}.$

35. $f(x) = \dfrac{1}{25-x^2}$,

$a = 3.1 - \frac{0.2}{2} = 3,\ b = 4.9 + \frac{0.2}{2} = 5,\ n = \frac{5-3}{0.2} = 10$

36. $f(x) = \sin(x^2),\ a = 2,\ b = 6,\ n = \frac{6-2}{0.5} = 8$

37. $\int_1^{11}\frac{1}{x}dx;\ \Delta x = \frac{(11-1)}{10} = 1$

Midpoint rule:

$x_1 = 1+\frac{1}{2} = 1.5,\ x_2 = 2.5,\ x_3 = 3.5,\ x_4 = 4.5,\ x_5 = 5.5,\ x_6 = 6.5,\ x_7 = 7.5,\ x_8 = 8.5,\ x_9 = 9.5,\ x_{10} = 10.5$

$$\int_1^{11}\frac{1}{x}dx \approx \left[\frac{1}{1.5}+\frac{1}{2.5}+\frac{1}{3.5}+\frac{1}{4.5}+\frac{1}{5.5}+\frac{1}{6.5}+\frac{1}{7.5}+\frac{1}{8.5}+\frac{1}{9.5}+\frac{1}{10.5}\right] \approx 2.361749156$$

Trapezoidal rule: $a_0 = 1,\ a_1 = 2,\ a_2 = 3,\ a_3 = 4,\ a_4 = 5,\ a_5 = 6,\ a_6 = 7,\ a_7 = 8,\ a_8 = 9,\ a_9 = 10,\ a_{10} = 11$

$$\int_1^{11}\frac{1}{x}dx \approx \left[\frac{1}{1}+(2)\frac{1}{2}+(2)\frac{1}{3}+(2)\frac{1}{4}+(2)\frac{1}{5}+(2)\frac{1}{6}+(2)\frac{1}{7}+(2)\frac{1}{8}+(2)\frac{1}{9}+(2)\frac{1}{10}+\frac{1}{11}\right]\frac{1}{2} \approx 2.474422799$$

Simpson's rule:

$$\int_1^{11}\frac{1}{x}dx \approx \left[\frac{1}{1}+(4)\frac{1}{1.5}+(2)\frac{1}{2}+(4)\frac{1}{2.5}+(2)\frac{1}{3}+(4)\frac{1}{3.5}+(2)\frac{1}{4}+(4)\frac{1}{4.5}+(2)\frac{1}{5}+(4)\frac{1}{5.5}+(2)\frac{1}{6}+(4)\frac{1}{6.5}\right.$$
$$\left.+(2)\frac{1}{7}+(4)\frac{1}{7.5}+(2)\frac{1}{8}+(4)\frac{1}{8.5}+(2)\frac{1}{9}+(4)\frac{1}{9.5}+(2)\frac{1}{10}+(4)\frac{1}{10.5}+\frac{1}{11}\right]\frac{1}{6}$$
$$\approx 2.399307037$$

Exact value: $\int_1^{11}\frac{1}{x}dx = \ln|x|\Big|_1^{11} = \ln 11 = 2.397895273$

Error using midpoint rule = .036146117
Error using trapezoidal rule = .076527526
Error using Simpson's rule = .001411764

38. $\int_0^{\pi/2} \cos x\, dx;\ \Delta x = \dfrac{\left(\frac{\pi}{2}-0\right)}{10} = \dfrac{\pi}{20}$

Midpoint rule: $x_1 = 0 + \dfrac{\left(\frac{\pi}{20}\right)}{2} = \dfrac{\pi}{40},\ x_2 = \dfrac{3\pi}{40},\ x_3 = \dfrac{5\pi}{40},\ x_4 = \dfrac{7\pi}{40},\ x_5 = \dfrac{9\pi}{40},\ x_6 = \dfrac{11\pi}{40},\ x_7 = \dfrac{13\pi}{40},\ x_8 = \dfrac{15\pi}{40}$

$x_9 = \dfrac{17\pi}{40},\ x_{10} = \dfrac{19\pi}{40}$

$$\int_0^{\pi/2} \cos x\, dx \approx \left[\cos\left(\frac{\pi}{40}\right) + \cos\left(\frac{3\pi}{40}\right) + \cos\left(\frac{5\pi}{40}\right) + \cos\left(\frac{7\pi}{40}\right) + \cos\left(\frac{9\pi}{40}\right) + \cos\left(\frac{11\pi}{40}\right) + \cos\left(\frac{13\pi}{40}\right) + \cos\left(\frac{15\pi}{40}\right) + \cos\left(\frac{17\pi}{40}\right) + \cos\left(\frac{19\pi}{40}\right)\right]\frac{\pi}{20} \approx 1.001028824$$

Trapezoidal rule: $a_0 = 0,\ a_1 = \dfrac{\pi}{20},\ a_2 = \dfrac{2\pi}{20},\ a_3 = \dfrac{3\pi}{20},\ a_4 = \dfrac{4\pi}{20},\ a_5 = \dfrac{5\pi}{20},\ a_6 = \dfrac{6\pi}{20},\ a_7 = \dfrac{7\pi}{20},\ a_8 = \dfrac{8\pi}{20},$

$a_9 = \dfrac{9\pi}{20},\ a_{10} = \dfrac{\pi}{2}$

$$\int_0^{\pi/2} \cos x\, dx \approx \left[\cos(0) + 2\cos\left(\frac{\pi}{20}\right) + 2\cos\left(\frac{2\pi}{20}\right) + 2\cos\left(\frac{3\pi}{20}\right) + 2\cos\left(\frac{4\pi}{20}\right) + 2\cos\left(\frac{5\pi}{20}\right) + 2\cos\left(\frac{6\pi}{20}\right) + 2\cos\left(\frac{7\pi}{20}\right) + 2\cos\left(\frac{8\pi}{20}\right) + 2\cos\left(\frac{9\pi}{20}\right) + \cos\left(\frac{\pi}{2}\right)\right]\frac{\pi}{20}\left(\frac{1}{2}\right)$$

≈ 0.9979429868

Simpson's rule:

$$\int_0^{\pi/2} \cos x\, dx = \left[\cos(0) + 4\cos\left(\frac{\pi}{40}\right) + 2\cos\left(\frac{\pi}{20}\right) + 4\cos\left(\frac{3\pi}{40}\right) + 2\cos\left(\frac{2\pi}{20}\right) + 4\cos\left(\frac{5\pi}{40}\right) + 2\cos\left(\frac{3\pi}{20}\right) + 4\cos\left(\frac{7\pi}{40}\right) + 2\cos\left(\frac{4\pi}{20}\right) + 4\cos\left(\frac{9\pi}{40}\right) + 2\cos\left(\frac{5\pi}{20}\right) + 4\cos\left(\frac{11\pi}{40}\right) + 2\cos\left(\frac{6\pi}{20}\right) + 4\cos\left(\frac{13\pi}{40}\right) + 2\cos\left(\frac{7\pi}{20}\right) + 4\cos\left(\frac{15\pi}{40}\right) + 2\cos\left(\frac{8\pi}{20}\right) + 4\cos\left(\frac{17\pi}{40}\right) + 2\cos\left(\frac{9\pi}{20}\right) + 4\cos\left(\frac{19\pi}{40}\right) + \cos\left(\frac{\pi}{2}\right)\right]\left(\frac{\pi}{20}\right)\left(\frac{1}{6}\right)$$

≈ 1.000000212

Exact value: $\int_0^{\pi/2} \cos x\, dx = \sin x\Big|_0^{\pi/2} = \sin\left(\dfrac{\pi}{2}\right) - \sin(0) = 1$

Error using midpoint rule = .001028824; error using trapezoidal rule = .0020570132
Error using Simpson's rule = .000000212

39. $\int_0^{\pi/4} \sec^2 x\,dx;\ \Delta x = \dfrac{\left(\frac{\pi}{4}-0\right)}{10} = \dfrac{\pi}{40}$

Midpoint rule:

$x_1 = 0 + \dfrac{\left(\frac{\pi}{40}\right)}{2} = \dfrac{\pi}{80},\ x_2 = \dfrac{3\pi}{80},\ x_3 = \dfrac{5\pi}{80},\ x_4 = \dfrac{7\pi}{80},\ x_5 = \dfrac{9\pi}{80},\ x_6 = \dfrac{11\pi}{80},\ x_7 = \dfrac{13\pi}{80}\ x_8 = \dfrac{15\pi}{80},\ x_9 = \dfrac{17\pi}{80},$

