Chapter 6

Functions of Several Variables

Exercise Set 6.1

1. $f(x,y)=x^2-3xy$

$f(0,-2)=(0)^2-3(0)(-2)$ Substituting 0 for x and -2 for y.

$=0-0$

$=0$

$f(2,3)=(2)^2-3(2)(3)$ Substituting 2 for x and 3 for y.

$=4-18$

$=-14$

$f(10,-5)=(10)^2-3(10)(-5)$ Substituting 10 for x and -5 for y.

$=100+150$

$=250$

2. $f(x,y)=\left(y^2+2xy\right)^3$

$f(-2,0)=\left((0)^2+2(-2)(0)\right)^3$

$=(0+0)^3=0$

$f(3,2)=\left((2)^2+2(3)(2)\right)^3$

$=(4+12)^3$

$=4096$

$f(-5,10)=\left((10)^2+2(-5)(10)\right)^3$

$=(100-100)^3$

$=0$

3. $f(x,y)=3^x+7xy$

$f(0,-2)=3^0+7(0)(-2)$ Substituting 0 for x and -2 for y.

$=1+0$

$=1$

$f(-2,1)=3^{-2}+7(-2)(1)$ Substituting -2 for x and 1 for y.

$=\dfrac{1}{9}-14$

$=-\dfrac{125}{9}$

$f(2,1)=3^2+7(2)(1)$ Substituting 2 for x and 1 for y.

$=9+14$

$=23$

4. $f(x,y)=\log_{10}(x+y)+3x^2$

$f(3,7)=\log_{10}(3+7)+3(3)^2$

$=\log_{10}(10)+27$

$=1+27$

$=28$

$f(1,99)=\log_{10}(1+99)+3(1)^2$

$=\log_{10}(100)+3$

$=2+3$

$=5$

$f(2,-1)=\log_{10}(2+(-1))+3(2)^2$

$=\log_{10}(1)+12$

$=0+12$

$=12$

5. $f(x,y)=\ln x+y^3$

$f(e,2)=\ln e+(2)^3$ Substituting e for x and 2 for y.

$=1+8$

$=9$

$f(e^2,4)=\ln e^2+(4)^3$ Substituting e^2 for x and 4 for y.

$=2+64$

$=66$

$f(e^3,4)=\ln e^3+(5)^3$ Substituting e^3 for x and 5 for y.

$=3+125$

$=128$

6. $f(x)=2^x-3^y$

$f(0,2)=2^0-3^2$

$=1-9$

$=-8$

$f(3,1)=2^3-3^1$

$=8-3$

$=5$

$f(2,3)=2^2-3^3$

$=4-27$

$=-23$

7. $f(x,y,z)=x^2-y^2+z^2$

We substitute -1 for x, 2 for y, and 3 for z.

$$\begin{aligned} f(-1,2,3) &= (-1)^2-(2)^2+(3)^2 \\ &= 1-4+9 \\ &= 6 \end{aligned}$$

We substitute 2 for x, -1 for y, and 3 for z.

$$\begin{aligned} f(2,-1,3) &= (2)^2-(-1)^2+(3)^2 \\ &= 4-1+9 \\ &= 12 \end{aligned}$$

8. $f(x,y,z)=2^x+5zy-x$

$$\begin{aligned} f(0,1,-3) &= 2^0+5(-3)(1)-(0) \\ &= 1-15-0 \\ &= -14 \end{aligned}$$

$$\begin{aligned} f(1,0,-3) &= 2^1+5(-3)(0)-(1) \\ &= 2+0-1 \\ &= 1 \end{aligned}$$

9. $R(P,E)=\dfrac{P}{E}$

Substituting 32.03 for P, and 1.25 for E, gives

$$\begin{aligned} R(32.03,1.25) &= \frac{32.03}{1.25} \\ &\approx 25.624 \\ &\approx 25.62. \end{aligned}$$

The price-earnings ration for Hewlett-Packard was 25.62.

10. $Y(D,P)=\dfrac{D}{P}$

$Y(0.12,30)=\dfrac{0.12}{30}\approx 0.0040$

Using percent notation $0.0040 = 0.40\%$.

11. From Example 3 we have $C_2=\left(\dfrac{V_2}{V_1}\right)^{0.6}C_1$.

Where C_1 is the cost of the original piece of equipment, V_1 is the capacity of the original piece of equipment, and V_2 is the capacity of the new piece of equipment.

We substitute 100,000 for C_1, 80,000 for V_1 and 160,000 for V_2.

$$\begin{aligned} C_2 &= \left(\frac{160{,}000}{80{,}000}\right)^{0.6}(100{,}000) \\ &= (2)^{0.6}(100{,}000) \\ &= 151{,}571.6567 \end{aligned}$$

We estimate the cost of the new tank to be \$151,571.66.

12. $V(L,p,R,r,v)=\dfrac{p}{4Lv}\left(R^2-r^2\right)$

$$\begin{aligned} &V(1,100,0.0075,0.0025,0.05) \\ &= \frac{(100)}{4(1)(0.05)}\left((0.0075)^2-(0.0025)^2\right) \\ &= \frac{100}{0.2}(0.00005625-0.00000625) \\ &= 0.025 \end{aligned}$$

13. $S(a,d,V)=\dfrac{aV}{0.51d^2}$

Substituting 100 for d, 1,600,000 for V, and 0.78 for a, we have:

$$\begin{aligned} S(0.78,100,1{,}600{,}000) &= \frac{(0.78)(1{,}600{,}000)}{0.51(100)^2} \\ &= \frac{1{,}248{,}000}{5100} \\ &\approx 244.70588 \end{aligned}$$

The approximate wind speed 100 ft from the center of the tornado is 244.7 miles per hour.

14. $S(h,w)=\dfrac{\sqrt{hw}}{60}$

$$\begin{aligned} S(165,80) &= \frac{\sqrt{(165)(80)}}{60} \\ &= \frac{\sqrt{13{,}200}}{60} \\ &\approx 1.9149 \end{aligned}$$

The approximate surface area for the person is 1.915 square meters.

15. $S(h,w)=0.024265h^{0.3964}w^{0.5378}$

Substituting 165 for h and 80 for w, we have:

$$\begin{aligned} S(165,80) &= 0.024265(165)^{0.3964}(80)^{0.5378} \\ &\approx 0.024265(7.56851)(10.55557) \\ &\approx 1.93852 \end{aligned}$$

The approximate surface area for the person is 1.939 square meters.

16. $Q(m,c) = 100 \cdot \dfrac{m}{c}$

$$Q(21,20) = 100 \cdot \frac{21}{20} = 5 \cdot 21 = 105$$

$$Q(19,20) = 100 \cdot \frac{19}{20} = 5 \cdot 19 = 95$$

17. Using the formula $S(a,d,V) = \dfrac{aV}{0.51d^2}$, we remember from problem 13, that $a = 0.78,\ V = 1{,}600{,}000$. We substitute 200 for *S*, and solve for *d*.

$$200 = \frac{(0.78)(1{,}600{,}000)}{0.51 \cdot d^2}$$

$$200 = \frac{1{,}248{,}000}{0.51 \cdot d^2}$$

$$102d^2 = 1{,}248{,}000$$

$$d^2 = \frac{1{,}248{,}000}{102}$$

$$d^2 \approx 12{,}235.29$$

$$d \approx \pm\sqrt{12{,}235.29} \quad \text{Taking the square root of both sides.}$$

$$d \approx 110.6132 \quad d \text{ Must be positive.}$$

The measurement was taken approximately 110.6 feet from the center of the tornado.

18. A person weight drops by 19% means that $w_N = w(1-0.19) = w(0.81) = 0.81w$

Giving the new surface area of

$S_N(h, w_N) = \dfrac{\sqrt{hw_N}}{60}$ Writing the new surface area as a function of the original weight w gives us: $S_N(h, 0.81w) = \dfrac{\sqrt{h \cdot 0.81w}}{60} = \dfrac{\sqrt{0.81hw}}{60}$

The percentage change from the original surface area is now.

$$\frac{S_N - S}{S} = \frac{\frac{\sqrt{0.81hw}}{60} - \frac{\sqrt{hw}}{60}}{\frac{\sqrt{hw}}{60}}$$

$$= \frac{\frac{\sqrt{0.81hw} - \sqrt{hw}}{60}}{\frac{\sqrt{hw}}{60}}$$

$$= \frac{\sqrt{0.81hw} - \sqrt{hw}}{\sqrt{hw}}$$

$$= \frac{0.9\sqrt{hw} - \sqrt{hw}}{\sqrt{hw}}$$

$$= \frac{(0.9-1)\sqrt{hw}}{\sqrt{hw}}$$

$$= 0.9 - 1$$

$$= -0.1$$

Therefore, the percentage decrease in the person's surface area resulting from a 19% decrease in body weight is about 10%.

19. $\boxed{tw}$ The output of a function of two variables depends on two inputs; where as the output of a function of one variable depends only on the input of a single variable. The domain of a function of two variables is a set of pairs of numbers, the domain of a function of one variable is a set of numbers.

20. $\boxed{tw}$ Answers will vary, but might include many examples from geometry such as $A = \dfrac{1}{2}bh$, $V = \pi r^2 h$. The decision of whether to fly or drive to another city depends on several variables such as the distance involved, the amount of time available, and the cost of each method of transportation. The demand for a good or service is depends on variables such as the price of the good or service, the price of substitute and complementary goods, the income of the consumer and the expectation of future prices, to name a few.

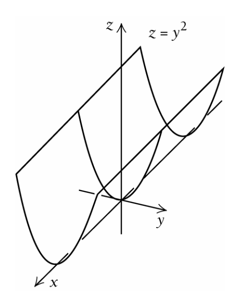
21. $W(v,T) =$
$$91.4 - \frac{\left(10.45 + 6.68\sqrt{v} - 0.447v\right)(457 - 5T)}{110}$$
$W(25,30)$
$$= 91.4 - \frac{\left(10.45 + 6.68\sqrt{25} - 0.447 \cdot 25\right)(457 - 5 \cdot 30)}{110}$$
$$= 91.4 - \frac{(10.45 + 33.4 - 11.175)(307)}{110}$$
$$\approx 91.4 - \frac{10,031.225}{110}$$
$$\approx 91.4 - 91.2$$
$$\approx 0.2$$
The wind chill, rounded to the nearest degree is $0°F$.

22. $W(v,T) =$
$$91.4 - \frac{\left(10.45 + 6.68\sqrt{v} - 0.447v\right)(457 - 5T)}{110}$$
$W(20,20)$
$$= 91.4 - \frac{\left(10.45 + 6.68\sqrt{20} - 0.447 \cdot 20\right)(457 - 5 \cdot 20)}{110}$$
$$\approx 91.4 - \frac{(10.45 + 29.874 - 8.94)(457 - 100)}{110}$$
$$\approx 91.4 - \frac{(31.384)(357)}{110}$$
$$\approx 91.4 - 101.855$$
$$\approx -10.455$$
The wind chill, rounded to the nearest degree is $-10°F$.

23. $W(v,T) =$
$$91.4 - \frac{\left(10.45 + 6.68\sqrt{v} - 0.447v\right)(457 - 5T)}{110}$$
$W(40,20)$
$$= 91.4 - \frac{\left(10.45 + 6.68\sqrt{40} - 0.447 \cdot 40\right)(457 - 5 \cdot 20)}{110}$$
$$\approx 91.4 - \frac{(10.45 + 42.248 - 17.88)(357)}{110}$$
$$\approx 91.4 - \frac{(34.818)(357)}{110}$$
$$\approx 91.4 - \frac{12,430.026}{110}$$
$$\approx 91.4 - 113.0$$
$$\approx -21.6$$
The wind chill, rounded to the nearest degree is $-22°F$.

24. $W(v,T) =$
$$91.4 - \frac{\left(10.45 + 6.68\sqrt{v} - 0.447v\right)(457 - 5T)}{110}$$
$W(30,-10)$
$$= 91.4 - \frac{\left(10.45 + 6.68\sqrt{30} - 0.447 \cdot 30\right)(457 - 5 \cdot (-10))}{110}$$
$$\approx 91.4 - \frac{(10.45 + 36.588 - 13.41)(457 + 50)}{110}$$
$$\approx 91.4 - \frac{(33.628)(507)}{110}$$
$$\approx 91.4 - 155.0$$
$$\approx -63.6$$
The wind chill, rounded to the nearest degree is $-64°F$.

25.



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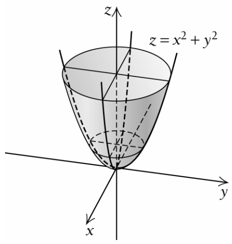

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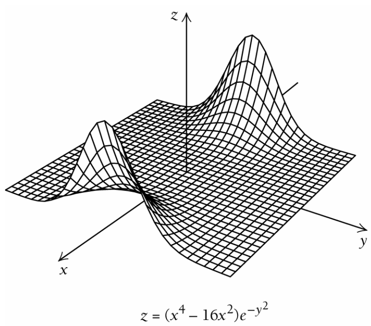

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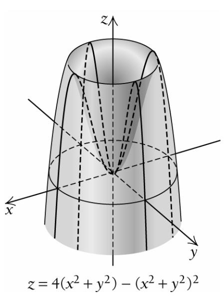

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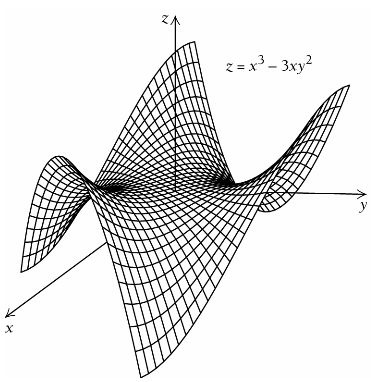

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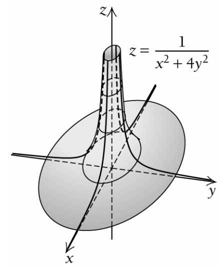


Exercise Set 6.2

1. $z = 2x - 3y$

Find $\frac{\partial z}{\partial x}$.

$z = 2\underline{x} - 3y$ The variable is underlined; y is treated as a constant.

$$\frac{\partial z}{\partial x} = 2$$

Find $\frac{\partial z}{\partial y}$.

$z = 2x - 3\underline{y}$ The variable is underlined; x is treated as a constant.

$$\frac{\partial z}{\partial y} = -3$$

Find $\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)}$

$\frac{\partial z}{\partial x} = 2$ The partial derivative is constant for all values of x and y. Therefore:

$$\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)} = 2$$

Find $\left.\frac{\partial z}{\partial y}\right|_{(0,-5)}$

$\frac{\partial z}{\partial x} = -3$ The partial derivative is constant for all values of x and y. Therefore:

$$\left.\frac{\partial z}{\partial y}\right|_{(0,-5)} = -3$$

2. $z = 7x - 5y$

$$\frac{\partial z}{\partial x} = 7$$

$$\frac{\partial z}{\partial y} = -5$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)} = 7$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,-5)} = -5$$

3. $z = 3x^2 - 2xy + y$

Find $\frac{\partial z}{\partial x}$.

$z = 3\underline{x}^2 - 2\underline{x}y + y$ The variable is underlined; y is treated as a constant.

$$\frac{\partial z}{\partial x} = 6x - 2y$$

Find $\frac{\partial z}{\partial y}$.

$z = 3x^2 - 2x\underline{y} + \underline{y}$ The variable is underlined; x is treated as a constant.

$$\frac{\partial z}{\partial x} = -2x + 1$$

Find $\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)}$

$$\frac{\partial z}{\partial x} = 6x - 2y$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)} = 6(-2) - 2(-3)$$ Substituting -2 for x and -3 for y.

$$= -12 + 6$$
$$= -6$$

Find $\left.\frac{\partial z}{\partial y}\right|_{(0,-5)}$

$$\frac{\partial z}{\partial y} = -2x + 1$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,-5)} = -2(0) + 1$$ Substituting 0 for x.

$$= 1$$

4. $z = 2x^3 + 3xy - x$

$$\frac{\partial z}{\partial x} = 6x^2 + 3y - 1$$

$$\frac{\partial z}{\partial y} = 3x$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-2,-3)} = 6(-2)^2 + 3(-3) - 1$$
$$= 24 - 9 - 1 = 14$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,-5)} = 3 \cdot 0 = 0$$

5. $f(x, y) = 2x - 5xy$

Find f_x.