$x_{10} = \dfrac{19\pi}{80}$

$$\int_0^{\pi/4} \sec^2 x\,dx = \left[\sec^2\left(\frac{\pi}{80}\right) + \sec^2\left(\frac{3\pi}{80}\right) + \sec^2\left(\frac{5\pi}{80}\right) + \sec^2\left(\frac{7\pi}{80}\right) + \sec^2\left(\frac{9\pi}{80}\right) + \sec^2\left(\frac{11\pi}{80}\right) + \sec^2\left(\frac{13\pi}{80}\right) + \sec^2\left(\frac{15\pi}{80}\right) + \sec^2\left(\frac{17\pi}{80}\right) + \sec^2\left(\frac{19\pi}{80}\right)\right]\frac{\pi}{40}$$

$$\approx 0.9989755866$$

Trapezoidal rule: $a_0 = 0,\ a_1 = \dfrac{\pi}{40},\ a_2 = \dfrac{2\pi}{40},\ a_3 = \dfrac{3\pi}{40},\ a_4 = \dfrac{4\pi}{40},\ a_5 = \dfrac{5\pi}{40},\ a_6 = \dfrac{6\pi}{40},\ a_7 = \dfrac{7\pi}{40},\ a_8 = \dfrac{8\pi}{40},$

$a_9 = \dfrac{9\pi}{40},\ a_{10} = \dfrac{\pi}{4}$

$$\int_0^{\pi/4} \sec^2 x\,dx \approx \left[\sec^2(0) + 2\sec^2\left(\frac{\pi}{40}\right) + 2\sec^2\left(\frac{2\pi}{40}\right) + 2\sec^2\left(\frac{3\pi}{40}\right) + 2\sec^2\left(\frac{4\pi}{40}\right) + 2\sec^2\left(\frac{5\pi}{40}\right) + 2\sec^2\left(\frac{6\pi}{40}\right) + 2\sec^2\left(\frac{7\pi}{40}\right) + 2\sec^2\left(\frac{8\pi}{40}\right) + 2\sec^2\left(\frac{9\pi}{40}\right) + \sec^2\left(\frac{\pi}{4}\right)\right]\left(\frac{\pi}{40}\right)\left(\frac{1}{2}\right)$$

$$\approx 1.00205197$$

Simpson's rule:

$$\int_0^{\pi/4} \sec^2 x\,dx \approx \left[\sec^2(0) + 4\sec^2\left(\frac{\pi}{80}\right) + 2\sec^2\left(\frac{\pi}{40}\right) + 4\sec^2\left(\frac{3\pi}{80}\right) + 2\sec^2\left(\frac{2\pi}{40}\right) + 4\sec^2\left(\frac{5\pi}{80}\right) + 2\sec^2\left(\frac{3\pi}{40}\right) + 4\sec^2\left(\frac{7\pi}{80}\right) + 2\sec^2\left(\frac{4\pi}{40}\right) + 4\sec^2\left(\frac{9\pi}{80}\right) + 2\sec^2\left(\frac{5\pi}{40}\right) + 4\sec^2\left(\frac{11\pi}{80}\right) + 2\sec^2\left(\frac{6\pi}{40}\right) + 4\sec^2\left(\frac{13\pi}{80}\right) + 2\sec^2\left(\frac{7\pi}{40}\right) + 4\sec^2\left(\frac{15\pi}{80}\right) + 2\sec^2\left(\frac{8\pi}{40}\right) + 4\sec^2\left(\frac{17\pi}{80}\right) + 2\sec^2\left(\frac{9\pi}{40}\right) + 4\sec^2\left(\frac{19\pi}{80}\right) + \sec^2\left(\frac{\pi}{4}\right)\right]\left(\frac{\pi}{40}\right)\left(\frac{1}{6}\right)$$

$$\approx 1.000001048$$

Exact value: $\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = \tan\left(\dfrac{\pi}{4}\right) - \tan(0) = 1$

Error using midpoint rule = .0010244134; error using trapezoidal rule = .00205197
Error using Simpson's rule = .000001048

40. $\int_0^1 2xe^{x^2}\,dx;\ \Delta x = \frac{(1-0)}{10} = ..1$

Midpoint rule: $x_1 = 0 + \frac{.1}{2} = .05,\ x_2 = .15,\ x_3 = .25,\ x_4 = .35,\ x_5 = .45,\ x_6 = .55,\ x_7 = .65,\ x_8 = .75,\ x_9 = .85,$
$x_{10} = .95$

$$\int_0^1 2xe^{x^2}\,dx = \Big[2(.05)e^{(0.05)^2} + 2(.15)e^{(0.15)^2} + 2(.25)e^{(0.25)^2} + 2(.35)e^{(0.35)^2} + 2(.45)e^{(0.45)^2}$$
$$+ 2(.55)e^{(0.55)^2} + 2(.65)e^{(.65)^2} + 2(.75)e^{(.75)^2} + 2(.85)e^{(.85)^2} + 2(.95)e^{(.95)^2}\Big].1$$
$$\approx 1.712342989$$

Trapezoidal rule:
$a = 0,\ a_1 = 0.1,\ a_2 = 0.2,\ a_3 = 0.3,\ a_4 = 0.4,\ a_5 = 0.5,\ a_6 = 0.6,\ a_7 = 0.7,\ a_8 = 0.8,\ a_9 = 0.9,\ a_{10} = 1$

$$\int_0^1 2xe^{x^2}\,dx \approx \Big[2(0)e^{(0)^2} + 4(0.1)e^{(0.1)^2} + 4(0.2)e^{(0.2)^2} + 4(0.3)e^{(0.3)^2} + 4(0.4)e^{(0.4)^2} + 4(0.5)e^{(0.5)^2}$$
$$+ 4(0.6)e^{(0.6)^2} + 4(0.7)e^{(0.7)^2} + 4(0.8)e^{(0.8)^2} + 4(0.9)e^{(0.9)^2} + 2(1)e^{(1)^2}\Big](0.1)\left(\frac{1}{2}\right)$$
$$\approx 1.730179664$$

Simpson's rule:

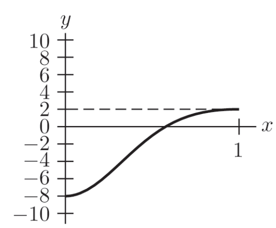
$$\int_0^1 2xe^{x^2}\,dx \approx \Big[2(0)e^{(0)^2} + 8(0.05)e^{(0.05)^2} + 4(0.1)e^{(0.1)^2} + 8(0.15)e^{(0.15)^2} + 4(0.2)e^{(0.2)^2} + 8(0.25)e^{(0.25)^2}$$
$$+ 4(0.3)e^{(0.3)^2} + 8(0.35)e^{(0.35)^2} + 4(0.4)e^{(0.4)^2} + 8(0.45)e^{(0.45)^2} + 4(0.5)e^{(0.5)^2}$$
$$+ 8(0.55)e^{(0.55)^2} + 4(0.6)e^{(0.6)^2} + 8(0.65)e^{(0.65)^2} + 4(0.7)e^{(0.7)^2} + 8(0.75)e^{(0.75)^2}$$
$$+ 4(0.8)e^{(0.8)^2} + 8(0.85)e^{(0.85)^2} + 4(0.9)e^{(0.9)^2} + 8(0.95)e^{(0.95)^2} + 2(1)e^{(1)^2}\Big](0.1)\left(\frac{1}{6}\right)$$
$$\approx 1.718288548$$

Exact value: $\int_0^1 2xe^{x^2}\,dx$; let $u = x^2$; $du = 2x\,dx$.
When $x = 0$, $u = 0$. When $x = 1$, $u = 1$.

$$\int_0^1 2xe^{x^2}\,dx = \int_0^1 e^u\,du = e^u\Big|_0^1 = e^1 - e^0 = e - 1 \approx 1.718281828$$

Error using midpoint rule = .005938839
Error using trapezoidal rule = .011897836
Error using Simpson's rule = .000006720

41. $f(x) = \frac{4}{1+x^2} = 4(1+x^2)^{-1}$



$$f'(x) = -4(1+x^2)^{-2}(2x) = -8x(1+x^2)^{-2}$$
$$f''(x) = -8x(-2)(1+x^2)^{-3}(2x) + (1+x^2)^{-2}(-8)$$
$$= \frac{32x^2}{(1+x^2)^3} - \frac{8}{(1+x^2)^2}$$

Let $A = 8$, $a = 0$, $b = 1$, $n = 20$.

A bound on the error of the estimate is $\frac{A(b-a)^3}{24n^2} = \frac{8(1-0)^3}{24(20)^2} = .0008333$.

42. Same graph as for Exercise 41. Let $A = 8$, $a = 0$, $b = 1$, $n = 15$.

A bound on the error of the estimate is $\frac{A(b-a)^3}{12n^2} = \frac{8(1-0)^3}{12(15)^2} = .002962963$.

9.5 Some Applications of the Integral

1. Present value $=\int_{T_1}^{T_2} K(t)e^{-rt}\,dt$;
$K(t)=35{,}000,\ T_1=0,\ T_2=5,\ r=.07.$
So
$$\text{P.V.}=\int_0^5 35{,}000e^{-.07t}\,dt$$
$$=-\frac{100}{7}(35{,}000)e^{-.07t}\Big|_0^5$$
$$=-500{,}000\left[e^{-.07(5)}-e^0\right]\approx \$147{,}656$$

2. $K(t)=60{,}000,\ T_1=2,\ T_2=6,\ r=.06$
So
$$\text{P.V.}=\int_2^6 60{,}000e^{-.06t}\,dt$$
$$=-\frac{100}{6}(60{,}000)e^{-.06t}\Big|_2^6\approx \$189{,}244$$

3. $K(t)=12{,}000,\ T_1=1,\ T_2=5,\ r=.1$
So
$$\text{P.V.}=\int_1^5 12{,}000e^{-0.1t}\,dt=-10(12{,}000)e^{-0.1t}\Big|_1^5$$
$$\approx \$35{,}797$$

4. $K(t)=25e^{-0.02t},\ T_1=0,\ T_2=4,\ r=.08$
The present value is
$$\int_0^4(25e^{-0.02t})e^{-0.08t}\,dt=\int_0^4 25e^{-0.1t}\,dt$$
$$=-10(25)e^{-0.1t}\Big|_0^4$$
$$=-250[e^{-0.4}-1]$$
$$\approx \$82.4 \text{ thousand}$$

5. $K(t)=80e^{-0.08t},\ T_1=0,\ T_2=3,\ r=0.11$
The present value is
$$\int_0^3(80e^{-0.08t})e^{-0.11t}\,dt=\int_0^3 80e^{-0.19}\,dt$$
$$=-\frac{100}{19}(80)e^{-0.19}\Big|_0^3$$
$$=-\frac{8000}{19}[e^{-0.57}-1]$$
$$\approx \$182{,}937$$

6. a. $K(t)=20e^{1-0.09t},\ T_1=2,\ T_2=5,$
$r=0.06.$
So $\text{P.V.}=\int_2^5(20e^{1-0.09t})e^{-0.06t}\,dt$.

b.
$$\int_2^5(20e^{1-0.09t})e^{-0.06t}\,dt$$
$$=\int_2^5 20e^{1-0.15t}\,dt$$
$$=-\frac{100}{15}(20)e^{1-0.15t}\Big|_2^5$$
$$=-\frac{2000}{15}[e^{0.25}-e^{0.7}]$$
$$\approx \$97.297 \text{ thousand}$$

7. a. $K(t)=30+5t, T_1=0, T_2=2, r=0.1$.
So $\text{P.V.}=\int_0^2(30+5t)e^{-0.10t}\,dt$.

b. Use integration by parts with $f(t)=30+5t$ and $g(t)=e^{-0.1t}$. Then $f'(t)=5$ and $G(t)=-10e^{-0.1t}$.
$$\int_0^2(30+5t)e^{-0.1t}\,dt$$
$$=(30+5t)(-10)e^{-0.1t}\Big|_0^2-\int_0^2 5(-10)e^{-0.1t}\,dt$$
$$=\left(-400e^{-0.2}+300\right)-500\,e^{-0.1t}\Big|_0^2$$
$$=\left(-400e^{-0.2}+300\right)-500e^{-0.2}+500$$
$$=-900e^{-0.2}+800$$
$$\approx 63.1 \text{ million dollars}$$

8. $\int_0^2(50+7t)e^{-0.1t}\,dt$
Use integration by parts with $f(t)=50+7t$, $g(t)=e^{-0.1t}$. Then
$f'(t)=7$ and $G(t)=-10e^{-.1t}$.
$$\int_0^2(50+7t)e^{-0.1t}\,dt$$
$$=(50+7t)(-10)e^{-0.1t}\Big|_0^2-\int_0^2(7)(-10)e^{-0.1t}\,dt$$
$$=500-e^{-0.2}(500+140)-700e^{-0.1t}\Big|_0^2$$
$$\approx \$102.9 \text{ thousand}$$

9. a. $D(t)=120e^{-0.65t}$
$$\text{Population}=\int_0^5(2\pi t)120e^{-0.65t}\,dt$$
$$=240\pi\int_0^5 te^{-0.65t}\,dt$$

b. Use integration by parts with $f(t) = t$, $g(t) = e^{-0.65t}$. Then $f'(t) = 1$ and $G(t) = \frac{-100}{65}e^{-0.65t} = -\frac{20}{13}e^{-0.65t}$.

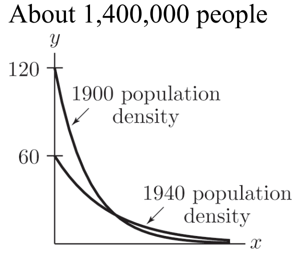
$$240\pi\int_0^5 te^{-0.65t}dt$$
$$= 240\pi\left[t\left(-\frac{20}{13}\right)e^{-0.65t}\Big|_0^5 - \int_0^5\left(-\frac{20}{13}\right)e^{-0.65t}dt\right]$$
≈ 1490.493 thousand or 1,490,493 people

10. $\int_3^5 (2\pi t)120e^{-0.65t}dt$
$$= 240\pi\int_3^5 te^{-0.65t}dt$$
$$= 240\pi\left[t\left(-\frac{20}{13}\right)e^{-0.65t}\Big|_3^5 - \int_3^5\left(-\frac{20}{13}\right)e^{-0.65t}dt\right]$$
(using the integration by parts from Exercise 9)
≈ 454.920 thousand of 454,920 people

11. $D(t) = 60e^{-0.4t}$

$$\text{Population} = \int_0^5 (2\pi t)60e^{-0.4t}dt$$
$$= 120\pi\int_0^5 te^{-0.4t}dt$$
Use integration by parts with $f(t) = t$, $g(t) = e^{-0.4t}$. Then $f'(t) = 1$ and $G(t) = -\frac{10}{4}e^{-0.4t} = -\frac{5}{2}e^{-0.4t}$.

$$120\pi\int_0^5 te^{-0.4t}dt$$
$$= 120\pi\left[t\left(-\frac{5}{2}\right)e^{-0.4t}\Big|_0^5 - \int_0^5\left(-\frac{5}{2}\right)e^{-0.4t}dt\right]$$
$\approx 120\pi(3.712)$
≈ 1399.391
About 1,400,000 people



People moved away from the center of the city

The graphs show that people moved away from the center of the city.

12. $D(t) = 11(t^2 + 10)^{-2}$
Amount of lava
$$\text{Amt of lava} = \int_1^{10} (2\pi t)(11)(t^2 + 10)^{-2}dt$$
$$= 22\pi\int_1^{10} t(t^2 + 10)^{-2}dt$$
$$= 22\pi\left(\frac{1}{2}\right)(-1)(t^2 + 10)^{-1}\Big|_1^{10}$$
$$= (-11\pi)\left[\frac{1}{110} - \frac{1}{11}\right]$$
$$= \frac{9}{10}\pi \text{ thousand tons}$$

13. a. The area of the ring is $2\pi t(\Delta t)$. The population density is $40e^{-0.5t}$. So the population is $2\pi t(\Delta t)40e^{-0.5t} = 80\pi t(\Delta t)e^{-0.5t}$ thousand.

b. $\frac{dP}{dt}$ or $P'(t)$

c. It represents the number of people who live between $(5 + \Delta t)$ miles from the city center and 5 miles from the city center.

d. $P(t + \Delta t) - P(t) = 80\pi t(\Delta t)e^{-0.5t}$ from (a).
So, $\frac{P(t + \Delta t) - P(t)}{\Delta t} \approx P'(t) = 80\pi te^{-0.5t}$

e. $\int_a^b P'(t)dt = P(b) - P(a) = \int_a^b 80\pi te^{-0.5t}dt$

9.6 Improper Integrals

1. As $b \to \infty$, $\frac{5}{b}$ approaches 0.
2. As $b \to \infty$, b^2 increases without bound.
3. As $b \to \infty$, $-3e^{2b}$ decreases without bound.
4. As $b \to \infty$, $\frac{1}{b} + \frac{1}{3}$ approaches $0 + \frac{1}{3} = \frac{1}{3}$.
5. As $b \to \infty$, $\frac{1}{4} - \frac{1}{b^2}$ approaches $\frac{1}{4} - 0 = \frac{1}{4}$.
6. As $b \to \infty$, $\frac{1}{2}\sqrt{b}$ increases without bound.
7. As $b \to \infty$, $2 - (b+1)^{-1/2}$ approaches $2 - 0 = 2$.