$f(x, y) = 2\underline{x} - 5\underline{x}y$ The variable is underlined; y is treated as a constant.

$$f_x = 2 - 5y$$

Find f_y.

$f(x, y) = 2x - 5x\underline{y}$ The variable is underlined; x is treated as a constant.

$$f_y = -5x$$

Find $f_x(-2, 4)$.

$$f_x = 2 - 5y$$

$$f_x(-2, 4) = 2 - 5(4)$$ Substituting 4 for y.

$$= 2 - 20$$
$$= -18$$

Find $f_y(4, -3)$.

$$f_y = -5x$$

$$f_y(4, -3) = -5(4)$$ Substituting 4 for x.

$$= -20$$

6. $f(x, y) = 5x + 7y$

$$f_x = 5$$
$$f_y = 7$$
$$f_x(-2, 4) = 5$$
$$f_y(4, -3) = 7$$

7. $f(x, y) = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$

Find f_x.

$f(x, y) = (\underline{x}^2 + y^2)^{1/2}$ The variable is underlined; y is treated as a constant.

$$f_x = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x$$

$$= x(x^2 + y^2)^{-1/2}, \text{ or } \frac{x}{\sqrt{x^2 + y^2}}$$

Find f_y.

$$f(x,y) = \left(x^2 + \underline{y}^2\right)^{1/2}$$ The variable is underlined; x is treated as a constant.

$$f_y = \frac{1}{2}\left(x^2+y^2\right)^{-1/2} \cdot 2y$$

$$= y\left(x^2+y^2\right)^{-1/2}, \text{ or } \frac{y}{\sqrt{x^2+y^2}}$$

Find $f_x(-2,1)$.

$$f_x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_x(-2,1) = \frac{(-2)}{\sqrt{(-2)^2+(1)^2}}$$ Substituting -2 for x and 1 for y.

$$= \frac{-2}{\sqrt{4+1}}$$

$$= \frac{-2}{\sqrt{5}}$$

Find $f_y(-3,-2)$.

$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_y(-3,-2) = \frac{(-2)}{\sqrt{(-3)^2+(-2)^2}}$$ Substituting -3 for x and -2 for y.

$$= \frac{-2}{\sqrt{9+4}}$$

$$= \frac{-2}{\sqrt{13}}$$

8. $f(x,y) = \sqrt{x^2-y^2} = \left(x^2-y^2\right)^{1/2}$

$$f(x,y) = \left(\underline{x}^2 - y^2\right)^{1/2}$$ The variable is underlined; y is treated as a constant.

$$f_x = \frac{1}{2}\left(x^2-y^2\right)^{-1/2} \cdot 2x$$

$$= x\left(x^2-y^2\right)^{-1/2}, \text{ or } \frac{x}{\sqrt{x^2-y^2}}$$

$$f(x,y) = \left(x^2 - \underline{y}^2\right)^{1/2}$$ The variable is underlined; x is treated as a constant.

$$f_y = \frac{1}{2}\left(x^2-y^2\right)^{-1/2} \cdot (-2y)$$

$$= -y\left(x^2-y^2\right)^{-1/2}, \text{ or } \frac{-y}{\sqrt{x^2-y^2}}$$

$$f_x(-2,1) = \frac{(-2)}{\sqrt{(-2)^2-(1)^2}}$$

$$= \frac{-2}{\sqrt{4-1}}$$

$$= \frac{-2}{\sqrt{3}}$$

$$f_y = \frac{-y}{\sqrt{x^2-y^2}}$$

$$f_y(-3,-2) = \frac{-(-2)}{\sqrt{(-3)^2-(-2)^2}}$$ Substituting -3 for x and -2 for y.

$$= \frac{2}{\sqrt{9-4}}$$

$$= \frac{2}{\sqrt{5}}$$

9. $f(x,y) = e^{2x-y}$

Find f_x.

$$f(x,y) = e^{2\underline{x}-y}$$ The variable is underlined; y is treated as a constant.

$$f_x = e^{2x-y} \cdot (2)$$

$$= 2e^{2x-y}$$

Find f_y.

$$f(x,y) = e^{2x-\underline{y}}$$ The variable is underlined; x is treated as a constant.

$$f_y = e^{2x-y} \cdot (-1)$$

$$= -e^{2x-y}$$

10. $f(x,y) = e^{3x-2y}$

$f_x = 3e^{3x-2y}$

$f_y = -2e^{3x-2y}$

11. $f(x,y) = e^{xy}$

Find f_x.

$$f(x,y) = e^{\underline{x}y}$$ The variable is underlined; y is treated as a constant.

$$f_x = e^{xy} \cdot (y)$$

$$= ye^{xy}$$

Find f_y.

$$f(x,y) = e^{x\underline{y}}$$ The variable is underlined; x is treated as a constant.

$$f_y = e^{xy} \cdot (x)$$

$$= xe^{xy}$$

12. $f(x,y)=e^{2xy}$

$f_x=2ye^{2xy}$

$f_y=2xe^{2xy}$

13. $f(x,y)=y\ln(x+2y)$

Find f_x.

$f(x,y)=y\ln(\underline{x}+2y)$ The variable is underlined.

$$f_x=y\cdot\frac{1}{x+2y}\cdot 1=\frac{y}{x+2y}$$

Find f_y.

$f(x,y)=\underline{y}\ln(x+2\underline{y})$ The variable is underlined.

$$f_y=y\cdot\frac{1}{x+2y}\cdot 2+1\cdot\ln(x+2y)=\frac{2y}{x+2y}+\ln(x+2y)$$

14. $f(x,y)=x\ln(x-y)$

$$f_x=x\cdot\frac{1}{x-y}+\ln(x-y)=\frac{x}{x-y}+\ln(x-y)$$

$$f_y=x\cdot\frac{1}{x-y}\cdot(-1)=\frac{-x}{x-y}$$

15. $f(x,y)=x\ln(xy)$

Find f_x.

$f(x,y)=\underline{x}\ln(\underline{x}y)$ The variable is underlined.

$$f_x=x\cdot\left(\frac{1}{xy}\cdot y\right)+1\cdot\ln(xy)=1+\ln(xy)$$

Find f_y.

$f(x,y)=x\ln(x\underline{y})$ The variable is underlined.

$$f_y=x\cdot\left(\frac{1}{xy}\cdot x\right)=\frac{x}{y}$$

16. $f(x,y)=y\ln(xy)$

$$f_x=y\cdot\left(\frac{1}{xy}\cdot y\right)=\frac{y}{x}$$

$$f_y=y\cdot\left(\frac{1}{xy}\cdot x\right)+1\cdot\ln(xy)=1+\ln(xy)$$

17. $f(x,y)=\frac{x}{y}-\frac{y}{3x}$

Find f_x.

$f(x,y)=\frac{1}{y}\underline{x}-\frac{y}{3}\cdot\underline{x}^{-1}$ The variable is underlined.

$$f_x=\frac{1}{y}-\frac{y}{3}(-1x^{-2})=\frac{1}{y}+\frac{y}{3x^2}$$

Find f_y.

$f(x,y)=x\underline{y}^{-1}-\frac{1}{3x}\cdot\underline{y}$ The variable is underlined.

$$f_y=x(-1y^{-2})-\frac{1}{3x}\cdot 1=-\frac{x}{y^2}-\frac{1}{3x}$$

18. $f(x,y)=\frac{x}{y}+\frac{y}{5x}$

$$f_x=\frac{1}{y}\cdot 1+\frac{y}{5}(-1x^{-2})=\frac{1}{y}-\frac{y}{5x^2}$$

$$f_y=x(-1y^{-2})+\frac{1}{5x}\cdot 1=-\frac{x}{y^2}+\frac{1}{5x}$$

19. $f(x,y)=3(2x+y-5)^2$

Find f_x.

$f(x,y)=3(2\underline{x}+y-5)^2$ The variable is underlined.

$$f_x=3[2(2x+y-5)\cdot 2]=12(2x+y-5)$$

Find f_y.

$f(x,y)=3(2x+\underline{y}-5)^2$ The variable is underlined.

$f_y = 3[2(2x+y-5)\cdot 1]$

$= 6(2x+y-5)$

20. $f(x,y)=4(3x+y-8)^2$

$f_x = 4[2(3x+y-8)\cdot 3]$

$= 24(3x+y-8)$

$f_y = 4[2(3x+y-8)\cdot 1]$

$= 8(3x+y-8)$

21. $f(b,m)=$

$m^3+4m^2b-b^2+(2m+b-5)^2+(3m+b-6)^2$

Find $\frac{\partial f}{\partial b}$.

$f(b,m)=$

$m^3+4m^2\underline{b}-\underline{b}^2+(2m+\underline{b}-5)^2+(3m+\underline{b}-6)^2$

The variable is underlined.

$\frac{\partial f}{\partial b}=4m^2-2b+2(2m+b-5)\cdot 1+$

$2(3m+b-6)\cdot 1$

$=4m^2-2b+4m+2b-10+6m+2b-12$

$=4m^2+10m+2b-22$

Find $\frac{\partial f}{\partial m}$.

$f(b,m)=$

$\underline{m}^3+4\underline{m}^2b-b^2+(2\underline{m}+b-5)^2+(3\underline{m}+b-6)^2$

The variable is underlined.

$\frac{\partial f}{\partial m}=3m^2+8mb+2(2m+b-5)\cdot 2+$

$2(3m+b-6)\cdot 3$

$=3m^2+8mb+8m+4b-20+$

$18m+6b-36$

$=3m^2+8mb+26m+10b-56$

22. $f(b,m)=$

$5m^2-mb^2-3b+(2m+b-8)^2+(3m+b-9)^2$

$\frac{\partial f}{\partial b}=-2mb-3+2(2m+b-8)\cdot 1+$

$2(3m+b-9)\cdot 1$

$=-2mb-3+4m+2b-16+6m+2b-18$

$=-2mb+10m+4b-37$

$\frac{\partial f}{\partial m}=10m-b^2+2(2m+b-8)\cdot 2+$

$2(3m+b-9)\cdot 3$

$=10m-b^2+8m+4b-32+$

$18m+6b-54$

$=-b^2+36m+10b-86$

23. $f(x,y,\lambda)=5xy-\lambda(2x+y-8)$

Find f_x.

$f(x,y,\lambda)=5\underline{x}y-\lambda(2\underline{x}+y-8)$ The variable is underlined.

$f_x=5y-\lambda\cdot 2$

$=5y-2\lambda$

Find f_y.

$f(x,y,\lambda)=5x\underline{y}-\lambda(2x+\underline{y}-8)$ The variable is underlined.

$f_y=5x-\lambda\cdot 1$

$=5x-\lambda$

Find f_λ.

$f(x,y,\lambda)=5xy-\underline{\lambda}(2x+y-8)$ The variable is underlined.

$f_\lambda=-1\cdot(2x+y-8)$

$=-(2x+y-8)$

24. $f(x,y,\lambda)=9xy-\lambda(3x-y+7)$

$f_x=9y-\lambda(3)=9y-3\lambda$

$f_y=9x-\lambda(-1)=9x+\lambda$

$f_\lambda=-1(3x-y+7)=-(3x-y+7)$

25. $f(x,y,\lambda)=x^2+y^2-\lambda(10x+2y-4)$

Find f_x.

$f(x,y,\lambda)=\underline{x}^2+y^2-\lambda(10\underline{x}+2y-4)$

The variable is underlined.

$f_x=2x-\lambda\cdot 10$

$=2x-10\lambda$

Find f_y.

$f(x,y,\lambda)=x^2+\underline{y}^2-\lambda(10x+2\underline{y}-4)$

The variable is underlined.

$$\begin{aligned} f_y &= 2y-\lambda\cdot 2 \\ &= 2y-2\lambda \end{aligned}$$

Find f_λ.

$f(x,y,\lambda)=x^2+y^2-\underline{\lambda}(10x+2y-4)$

The variable is underlined.

$$\begin{aligned} f_\lambda &= -1(10x+2y-4) \\ &= -(10x+2y-4) \end{aligned}$$

26. $f(x,y,\lambda)=x^2-y^2-\lambda(4x-7y-10)$

$f_x = 2x-\lambda(4)=2x-4\lambda$

$f_y = -2y-\lambda(-7)=-2y+7\lambda$

$f_\lambda = -1(4x-7y-10)=-(4x-7y-10)$

27. $f(x,y)=5xy$

First, we find the partial derivatives.
We find f_x first.

$f(x,y)=5\underline{x}y$ The variable is underlined.

$f_x = 5y$

Then we find f_y.

$f(x,y)=5x\underline{y}$ The variable is underlined.

$f_y = 5x$

We find f_{xx} by taking the partial derivative with respect to *x* of f_x.

$$f_{xx}=\frac{\partial}{\partial x}(f_x)=\frac{\partial}{\partial x}(5y)=0$$

We find f_{xy} by taking the partial derivative with respect to *y* of f_x.

$$f_{xy}=\frac{\partial}{\partial y}(f_x)=\frac{\partial}{\partial y}(5y)=5$$

We find f_{yx} by taking the partial derivative with respect to *x* of f_y.

$$f_{yx}=\frac{\partial}{\partial x}(f_y)=\frac{\partial}{\partial x}(5x)=5$$

We find f_{yy} by taking the partial derivative with respect to *y* of f_y.

$$f_{yy}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}(5x)=0$$

28. $f(x,y)=2xy$

We find the first-order partial derivatives.

$f_x = 2y$

$f_y = 2x$

Then we find the second-order partial derivatives.

$$f_{xx}=\frac{\partial}{\partial x}(f_x)=\frac{\partial}{\partial x}(2y)=0$$

$$f_{xy}=\frac{d}{\partial y}(f_x)=\frac{\partial}{\partial y}(2y)=2$$

$$f_{yx}=\frac{\partial}{\partial x}(f_y)=\frac{\partial}{\partial x}(2x)=2$$

$$f_{yy}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}(2x)=0$$

29. $f(x,y)=7xy^2+5xy-2y$

First, we find the partial derivatives.
We find f_x first.

$f(x,y)=7\underline{x}y^2+5\underline{x}y-2y$ The variable is underlined.

$f_x = 7y^2+5y$

Then we find f_y.

$f(x,y)=7x\underline{y}^2+5x\underline{y}-2\underline{y}$ The variable is underlined.

$f_y = 14xy-5x-2$

We find f_{xx} by taking the partial derivative with respect to *x* of f_x.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}(7y^2+5y) \quad \text{y is treated as a constant.} \\ &= 0 \end{aligned}$$

We find f_{xy} by taking the partial derivative with respect to *y* of f_x.

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}(7\underline{y}^2+5\underline{y}) \quad \text{The variable is underlined.} \\ &= 14y+5 \end{aligned}$$

We find f_{yx} by taking the partial derivative with respect to *x* of f_y.

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) \\ &= \frac{\partial}{\partial x}(14\underline{x}y+5\underline{x}-2) \quad \text{The variable is underlined.} \\ &= 14y+5 \end{aligned}$$

We find f_{yy} by taking the partial derivative with respect to y of f_y.

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}(14x\underline{y} + 5x - 2) \quad \text{The variable is underlined.} \\ &= 14x \end{aligned}$$

30. $f(x,y) = 3x^2y - 2xy + 4y$

We find the first-order partial derivatives.

$f_x = 6xy - 2y$

$f_y = 3x^2 - 2x + 4$

Then we find the second-order partial derivatives.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{dx}(6xy - 2y) = 6y$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{dy}(6xy - 2y) = 6x - 2$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(3x^2 - 2x) = 6x - 2$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(3x^2 - 2x) = 0$$

31. $f(x,y) = x^5y^4 + x^3y^2$

First, we find the partial derivatives.