8. As $b \to \infty$, $5 - \dfrac{1}{b-1}$ approaches 5 – 0 = 5.

9. As $b \to \infty$, $5(b^2+3)^{-1}$ approaches 0.

10. As $b \to \infty$, $4(1-b^{-3/4})$ approaches 4.

11. As $b \to \infty$, $e^{-b/2}+5$ approaches 0 + 5 = 5.

12. As $b \to \infty$, $2-e^{-3b}$ approaches 2 – 0 = 2.

13. The given area is $\int_2^\infty \frac{1}{x^2}\,dx.$

$$\int_2^b \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_2^b = -\frac{1}{b}+\frac{1}{2}$$

As $b \to \infty$, $-\frac{1}{b}+\frac{1}{2}$ approaches $\frac{1}{2}.$

Thus, $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \frac{1}{2}.$

14. The given area is $\int_0^\infty (x+1)^{-2}\,dx.$

$$\int_0^b (x+1)^{-2}\,dx = -\frac{1}{x+1}\bigg|_0^b = -\frac{1}{b+1}+1$$

As $b \to \infty$, $-\frac{1}{b+1}+1$ approaches 1.

Thus, $\int_0^\infty (x+1)^{-2}\,dx = \lim_{b\to\infty}\int_0^b (x+1)^{-2}\,dx = 1.$

15. The given area is $\int_0^\infty e^{-x/2}\,dx.$

$$\int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2}+2$$

As $b \to \infty$, $-2e^{-b/2}+2$ approaches 2.

Thus, $\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$

16. The given area is $\int_0^\infty 4e^{-4x}\,dx.$

$$\int_0^b 4e^{-4x}\,dx = -e^{-4x}\Big|_0^b = -e^{-4b}+1$$

As $b \to \infty$, $-e^{-4b}+1$ approaches 1.

Thus, $\int_0^\infty 4e^{-4x}\,dx = \lim_{b\to\infty}\int_0^b 4e^{-4x}\,dx = 1.$

17. The given area is $\int_3^\infty (x+1)^{-3/2}\,dx.$

$$\int_3^b (x+1)^{-3/2}\,dx = -2(x+1)^{-1/2}\Big|_3^b = -2(b+1)^{-1/2}+1$$

As $b \to \infty$, $-2(b+1)^{-1/2}+1$ approaches 1.

Thus,

$$\int_3^\infty (x+1)^{-3/2}\,dx = \lim_{b\to\infty}\int_3^b (x+1)^{-3/2}\,dx = 1.$$

18. The given area is $\int_1^\infty (2x+6)^{-4/3}\,dx.$

$$\int_1^b (2x+6)^{-4/3}\,dx = -\frac{3}{2}(2x+6)^{-1/3}\bigg|_1^b = -\frac{3}{2}(2b+6)^{-1/3}+\frac{3}{4}$$

As $b \to \infty$, $-\frac{3}{2}(2b+6)^{-1/3}+\frac{3}{4}$ approaches $\frac{3}{4}.$

Thus, $\int_1^\infty (2x+6)^{-4/3}\,dx$

$$= \lim_{b\to\infty}\int_1^b (2x+6)^{-4/3}\,dx = \frac{3}{4}.$$

19. $\int_1^b (14x+18)^{-4/5}\,dx = \frac{5}{14}(14x+18)^{1/5}\Big|_1^b$

$$= \frac{5}{14}(14b+18)^{1/5}-\frac{5}{7}$$

As $b \to \infty$, $\frac{5}{14}(14b+18)^{1/5}-\frac{5}{7}$ increases without bound. Thus, $\int_1^\infty (14x+18)^{-4/5}\,dx$ diverges and the area under the curve $y=(14x+18)^{-4/5}$ for $x \ge 1$ cannot be assigned any finite number.

20. $\int_2^b (x-1)^{-1/3}\,dx = \frac{3}{2}(x-1)^{2/3}\Big|_2^b$

$$= \frac{3}{2}(b-1)^{2/3}-\frac{3}{2}$$

As $b \to \infty$, $\frac{3}{2}(b-1)^{2/3}-\frac{3}{2}$ increases without bound. Thus, $\int_2^\infty (x-1)^{-1/3}\,dx$ diverges and the area under the curve $y=(x-1)^{-1/3}$ for $x \ge 2$ cannot be assigned a finite number.

21. $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}x^{-2}\Big|_1^b\right]$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}b^{-2}+\frac{1}{2}\right] = \frac{1}{2}$$

22. $\int_1^{\infty} \frac{2}{x^{3/2}}\,dx = \lim_{b\to\infty} \int_1^b \frac{2}{x^{3/2}}\,dx$
$$= \lim_{b\to\infty} \left[-4x^{-1/2} \Big|_1^b \right]$$
$$= \lim_{b\to\infty} [-4b^{-1/2} + 4] = 4$$

23. $\int_0^{\infty} \frac{1}{(2x+3)^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{(2x+3)^2}\,dx$
$$= \lim_{b\to\infty} \left[-\frac{1}{2}(2x+3)^{-1} \Big|_0^b \right]$$
$$= \lim_{b\to\infty} \left[-\frac{1}{2}(2b+3)^{-1} + \frac{1}{6} \right] = \frac{1}{6}$$

24. $\int_0^{\infty} e^{-3x}\,dx = \lim_{b\to\infty} \int_0^b e^{-3x}\,dx = \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3x} \Big|_0^b \right]$
$$= \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3b} + \frac{1}{3} \right] = \frac{1}{3}$$

25. $\int_0^{\infty} e^{2x}\,dx = \lim_{b\to\infty} \int_0^b e^{2x}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}e^{2x} \Big|_0^b \right]$
$$= \lim_{b\to\infty} \left[\frac{1}{2}e^{2b} - \frac{1}{2} \right]$$
As $b \to \infty$, $\frac{1}{2}e^{2b} - \frac{1}{2}$ increases without bound. Therefore, $\int_0^{\infty} e^{2x}\,dx$ diverges.

26. $\int_0^{\infty} (x^2+1)dx = \lim_{b\to\infty} \int_0^b (x^2+1)dx$
$$= \lim_{b\to\infty} \left[\frac{x^3}{3} + x \Big|_0^b \right]$$
$$= \lim_{b\to\infty} \left[\frac{b^3}{3} + b - 0 \right]$$
As $b \to \infty$, $\frac{b^3}{3} + b$ increases without bound.
Therefore, $\int_0^{\infty} (x^2+1)dx$ diverges.

27. $\int_2^{\infty} \frac{1}{(x-1)^{5/2}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{(x-1)^{5/2}}\,dx$
$$= \lim_{b\to\infty} \left[-\frac{2}{3}(x-1)^{-3/2} \Big|_2^b \right]$$
$$= \lim_{b\to\infty} \left[-\frac{2}{3}(b-1)^{-3/2} + \frac{2}{3} \right] = \frac{2}{3}$$

28. $\int_2^{\infty} e^{2-x}\,dx = \lim_{b\to\infty} \int_2^b e^{2-x}\,dx = \lim_{b\to\infty} \left[-e^{2-x} \Big|_2^b \right]$
$$= \lim_{b\to\infty} [-e^{2-b} + 1] = 1$$

29. $\int_0^{\infty} 0.01e^{-0.01x}\,dx = \lim_{b\to\infty} \int_0^b 0.01e^{-0.01x}\,dx$
$$= \lim_{b\to\infty} \left[-e^{-0.01x} \Big|_0^b \right]$$
$$= \lim_{b\to\infty} [-e^{-0.01b} + 1] = 1$$

30. $\int_0^{\infty} \frac{4}{(2x+1)^3}\,dx = \lim_{b\to\infty} \int_0^b \frac{4}{(2x+1)^3}\,dx$
$$= \lim_{b\to\infty} \left[-(2x+1)^{-2} \Big|_0^b \right]$$
$$= \lim_{b\to\infty} [-(2b+1)^{-2} + 1] = 1$$

31. $\int_0^{\infty} 6e^{1-3x}\,dx = \lim_{b\to\infty} \int_0^b 6e^{1-3x}\,dx$
$$= \lim_{b\to\infty} \left[-2e^{1-3x} \Big|_0^b \right]$$
$$= \lim_{b\to\infty} [-2e^{1-3b} + 2e] = 2e$$

32. $\int_1^{\infty} e^{-0.2x}\,dx = \lim_{b\to\infty} \int_1^b e^{-0.2x}\,dx$
$$= \lim_{b\to\infty} \left[-5e^{-0.2x} \Big|_1^b \right]$$
$$= \lim_{b\to\infty} \left[-5e^{-0.2b} + 5e^{-0.2} \right]$$
$$= 5e^{-0.2} \approx 4.09365$$

33. $\int_3^{\infty} \frac{x^2}{\sqrt{x^3-1}}\,dx = \lim_{b\to\infty} \int_3^b \frac{x^2}{\sqrt{x^3-1}}\,dx$

To evaluate $\int_3^b \frac{x^2}{\sqrt{x^3-1}}\,dx$, use the substitution $u = x^3 - 1$, $du = 3x^2dx$. When $x = 3$, $u = 3^3 - 1 = 26$; when $x = b$, $u = b^3 - 1$. Thus,
$$\int_3^b \frac{x^2}{\sqrt{x^3-1}}\,dx = \frac{1}{3}\int_{26}^{b^3-1} \frac{1}{\sqrt{u}}\,du = \left(\frac{1}{3}\right)2u^{1/2}\Big|_{26}^{b^3-1}$$
$$= \frac{2}{3}(b^3-1) - \frac{2}{3}\sqrt{26}$$
As $b \to \infty$, $\frac{2}{3}(b^3-1)^{1/2} - \frac{2}{3}\sqrt{26}$ increases without bound. Therefore, $\int_3^{\infty} \frac{x^2}{\sqrt{x^3-1}}\,dx$ diverges.

34. $\int_2^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}dx$

To evaluate $\int_2^b \frac{1}{x\ln x}dx,$ use the substitution $u = \ln x,\ du = \frac{1}{x}dx.$ When $x = 2$, $u = \ln 2$; when $x = b$, $u = \ln b$. Thus,

$$\int_2^b \frac{1}{x\ln x}dx = \int_{\ln 2}^{\ln b}\frac{1}{u}du = \ln|u|\Big|_{\ln 2}^{\ln b} = \ln(\ln b) - \ln(\ln 2)$$

As $b \to \infty$, $\ln(\ln b) - \ln(\ln 2)$ increases without bound. Therefore, $\int_2^\infty \frac{1}{x\ln x}dx$ diverges.

35. $\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$

To evaluate $\int_0^b xe^{-x^2}dx,$ use the substitution $u = -x^2,\ du = -2x\,dx.$ When $x = 0$, $u = 0$; when $x = b,\ u = -b^2.$

$$\int_0^b xe^{-x^2}dx = -\frac{1}{2}\int_0^{-b^2} e^u du = -\frac{1}{2}e^u\Big|_0^{-b^2} = -\frac{1}{2}e^{-b^2} + \frac{1}{2}$$

Thus, $\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right] = \frac{1}{2}.$

36. $\int_0^\infty \frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}dx$

To evaluate $\int_0^b \frac{x}{x^2+1}dx,$ use the substitution $u = x^2+1,\ du = 2x\,dx.$ When $x = 0$, $u = 1$; when $x = b,\ u = b^2+1.$

$$\int_0^b \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^{b^2+1}\frac{1}{u}du = \frac{1}{2}\ln|u|\Big|_1^{b^2+1} = \frac{1}{2}\ln(b^2+1) - 0$$

As $b \to \infty,\ \frac{1}{2}\ln(b^2+1)$ increases without bound. Thus, $\int_0^\infty \frac{x}{x^2+1}dx$ diverges.

37. $\int_0^\infty 2x(x^2+1)^{-3/2}dx = \lim_{b\to\infty}\int_0^b 2x(x^2+1)^{-3/2}dx$

To evaluate $\int_0^b 2x(x^2+1)^{-3/2}dx,$ use the substitution $u = x^2+1,\ du = 2x\,dx.$ When $x = 0$, $u = 1$; when $x = b,\ u = b^2+1.$

$$\int_0^b 2x(x^2+1)^{-3/2}dx = \int_1^{b^2+1}u^{-3/2}du = -2u^{-1/2}\Big|_1^{b^2+1} = -2(b^2+1)^{-1/2} + 2$$

Thus,

$$\int_0^\infty 2x(x^2+1)^{-3/2}dx = \lim_{b\to\infty}[-2(b^2+1)^{-1/2} + 2] = 2$$

38. $\int_1^\infty (5x+1)^{-4}dx = \lim_{b\to\infty}\int_1^b (5x+1)^{-4}dx$

$$= \lim_{b\to\infty}\left[-\frac{1}{15}(5x+1)^{-3}\Big|_1^b\right] = \lim_{b\to\infty}\left[-\frac{1}{15}(5b+1)^{-3} + \frac{1}{3240}\right] = \frac{1}{3240} \approx .00031$$

39. $\int_{-\infty}^0 e^{4x}dx = \lim_{b\to-\infty}\int_b^0 e^{4x}dx = \lim_{b\to-\infty}\left[\frac{1}{4}e^{4x}\Big|_b^0\right]$

$$= \lim_{b\to-\infty}\left[\frac{1}{4} - \frac{1}{4}e^{4b}\right] = \frac{1}{4}$$

40. $\int_{-\infty}^0 \frac{8}{(x-5)^2}dx = \lim_{b\to-\infty}\int_b^0 \frac{8}{(x-5)^2}dx$

$$= \lim_{b\to-\infty}\left[-\frac{8}{(x-5)}\Big|_b^0\right] = \lim_{b\to-\infty}\left[\frac{8}{5} + \frac{8}{b-5}\right] = \frac{8}{5}$$

41. $\int_{-\infty}^0 \frac{6}{(1-3x)^2}dx = \lim_{b\to-\infty}\int_b^0 \frac{6}{(1-3x)^2}dx$

$$= \lim_{b\to-\infty}\left[\frac{2}{(1-3x)}\Big|_b^0\right] = \lim_{b\to-\infty}\left[2 - \frac{2}{1-3b}\right] = 2$$

42. $\int_{-\infty}^0 \frac{1}{\sqrt{4-x}}dx = \lim_{b\to-\infty}\int_b^0 \frac{1}{\sqrt{4-x}}dx$

$$= \lim_{b\to-\infty}\left[-2\sqrt{4-x}\Big|_b^0\right] = \lim_{b\to-\infty}\left[-4 + 2\sqrt{4-b}\right]$$

As $b \to -\infty,\ -4 + 2\sqrt{4-b}$ increases without bound. Thus, $\int_{-\infty}^0 \frac{1}{\sqrt{4-x}}dx$ diverges.

43. $\int_0^{\infty} \frac{e^{-x}}{(e^{-x}+2)^2}dx = \lim_{b\to\infty} \int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}dx$

To evaluate $\int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}dx,$ use the substitution $u = e^{-x}+2,\ du = -e^{-x}dx.$

When $x = 0$, $u = 3$; when $x = b,\ u = e^{-b}+2.$

$$\int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}dx = -\int_3^{e^{-b}+2} \frac{du}{u^2} = u^{-1}\Big|_3^{e^{-b}+2} = \frac{1}{e^{-b}+2} - \frac{1}{3}$$

Thus,

$$\int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}dx = \lim_{b\to\infty}\left[\frac{1}{e^{-b}+2} - \frac{1}{3}\right]. = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

44. $\int_{-\infty}^{\infty} \frac{e^{-x}}{(e^{-x}+2)^2}dx$

$$= \int_0^{\infty} \frac{e^{-x}}{(e^{-x}+2)^2}dx + \int_{-\infty}^0 \frac{e^{-x}}{(e^{-x}+2)^2}dx$$

$\int_0^{\infty} \frac{e^{-x}}{(e^{-x}+2)^2}dx = \frac{1}{6}$ (See exercise 43.)

$$\int_b^0 \frac{e^{-x}}{(e^{-x}+2)^2}dx = -\int_0^b \frac{e^{-x}}{(e^{-x}+2)^2}dx = \frac{1}{3} - \frac{1}{e^{-b}+2}$$

(See exercise 43.) Thus,

$$\int_{-\infty}^0 \frac{e^{-x}}{(e^{-x}+2)^2}dx = \lim_{b\to-\infty}\left[\frac{1}{3} - \frac{1}{e^{-b}+2}\right] = \frac{1}{3} - 0 = \frac{1}{3}.$$

Combining these gives

$$\int_{-\infty}^{\infty} \frac{e^{-x}}{(e^{-x}+2)^2}dx = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

45. $\int_0^{\infty} ke^{-kx}dx = \lim_{b\to\infty}\int_0^b ke^{-kx}dx$

$$= \lim_{b\to\infty}\left[-e^{-kx}\Big|_0^b\right] = \lim_{b\to\infty}[-e^{-kb}+1]$$

If $k > 0$, as $b \to \infty,\ -e^{-kb}$ approaches 0. Thus, in this case $\int_0^{\infty} ke^{-kx}dx = 1.$

46. $\int_1^{\infty} \frac{k}{x^{k+1}}dx = \lim_{b\to\infty}\int_1^b \frac{k}{x^{k+1}}dx = \lim_{b\to\infty}\left[-x^{-k}\Big|_1^b\right]$

$$= \lim_{b\to\infty}[-b^{-k}+1]$$

If $k > 0$, as $b \to \infty,\ -b^{-k}$ approaches 0.

Thus, in this case $\int_1^{\infty} \frac{k}{x^{k+1}}dx = 1.$

47. $\int_e^{\infty} \frac{k}{x(\ln x)^{k+1}}dx = \lim_{b\to\infty}\int_e^b \frac{k}{x(\ln x)^{k+1}}dx$

$$= \lim_{b\to\infty}\left[(k)\frac{(\ln x)^{-k}}{-k}\Big|_e^b\right] = \lim_{b\to\infty}[-(\ln b)^{-k}+1]$$

If $k > 0$, as $b \to \infty,\ (\ln b)^{-k}$ approaches 0.