We find f_x first.

$$f(x,y) = \underline{x}^5y^4 + \underline{x}^3y^2 \quad \text{The variable is underlined.}$$

$$f_x = 5x^4y^4 + 3x^2y^2$$

Then we find f_y.

$$f(x,y) = x^5\underline{y}^4 + x^3\underline{y}^2 \quad \text{The variable is underlined.}$$

$$f_y = 4x^5y^3 + 2x^3y$$

We find f_{xx} by taking the partial derivative with respect to x of f_x.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}(5\underline{x}^4y^4 + 3\underline{x}^2y^2) \quad \text{The variable is underlined.} \\ &= 20x^3y^4 + 6xy^2 \end{aligned}$$

We find f_{xy} by taking the partial derivative with respect to y of f_x.

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}(5x^4\underline{y}^4 + 3x^2\underline{y}^2) \quad \text{The variable is underlined.} \\ &= 20x^4y^3 + 6x^2y \end{aligned}$$

We find f_{yx} by taking the partial derivative with respect to x of f_y.

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) \\ &= \frac{\partial}{\partial x}(4\underline{x}^5y^3 + 2\underline{x}^3y) \quad \text{The variable is underlined.} \\ &= 20x^4y^3 + 6x^2y \end{aligned}$$

We find f_{yy} by taking the partial derivative with respect to y of f_y.

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}(4x^5\underline{y}^3 + 2x^3\underline{y}) \quad \text{The variable is underlined.} \\ &= 12x^5y^2 + 2x^3 \end{aligned}$$

32. $f(x,y) = x^4y^3 - x^2y^3$

We find the first-order partial derivatives.

$f_x = 4x^3y^3 - 2xy^3$

$f_y = 3x^4y^2 - 3x^2y^2$

Then we find the second-order partial derivatives.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}(4x^3y^3 - 2xy^3) = 12x^2y^3 - 2y^3 \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}(4x^3y^3 - 2xy^3) = 12x^3y^2 - 6xy^2 \end{aligned}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) \\ &= \frac{\partial}{\partial x}(3x^4y^2 - 3x^2y^2) = 12x^3y^2 - 6xy^2 \end{aligned}$$

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}(3x^4y^2 - 3x^2y^2) = 6x^4y - 6x^2y \end{aligned}$$

33. $f(x,y)=2x-3y$

First, we find the partial derivatives.
We find f_x first.

$f(x,y)=2\underline{x}-3y$ The variable is underlined.

$f_x=2$

Then we find f_y.

$f(x,y)=2x-3\underline{y}$ The variable is underlined.

$f_y=-3$

We find f_{xx} by taking the partial derivative with respect to *x* of f_x.

$$f_{xx}=\frac{\partial}{\partial x}(f_x)=\frac{\partial}{\partial x}(2)=0$$

We find f_{xy} by taking the partial derivative with respect to *y* of f_x.

$$f_{xy}=\frac{\partial}{\partial y}(f_x)=\frac{\partial}{\partial y}(2)=0$$

We find f_{yx} by taking the partial derivative with respect to *x* of f_y.

$$f_{yx}=\frac{\partial}{\partial x}(f_y)=\frac{\partial}{\partial x}(-3)=0$$

We find f_{yy} by taking the partial derivative with respect to *y* of f_y.

$$f_{yy}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}(-3)=0$$

34. $f(x,y)=3x+5y$

We find the first-order partial derivatives.

$f_x=3$

$f_y=5$

Then we find the second-order partial derivatives.

$$f_{xx}=\frac{\partial}{dx}(f_x)=\frac{\partial}{dx}(3)=0$$

$$f_{xy}=\frac{\partial}{\partial y}(f_x)=\frac{\partial}{dy}(3)=0$$

$$f_{yx}=\frac{\partial}{\partial x}(f_y)=\frac{\partial}{\partial x}(5)=0$$

$$f_{yy}=\frac{\partial}{\partial y}(f_y)=\frac{\partial}{\partial y}(5)=0$$

35. $f(x,y)=e^{2xy}$

First, we find the partial derivatives.
We find f_x first.

$f(x,y)=e^{2\underline{x}y}$ The variable is underlined.

$f_x=2ye^{2xy}$

Then we find f_y.

$f(x,y)=e^{2x\underline{y}}$ The variable is underlined.

$f_y=2xe^{2xy}$

We find f_{xx} by taking the partial derivative with respect to *x* of f_x.

$$\begin{aligned}f_{xx}&=\frac{\partial}{\partial x}(f_x)\\&=\frac{\partial}{\partial x}\left(2ye^{2\underline{x}y}\right)\quad\text{The variable is underlined.}\\&=2y\left(2ye^{2xy}\right)\\&=4y^2e^{2xy}\end{aligned}$$

We find f_{xy} by taking the partial derivative with respect to *y* of f_x.

$$\begin{aligned}f_{xy}&=\frac{\partial}{\partial y}(f_x)\\&=\frac{\partial}{\partial y}\left(2\underline{y}e^{2x\underline{y}}\right)\quad\text{The variable is underlined.}\\&=2y\left(2xe^{2xy}\right)+2\left(e^{2xy}\right)\\&=4xye^{2xy}+2e^{2xy}\end{aligned}$$

We find f_{yx} by taking the partial derivative with respect to *x* of f_y.

$$\begin{aligned}f_{yx}&=\frac{\partial}{\partial x}(f_y)\\&=\frac{\partial}{\partial x}\left(2\underline{x}e^{2\underline{x}y}\right)\quad\text{The variable is underlined.}\\&=2x\left(2ye^{2xy}\right)+2\left(e^{2xy}\right)\\&=4xye^{2xy}+2e^{2xy}\end{aligned}$$

We find f_{yy} by taking the partial derivative with respect to *y* of f_y.

$$\begin{aligned}f_{yy}&=\frac{\partial}{\partial y}(f_y)\\&=\frac{\partial}{\partial y}\left(2xe^{2x\underline{y}}\right)\quad\text{The variable is underlined.}\\&=2x\left(2xe^{2xy}\right)\\&=4x^2e^{2xy}\end{aligned}$$

36. $f(x,y)=e^{xy}$

We find the first-order partial derivatives.

$f_x = ye^{xy}$

$f_y = xe^{xy}$

Then we find the second-order partial derivatives.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(ye^{xy}) = y^2e^{xy}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(ye^{xy}) = xye^{xy} + e^{xy}$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(xe^{xy}) = xye^{xy} + e^{xy}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(xe^{xy}) = x^2e^{xy}$$

37. $f(x,y)=x+e^y$

First, we find the partial derivatives.

We find f_x first.

$f(x,y)=\underline{x}+e^y$ The variable is underlined.

$f_x = 1$

Then we find f_y.

$f(x,y)=x+e^{\underline{y}}$ The variable is underlined.

$f_y = e^y$

We find f_{xx} by taking the partial derivative with respect to x of f_x.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(1) = 0$$

We find f_{xy} by taking the partial derivative with respect to y of f_x.

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(1) = 0$$

We find f_{yx} by taking the partial derivative with respect to x of f_y.

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(e^y) = 0$$

e^y is treated as a constant.

We find f_{yy} by taking the partial derivative with respect to y of f_y.

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(e^{\underline{y}}) = e^y$$

The variable is underlined.

38. $f(x,y)=y-e^x$

We find the first-order partial derivatives.

$f_x = -e^x$

$f_y = 1$

Then we find the second-order partial derivatives.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(-e^x) = -e^x$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(-e^x) = 0$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(1) = 0$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(1) = 0$$

39. $f(x,y)=y\ln x$

First, we find the partial derivatives.

We find f_x first.

$f(x,y)=y\ln \underline{x}$ The variable is underlined.

$$f_x = y\cdot\frac{1}{x} = \frac{y}{x}$$

Then we find f_y.

$f(x,y)=\underline{y}\ln x$ The variable is underlined.

$f_y = 1\cdot\ln x = \ln x$

We find f_{xx} by taking the partial derivative with respect to x of f_x.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}\left(\frac{y}{\underline{x}}\right) = \frac{\partial}{\partial x}\left(y\underline{x}^{-1}\right) \quad \text{The variable is underlined.} \\ &= -yx^{-2} \\ &= \frac{-y}{x^2} \end{aligned}$$

We find f_{xy} by taking the partial derivative with respect to y of f_x.

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}\left(\underline{y}\cdot\frac{1}{x}\right) \quad \text{The variable is underlined.} \\ &= \frac{1}{x} \end{aligned}$$

We find f_{yx} by taking the partial derivative with respect to x of f_y.

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) \\ &= \frac{\partial}{\partial x}(\ln \underline{x}) \quad \text{The variable is underlined.} \\ &= \frac{1}{x} \end{aligned}$$

We find f_{yy} by taking the partial derivative with respect to y of f_y.

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}(\ln x) \quad \ln x \text{ is treated as a constant.} \\ &= 0 \end{aligned}$$

40. $f(x,y) = x\ln y$

We find the first-order partial derivatives.

$$f_x = \ln y$$
$$f_y = \frac{x}{y}$$

Then we find the second-order partial derivatives.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(\ln y) = 0$$
$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(\ln y) = \frac{1}{y}$$
$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$$
$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-x}{y^2}$$

41. $p(x,y) = 2400x^{2/5}y^{3/5}$

a) Substituting 32 for x and 1024 for y, we have:

$$\begin{aligned} p(32,1024) &= 2400(32)^{2/5}(1024)^{3/5} \\ &= 2400(4)(64) \\ &= 614,400. \end{aligned}$$

Using 32 units of labor and 1024 units of capital, Lincolnville Sporting goods will produce 614,400 units.

b) We find the marginal productivity of labor by taking the partial derivative with respect to x.

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x}\left(2400\underline{x}^{2/5}y^{3/5}\right) \quad \text{The variable is underlined.} \\ &= 2400\left(\frac{2}{5}x^{-3/5}\right)y^{3/5} \\ &= 960x^{-3/5}y^{3/5} \\ &= 960\left(\frac{y}{x}\right)^{3/5} \end{aligned}$$

We find the marginal productivity of capital by taking the partial derivative with respect to y.

$$\begin{aligned} \frac{\partial p}{\partial y} &= \frac{\partial}{\partial y}\left(2400x^{2/5}\underline{y}^{3/5}\right) \quad \text{The variable is underlined.} \\ &= 2400x^{2/5}\left(\frac{3}{5}y^{-2/5}\right) \\ &= 1440x^{2/5}y^{-2/5} \\ &= 1440\left(\frac{x}{y}\right)^{2/5} \end{aligned}$$

c) Substituting 32 for x and 1024 for y into $\frac{\partial p}{\partial x}$, we have:

$$\left.\frac{\partial p}{\partial x}\right|_{(32,1024)} = 960\left(\frac{1024}{32}\right)^{3/5}$$
$$= 960(32)^{3/5}$$
$$= 960(8)$$
$$= 7680$$

The marginal productivity of labor when 32 units of labor and 1024 units of capital are currently being used is 7680 units per unit labor.

Substituting 32 for x and 1024 for y into $\frac{\partial p}{\partial y}$, we have:

$$\left.\frac{\partial p}{\partial y}\right|_{(32,1024)} = 1440\left(\frac{32}{1024}\right)^{2/5}$$
$$= 1440\left(\frac{1}{32}\right)^{2/5}$$
$$= 1440\left(\tfrac{1}{4}\right)$$
$$= 360$$

The marginal productivity of capital when 32 units of labor and 1024 units of capital are currently being used is 360 units per unit capital.

d) $\boxed{tw}$ The marginal productivity of labor, $\frac{\partial p}{\partial x}$, gives the rate of change of productivity with respect to labor. In other words, when 32 units of labor and 1024 units of capital are currently being used, employing one additional unit of labor to 33 units and keeping capital fixed will result in approximately 7680 additional units of production.

The marginal productivity of capital, $\frac{\partial p}{\partial y}$, gives the rate of change of productivity with respect to capital. In other words, when 32 units of labor and 1024 units of capital are currently being used, employing one additional unit of capital to 1025 units and keeping labor fixed will result in approximately 360 additional units of production.

42. $p(x, y) = 1800x^{0.621}y^{0.379}$

a) $p(2500,1700)$

$$= 1800(2500)^{0.621}(1700)^{0.379}$$
$$\approx 3,888,064$$

Using 2500 units of labor and 1700 units of capital, Riverside Appliances will produce about 3,888,064 units.

b) We find the marginal productivity of labor by taking the partial derivative with respect to x.

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(1800\underline{x}^{0.621}y^{0.379}\right)$$
$$= 1800\left(0.621x^{-0.379}\right)y^{0.379}$$
$$= 1117.8\left(\frac{y}{x}\right)^{0.379}$$

We find the marginal productivity of capital by taking the partial derivative with respect to y.

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(1800x^{0.621}\underline{y}^{0.379}\right)$$
$$= 1800x^{0.621}\left(0.379y^{-0.621}\right)$$
$$= 682.2\left(\frac{x}{y}\right)^{0.621}$$

c) $$\left.\frac{\partial p}{\partial x}\right|_{(2500,1700)} = 1117.8\left(\frac{1700}{2500}\right)^{0.379}$$
$$\approx 1117.8(0.86401420)$$
$$\approx 966$$

The marginal productivity of labor when 2500 units of labor and 1700 units of capital are currently being used is 966 units per unit labor.

$$\left.\frac{\partial p}{\partial y}\right|_{(2500,1700)} = 682.2\left(\frac{2500}{1700}\right)^{0.621}$$
$$\approx 682.2(1.27060911)$$
$$\approx 867$$

The marginal productivity of capital when 2500 units of labor and 1700 units of capital are currently being used is 867 units per unit capital.

d) $\boxed{tw}$ The marginal productivity of labor, $\frac{\partial p}{\partial x}$, gives the rate of change of productivity with respect to labor. In other words, when 2500 units of labor and 1700 units of capital are currently being used, employing one additional unit of labor to 2501 units and keeping capital fixed will result in an additional 966 units of production.

The marginal productivity of capital, $\frac{\partial p}{\partial y}$, gives the rate of change of productivity with respect to capital. In other words, when 2500 units of labor and 1700 units of capital are currently being used, employing one additional unit of capital to 1701 units and keeping labor fixed will result in an additional 867 units of production.

43. $P(w,r,s,t) = 0.007955w^{-0.638}r^{1.038}s^{0.873}t^{2.468}$

a) We substitute 20 for w, 70 for r, 400,000 for s, and 8 for t.

$$P(20,70,400,000,8)$$
$$= 0.007955(20)^{-0.638}(70)^{1.038}(400,000)^{0.873}(8)^{2.468}$$
$$\approx 1,274,146$$

The nursing home's annual profit is approximately $1,274,146.

b) Taking the partial derivative with respect to each variable, we have:

$$\frac{\partial P}{\partial w} = 0.007955\left(-0.638w^{-1.638}\right)r^{1.038}s^{0.873}t^{2.468}$$
$$= -0.005075w^{-1.638}r^{1.038}s^{0.873}t^{2.468}$$
$$\frac{\partial P}{\partial r} = 0.007955w^{-0.638}\left(1.038r^{1-1.038}\right)s^{0.873}t^{2.468}$$
$$= 0.008257w^{-0.638}r^{0.038}s^{0.873}t^{2.468}$$
$$\frac{\partial P}{\partial s} = 0.007955w^{-0.638}r^{1.038}\left(0.873s^{-0.127}\right)t^{2.468}$$
$$= 0.006945w^{-0.638}r^{1.038}s^{-0.127}t^{2.468}$$
$$\frac{\partial P}{\partial t} = 0.007955w^{-0.638}r^{1.038}s^{0.873}\left(2.468t^{1.468}\right)$$
$$= 0.019633w^{-0.638}r^{1.038}s^{0.873}t^{1.468}$$

c) $\boxed{tw}$ The partial derivative with respect to a particular variable represents the rate of change in profit with respect to that variable, holding all other variables constant.

44. $\Delta P \approx \frac{\partial P}{\partial w}\Delta w$

$$\approx \Big[\left(-0.005075\right)(20)^{-1.638}(70)^{1.038}\cdot (400,000)^{0.873}(8)^{2.468}\Big]\cdot(0.25)$$
$$\approx -10,161$$

The nursing homes profit will decrease by about $10,161.

45. $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

Substituting 85 for T, and 60%=0.60 for H, we have:

$$T_h = 1.98(85) - 1.09(1-0.6)(85-58) - 56.9$$
$$= 168.3 - 1.09(0.4)(27) - 56.9$$
$$= 168.3 - 11.772 - 56.9$$
$$= 99.628$$
$$\approx 99.6$$

The temperature-humidity index is about $99.6°F$.

46. $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

Substituting 90 for T, and 90%=0.9 for H, we have:

$$T_h = 1.98(90) - 1.09(1-0.9)(90-58) - 56.9$$
$$= 178.2 - 1.09(0.1)(32) - 56.9$$
$$= 178.2 - 3.488 - 56.9$$
$$= 117.812$$
$$\approx 117.8$$

The temperature-humidity index is about $117.8°F$.

47. $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

Substituting 90 for T, and 100%=1.0 for H, we have:

$$T_h = 1.98(90) - 1.09(1-1.0)(90-58) - 56.9$$
$$= 178.2 - 1.09(0)(32) - 56.9$$
$$= 178.2 - 0 - 56.9$$
$$= 121.3$$

The temperature-humidity index is $121.3°F$.

48. $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

Substituting 78 for T, and 100%=1.0 for H, we have:

$$\begin{aligned} T_h &= 1.98(78) - 1.09(1-1.0)(78-58) - 56.9 \\ &= 154.44 - 1.09(0)(20) - 56.9 \\ &= 154.44 - 0 - 56.9 \\ &= 97.54 \\ &\approx 97.5 \end{aligned}$$

The temperature-humidity index is about $97.5°F$.

49. $\boxed{tw}$ $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

$$\begin{aligned} \frac{\partial T_h}{\partial H} &= -1.09(-1)(T-58) \\ &= 1.09(T-58) \end{aligned}$$

$\frac{\partial T_h}{\partial H}$ gives the rate of change of the temperature-humidity index with respect to humidity, holding temperature constant.

50. $\boxed{tw}$ $T_h = 1.98T - 1.09(1-H)(T-58) - 56.9$

$$\begin{aligned} \frac{\partial T_h}{\partial T} &= 1.98 \cdot 1 - 1.09(1-H)\cdot(1) \\ &= 1.98 - 1.09(1-H) \end{aligned}$$

$\frac{\partial T_h}{\partial T}$ gives the rate of change of the temperature-humidity index with respect to temperature, holding humidity constant.

51. $S = \frac{\sqrt{hw}}{60} = \frac{(hw)^{1/2}}{60} = \frac{h^{1/2}w^{1/2}}{60}$

a) $$\begin{aligned} \frac{\partial S}{\partial h} &= \frac{1}{60}\left(\frac{1}{2}h^{-1/2}w^{1/2}\right) \\ &= \frac{1}{120}\left(\frac{w^{1/2}}{h^{1/2}}\right) \\ &= \frac{\sqrt{w}}{120\sqrt{h}} \end{aligned}$$

b) $$\begin{aligned} \frac{\partial S}{\partial w} &= \frac{1}{60}h^{1/2}\left(\frac{1}{2}w^{-1/2}\right) \\ &= \frac{1}{120}\left(\frac{h^{1/2}}{w^{1/2}}\right) \\ &= \frac{\sqrt{h}}{120\sqrt{w}} \end{aligned}$$

c) $$\begin{aligned} \Delta S &\approx \frac{\partial S}{\partial w}\Delta w \\ \Delta S &\approx \left[\frac{\sqrt{h}}{120\sqrt{w}}\right]\Delta w \\ &\approx \frac{\sqrt{170}}{120\sqrt{80}}(-2) \\ &\approx -0.0243 \end{aligned}$$

The change in the surface area is approximately -0.0243 m^2.

52. $S = 0.024265h^{0.3964}w^{0.5378}$

a) $$\begin{aligned} \frac{\partial S}{\partial h} &= 0.024265\left(0.3964h^{-0.6036}\right)w^{0.5378} \\ &= 0.009619h^{-0.6036}w^{0.5378} \end{aligned}$$

b) $$\begin{aligned} \frac{\partial S}{\partial w} &= 0.024265h^{0.3964}\left(0.5378w^{-0.4622}\right) \\ &= 0.01305h^{0.3964}w^{-0.4622} \end{aligned}$$

c) $$\begin{aligned} \Delta S &\approx \frac{\partial S}{\partial w}\Delta w \\ \Delta S &\approx \left[0.01305h^{0.3964}w^{-0.4622}\right]\Delta w \\ &\approx \left[0.01305(170)^{0.3964}(80)^{-0.4622}\right](-2) \\ &\approx -0.0264 \end{aligned}$$

The change in the surface area is approximately -0.0264 m^2.

53. $E = 206.835 - 0.846w - 1.015s$

Substituting 146 for w and 5 for s, we have:

$$\begin{aligned} E &= 206.835 - 0.846(146) - 1.015(5) \\ &= 206.835 - 123.516 - 5.075 \\ &= 78.244 \end{aligned}$$

The reading ease is 78.244.

54. $E = 206.835 - 0.846w - 1.015s$

$$\begin{aligned} E &= 206.835 - 0.846(180) - 1.015(6) \\ &= 206.835 - 152.28 - 6.09 \\ &= 48.465 \end{aligned}$$

The reading ease is 48.465.

55. $E = 206.835 - 0.846w - 1.015s$

$$\frac{\partial E}{\partial w} = -0.846$$

56. $E = 206.835 - 0.846w - 1.015s$

$$\frac{\partial E}{\partial s} = -1.015$$

57. $f(x,t)=\dfrac{x^2+t^2}{x^2-t^2}$

Find f_x.

$$f(x,t)=\frac{\underline{x}^2+t^2}{\underline{x}^2-t^2}$$

The variable is underlined.

$$f_x=\frac{(x^2-t^2)(2x)-(x^2+t^2)(2x)}{(x^2-t^2)^2}$$
$$=\frac{2x^3-2xt^2-2x^3-2xt^2}{(x^2-t^2)^2}$$
$$=\frac{-4xt^2}{(x^2-t^2)^2}$$

Find f_t.

$$f(x,t)=\frac{x^2+\underline{t}^2}{x^2-\underline{t}^2}$$

The variable is underlined.

$$f_t=\frac{(x^2-t^2)(2t)-(x^2+t^2)(-2t)}{(x^2-t^2)^2}$$
$$=\frac{2x^2t-2t^3+2x^2t+2t^3}{(x^2-t^2)^2}$$
$$=\frac{4x^2t}{(x^2-t^2)^2}$$

58. $f(x,t)=\dfrac{x^2-t}{x^3+t}$

Find f_x.

$$f(x,t)=\frac{\underline{x}^2-t}{\underline{x}^3+t}$$

The variable is underlined.

$$f_x=\frac{(x^3+t)(2x)-(x^2-t)(3x^2)}{(x^3+t)^2}$$
$$=\frac{2x^4+2xt-3x^4+3x^2t}{(x^3+t)^2}$$
$$=\frac{-x^4+3x^2t+2xt}{(x^3+t)^2}$$

Find f_t.

$$f(x,t)=\frac{x^2-\underline{t}}{x^3+\underline{t}}$$

The variable is underlined.

$$f_t=\frac{(x^3+t)(-1)-(x^2-t)(1)}{(x^3+t)^2}$$
$$=\frac{-x^3-t-x^2+t}{(x^3+t)^2}$$
$$=\frac{-x^3-x^2}{(x^3+t)^2}$$

59. $f(x,t)=\dfrac{2\sqrt{x}-2\sqrt{t}}{1+2\sqrt{t}}=\dfrac{2x^{1/2}-2t^{1/2}}{1+2t^{1/2}}$

Find f_x.

$$f(x,t)=\frac{2\underline{x}^{1/2}-2t^{1/2}}{1+2t^{1/2}}$$

The variable is underlined.

$$=\frac{2x^{1/2}}{1+2t^{1/2}}-\frac{2t^{1/2}}{1+2t^{1/2}}$$
$$f_x=\frac{2\left(\frac{1}{2}x^{-1/2}\right)}{1+2t^{1/2}}-0$$
$$=\frac{x^{-1/2}}{1+2t^{1/2}}$$
$$=\frac{1}{x^{1/2}\left(1+2t^{1/2}\right)}$$
$$=\frac{1}{\sqrt{x}\left(1+2\sqrt{t}\right)}$$

Find f_t.

$$f(x,t)=\frac{2x^{1/2}-2\underline{t}^{1/2}}{1+2\underline{t}^{1/2}}$$

The variable is underlined.

$$f_t=\frac{\left(1+2t^{1/2}\right)\left(-1t^{-1/2}\right)-\left(2x^{1/2}-2t^{1/2}\right)\left(t^{-1/2}\right)}{\left(1+2t^{1/2}\right)^2}$$
$$=\frac{-t^{-1/2}-2t^0-2x^{1/2}t^{-1/2}+2t^0}{\left(1+2t^{1/2}\right)^2}$$
$$=\frac{t^{-1/2}\left(-1-2x^{1/2}\right)}{\left(1+2t^{1/2}\right)^2}$$
$$=\frac{-1-2\sqrt{x}}{\sqrt{t}\left(1+2\sqrt{t}\right)^2}$$

60. $f(x,t)=\sqrt[4]{x^3t^5}=x^{3/4}t^{5/4}$

Find f_x.

$f(x,t)=\underline{x}^{3/4}t^{5/4}$ The variable is underlined.

$f_x=\frac{3}{4}x^{-1/4}t^{5/4}$

$=\frac{3}{4}\cdot\sqrt[4]{\frac{t^5}{x}}$

Find f_t.

$f(x,t)=x^{3/4}\underline{t}^{5/4}$ The variable is underlined.

$f_t=x^{3/4}\left(\frac{5}{4}t^{1/4}\right)$

$=\frac{5}{4}\cdot\sqrt[4]{x^3t}$

61. $f(x,t)=6x^{2/3}-8x^{1/4}t^{1/2}-12x^{-1/2}t^{3/2}$

Find f_x.

$f(x,t)=6\underline{x}^{2/3}-8\underline{x}^{1/4}t^{1/2}-12\underline{x}^{-1/2}t^{3/2}$

The variable is underlined.

$$f_x=6\left(\frac{2}{3}x^{-1/3}\right)-8\left(\frac{1}{4}x^{-3/4}\right)t^{1/2}-12\left(\frac{-1}{2}x^{-3/2}\right)t^{3/2}$$

$$=4x^{-1/3}-2x^{-3/4}t^{1/2}+6x^{-3/2}t^{3/2}$$

Find f_t.

$f(x,t)=6x^{2/3}-8x^{1/4}\underline{t}^{1/2}-12x^{-1/2}\underline{t}^{3/2}$

The variable is underlined.

$$f_t=-8x^{1/4}\left(\frac{1}{2}t^{-1/2}\right)-12x^{-1/2}\left(\frac{3}{2}t^{1/2}\right)$$

$$=-4x^{1/4}t^{-1/2}-18x^{-1/2}t^{1/2}$$

62. $f(x,t)=\left(\frac{x^2+t^2}{x^2-t^2}\right)^5$

Find f_x.

$f(x,t)=\left(\frac{\underline{x}^2+t^2}{\underline{x}^2-t^2}\right)^5$ The variable is underlined.

$$f_x=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{(x^2-t^2)(2x)-(x^2+t^2)(2x)}{(x^2-t^2)^2}$$

$$=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{2x^3-2xt^2-2x^3-2xt^2}{(x^2-t^2)^2}$$

$$=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{-4xt^2}{(x^2-t^2)^2}$$

$$=\frac{-20xt^2(x^2+t^2)^4}{(x^2-t^2)^6}$$

Find f_t.

$f(x,t)=\left(\frac{x^2+\underline{t}^2}{x^2-\underline{t}^2}\right)^5$ The variable is underlined.

$$f_t=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{(x^2-t^2)(2t)-(x^2+t^2)(-2t)}{(x^2-t^2)^2}$$

$$=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{2x^2t-2t^3+2x^2t+2t^3}{(x^2-t^2)^2}$$

$$=5\left(\frac{x^2+t^2}{x^2-t^2}\right)^4\cdot\frac{4x^2t}{(x^2-t^2)^2}$$

$$=\frac{20x^2t(x^2+t^2)^4}{(x^2-t^2)^6}$$

63. $f(x,y)=\frac{x}{y^2}-\frac{y}{x^2}=xy^{-2}-yx^{-2}$

First, we find the partial derivatives.

We find f_x first.

$f(x,y)=\underline{x}y^{-2}-y\underline{x}^{-2}$ The variable is underlined.

$f_x=y^{-2}+2yx^{-3}$

Then we find f_y.

$$f(x,y) = x\underline{y}^{-2} - \underline{y}x^{-2}$$ The variable is underlined.

$$f_y = -2xy^{-3} - x^{-2}$$

We find f_{xx} by taking the partial derivative with respect to x of f_x.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(f_x) \\ &= \frac{\partial}{\partial x}\left(y^{-2} + 2yx^{-3}\right) \\ &= -6yx^{-4} \\ &= \frac{-6y}{x^4} \end{aligned}$$

We find f_{xy} by taking the partial derivative with respect to y of f_x.

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(f_x) \\ &= \frac{\partial}{\partial y}\left(y^{-2} + 2yx^{-3}\right) \\ &= -2y^{-3} + 2x^{-3} \\ &= -\frac{2}{y^3} + \frac{2}{x^3} \end{aligned}$$

We find f_{yx} by taking the partial derivative with respect to x of f_y.

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(f_y) \\ &= \frac{\partial}{\partial x}\left(-2xy^{-3} - x^{-2}\right) \\ &= -2y^{-3} + 2x^{-3} \\ &= -\frac{2}{y^3} + \frac{2}{x^3} \end{aligned}$$

We find f_{yy} by taking the partial derivative with respect to y of f_y.

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(f_y) \\ &= \frac{\partial}{\partial y}\left(-2xy^{-3} - x^{-2}\right) \\ &= 6xy^{-4} \\ &= \frac{6x}{y^4} \end{aligned}$$

64. $f(x,y) = \dfrac{xy}{x-y}$

We find the first-order partial derivatives.

$$\begin{aligned} f_x &= \frac{(x-y)y - (xy)(1)}{(x-y)^2} \\ &= \frac{-y^2}{(x-y)^2} \end{aligned}$$

$$\begin{aligned} f_y &= \frac{(x-y)x - (xy)(-1)}{(x-y)^2} \\ &= \frac{x^2}{(x-y)^2} \end{aligned}$$

Then we find the second-order partial derivatives.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}\left(\frac{-y^2}{(x-y)^2}\right) \\ &= \frac{(x-y)^2(0) - (-y^2)\left(2(x-y)(1)\right)}{(x-y)^4} \\ &= \frac{(2y^2)(x-y)}{(x-y)^4} \\ &= \frac{2y^2}{(x-y)^3} \end{aligned}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}\left(\frac{-y^2}{(x-y)^2}\right) \\ &= \frac{(x-y)^2(-2y) - (-y^2)\left(2(x-y)(-1)\right)}{(x-y)^4} \\ &= \frac{(x-y)\left[(-2y)(x-y) - 2y^2\right]}{(x-y)^4} \\ &= \frac{\left[-2xy + 2y^2 - 2y^2\right]}{(x-y)^3} \\ &= \frac{-2xy}{(x-y)^3} \end{aligned}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{x^2}{(x-y)^2}\right)$$
$$= \frac{(x-y)^2(2x) - (x^2)(2(x-y)(1))}{(x-y)^4}$$
$$= \frac{(x-y)\left[(2x)(x-y) - 2x^2\right]}{(x-y)^4}$$
$$= \frac{\left[2x^2 - 2xy - 2x^2\right]}{(x-y)^3}$$
$$= \frac{-2xy}{(x-y)^3}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{x^2}{(x-y)^2}\right)$$
$$= \frac{(x-y)^2(0) - (x^2)(2(x-y)(-1))}{(x-y)^4}$$
$$= \frac{2x^2(x-y)}{(x-y)^4}$$
$$= \frac{2x^2}{(x-y)^3}$$

65. $\boxed{tw}$ Charles W. Cobb and Paul M. Douglas sought a function that would relate the total production of an economy to the size of its labor force and the amount of its invested capital. They started with various economic assumptions, used these assumptions and mathematics to derive a production function, and then checked the function against tabulated data.