Thus, in this case $\int_e^{\infty} \frac{k}{x(\ln x)^{k+1}}dx = 1.$

48. $\int_0^{\infty} 5000e^{-0.1t}dt$

$$= \lim_{b\to\infty}\int_0^b 5000e^{-0.1t}dt = \lim_{b\to\infty}\left[-50,000e^{-0.1t}\Big|_0^b\right] = \lim_{b\to\infty}[-50,000e^{-0.1b}+50,000] = \$50,000$$

49. Capital value $= \int_0^{\infty} Ke^{-rt}dt$

$$= \lim_{b\to\infty} K\left(-\frac{1}{r}\right)\int_0^b -re^{-rt}dt = -\frac{K}{r}\lim_{b\to\infty}\left[e^{-rt}\Big|_0^b\right] = -\frac{K}{r}\lim_{b\to\infty}[e^{-rb}-1] = \frac{K}{r}$$

50. Present value

$$= \int_0^{\infty} 10,000e^{0.04t}\left(e^{-0.12t}\right)dt = \lim_{b\to\infty}\int_0^b 10,000e^{-0.08t}dt = \lim_{b\to\infty}\left[10,000\left(-\frac{100}{8}\right)e^{-0.08t}\Big|_0^b\right]$$

$$= \lim_{b\to\infty}\left[-125,000e^{-0.08b}+125,000\right] = \$125,000$$

Chapter 9 Fundamental Concept Check Exercises

1. Integration by substitution is an integration process that is based on reversing the chain rule.

2. Integration by parts is an integration process that is based on reversing the product rule.

3. Suppose that we use the substitution $u = g(x)$ when evaluating the integral $\int_a^b f(g(x))g'(x)\,dx$. The limits of the new integral in u become $g(a)$ and $g(b)$. More precisely, the integral becomes $\int_{g(a)}^{g(b)} f(u)\,du$.

4. $\int_a^b f(x)g(x)\,dx$
$= f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx.$

5. $\int_a^b f(x)\,dx \approx [f(x_1)+f(x_2)+\cdots+f(x_n)]\Delta x,$ where the x_i's are the midpoints of the subintervals and Δx is the length of a subinterval.

6. $\int_a^b f(x)\,dx$
$\approx [f(a_0)+2f(a_1)+\cdots$
$+2f(a_{n-1})+f(a_n)]\Delta x,$
where the a_i's are the endpoints of the subintervals and Δx is the length of a subinterval.

7. Simpson's method for approximating the definite integral is a weighted average of the approximation from the midpoint rule, *M*, and the approximation from the trapezoidal rule *T*.

8. The error for the midpoint rule is at most $\frac{A(b-a)^3}{24n^2}$, where A is a number such that $|f''(x)| \le A$ for all $a \le x \le b$. The error for the trapezoidal rule is at most $\frac{A(b-a)^3}{12n^2}$, where A is a number such that $|f''(x)| \le A$ for all $a \le x \le b$. The error for Simpson's rule is at most $\frac{A(b-a)^5}{2880n^4}$, where A is a number such that $|f''''(x)| \le A$ for all $a \le x \le b$.

9. a. Present value $= \int_{T_1}^{T_2} K(t)e^{-rt}\,dt$, where $K(t)$ dollars per year is the annual rate of income at time t, \$ is the annual interest rate of invested money, T_1 to T_2 (years) is the time period of the income stream.

b. Population $= \int_a^b 2\pi t D(t)\,dt$, where $D(t)$ is the density of population (in persons per square mile) at distance t miles from the city center. The ring includes all persons who live between a and b miles from the city center.

10. $\int_a^\infty f(x)\,dx$ is convergent if $\lim\limits_{b\to\infty}\int_a^b f(x)\,dx$ exists.

Chapter 9 Review Exercises

1. $\int x\sin 3x^2\,dx$

Let $u = 3x^2,\ du = 6x\,dx$.
Then

$$\int x\sin 3x^2\,dx = \frac{1}{6}\int \sin u\,du = -\frac{1}{6}\cos u + C$$
$$= -\frac{1}{6}\cos 3x^2 + C$$

2. $\int \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2} + C$

3. $\int x(1-3x^2)^5\,dx$

Let $u = 1-3x^2,\ du = -6x\,dx$. Then

$$\int x(1-3x^2)^5\,dx = -\frac{1}{6}\int u^5\,du = -\frac{1}{36}u^6 + C$$
$$= -\frac{1}{36}(1-3x^2)^6 + C$$

4. $\int \frac{(\ln x)^5}{x}\,dx$

Let $u = \ln x,\ du = \frac{1}{x}dx$. Then

$$\int \frac{(\ln x)^5}{x}\,dx = \int u^5\,du = \frac{1}{6}u^6 + C$$
$$= \frac{1}{6}(\ln x)^6 + C$$

5. $\int \frac{(\ln x)^2}{x}dx$

Let $u = \ln x$, $du = \frac{1}{x}dx$. Then

$$\int \frac{(\ln x)^2}{x}dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$$

6. $\int \frac{1}{\sqrt{4x+3}}dx = \frac{1}{2}(4x+3)^{1/2} + C$

7. $\int x\sqrt{4-x^2}dx$

Let $u = 4 - x^2$, $du = -2x\,dx$. Then

$$\int x\sqrt{4-x^2}dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} + C$$

8. $\int x \sin 3x\,dx$

Use integration by parts with $f(x) = x$, $g(x) = \sin 3x$. Then

$f'(x) = 1$, $G(x) = -\frac{1}{3}\cos 3x$ and

$$\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

9. $\int x^2 e^{-x^3}dx$

Let $u = -x^3$, $du = -3x^2dx$. Then

$$\int x^2 e^{-x^3}dx = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C$$

10. $\int \frac{x\ln(x^2+1)}{x^2+1}dx$

Let $u = \ln(x^2+1)$, $du = \frac{2x}{x^2+1}dx$.

Then

$$\int \frac{x\ln(x^2+1)}{x^2+1}dx = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}(\ln(x^2+1))^2 + C$$

11. $\int x^2 \cos 3x\,dx$

Use integration by parts with $f(x) = x^2$, $g(x) = \cos 3x$. Then

$f'(x) = 2x$, $G(x) = \frac{1}{3}\sin 3x$ and

$$\int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\int x\sin 3x\,dx.$$

To evaluate $\int x\sin 3x\,dx$ integrate by parts again:

$$\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C_1.$$

Thus,

$$\int x^2\cos 3x\,dx = \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$$

12. $\int \frac{\ln(\ln x)}{x\ln x}dx$

Let $u = \ln(\ln x)$, $du = \frac{1}{x\ln x}dx$.

$$\int \frac{\ln(\ln x)}{x\ln x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln(\ln x))^2 + C$$

13. $\int \ln x^2 dx = \int 2\ln x\,dx = 2\int \ln x\,dx$

To evaluate $\int \ln x\,dx$, use integration by parts with $f(x) = \ln x$, $g(x) = 1$. Then $f'(x) = \frac{1}{x}$, $G(x) = x$ and

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C_1.$$

Thus, $\int \ln x^2 dx = 2x\ln x - 2x + C.$

14. $\int x\sqrt{x+1}dx$

Use integration by parts with $f(x) = x$, $g(x) = \sqrt{x+1}$. Then

$f'(x) = 1$, $G(x) = \frac{2}{3}(x+1)^{3/2}$ and

$$\int x\sqrt{x+1}dx = \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}dx = \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C.$$

15. $\int \frac{x}{\sqrt{3x-1}}dx$

Use integration by parts with $f(x) = x$, $g(x) = (3x-1)^{-1/2}$. Then $f'(x) = 1$, $G(x) = \frac{2}{3}(3x-1)^{1/2}$ and

$$\int \frac{x}{\sqrt{3x-1}}dx = \frac{2}{3}x(3x-1)^{1/2} - \frac{2}{3}\int (3x-1)^{1/2}dx = \frac{2}{3}x(3x-1)^{1/2} - \frac{4}{27}(3x-1)^{3/2} + C$$

16. $\int x^2 \ln x^2 dx$

Use integration by parts with $f(x) = \ln x^2$, $g(x) = x^2$. Then $f'(x) = \frac{2x}{x^2} = \frac{2}{x}$, and $G(x) = \frac{x^3}{3}$.

$$\int x^2 \ln x^2 dx = \frac{x^3}{3}\ln x^2 - \int \frac{2x^2}{3}dx = \frac{x^3}{3}\ln x^2 - \frac{2}{9}x^3 + C.$$

17. $\int \frac{x}{(1-x)^5}dx$

Use integration by parts with $f(x) = x$, $g(x) = (1-x)^{-5}$. Then $f'(x) = 1$, $G(x) = \frac{1}{4}(1-x)^{-4}$ and

$$\int \frac{x}{(1-x)^5}dx = \frac{1}{4}x(1-x)^{-4} - \frac{1}{4}\int (1-x)^{-4}dx = \frac{1}{4}x(1-x)^{-4} - \frac{1}{12}(1-x)^{-3} + C.$$