66. $\boxed{tw}$ The graph of a function of two variables $z = f(x, y)$ is a 3-dimensional surface. when x is fixed at some value a, the plane $x = a$ intersects the surface in a curve. Then f_y gives the slopes of lines tangent to this curve in the plane $x = a$. Similarly, if y is fixed at some value b, the plane $y = b$ intersects the surface in a curve. Then f_x gives the slopes of lines tangent to the curve in the plane $y = b$.

67. $f(x, y) = \ln(x^2 + y^2)$

We need to find the second-order partial derivatives.

First, we find f_x.

$$f(x, y) = \ln(\underline{x}^2 + y^2)$$
$$f_x = \frac{1}{x^2+y^2}\cdot 2x = \frac{2x}{x^2+y^2}$$

Then we find f_{xx}.

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$
$$= \frac{\partial}{\partial x}\left(\frac{2\underline{x}}{\underline{x}^2+y^2}\right)$$
$$= \frac{(x^2+y^2)(2) - (2x)(2x)}{(x^2+y^2)^2}$$
$$= \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

Now we find f_y.

$$f(x, y) = \ln(x^2 + \underline{y}^2)$$
$$f_x = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

Then we find f_{yy}.

$$f_{yy} = \frac{\partial}{\partial y}(f_y)$$
$$= \frac{\partial}{\partial y}\left(\frac{2\underline{y}}{x^2+\underline{y}^2}\right)$$
$$= \frac{(x^2+y^2)(2) - (2y)(2y)}{(x^2+y^2)^2}$$
$$= \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

Therefore,

$$\begin{array}{r|l}
\multicolumn{2}{c}{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0} \\
\hline
\dfrac{-2x^2+2y^2}{(x^2+y^2)^2} + \dfrac{2x^2-2y^2}{(x^2+y^2)^2} & 0 \\
\dfrac{0}{(x^2+y^2)^2} & 0 \\
0 & 0
\end{array}$$

Thus, f is a solution to $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0.$

68. $f(x,y)=x^3-5xy^2$

$f_x=3x^2-5y^2$

$f_{xy}=-10y$

$f_y=-10xy$

$$\begin{array}{r|l} xf_{xy}-f_y & 0 \\ \hline x(-10y)-(-10xy) & 0 \\ -10xy+10xy & 0 \\ 0 & 0 \end{array}$$

69.

$$f(x,y)=\begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{for } (x,y)\neq(0,0), \\ 0, & \text{for } (x,y)=(0,0). \end{cases}$$

a) Find $f_x(0,y)$.

$$\lim_{h\to0}\frac{f(h,y)-f(0,y)}{h}$$

$$=\lim_{h\to0}\frac{\dfrac{hy(h^2-y^2)}{h^2+y^2}-\dfrac{0\cdot y(0^2-y^2)}{0^2+y^2}}{h}$$

$$=\lim_{h\to0}\frac{hy(h^2-y^2)}{h(h^2+y^2)}$$

$$=\lim_{h\to0}\frac{y(h^2-y^2)}{(h^2+y^2)}$$

$$=\frac{y(-y^2)}{y^2}$$

$$=-y$$

Thus, $f_x(0,y)=-y$.

b) Find $f_y(x,0)$.

$$\lim_{h\to0}\frac{f(x,h)-f(x,0)}{h}$$

$$=\lim_{h\to0}\frac{\dfrac{xh(x^2-h^2)}{x^2+h^2}-\dfrac{x\cdot0(x^2-0^2)}{x^2+0^2}}{h}$$

$$=\lim_{h\to0}\frac{xh(x^2-h^2)}{h(x^2+h^2)}$$

$$=\lim_{h\to0}\frac{x(x^2-h^2)}{(x^2+h^2)}$$

$$=\frac{x(x^2)}{x^2}$$

$$=x$$

Thus, $f_y(x,0)=x$.

c) Find $f_{yx}(0,0)$

$$\lim_{h\to0}\frac{f_y(h,0)-f_y(0,0)}{h}$$

$$=\lim_{h\to0}\frac{h-0}{h}\qquad \text{Substituting } f_y(x,0)=x.$$

$$=\lim_{h\to0}1$$

$$=1$$

Find $f_{xy}(0,0)$

$$\lim_{h\to0}\frac{f_x(0,h)-f_x(0,0)}{h}$$

$$=\lim_{h\to0}\frac{-h-(0)}{h}\qquad \text{Substituting } f_x(0,y)=-y.$$

$$=\lim_{h\to0}(-1)$$

$$=-1$$

Thus, $f_{yx}(0,0)\neq f_{xy}(0,0)$. The mixed partials are not equal at $(0,0)$

Exercise Set 6.3

1. $f(x,y)=x^2+xy+y^2-y$

Find f_x.

$f(x,y)=\underline{x}^2+\underline{x}y+y^2-y,$ The variable is underlined.

$f_x=2x+y.$

Find f_y.

$f(x,y)=x^2+x\underline{y}+\underline{y}^2-\underline{y},$ The variable is underlined.

$f_y=x+2y-1.$

Find f_{xx} and f_{xy}.

$f_x=2\underline{x}+y,\quad f_x=2x+\underline{y},$

$f_{xx}=2.\qquad f_{xy}=1.$

Find f_{yy}.

$f_y=x+2\underline{y}-1,$

$f_{yy}=2.$

Solve the system of equations $f_x=0$ and $f_y=0$:

$2x+y=0,\qquad (1)$

$x+2y-1=0.\qquad (2)$

Solving Eq. (1) for y, we get $y=-2x$.

Substituting $-2x$ for y in Eq. (2) and solving, we get

$$x+2(-2x)-1=0$$
$$-3x-1=0$$
$$-3x=1$$
$$x=-\frac{1}{3}.$$

To find y when $x=-\frac{1}{3}$, we substitute $-\frac{1}{3}$ for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$$2\left(-\frac{1}{3}\right)+y=0$$
$$-\frac{2}{3}+y=0$$
$$y=\frac{2}{3}.$$

Thus, $\left(-\frac{1}{3},\frac{2}{3}\right)$ is our candidate for a maximum or minimum.

We have to check to see if $f\left(-\frac{1}{3},\frac{2}{3}\right)$ is a maximum or minimum:

$$D=f_{xx}(a,b)\cdot f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$
$$D=f_{xx}\left(-\frac{1}{3},\frac{2}{3}\right)\cdot f_{yy}\left(-\frac{1}{3},\frac{2}{3}\right)-\left[f_{xy}\left(-\frac{1}{3},\frac{2}{3}\right)\right]^2$$
$$D=2\cdot 2-1^2 \quad \text{For all values of } x \text{ and } y,\ f_{xx}=2,\ f_{yy}=2, \text{ and } f_{xy}=1.$$
$$D=3.$$

Thus, $D=3$ and $f_{xx}=2.$ Since $D>0$ and $f_{xx}\left(-\frac{1}{3},\frac{2}{3}\right)=2>0$, it follows that f has a relative minimum at $\left(-\frac{1}{3},\frac{2}{3}\right)$. The minimum is found as follows:

$$f(x,y)=x^2+xy+y^2-y$$
$$f\left(-\frac{1}{3},\frac{2}{3}\right)=\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)$$
$$=\frac{1}{9}-\frac{2}{9}+\frac{4}{9}-\frac{2}{3}$$
$$=\frac{1}{9}-\frac{2}{9}+\frac{4}{9}-\frac{6}{9}$$
$$=-\frac{3}{9}$$
$$=-\frac{1}{3}.$$

The relative minimum value of f is $-\frac{1}{3}$ at $\left(-\frac{1}{3},\frac{2}{3}\right)$.

2. $f(x,y)=x^2+xy+y^2-5y$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

$$f_x=2x+y, \qquad f_y=x+2y-5,$$
$$f_{xx}=2; \qquad f_{yy}=2;$$
$$f_{xy}=1.$$

2. Solve the system of equations $f_x=0$ and $f_y=0$:

$$2x+y=0 \qquad (1)$$
$$x+2y-5=0 \qquad (2)$$

Solving Eq. (1) for y, we get $y=-2x$.

Substituting $-2x$ for y in Eq. (2) and solving, we get

$$x+2(-2x)-5=0$$
$$-3x-5=0$$
$$-3x=5$$
$$x=-\frac{5}{3}.$$

To find y when $x=-\frac{5}{3}$, we substitute $-\frac{5}{3}$ for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$$2\left(-\frac{5}{3}\right)+y=0$$
$$-\frac{10}{3}+y=0$$
$$y=\frac{10}{3}.$$

Thus, $\left(-\frac{5}{3}, \frac{10}{3}\right)$ is our candidate for a maximum or minimum.

3. We must check to see if $f\left(-\frac{5}{3}, \frac{10}{3}\right)$ is a maximum or minimum:

$$D = f_{xx}\left(-\frac{5}{3}, \frac{10}{3}\right) \cdot f_{yy}\left(-\frac{5}{3}, \frac{10}{3}\right) - \left[f_{xy}\left(-\frac{5}{3}, \frac{10}{3}\right)\right]^2$$

$$D = 2 \cdot 2 - 1^2$$ For all values of x and y, $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = 1$.

$$D = 3.$$

4. Since $D > 0$ and $f_{xx}\left(-\frac{5}{3}, \frac{10}{3}\right) = 2 > 0$, it follows that f has a relative minimum at $\left(-\frac{5}{3}, \frac{10}{3}\right)$. The minimum is found as follows:

$$f\left(-\frac{5}{3}, \frac{10}{3}\right) = \left(-\frac{5}{3}\right)^2 + \left(-\frac{5}{3}\right)\left(\frac{10}{3}\right) + \left(\frac{10}{3}\right)^2 - 5\left(\frac{10}{3}\right)$$

$$= \frac{25}{9} - \frac{50}{9} + \frac{100}{9} - \frac{50}{3}$$

$$= -\frac{75}{9}$$

$$= -\frac{25}{3}.$$

The relative minimum value of f is $-\frac{25}{3}$ at $\left(-\frac{5}{3}, \frac{10}{3}\right)$.

3. $f(x,y) = 2xy - x^3 - y^2$

Find f_x.

$f(x,y) = 2\underline{x}y - \underline{x}^3 - y^2$, The variable is underlined.

$f_x = 2y - 3x^2.$

Find f_y.

$f(x,y) = 2x\underline{y} - x^3 - \underline{y}^2$, The variable is underlined.

$f_y = 2x - 2y.$

Find f_{xx} and f_{xy}.

$f_x = 2y - 3\underline{x}^2$, $\quad f_x = 2\underline{y} - 3x^2$,

$f_{xx} = -6x.$ $\quad f_{xy} = 2.$

Find f_{yy}.

$f_y = 2x - 2\underline{y}$,

$f_{yy} = -2.$

Solve the system of equations $f_x = 0$ and $f_y = 0$:

$$2y - 3x^2 = 0, \qquad (1)$$

$$2x - 2y = 0. \qquad (2)$$

Solving Eq. (1) for $2y$, we get $2y = 3x^2$. Substituting $3x^2$ for $2y$ in Eq. (2) and solving, we get

$$2x - 3x^2 = 0$$

$$x(2 - 3x) = 0$$

$$x = 0 \quad \text{or} \quad 2 - 3x = 0$$

$$x = 0 \quad \text{or} \quad -3x = -2$$

$$x = 0 \quad \text{or} \quad x = \frac{2}{3}$$

To find y when $x = 0$, we substitute 0 for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$$2y - 3(0)^2 = 0$$

$$2y = 0$$

$$y = 0.$$

Thus, $(0,0)$ is one critical point, and $f(0,0)$ is a candidate for a maximum or minimum value.

To find the other critical point we substitute $\frac{2}{3}$ for x in either Eq. (1) or Eq. (2). We use Eq. (2):

$$2\left(\frac{2}{3}\right) - 2y = 0$$

$$\frac{4}{3} - 2y = 0$$

$$-2y = -\frac{4}{3}$$

$$y = \frac{2}{3}.$$

Thus, $\left(\frac{2}{3}, \frac{2}{3}\right)$ is the other critical point, and $f\left(\frac{2}{3}, \frac{2}{3}\right)$ is another candidate for maximum or minimum value.

We must check both $(0,0)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$ to see whether they yield maximum or minimum values.

For $(0,0)$

$$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$

$$D = 0\cdot(-2) - 2^2 \qquad \begin{bmatrix} f_{xx}(0,0) = -6\cdot 0 = 0 \\ f_{yy}(0,0) = -2 \\ f_{xy}(0,0) = 2 \end{bmatrix}$$

$$D = -4.$$

Since $D<0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

For $\left(\frac{2}{3},\frac{2}{3}\right)$

$$D = f_{xx}\left(\frac{2}{3},\frac{2}{3}\right)\cdot f_{yy}\left(\frac{2}{3},\frac{2}{3}\right) - \left[f_{xy}\left(\frac{2}{3},\frac{2}{3}\right)\right]^2$$

$$D = (-4)\cdot(-2) - 2^2 \qquad \begin{bmatrix} f_{xx}\left(\frac{2}{3},\frac{2}{3}\right) = -6\cdot\frac{2}{3} = -4 \\ f_{yy}\left(\frac{2}{3},\frac{2}{3}\right) = -2 \\ f_{xy}\left(\frac{2}{3},\frac{2}{3}\right) = 2 \end{bmatrix}$$

$$D = 8-4$$
$$D = 4.$$

Thus, $D=4$ and $f_{xx}\left(\frac{2}{3},\frac{2}{3}\right) = -4$. Since $D>0$ and $f_{xx}\left(\frac{2}{3},\frac{2}{3}\right)<0$, it follows that *f* has a relative maximum at $\left(\frac{2}{3},\frac{2}{3}\right)$. The maximum is found as follows:

$$f(x,y) = 2xy - x^3 - y^2$$

$$f\left(\frac{2}{3},\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2$$

$$= \frac{8}{9} - \frac{8}{27} - \frac{4}{9}$$

$$= \frac{4}{9} - \frac{8}{27}$$

$$= \frac{12}{27} - \frac{8}{27}$$

$$= \frac{4}{27}.$$

The relative maximum value of *f* is $\frac{4}{27}$ at $\left(\frac{2}{3},\frac{2}{3}\right)$.

4. $f(x,y) = 4xy - x^3 - y^2$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

   $f_x = 4y - 3x^2 \qquad f_y = 4x - 2y$

   $f_{xx} = -6x; \qquad f_{yy} = -2;$

   $f_{xy} = 4.$

2. Solve the system of equations $f_x = 0$ and $f_y = 0$:

   $4y - 3x^2 = 0, \qquad (1)$

   $4x - 2y = 0. \qquad (2)$

   Solving Eq. (2) for *y*, we get $y = 2x$.

   Substituting $2x$ for *y* in Eq. (1) and solving, we get

   $$4(2x) - 3x^2 = 0$$
   $$8x - 3x^2 = 0$$
   $$x(8-3x) = 0$$
   $$x = 0 \quad \text{or} \quad 8 - 3x = 0$$
   $$x = 0 \quad \text{or} \quad x = \frac{8}{3}$$

   To find *y* when $x=0$, we substitute 0 for *x* in either Eq. (1) or Eq. (2). We use Eq. (1):

   $$4y - 3(0)^2 = 0$$
   $$y = 0.$$

   Thus, $(0,0)$ is one critical point, and $f(0,0)$ is a candidate for a maximum or minimum value.

   To find the other critical point we substitute $\frac{8}{3}$ for *x* in either Eq. (1) or Eq. (2). We use Eq. (2):

   $$4\left(\frac{8}{3}\right) - 2y = 0$$
   $$\frac{32}{3} - 2y = 0$$
   $$-2y = -\frac{32}{3}$$
   $$y = \frac{16}{3}.$$

   Thus, $\left(\frac{8}{3},\frac{16}{3}\right)$ is the other critical point, and $f\left(\frac{8}{3},\frac{16}{3}\right)$ is another candidate for maximum or minimum value.

3. We must check both $(0,0)$ and $\left(\frac{8}{3},\frac{16}{3}\right)$ to see whether they yield maximum or minimum values.

For $(0,0)$

$$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$
$$D = (-6\cdot 0)\cdot(-2) - 4^2$$
$$D = -16.$$

Since $D<0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

For $\left(\frac{8}{3},\frac{16}{3}\right)$

$$D = f_{xx}\left(\frac{8}{3},\frac{16}{3}\right)\cdot f_{yy}\left(\frac{8}{3},\frac{16}{3}\right) - \left[f_{xy}\left(\frac{8}{3},\frac{16}{3}\right)\right]^2$$
$$D = \left(-6\cdot\frac{8}{3}\right)\cdot(-2) - 4^2$$
$$D = 32 - 16$$
$$D = 16.$$

4. Thus, $D=16$ and $f_{xx}\left(\frac{8}{3},\frac{16}{3}\right) = -16$. Since $D>0$ and $f_{xx}\left(\frac{8}{3},\frac{16}{3}\right)<0$, it follows that f has a relative maximum at $\left(\frac{8}{3},\frac{16}{3}\right)$. The maximum is found as follows:

$$f\left(\frac{8}{3},\frac{16}{3}\right) = 4\left(\frac{8}{3}\right)\left(\frac{16}{3}\right) - \left(\frac{8}{3}\right)^3 - \left(\frac{16}{3}\right)^2$$
$$= \frac{512}{9} - \frac{512}{27} - \frac{256}{9}$$
$$= \frac{256}{27}.$$

The relative maximum value of f is $\frac{256}{27}$ at $\left(\frac{8}{3},\frac{16}{3}\right)$.

5. $f(x,y) = x^3 + y^3 - 3xy$

Find f_x.

$f(x,y) = \underline{x}^3 + y^3 - 3\underline{x}y,$ The variable is underlined.

$f_x = 3x^2 - 3y.$

Find f_y.

$f(x,y) = x^3 + \underline{y}^3 - 3x\underline{y},$ The variable is underlined.

$f_y = 3y^2 - 3x.$

Find f_{xx} and f_{xy}.

$f_x = 3\underline{x}^2 - 3y, \quad f_x = 3x^2 - 3\underline{y},$

$f_{xx} = 6x. \qquad f_{xy} = -3.$

Find f_{yy}.

$f_y = 3\underline{y}^2 - 3x,$

$f_{yy} = 6y.$

Solve the system of equations $f_x = 0$ and $f_y = 0$:

$$3x^2 - 3y = 0, \qquad (1)$$
$$3y^2 - 3x = 0. \qquad (2)$$

We multiply each equation by $\frac{1}{3}$.

$$x^2 - y = 0, \qquad (1)$$
$$y^2 - x = 0. \qquad (2)$$

Solving Eq. (1) for y, we get $y = x^2$.

Substituting x^2 for y in Eq. (2) and solving, we get

$$(x^2)^2 - x = 0$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0$$
$$x = 0 \quad\text{or}\quad x^3 - 1 = 0$$
$$x = 0 \quad\text{or}\quad x^3 = 1$$
$$x = 0 \quad\text{or}\quad x = 1$$

To find y when $x=0$, we substitute 0 for x in either Eq. (1) or Eq. (2). We use Eq. (2):

$$y^2 - (0) = 0$$
$$y^2 = 0$$
$$y = 0.$$

Thus, $(0,0)$ is one critical point.

To find the other critical point we substitute 1 for x in either Eq. (1) or Eq. (2). We use Eq. (2):

$$y^2 - 1 = 0$$
$$y^2 = 1$$
$$y = \pm 1.$$

However, we notice that $(1,-1)$ is not a solution to Eq. (1) $1^2 - (-1) = 2 \neq 0$.

Thus, $(1,1)$ is the other critical point.

We must check both $(0,0)$ and $(1,1)$ to see whether they yield maximum or minimum values.

For $(0,0)$

$$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$

$$D = 0\cdot(0) - (-3)^2 \qquad \begin{bmatrix} f_{xx}(0,0) = 6\cdot 0 = 0 \\ f_{yy}(0,0) = 6\cdot 0 = 0 \\ f_{xy}(0,0) = -3 \end{bmatrix}$$

$$D = -9.$$

Since $D < 0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

For $(1,1)$

$$D = f_{xx}(1,1)\cdot f_{yy}(1,1) - \left[f_{xy}(1,1)\right]^2$$

$$D = 6\cdot(6) - (-3)^2 \qquad \begin{bmatrix} f_{xx}(1,1) = 6\cdot 1 = 6 \\ f_{yy}(1,1) = 6\cdot(1) = 6 \\ f_{xy}(1,1) = -3 \end{bmatrix}$$

$$D = 36 - 9$$
$$D = 27.$$

Thus, $D = 27$ and $f_{xx}(1,1) = 6$. Since $D > 0$ and $f_{xx}(1,1) > 0$, it follows that f has a relative minimum at $(1,1)$. The minimum is found as follows:

$$f(x,y) = x^3 + y^3 - 3xy$$
$$f(1,1) = 1^3 + 1^3 - 3(1)(1)$$
$$= 1 + 1 - 3$$
$$= -1.$$

The relative minimum value of f is -1 at $(1,1)$.

6. $f(x,y) = x^3 + y^3 - 6xy$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

$$f_x = 3x^2 - 6y \qquad f_y = 3y^2 - 6x$$
$$f_{xx} = 6x; \qquad f_{yy} = 6y;$$
$$f_{xy} = -6.$$

2. Solve the system of equations $f_x = 0$ and $f_y = 0$:

$$3x^2 - 6y = 0, \qquad (1)$$
$$3y^2 - 6x = 0. \qquad (2)$$

Solving Eq. (1) for y, we get $y = \dfrac{x^2}{2}$.

Substitute into Eq. (2) and solve for x:

$$3\left(\frac{x^2}{2}\right)^2 - 6x = 0$$
$$\frac{3}{4}x^4 - 6x = 0$$
$$3x^4 - 24x = 0$$
$$3x\left(x^3 - 8\right) = 0$$

$$3x = 0 \quad \text{or} \quad x^3 - 8 = 0$$
$$x = 0 \quad \text{or} \quad x^3 = 8$$
$$x = 0 \quad \text{or} \quad x = 2$$

To find y when $x = 0$, we substitute 0 for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$$3(0)^2 - 6y = 0$$
$$y = 0.$$

Thus, $(0,0)$ is one critical point. To find the other critical point we substitute 2 for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$$3\cdot 2^2 - 6y = 0$$
$$12 - 6y = 0$$
$$-6y = -12$$
$$y = 2.$$

Thus, $(2,2)$ is the other critical point.

3. We must check both $(0,0)$ and $(2,2)$ to see whether they yield maximum or minimum values.

For $(0,0)$

$$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$
$$D = (6\cdot 0)\cdot(6\cdot 0) - (-6)^2$$
$$D = -36.$$

Since $D < 0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

For $(2,2)$

$D = f_{xx}(2,2) \cdot f_{yy}(2,2) - \left[f_{xy}(2,2) \right]^2$

$D = (6 \cdot 2) \cdot (6 \cdot 2) - (-6)^2$

$D = 144 - 36$

$D = 108.$

4. Since $D > 0$ and $f_{xx}(2,2) = 12 > 0$, it follows that f has a relative minimum at $(2,2)$. The minimum is found as follows:

$f(2,2) = 2^3 + 2^3 - 6(2)(2)$
$= 8 + 8 - 24$
$= -8.$

The relative minimum value of f is -8 at $(2,2)$.

7. $f(x,y) = x^2 + y^2 - 2x + 4y - 2$

Find f_x.

$f(x,y) = \underline{x}^2 + y^2 - 2\underline{x} + 4y - 2,$

$f_x = 2x - 2.$

Find f_y.

$f(x,y) = x^2 + \underline{y}^2 - 2x + 4\underline{y} - 2,$

$f_y = 2y + 4.$

Find f_{xx} and f_{xy}.

$f_x = 2\underline{x} - 2, \quad f_x = 2x - 2,$

$f_{xx} = 2. \quad f_{xy} = 0.$

Find f_{yy}.

$f_y = 2\underline{y} + 4,$

$f_{yy} = 2.$

Solve the system of equations $f_x = 0$ and $f_y = 0$:

$2x - 2 = 0, \quad 2y + 4 = 0,$

$2x = 2, \quad 2y = -4,$

$x = 1. \quad y = -2.$

The only critical point is $(1,-2)$.

We must check $(1,-2)$ to see whether it yields a maximum or minimum value.

For $(1,-2)$

$D = f_{xx}(1,-2) \cdot f_{yy}(1,-2) - \left[f_{xy}(1,-2) \right]^2$

$D = 2 \cdot (2) - (0)^2 \qquad \begin{bmatrix} f_{xx}(1,-2) = 2 \\ f_{yy}(1,-2) = 2 \\ f_{xy}(1,-2) = 0 \end{bmatrix}$

$D = 4.$

Thus, $D = 4$ and $f_{xx}(1,-2) = 2$. Since $D > 0$ and $f_{xx}(1,-2) > 0$, it follows that f has a relative minimum at $(1,-2)$. The minimum is found as follows:

$f(x,y) = x^2 + y^2 - 2x + 4y - 2$

$f(1,-2) = 1^2 + (-2)^2 - 2(1) + 4(-2) - 2$
$= 1 + 4 - 2 - 8 - 2$
$= -7.$

The relative minimum value of f is -7 at $(1,-2)$.

8. $f(x,y) = x^2 + 2xy + 2y^2 - 6y + 2$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

$f_x = 2x + 2y, \qquad f_y = 2x + 4y - 6,$

$f_{xx} = 2; \qquad f_{yy} = 4;$

$f_{xy} = 2.$

2. Solve the system of equations $f_x = 0$ and $f_y = 0$:

$2x + 2y = 0 \qquad (1)$

$2x + 4y - 6 = 0 \qquad (2)$

Solving Eq. (1) for y, we get $y = -x$.

Substituting $-x$ for y in Eq. (2) and solving, we get

$2x + 4(-x) - 6 = 0$

$-2x - 6 = 0$

$-2x = 6$

$x = -3.$

To find y when $x = -3$, we substitute -3 for x in either Eq. (1) or Eq. (2). We use Eq. (1):

$2(-3) + 2y = 0$

$2y = 6$

$y = 3.$

Thus, $(-3,3)$ is a critical point, and $f(-3,3)$ is a candidate for a maximum or minimum.

3. We must check to see if $f(-3,3)$ is a maximum or minimum value:

$D = f_{xx}(-3,3)\cdot f_{yy}(-3,3) - \left[f_{xy}(-3,3)\right]^2$

$D = 2\cdot 4 - 2^2$

$D = 4.$

4. Since $D > 0$ and $f_{xx}(-3,3) = 2 > 0$, it follows that *f* has a relative minimum at $(-3,3)$. The minimum is found as follows:

$$f(-3,3) = (-3)^2 + 2(-3)(3) + 2(3)^2 - 6(3) + 2$$
$$= 9 - 18 + 18 - 18 + 2$$
$$= -7.$$

The relative minimum value of *f* is -7 at $(-3,3)$.

9. $f(x,y) = x^2 + y^2 + 2x - 4y$

Find f_x.

$f(x,y) = \underline{x}^2 + y^2 + 2\underline{x} - 4y,$

$f_x = 2x + 2.$

Find f_y.

$f(x,y) = x^2 + \underline{y}^2 + 2x - 4\underline{y},$

$f_y = 2y - 4.$

Find f_{xx} and f_{xy}.

$f_x = 2\underline{x} + 2, \quad f_x = 2x + 2,$

$f_{xx} = 2. \quad f_{xy} = 0.$

Find f_{yy}.

$f_y = 2\underline{y} - 4,$

$f_{yy} = 2.$

Solve the system of equations $f_x = 0$ and $f_y = 0$:

$2x + 2 = 0, \quad 2y - 4 = 0,$

$2x = -2, \quad 2y = 4,$

$x = -1. \quad y = 2.$

The only critical point is $(-1,2)$.

We must check $(-1,2)$ to see whether it yields a maximum or minimum value.

For $(-1,2)$, we have:

$D = f_{xx}(-1,2)\cdot f_{yy}(-1,2) - \left[f_{xy}(-1,2)\right]^2$

$D = 2\cdot(2) - (0)^2 \qquad \begin{bmatrix} f_{xx}(-1,2) = 2 \\ f_{yy}(-1,2) = 2 \\ f_{xy}(-1,2) = 0 \end{bmatrix}$

$D = 4.$

Thus, $D = 4$ and $f_{xx}(-1,2) = 2.$ Since $D > 0$ and $f_{xx}(-1,2) > 0$, it follows that *f* has a relative minimum at $(-1,2)$. The minimum is found as follows:

$f(x,y) = x^2 + y^2 + 2x - 4y$

$$f(-1,2) = (-1)^2 + (2)^2 + 2(-1) - 4(2)$$
$$= 1 + 4 - 2 - 8$$
$$= -5.$$

The relative minimum value of *f* is -5 at $(-1,2)$.

10. $f(x,y) = 4y + 6x - x^2 - y^2$

1. Find $f_x, f_y, f_{xx}, f_{yy},$ and f_{xy}:

$f_x = 6 - 2x, \qquad f_y = 4 - 2y,$

$f_{xx} = -2; \qquad f_{yy} = -2;$

$f_{xy} = 0.$

2. Solve the system of equations $f_x = 0$ and $f_y = 0$:

$6 - 2x = 0 \qquad 4 - 2y = 0$

$-2x = -6 \qquad -2y = -4$

$x = 3 \qquad y = 2$

Thus, $(3,2)$ is a critical point, and $f(3,2)$ is a candidate for a maximum or minimum.

3. We must check to see if $f(3,2)$ is a maximum or minimum value:

$D = f_{xx}(3,2)\cdot f_{yy}(3,2) - \left[f_{xy}(3,2)\right]^2$

$D = (-2)\cdot(-2) - 0^2$

$D = 4.$

4. Since $D > 0$ and $f_{xx}(3,2) = -2 < 0$, it follows that *f* has a relative maximum at $(3,2)$. The maximum is found as follows:

$$f(3,2) = 4(2) + 6(3) - (3)^2 - (2)^2$$
$$= 8 + 18 - 9 - 4$$
$$= 13$$

The relative maximum value of *f* is 13 at $(3,2)$.

11. $f(x,y)=4x^2-y^2$

Find f_x.

$f(x,y)=4\underline{x}^2-y^2,$

$f_x=8x.$

Find f_y.

$f(x,y)=4x^2-\underline{y}^2,$

$f_y=-2y.$

Find f_{xx} and f_{xy}.

$f_x=8\underline{x}, \quad f_x=8x,$

$f_{xx}=8. \quad f_{xy}=0.$

Find f_{yy}.

$f_y=-2\underline{y},$

$f_{yy}=-2.$

Solve the system of equations

$f_x=0$ and $f_y=0$:

$8x=0, \qquad -2y=0,$

$x=0. \qquad y=0.$

The only critical point is $(0,0)$.

We must check $(0,0)$ to see whether it yields a maximum or minimum value.

For $(0,0)$

$$D=f_{xx}(0,0)\cdot f_{yy}(0,0)-\left[f_{xy}(0,0)\right]^2$$

$$D=8\cdot(-2)-(0)^2 \qquad \begin{bmatrix} f_{xx}(0,0)=8 \\ f_{yy}(0,0)=-2 \\ f_{xy}(0,0)=0 \end{bmatrix}$$

$D=-16.$

Since $D<0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

12. $f(x,y)=x^2-y^2$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

   $f_x=2x, \qquad f_y=-2y,$

   $f_{xx}=2; \qquad f_{yy}=-2;$

   $f_{xy}=0.$

2. Solve the system of equations

   $f_x=0$ and $f_y=0$:

   $2x=0 \qquad -2y=0$

   $x=0 \qquad y=0$

   Thus, $(0,0)$ is a critical point, and $f(0,0)$ is a candidate for a maximum or minimum.

3. We must check to see if $f(0,0)$ is a maximum or minimum value:

   $D=f_{xx}(0,0)\cdot f_{yy}(0,0)-\left[f_{xy}(0,0)\right]^2$

   $D=(2)\cdot(-2)-0^2$

   $D=-4.$

4. Since $D<0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

13. $f(x,y)=e^{x^2+y^2+1}$

Find f_x.

$f(x,y)=e^{\underline{x}^2+y^2+1},$

$f_x=2xe^{x^2+y^2+1}.$

Find f_y.