18. $\int x(\ln x)^2 dx$

Use integration by parts with $f(x) = (\ln x)^2$, $g(x) = x$. Then $f'(x) = \frac{2\ln x}{x}$, $G(x) = \frac{x^2}{2}$ and

$$\int x(\ln x)^2 dx = \frac{x^2(\ln x)^2}{2} - \int x\ln x\, dx$$

$$= \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2}{2}\ln x - \int \frac{x}{2}dx\right]$$

(using parts again)

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C = \frac{x^2}{2}\left[(\ln x)^2 - \ln x + \frac{1}{2}\right] + C.$$

19. Integration by parts: $f(x) = x$, $g(x) = e^{2x}$

20. Integration by parts: $f(x) = x - 3$, $g(x) = e^{-x}$

21. Substitution: $u = \sqrt{x+1}$

22. Substitution: $u = x^3 - 1$

23. Substitution: $u = x^4 - x^2 + 4$

24. Integration by parts: $f(x) = \ln\sqrt{5-x} = \frac{1}{2}\ln(5-x)$, $g(x) = 1$

25. Repeated integration by parts, starting with $f(x) = (3x-1)^2$, $g(x) = e^{-x}$

26. Substitution: $u = 3 - x^2$

27. Integration by parts: $f(x) = 500 - 4x$, $g(x) = e^{-x/2}$

28. Integration by parts: $f(x) = \ln x$, $g(x) = x^{5/2}$

29. Integration by parts: $f(x) = \ln(x + 2)$, $g(x) = \sqrt{x+2}$

30. Repeated integration by parts, starting with $f(x) = (x+1)^2$, $g(x) = e^{3x}$

31. Substitution: $u = x^2 + 6x$

32. Substitution: $u = \sin x$

33. Substitution: $u = x^2 - 9$

34. Integration by parts: $f(x) = 3 - x$, $g(x) = \sin 3x$

35. Substitution: $u = x^3 - 6x$

36. Substitution: $u = \ln x$

37. $\int_0^1 \frac{2x}{(x^2+1)^3}dx$

Let $u = x^2 + 1$, $du = 2x\, dx$.
When $x = 0$, $u = 1$; when $x = 1$, $u = 2$.

$$\int_0^1 \frac{2x}{(x^2+1)^3}dx = \int_1^2 \frac{du}{u^3} = -\frac{1}{2}u^{-2}\Big|_1^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$$

38. $\int_0^{\pi/2} x\sin 8x\,dx$

Using integration by parts with $f(x) = x$,

$g(x) = \sin 8x$, we have $f'(x) = 1,\ G(x) = -\frac{1}{8}\cos 8x$ and

$$\int_0^{\pi/2} x\sin 8x\,dx = -\frac{1}{8}x\cos 8x\Big|_0^{\pi/2} + \frac{1}{8}\int_0^{\pi/2}\cos 8x\,dx = \frac{1}{8}\left[-x\cos 8x + \frac{1}{8}\sin 8x\right]\Big|_0^{\pi/2} = \frac{1}{8}\left[-\frac{\pi}{2} - 0\right] = -\frac{\pi}{16}$$

39. $\int_0^2 xe^{-(1/2)x^2}\,dx$

Let $u = -\frac{1}{2}x^2,\ du = -x\,dx$. When $x = 0$, $u = 0$; when $x = 2$, $u = -2$.

$$\int_0^2 xe^{-(1/2)x^2}\,dx = -\int_0^{-2} e^u\,du = -e^u\Big|_0^{-2} = 1 - e^{-2}$$

40. $\int_{1/2}^1 \frac{\ln(2x+3)}{2x+3}\,dx$

Let $u = \ln(2x + 3),\ du = \frac{2}{2x+3}dx$. When $x = \frac{1}{2},\ u = \ln(4)$, when $x = 1$, $u = \ln 5$.

$$\int_{1/2}^1 \frac{\ln(2x+3)}{2x+3}\,dx = \frac{1}{2}\int_{\ln 4}^{\ln 5} u\,du = \frac{u^2}{4}\Bigg|_{\ln 4}^{\ln 5} = \frac{1}{4}\left[(\ln 5)^2 - (\ln 4)^2\right]$$

41. $\int_1^2 xe^{-2x}dx$

Use integration by parts with $f(x) = x,\ g(x) = e^{-2x}$. Then $f'(x) = 1,\ G(x) = -\frac{1}{2}e^{-2x}$

$$\int_1^2 xe^{-2x}dx = -\frac{1}{2}xe^{-2x}\Big|_1^2 + \frac{1}{2}\int_1^2 e^{-2x}dx = \left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)\Big|_1^2 = -\frac{1}{2}\left[e^{-2x}\left(x+\frac{1}{2}\right)\Big|_1^2\right]$$
$$= -\frac{1}{2}\left[\frac{5}{2}e^{-4} - \frac{3}{2}e^{-2}\right] = \frac{3}{4}e^{-2} - \frac{5}{4}e^{-4}$$

42. $\int_1^2 x^{-3/2}\ln x\,dx$

Use integration by parts with $f(x) = \ln x,\ g(x) = x^{-3/2}$. Then $f'(x) = \frac{1}{x},\ G(x) = -2x^{-1/2}$ and

$$\int_1^2 x^{-3/2}\ln x\,dx = -2x^{-1/2}\ln x\Big|_1^2 + \int_1^2 2x^{-3/2}dx = \left(-2x^{-1/2}\ln x - 4x^{-1/2}\right)\Big|_1^2 = -2\left[x^{-1/2}(\ln x + 2)\Big|_1^2\right]$$
$$= -2\left[\frac{\ln 2 + 2}{\sqrt{2}} - 2\right] = -\sqrt{2}\ln 2 - 2\sqrt{2} + 4$$

43. $\int_1^9 \frac{1}{\sqrt{x}}dx$; $n=4$, $\Delta x = \frac{(9-1)}{4} = 2$

Midpoint rule: $x_1 = 1 + \frac{2}{2} = 2,\ x_2 = 4,\ x_3 = 6,\ x_4 = 8$

$$\int_1^9 \frac{1}{\sqrt{x}}dx \approx \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}}\right](2) \approx 3.93782$$

Trapezoidal rule: $a_0 = 1,\ a_1 = 3,\ a_2 = 5,\ a_3 = 7,\ a_4 = 9$

$$\int_1^9 \frac{1}{\sqrt{x}}dx \approx \left[\frac{1}{\sqrt{1}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{7}} + \frac{1}{\sqrt{9}}\right]\frac{2}{2} \approx 4.13839$$

Simpson's rule: $\int_1^9 \frac{1}{\sqrt{x}}dx \approx \frac{2(3.93782) + 4.13839}{3} \approx 4.00468$

44. $\int_0^{10} e^{\sqrt{x}}dx$; $n=5$, $\Delta x = \frac{(10-0)}{5} = 2$

Midpoint rule: $x_1 = 1,\ x_2 = 3,\ x_3 = 5,\ x_4 = 7,\ x_5 = 9$

$$\int_0^{10} e^{\sqrt{x}}dx \approx \left[e^{\sqrt{1}} + e^{\sqrt{3}} + e^{\sqrt{7}} + e^{\sqrt{9}}\right]2 \approx 103.81310$$

Trapezoidal rule: $a_0 = 0,\ a_1 = 2,\ a_2 = 4,\ a_3 = 6,\ a_4 = 8,\ a_5 = 10$

$$\int_0^{10} e^{\sqrt{x}}dx \approx \left[e^{\sqrt{0}} + 2e^{\sqrt{2}} + 2e^{\sqrt{4}} + 2e^{\sqrt{6}} + 2e^{\sqrt{8}} + e^{\sqrt{10}}\right]\frac{2}{2} \approx 104.63148$$

Simpson's rule: $\int_0^{10} e^{\sqrt{x}}dx \approx \frac{2(103.8130) + 104.63148}{3} \approx 104.08589$

45. $\int_1^4 \frac{e^x}{x+1}dx$; $n=5$, $\Delta x = \frac{3}{5} = 0.6$

Midpoint rule: $x_1 = 1.3,\ x_2 = 1.9,\ x_3 = 2.5,\ x_4 = 3.1,\ x_5 = 3.7$

$$\int_1^4 \frac{e^x}{x+1}dx \approx \left[\frac{e^{1.3}}{2.3} + \frac{e^{1.9}}{2.9} + \frac{e^{2.5}}{3.5} + \frac{e^{3.1}}{4.1} + \frac{e^{3.7}}{4.7}\right](0.6) \approx 12.84089$$