$f(x,y)=e^{x^2+\underline{y}^2+1},$

$f_y=2ye^{x^2+y^2+1}.$

Find f_{xx}.

$f_x=2\underline{x}e^{\underline{x}^2+y^2+1}$

$f_{xx}=2x\left(2xe^{x^2+y^2+1}\right)+2e^{x^2+y^2+1}$

$=4x^2e^{x^2+y^2+1}+2e^{x^2+y^2+1}.$

Find f_{xy}.

$f_x=2xe^{x^2+\underline{y}^2+1}$

$f_{xy}=2x\left(2ye^{x^2+y^2+1}\right)$

$=4xye^{x^2+y^2+1}.$

Find f_{yy}.

$f_y=2\underline{y}e^{x^2+\underline{y}^2+1}$

$f_{yy}=2y\left(2ye^{x^2+y^2+1}\right)+2e^{x^2+y^2+1}$

$=4y^2e^{x^2+y^2+1}+2e^{x^2+y^2+1}.$

Solve the system of equations

$f_x=0$ and $f_y=0$:

$2xe^{x^2+y^2+1}=0, \qquad 2ye^{x^2+y^2+1}=0,$

$x=0. \qquad y=0.$

The only critical point is $(0,0)$.

We must check $(0,0)$ to see whether it yields a maximum or minimum value.

For $(0,0)$

$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$

$D = 2e\cdot(2e) - (0)^2 \quad \begin{bmatrix} f_{xx}(0,0) = 2e \\ f_{yy}(0,0) = 2e \\ f_{xy}(0,0) = 0 \end{bmatrix}$

$D = 4e^2.$

Thus, $D > 0$ and $f_{xx}(0,0) = 2e > 0$, it follows that f has a relative minimum at $(0,0)$. The minimum is found as follows:

$f(x,y) = e^{x^2+y^2+1}$

$f(0,0) = e^{0^2+0^2+1}$

$= e.$

The relative minimum value of f is e at $(0,0)$.

14. $f(x,y) = e^{x^2-2x+y^2-4y+2}$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:

$f_x = (2x-2)e^{x^2-2x+y^2-4y+2}$

$f_{xx} = (2x-2)(2x-2)e^{x^2-2x+y^2-4y+2} + 2e^{x^2-2x+y^2-4y+2}$

$= (4x^2-8x+6)e^{x^2-2x+y^2-4y+2}$

$f_y = (2y-4)e^{x^2-2x+y^2-4y+2}$

$f_{yy} = (2y-4)(2y-4)e^{x^2-2x+y^2-4y+2} + 2e^{x^2-2x+y^2-4y+2}$

$= (4y^2-16y+18)e^{x^2-2x+y^2-4y+2}$

$f_{xy} = (2x-2)(2y-4)e^{x^2-2x+y^2-4y+2}$

2. Solve the system of equations $f_x = 0$ and $f_y = 0$:

$(2x-2)e^{x^2-2x+y^2-4y+2} = 0$

$2x-2 = 0$

$x = 1.$

$(2y-4)e^{x^2-2x+y^2-4y+2} = 0$

$2y-4 = 0$

$y = 2.$

Thus, $(1,2)$ is a critical point, and $f(1,2)$ is a candidate for a maximum or minimum.

3. We must check to see if $f(1,2)$ is a maximum or minimum value:

$D = f_{xx}(1,2)\cdot f_{yy}(1,2) - \left[f_{xy}(1,2)\right]^2$

$D = 2e^{-3}\cdot 2e^{-3} - 0^2$

$D = 4e^{-6}.$

4. Since $D > 0$ and $f_{xx}(1,2) = 2e^{-3} > 0$, it follows that f has a relative minimum at $(1,2)$. The minimum is found as follows:

$f(1,2) = e^{1^2-2\cdot1+2^2-4\cdot2+2}$

$= e^{-3}$

The relative minimum value of f is e^{-3} at $(1,2)$.

15. $R(x,y) = 17x + 21y$

$C(x,y) = 4x^2 - 4xy + 2y^2 - 11x + 25y - 3$

Total profit, $P(x,y)$ is given by

$P(x,y)$

$= R(x,y) - C(x,y)$

$= (17x+21y) - (4x^2 - 4xy + 2y^2 - 11x + 25y - 3)$

$= -4x^2 + 4xy - 2y^2 + 28x - 4y + 3$

Find P_x.

$P(x,y) = -4\underline{x}^2 + 4\underline{x}y - 2y^2 + 28\underline{x} - 4y + 3$

$P_x = -8x + 4y + 28$

Find P_y.

$P(x,y) = -4x^2 + 4x\underline{y} - 2\underline{y}^2 + 28x - 4\underline{y} + 3$

$P_y = 4x - 4y - 4$

Find P_{xx} and P_{xy}.

$P_x = -8\underline{x} + 4y + 28 \qquad P_x = -8x + 4\underline{y} + 28$

$P_{xx} = -8. \qquad P_{xy} = 4.$

Find P_{yy}.

$P_y = 4x - 4\underline{y} - 4$

$P_{yy} = -4.$

Solve the system of equations $P_x = 0$ and $P_y = 0$:

$-8x + 4y + 28 = 0, \qquad (1)$

$4x - 4y - 4 = 0. \qquad (2)$

Adding these equations, we get:

$-4x + 24 = 0.$

Then,

$-4x = -24$

$x = 6.$

To find y when $x = 6$, we substitute 6 for x into either Eq. (1) or Eq. (2). We use Eq. (1):

$$\begin{aligned} -8(6) + 4y + 28 &= 0 \\ 4y - 20 &= 0 \\ 4y &= 20 \\ y &= 5. \end{aligned}$$

Thus, $(6,5)$ is the only critical point, and $P(6,5)$ is a candidate for a maximum or minimum value.

We must check to see whether $P(6,5)$ is a maximum or minimum value:

$$D = P_{xx}(6,5) \cdot P_{yy}(6,5) - \left[P_{xy}(6,5)\right]^2$$

$$= (-8)(-4) - 4^2 \qquad \begin{bmatrix} P_{xx}(6,5) = -8 \\ P_{yy}(6,5) = -4 \\ P_{xy}(6,5) = 4 \end{bmatrix}$$

$$= 32 - 16$$

$$= 16.$$

Thus, $D = 16$ and $P_{xx}(6,5) = -8$. Since $D > 0$ and $P_{xx}(6,5) < 0$, it follows that P has a relative maximum at $(6,5)$. Thus, to maximize profit, the company must produce and sell 6 thousand of the \$17 sunglasses and 5 thousand of the \$21 sunglasses.

16. $R(x,y) = 18x + 25y$

$C(x,y) = 4x^2 - 6xy + 3y^2 + 20x + 19y - 12$

$P(x,y) = R(x,y) - C(x,y)$

$= (18x + 25y) -$

$\left(4x^2 - 6xy + 3y^2 + 20x + 19y - 12\right)$

$= -4x^2 + 6xy - 3y^2 - 2x + 6y + 12$

1. Find P_x, P_y, P_{xx}, P_{yy}, and P_{xy}:

   $P_x = -8x + 6y - 2, \qquad P_y = 6x - 6y + 6,$

   $P_{xx} = -8; \qquad P_{yy} = -6;$

   $P_{xy} = 6.$

2. Solve the system of equations $P_x = 0$ and $P_y = 0$:

   $-8x + 6y - 2 = 0, \qquad (1)$

   $6x - 6y + 6 = 0. \qquad (2)$

   Adding these equations, we get
   $-2x + 4 = 0.$

   Then,
   $$\begin{aligned} -2x &= -4 \\ x &= 2. \end{aligned}$$
   To find y when $x = 2$, we substitute 2 for x into Eq. (1):
   $$\begin{aligned} -8(2) + 6y - 2 &= 0 \\ 6y &= 18 \\ y &= 3. \end{aligned}$$
   Thus, $(2,3)$ is the only critical point, and $P(2,3)$ is a candidate for a maximum or minimum value.

3. We must check to see whether $P(2,3)$ is a maximum or minimum value:

   $$\begin{aligned} D &= P_{xx}(2,3) \cdot P_{yy}(2,3) - \left[P_{xy}(2,3)\right]^2 \\ &= (-8)(-6) - 6^2 \qquad \text{Using step 1 above} \\ &= 48 - 36 \\ &= 12. \end{aligned}$$

4. Since $D > 0$ and $P_{xx}(2,3) = -8 < 0$, it follows that P has a relative maximum at $(2,3)$. Thus, to maximize profit, the company must produce and sell 2 thousand of the \$18 shirt and 3 thousand of the \$25 shirt.

17. $P(a,p) = 2ap + 80p - 15p^2 - \frac{1}{10}a^2p - 80$

Find P_a.

$$P(a,p) = 2\underline{a}p + 80p - 15p^2 - \frac{1}{10}\underline{a}^2p - 80,$$

$$P_a = 2p - \frac{1}{5}ap.$$

Find P_p.

$$P(a,p) = 2a\underline{p} + 80\underline{p} - 15\underline{p}^2 - \frac{1}{10}a^2\underline{p} - 80,$$

$$P_p = 2a + 80 - 30p - \frac{1}{10}a^2.$$

Find P_{aa} and P_{ap}.

$$P_a = 2p - \frac{1}{5}\underline{a}p \qquad P_a = 2\underline{p} - \frac{1}{5}a\underline{p}$$

$$P_{aa} = -\frac{1}{5}p. \qquad P_{ap} = 2 - \frac{1}{5}a.$$

Find P_{pp}.

$$P_p = 2a + 80 - 30\underline{p} - \frac{1}{10}a^2$$

$$P_{pp} = -30.$$

Solve the system of equations
$P_a = 0$ and $P_p = 0$:

$$2p - \frac{1}{5}ap = 0, \qquad (1)$$

$$2a + 80 - 30p - \frac{1}{10}a^2 = 0. \qquad (2)$$

Solving Eq. (1) by factoring, we see that $a = 10$ or $p = 0$. But p cannot equal 0 in the original equation and yield a positive profit. Substituting 10 for a in Eq. (2) and solving for p, we get

$$\begin{aligned} 2(10) + 80 - 30p - \frac{1}{10}(10)^2 &= 0 \\ 20 + 80 - 10 &= 30p \\ 90 &= 30p \\ 3 &= p. \end{aligned}$$

Thus, $(10,3)$ is the only critical point to consider, and $P(10,3)$ is a candidate for a maximum or minimum value.
We must check to see whether $P(10,3)$ is a maximum or minimum value:

$$\begin{aligned} D &= P_{aa}(10,3) \cdot P_{pp}(10,3) - \left[P_{ap}(10,3)\right]^2 \\ &= \left(-\frac{3}{5}\right)(-30) - 0^2 \\ &\qquad \begin{bmatrix} P_{aa}(10,3) = -\frac{1}{5} \cdot 3 = \frac{-3}{5} \\ P_{pp}(10,3) = -30 \\ P_{ap}(10,3) = 2 - \frac{1}{5} \cdot 10 = 0 \end{bmatrix} \\ &= 18. \end{aligned}$$

Since $D > 0$ and $P_{aa}(10,3) = -\frac{3}{5} < 0$, it follows that P has a relative maximum at $(10,3)$. Thus, to maximize profit, the company must spend 10 million dollars on advertising and charge \$3 per item. The maximum profit is found as follows:

$$\begin{aligned} P(10,3) &= 2(10)(3) + 80(3) - 15(3)^2 - \\ &\qquad \frac{1}{10}(10)^2(3) - 80 \\ &= 60 + 240 - 135 - 30 - 80 \\ &= 55. \end{aligned}$$

The maximum profit is \$55 million.

18. $P(a,n) = -5a^2 - 3n^2 + 48a - 4n + 2an + 290$

1. Find P_a, P_n, P_{aa}, P_{nn}, and P_{an}:

   $P_a = -10a + 48 + 2n, \quad P_n = -6n - 4 + 2a,$
   $P_{aa} = -10; \quad P_{nn} = -6;$
   $P_{an} = 2.$

2. Solve the system of equations
   $P_a = 0$ and $P_n = 0$:

   $$-10a + 48 + 2n = 0, \qquad (1)$$
   $$-6n - 4 + 2a = 0. \qquad (2)$$

   Multiply Eq. (2) by 5 and add it to Eq. (1):

   $$\begin{aligned} -28n + 28 &= 0 \\ -28n &= -28 \\ n &= 1 \end{aligned}$$

   To find a when $n = 1$, we substitute 1 for n into Eq. (2):

   $$\begin{aligned} -6(1) - 4 + 2a &= 0 \\ 2a &= 10 \\ a &= 5. \end{aligned}$$

   Thus, $(5,1)$ is the only critical point, and $P(5,1)$ is a candidate for a maximum or minimum value.

3. We must check to see whether $P(5,1)$ is a maximum or minimum value:

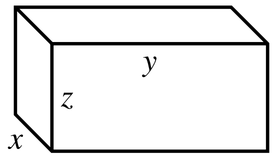
   $$\begin{aligned} D &= P_{aa}(5,1) \cdot P_{nn}(5,1) - \left[P_{an}(5,1)\right]^2 \\ &= (-10)(-6) - 2^2 \qquad \text{Using step 1 above} \\ &= 60 - 4 \\ &= 56. \end{aligned}$$

4. Since $D > 0$ and $P_{aa}(5,1) = -10 < 0$, it follows that P has a relative maximum at $(5,1)$. Thus, to maximize profit, the company must spend 5 million dollars in advertising and sell 1 thousand items. The maximum profit is:

   $$\begin{aligned} P(5,1) &= -5(5)^2 - 3(1)^2 + 48(5) - 4(1) + \\ &\qquad 2(5)(1) + 290 \\ &= 408. \end{aligned}$$

   The maximum profit is \$408 million.

19. Sketch a drawing of the container.



Let x, y and z represent the dimensions of the container as shown in the drawing.

$V = x \cdot y \cdot z$

$320 = x \cdot y \cdot z \qquad [V = 320 \text{ ft}^3]$

$\frac{320}{x \cdot y} = z$

Now we can express the cost as a function of two variables x and y. The area of the bottom is $xy \text{ ft}^2$, so the cost of the bottom is $5xy$, two of the sides have area xz, or $x\left(\frac{320}{xy}\right) = \frac{320}{y}$ each. The area of each of the remaining two sides is yz, or $y\left(\frac{320}{xy}\right) = \frac{320}{x}$. Then, the total area of all four sides is $2\left(\frac{320}{y} + \frac{320}{x}\right) = \frac{640}{y} + \frac{640}{x}$, and the cost of the four sides is $4\left(\frac{640}{y} + \frac{640}{x}\right)$, or $\frac{2560}{y} + \frac{2560}{x}$.

Now we can write the total cost function.

$\underset{\text{cost}}{\text{Total}} = \underset{\text{bottom}}{\text{Cost of}} + \underset{\text{sides}}{\text{Cost of}}$

$$C(x, y) = 5xy + \left(\frac{2560}{y} + \frac{2560}{x}\right).$$

Now, we try to find a minimum for $C(x, y)$

1. Find C_x, C_y, C_{xx}, C_{yy}, and C_{xy}:

$$C_x = 5y - \frac{2560}{x^2}, \qquad C_y = 5x - \frac{2560}{y^2},$$

$$C_{xx} = \frac{5120}{x^3}; \qquad C_{yy} = \frac{5120}{y^3};$$

$$C_{xy} = 5.$$

2. Solve the system of equations $C_x = 0$ and $C_y = 0$:

$$5y - \frac{2560}{x^2} = 0, \qquad (1)$$

$$5x - \frac{2560}{y^2} = 0. \qquad (2)$$

Solving Eq. (1) for y:

$$5y - \frac{2560}{x^2} = 0$$

$$5y = \frac{2560}{x^2}$$

$$y = \frac{512}{x^2}$$

Substitute $\frac{512}{x^2}$ for y into Eq. (2) and solve for x:

$$5x - \frac{2560}{\left(\frac{512}{x^2}\right)^2} = 0$$

$$5x - \frac{2560}{\frac{262{,}144}{x^4}} = 0$$

$$5x - \frac{2560x^4}{262{,}144} = 0$$

$$5x - \frac{5x^4}{512} = 0$$

$$2560x - 5x^4 = 0 \qquad \text{Multiplying by 512.}$$

$$5x(512 - x^3) = 0$$

$5x = 0$ or $512 - x^3 = 0$

$x = 0$ or $x^3 = 512$

$x = 0$ or $x = 8$.

Since none of the dimensions can be 0, only $x = 8$ has meaning in this application.

Substitute 8 for x into Eq. (1) to find y:

$$5y - \frac{2560}{(8)^2} = 0$$

$$5y - \frac{2560}{64} = 0$$

$$5y - 40 = 0$$

$$5y = 40$$

$$y = 8.$$

Thus, $(8,8)$ is the only critical point, and $C(8,8)$ is a candidate for a maximum or minimum value.

3. We must check to see whether $C(8,8)$ is a maximum or minimum value:

$$D = C_{xx}(8,8)\cdot C_{yy}(8,8) - \left[C_{xy}(8,8)\right]^2$$
$$= \left(\frac{5120}{8^3}\right)\left(\frac{5120}{8^3}\right) - 5^2 \quad \text{Using step 1 above}$$
$$= \frac{5120}{512}\cdot\frac{5120}{512} - 25$$
$$= 10\cdot 10 - 25$$
$$= 100 - 25$$
$$= 75.$$

4. Since $D > 0$ and $C_{xx}(8,8) = 10 > 0$, it follows that C has a relative minimum at $(8,8)$. Thus, to minimize cost, the dimensions of the bottom of the container should be 8 ft. by 8 ft. The height of the container should be $\frac{320}{8\cdot 8}$, or 5 ft.

20. a) $q_1 = 78 - 6p_1 - 3p_2 \quad (1)$

$q_2 = 66 - 3p_1 - 6p_2 \quad (2)$

$$R(p_1, p_2)$$
$$= p_1q_1 + p_2q_2$$
$$= p_1(78 - 6p_1 - 3p_2) + p_2(66 - 3p_1 - 6p_2)$$
$$= 78p_1 - 6p_1^2 - 3p_1p_2 + 66p_2 - 3p_1p_2 - 6p_2^2$$
$$= 78p_1 - 6p_1^2 - 6p_1p_2 + 66p_2 - 6p_2^2.$$

b) We now find the values of p_1 and p_2 to maximize total revenue.

$R_{p_1} = 78 - 12p_1 - 6p_2,$

$R_{p_2} = -6p_1 + 66 - 12p_2,$

$R_{p_1p_1} = -12,$

$R_{p_2p_2} = -12,$

$R_{p_1p_2} = -6.$

Solve the system of equations $R_{p_1} = 0$ and $R_{p_2} = 0$:

$78 - 12p_1 - 6p_2 = 0$

$-6p_1 + 66 - 12p_2 = 0$

The solution to this system is $p_1 = 5$ and $p_2 = 3$.

We check to see if $R(5,3)$ is a maximum or a minimum value.

$$D = R_{p_1p_1}(5,3)\cdot R_{p_2p_2}(5,3) - \left[R_{p_1p_2}(5,3)\right]^2$$
$$= (-12)(-12) - (-6)^2$$
$$= 144 - 36$$
$$= 108.$$

Since $D > 0$ and $R_{p_1p_1}(5,3) = -12 < 0$, it follows that R has a relative maximum at $(5,3)$. Thus, in order to maximize revenue, p_1 must be $5\cdot 10 = \$50$ and p_2 must be $3\cdot 10 = \$30$.

c) We substitute 5 for p_1 and 3 for p_2 into the demand equations to find q_1 and q_2.

$$q_1 = 78 - 6p_1 - 3p_2$$
$$q_1 = 78 - 6(5) - 3(3)$$
$$= 78 - 30 - 9$$
$$= 39$$
$$q_2 = 66 - 3p_1 - 6p_2$$
$$q_2 = 66 - 3(5) - 6(3)$$
$$= 66 - 15 - 18$$
$$= 33$$

39 hundred units of q_1 will be demanded and 33 hundred units of q_2 will be demanded.

d) To maximize revenue 3900 units of the \$50 calculator and 3300 units of the \$30 calculator must be produced and sold. The maximum revenue is found as follows:

$$R = 50\cdot 3900 + 30\cdot 3300$$
$$= 195{,}000 + 99{,}000$$
$$= 294{,}000.$$

The maximum revenue is \$294,000.

21. a) $q_1 = 64 - 4p_1 - 2p_2 \quad (1)$

$q_2 = 56 - 2p_1 - 4p_2 \quad (2)$

$$R(p_1, p_2)$$
$$= p_1q_1 + p_2q_2$$
$$= p_1(64 - 4p_1 - 2p_2) + p_2(56 - 2p_1 - 4p_2)$$
$$= 64p_1 - 4p_1^2 - 2p_1p_2 + 56p_2 - 2p_1p_2 - 4p_2^2$$
$$= 64p_1 - 4p_1^2 - 4p_1p_2 + 56p_2 - 4p_2^2.$$

b) We now find the values of p_1 and p_2 to maximize total revenue.

$R_{p_1} = 64 - 8p_1 - 4p_2,$

$R_{p_2} = -4p_1 + 56 - 8p_2,$

$R_{p_1p_1} = -8,$

$R_{p_2p_2} = -8,$

$R_{p_1p_2} = -4.$

Solve the system of equations

$R_{p_1} = 0$ and $R_{p_2} = 0$:

$64 - 8p_1 - 4p_2 = 0$

$-4p_1 + 56 - 8p_2 = 0$

The solution to this system is

$p_1 = 6$ and $p_2 = 4.$

We check to see if $R(6,4)$ is a maximum or a minimum value.

$$\begin{aligned}D &= R_{p_1p_1}(6,4) \cdot R_{p_2p_2}(6,4) - \left[R_{p_1p_2}(6,4)\right]^2 \\ &= (-8)(-8) - (-4)^2 \\ &= 64 - 16 \\ &= 48.\end{aligned}$$

Since $D > 0$ and $R_{p_1p_1}(6,4) = -8 < 0$, it follows that R has a relative maximum at $(6,4)$. Thus, in order to maximize revenue, p_1 must be $6 \cdot 10 = \$60$ and p_2 must be $4 \cdot 10 = \$40$.

c) We substitute 6 for p_1 and 4 for p_2 into the demand equations to find q_1 and q_2.

$$\begin{aligned}q_1 &= 64 - 4p_1 - 2p_2 \\ q_1 &= 64 - 4(6) - 2(4) \\ &= 64 - 24 - 8 \\ &= 32\end{aligned}$$

$$\begin{aligned}q_2 &= 56 - 2p_1 - 4p_2 \\ q_2 &= 56 - 2(6) - 4(4) \\ &= 56 - 12 - 16 \\ &= 28\end{aligned}$$

32 hundred units of q_1 will be demanded and 28 hundred units of q_2 will be demanded.

d) To maximize revenue 3200 units of the \$60 calculator and 2800 units of the \$40 calculator must be produced and sold. The maximum revenue is found as follows:

$$\begin{aligned}R &= 60 \cdot 3200 + 40 \cdot 2800 \\ &= 192,000 + 112,000 \\ &= 304,000.\end{aligned}$$

The maximum revenue is \$304,000.

22. $T(x,y) = x^2 + 2y^2 - 8x + 4y$

1. Find $T_x, T_y, T_{xx}, T_{yy},$ and T_{xy}:

$T_x = 2x - 8, \qquad T_y = 4y + 4,$

$T_{xx} = 2; \qquad T_{yy} = 4;$

$T_{xy} = 0.$

2. Solve the system of equations $T_x = 0$ and $T_y = 0$:

$2x - 8 = 0 \quad 4y + 4 = 0$

$2x = 8 \qquad 4y = -4$

$x = 4 \qquad y = -1$

Thus, $(4,-1)$ is the only critical point, and $T(4,-1)$ is a candidate for a maximum or minimum value.

3. We must check to see whether $T(4,-1)$ is a maximum or minimum value:

$$\begin{aligned}D &= T_{xx}(4,-1) \cdot T_{yy}(4,-1) - \left[T_{xy}(4,-1)\right]^2 \\ &= (2)(4) - 0^2 \qquad \text{Using step 1 above.} \\ &= 8.\end{aligned}$$

4. Since $D > 0$ and $T_{xx}(4,-1) = 2 > 0$, it follows that T has a relative minimum at $(4,-1)$. The minimum value is found as follows:

$$\begin{aligned}T(4,-1) &= 4^2 + 2(-1)^2 - 8 \cdot 4 + 4(-1) \\ &= 16 + 2 - 32 - 4 \\ &= -18.\end{aligned}$$

Thus, the minimum temperature is $-18°F$ at $(4,-1)$. There is no maximum temperature.

23. $f(x,y) = e^x + e^y - e^{x+y}$

1. Find $f_x, f_y, f_{xx}, f_{yy},$ and f_{xy}:

$f_x = e^x - e^{x+y}$

$f_y = e^y - e^{x+y}$

$f_{xx} = e^x - e^{x+y}$

$f_{yy} = e^y - e^{x+y}$

$f_{xy} = -e^{x+y}$

2. Solve the system of equations
$f_x = 0$ and $f_y = 0$:
$$e^x - e^{x+y} = 0 \qquad (1)$$
$$e^y - e^{x+y} = 0 \qquad (2)$$
We can solve the first equation for *y*:
$$e^x - e^{x+y} = 0$$
$$e^x = e^{x+y}$$
$$x = x + y$$
$$y = 0.$$
We can solve the second equation for *x*:
$$e^y - e^{x+y} = 0$$
$$e^y = e^{x+y}$$
$$y = x + y$$
$$x = 0.$$
Thus, $(0,0)$ is a critical point, and $f(0,0)$ is a candidate for a maximum or minimum.
3. We must check to see if $f(0,0)$ is a maximum or minimum value:
$$D = f_{xx}(0,0)\cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$
$$D = 0\cdot 0 - (-1)^2$$
$$D = -1.$$
4. Since $D < 0$, it follows that $f(0,0)$ is neither a maximum nor a minimum, but a saddle point.

24. $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:
$$f_x = y - \frac{2}{x^2}, \qquad f_y = x - \frac{4}{y^2},$$
$$f_{xx} = \frac{4}{x^3}; \qquad f_{yy} = \frac{8}{y^3};$$
$$f_{xy} = 1.$$
2. Solve the system of equations
$f_x = 0$ and $f_y = 0$:
$$y - \frac{2}{x^2} = 0 \qquad (1)$$
$$x - \frac{4}{y^2} = 0 \qquad (2)$$
Solving Eq. (1) for *y*, we get $y = \frac{2}{x^2}$.
Substituting $\frac{2}{x^2}$ for *y* in Eq. (2) and solving, we get
$$x - \frac{4}{\left(\frac{2}{x^2}\right)^2} = 0$$
$$x - \frac{4}{\frac{4}{x^4}} = 0$$
$$x - x^4 = 0$$
$$x\left(1 - x^3\right) = 0$$
$$x = 0 \text{ or } x = 1.$$
Note that $x = 0$ is not in the domain of $f(x,y)$. We consider only $x = 1$.
Substitute 1 for *x* in either Eq. (1) and solve for y:
$$y - \frac{2}{1^2} = 0$$
$$y - 2 = 0$$
$$y = 2.$$
Thus, $(1,2)$ is a critical point, and $f(1,2)$ is a candidate for a maximum or minimum.
3. We must check to see if $f(1,2)$ is a maximum or minimum value:
$$D = f_{xx}(1,2)\cdot f_{yy}(1,2) - \left[f_{xy}(1,2)\right]^2$$
$$D = \frac{4}{1^3}\cdot\frac{8}{2^3} - 1^2$$
$$= 4\cdot 1 - 1$$
$$= 3.$$
4. Since $D > 0$ and $f_{xx}(1,2) = 4 > 0$, it follows that *f* has a relative minimum at $(1,2)$. The minimum is found as follows:
$$f(1,2) = 1\cdot 2 + \frac{2}{1} + \frac{4}{2}$$
$$= 2 + 2 + 2$$
$$= 6.$$
The relative minimum value of *f* is 6 at $(1,2)$.

25. $f(x,y) = 2y^2 + x^2 - x^2 y$

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}:
$$f_x = 2x - 2xy \qquad f_y = 4y - x^2$$
$$f_{xx} = 2 - 2y; \qquad f_{yy} = 4;$$
$$f_{xy} = -2x.$$

2. Solve the system of equations
   $f_x = 0$ and $f_y = 0$:
   $$2x - 2xy = 0, \qquad (1)$$
   $$4y - x^2 = 0. \qquad (2)$$
   Solving Eq. (2) for y, we get
   $$4y = x^2$$
   $$y = \frac{x^2}{4}$$
   Substituting $\frac{x^2}{4}$ for y in Eq. (1) and solving, we get
   $$2x - 2x \cdot \frac{x^2}{4} = 0$$
   $$2x - \frac{x^3}{2} = 0$$
   $$4x - x^3 = 0$$
   $$x(4 - x^2) = 0$$
   $$x = 0 \quad \text{or} \quad 4 - x^2 = 0$$
   $$x = 0 \quad \text{or} \quad x^2 = 4$$
   $$x = 0 \quad \text{or} \quad x = \pm 2$$
   When $x = 0,\ y = \frac{0^2}{4} = 0.$
   When $x = 2,\ y = \frac{2^2}{4} = 1.$
   When $x = -2,\ y = \frac{(-2)^2}{4} = 1.$
   The critical points are $(0,0), (2,1),$ and $(-2,1)$
3. We must check all the critical points to determine whether they yield maximum or minimum values.
   For $(0,0)$
   $$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2$$
   $$D = (2)\cdot(4) - 0^2 \qquad \begin{bmatrix} f_{xx}(0,0) = 2 \\ f_{yy}(0,0) = 4 \\ f_{xy}(0,0) = 0 \end{bmatrix}$$
   $$D = 8.$$
   Since $D > 0$ and $f_{xx}(0,0) = 2 > 0$, it follows that f has a relative minimum at $(0,0)$. The minimum is found as follows:
   $$f(x,y) = 2y^2 + x^2 - x^2 y$$
   $$f(0,0) = 2\cdot 0^2 + 0^2 - 0^2 \cdot 0 = 0$$
   The relative minimum value of f is 0 at $(0,0)$.
   For $(2,1)$
   $$D = f_{xx}(2,1) \cdot f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2$$
   $$D = (0)\cdot(4) - (-4)^2 \qquad \begin{bmatrix} f_{xx}(2,1) = 0 \\ f_{yy}(2,1) = 4 \\ f_{xy}(2,1) = -4 \end{bmatrix}$$
   $$D = -16.$$
   For $(-2,1)$
   $$D = f_{xx}(-2,1) \cdot f_{yy}(-2,1) - \left[f_{xy}(-2,1)\right]^2$$
   $$D = (0)\cdot(4) - (4)^2 \qquad \begin{bmatrix} f_{xx}(-2,1) = 0 \\ f_{yy}(-2,1) = 4 \\ f_{xy}(-2,1) = 4 \end{bmatrix}$$
   $$D = -16.$$
   Since $D < 0$ for both $(2,1)$ and $(-2,1)$, it follows that f has neither a maximum nor a minimum, but a saddle point at both of these points. Therefore, the only relative extrema of f is a relative minimum of 0 occurring at $(0,0)$.

26. $S(b,m) = (m + b - 72)^2 + (2m + b - 73)^2 + (3m + b - 75)^2$

1. Find S_b, S_m, S_{bb}, S_{mm}, and S_{bm}:
   $$\begin{aligned} S_b &= 2(m + b - 72) + 2(2m + b - 73) + 2(3m + b - 75) \\ &= 2m + 2b - 144 + 4m + 2b - 146 + 6m + 2b - 150 \\ &= 12m + 6b - 440 \end{aligned}$$
   $$S_{bb} = 6$$
   $$S_{bm} = 12$$
   $$\begin{aligned} S_m &= 2(m + b - 72)\cdot 1 + 2(2m + b - 73)\cdot 2 + 2(3m + b - 75)\cdot 3 \\ &= 2m + 2b - 144 + 8m + 4b - 292 + 18m + 6b - 450 \\ &= 28m + 12b - 886 \end{aligned}$$
   $$S_{mm} = 28.$$