Trapezoidal rule: $a_0 = 1,\ a_1 = 1.6,\ a_2 = 2.2,\ a_3 = 2.8,\ a_4 = 3.4,\ a_5 = 4$

$$\int_1^4 \frac{e^x}{x+1}dx \approx \left[\frac{e}{2} + \frac{2e^{1.6}}{2.6} + \frac{2e^{2.2}}{3.2} + \frac{2e^{2.8}}{3.8} + \frac{2e^{3.4}}{4.4} + \frac{e^4}{5}\right](0.3) \approx 13.20137$$

Simpson's rule: $\int_1^4 \frac{e^x}{x+1}dx \approx \frac{2(12.84089) + 13.20137}{3} \approx 12.96105$

46. $\int_{-1}^1 \frac{1}{1+x^2}dx$; $n=5$, $\Delta x = \frac{2}{5} = 0.4$

Midpoint rule: $x_1 = -0.8,\ x_2 = -0.4,\ x_3 = 0,\ x_4 = 0.4,\ x_5 = 0.8$

$$\int_{-1}^1 \frac{1}{1+x^2}dx \approx \left[\frac{1}{1+(-0.8)^2} + \frac{1}{1+(-0.4)^2} + \frac{1}{1+0^2} + \frac{1}{1+(0.4)^2} + \frac{1}{1+(0.8)^2}\right](0.4) \approx 1.57746$$

Trapezoidal rule: $a_0 = -1,\ a_1 = -0.6,\ a_2 = -0.2,\ a_3 = 0.2,\ a_4 = 0.6,\ a_5 = 1$

$$\int_{-1}^1 \frac{1}{1+x^2}dx \approx \left[\frac{1}{1+(-1)^2} + \frac{2}{1+(-0.6)^2} + \frac{2}{1+(-0.2)^2} + \frac{2}{1+(0.2)^2} + \frac{2}{1+(0.6)^2} + \frac{1}{1+1^2}\right](0.2) \approx 1.55747$$

Simpson's rule:

$$\int_{-1}^1 \frac{1}{1+x^2}dx \approx \frac{2(1.57746) + 1.55747}{3} \approx 1.57080$$

47. $\int_0^\infty e^{6-3x}dx = \lim_{b\to\infty}\int_0^b e^{6-3x}dx$

Let $u = 6 - 3x$, $du = -3dx$. When $x = 0$, $u = 6$; when $x = b$, $u = 6 - 3b$.

$$\int_0^b e^{6-3x}dx = -\frac{1}{3}\int_6^{6-3b} e^u du = -\frac{1}{3}e^u\Big|_6^{6-3b} = -\frac{1}{3}e^{6-3b} + \frac{1}{3}e^6$$

Thus $\int_0^\infty e^{6-3x}dx = \lim_{b\to\infty}\left[-\frac{1}{3}e^{6-3b} + \frac{e^6}{3}\right] = \frac{e^6}{3}.$

48. $\int_1^\infty x^{-2/3}dx = \lim_{b\to\infty}\int_1^b x^{-2/3}dx$

$$= \lim_{b\to\infty}\left[3x^{1/3}\Big|_1^b\right] = \lim_{b\to\infty}[3b^{1/3} - 3]$$

As $b\to\infty$, $3b^{1/3}$ increases without bound.

Thus, $\int_1^\infty x^{-2/3}dx$ diverges.

49. $\int_1^\infty \frac{x+2}{x^2+4x-2}dx = \lim_{b\to\infty}\int_1^b \frac{x+2}{x^2+4x-2}dx$

Let $u = x^2 + 4x - 2$, $du = (2x + 4)dx = 2(x + 2)dx$. When $x = 1$, $u = 3$; when $x = b$, $u = b^2 + 4b - 2$.

$$\int_1^b \frac{x+2}{x^2+4x-2}dx = \frac{1}{2}\int_3^{b^2+4b-2}\frac{du}{u} = \frac{1}{2}\ln|u|\Big|_3^{b^2+4b-2} = \frac{1}{2}\left[\ln\left|b^2+4b-2\right| - \ln 3\right]$$

As $b\to\infty$, $\ln\left|b^2+4b-2\right|$ increases without bound. Thus $\int_1^\infty \frac{x+2}{x^2+4x-2}dx$ diverges.

50. $\int_0^\infty x^2e^{-x^3}dx = \lim_{b\to\infty}\int_0^b x^2e^{-x^3}dx$

Let $u = -x^3$, then $du = -3x^2dx$.

When $x = 0$, $u = 0$; when $x = b$, $u = -b^3$.

$$\int_0^b x^2e^{-x^3}dx = -\frac{1}{3}\int_0^{-b^3} e^u du = -\frac{1}{3}\left[e^{-b^3} - 1\right] = \frac{1}{3} - \frac{e^{-b^3}}{3}$$

Thus $\int_0^\infty x^2e^{-x^3}dx = \lim_{b\to\infty}\left[\frac{1}{3} - \frac{e^{-b^3}}{3}\right] = \frac{1}{3}.$

51. $\int_{-1}^\infty (x+3)^{-5/4}dx = \lim_{b\to\infty}\int_{-1}^b (x+3)^{-5/4}dx$

$$= \lim_{b\to\infty}\left[-4(x+3)^{-1/4}\Big|_{-1}^b\right] = \lim_{b\to\infty}\left[2^{7/4} - 4(b+3)^{-1/4}\right] = 2^{7/4}$$

52. $\int_{-\infty}^0 \frac{8}{(5-2x)^3}dx = \lim_{b\to-\infty}\int_b^0 \frac{8}{(5-2x)^3}dx$

$$= \lim_{b\to-\infty}\left[2(5-2x)^{-2}\Big|_b^0\right] = \lim_{b\to-\infty}\left[\frac{2}{25} - 2(5-2b)^{-2}\right] = \frac{2}{25}$$

53. $\int_1^\infty xe^{-x}dx = \lim_{b\to\infty}\int_1^b xe^{-x}dx$

Using integration by parts with $f(x) = x$, $g(x) = e^{-x}$; $f'(x) = 1$, $G(x) = -e^{-x}$ gives

$$\int_1^b xe^{-x}dx = -xe^{-x}\Big|_1^b + \int_1^b e^{-x}dx = (-xe^{-x} - e^{-x})\Big|_1^b = 2e^{-1} - be^{-b} - e^{-b}$$

Thus

$$\int_1^\infty xe^{-x}dx = \lim_{b\to\infty}\left[\frac{2}{e} - be^{-b} - e^{-b}\right] = \frac{2}{e}$$

$\left(\text{since } \lim_{b\to\infty} be^{-b} = 0\right)$.

54. $\int_0^\infty xe^{-kx}dx = \lim_{b\to\infty}\int_0^b xe^{-kx}dx$

Using integration by parts with $f(x) = x$, $g(x) = e^{-kx}$; $f'(x) = 1$, $G(x) = -\frac{1}{k}e^{-kx}$, we have

$$\int_0^b xe^{-kx}dx = \frac{-x}{k}e^{-kx}\Big|_0^b + \frac{1}{k}\int_0^b e^{-kx}dx = \left(-\frac{x}{k}e^{-kx} - \frac{1}{k^2}e^{-kx}\right)\Big|_0^b = \frac{1}{k^2} - \frac{b}{k}e^{-kb} - \frac{1}{k^2}e^{-kb}$$

Thus

$$\int_0^\infty xe^{-kx}dx = \lim_{b\to\infty}\left[\frac{1}{k}\left(\frac{1}{k} - be^{-kb} - \frac{1}{k}e^{-kb}\right)\right] = \frac{1}{k^2}\left(\text{since } \lim_{b\to\infty} be^{-kb} = 0\right).$$

55. The present value is

$$\int_0^4 50e^{-0.08t}e^{-0.12t}\,dt = \int_0^4 50e^{-0.2t}\,dt$$
$$= -250e^{-0.2t}\Big|_0^4$$
$$= 250 - 250e^{-0.8}$$
$$\approx \$137{,}668$$

56. Using the method of Example 3, section 9.5, the total tax revenue is

$$\int_0^{10} (2\pi t)50e^{-t/20}\,dt$$
$$= 100\pi\int_0^{10} te^{-t/20}\,dt$$
$$= 100\pi\left[-20te^{-t/20}\Big|_0^{10} + \int_0^{10} 20e^{-t/20}\,dt\right]$$
$$= 100\pi\left[-20te^{-t/20} - 400e^{-t/20}\Big|_0^{10}\right]$$
$$= 100\pi[400 - e^{-0.5}(600)]$$
$$\approx \$11{,}335 \text{ thousand}$$

57. **a.** $M(t_1)\Delta t + \cdots + M(t_n)\Delta t \approx \int_0^2 M(t)\,dt$

b. $M(t_1)e^{-0.1t_1}\Delta t + \cdots + M(t_n)e^{-0.1t_n}\Delta t$
$$\approx \int_0^2 M(t)e^{-0.1t}\,dt$$

58. $80{,}000 + \int_0^\infty 50{,}000e^{-rt}\,dt$