Exercise Set 2.5

1. Express $Q = xy$ as a function of one variable. First, we solve $x + y = 50$ for y.

$x + y = 50$
$y = 50 - x$

Next, we substitute $50 - x$ for y in $Q = xy$.

$Q = xy$
$Q = x(50 - x)$ Substituting
$= 50x - x^2$

Now that Q is a function of one variable we can find the maximum. First, we find the critical values.

$Q'(x) = 50 - 2x$. Since $Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We solve;

$50 - 2x = 0$
$50 = 2x$
$25 = x$

There is only one critical value. We use the second derivative to determine if the critical value is a maximum. Note that:

$Q''(x) = -2 < 0$. The second derivative is negative for all values of x. Therefore, a maximum occurs at $x = 25$.

Now,

$Q(25) = 50(25) - (25)^2 = 625$

Therefore, the maximum product is 625, which occurs when $x = 25$. If $x = 25$, then $y = 50 - 25 = 25$. The two numbers are 25 and 25.

2. $x + y = 70$, so $y = 70 - x$.

$Q = xy = x(70 - x) = 70x - x^2$
$Q(x) = 70x - x^2$
$Q'(x) = 70 - 2x$

$Q'(x)$ exists for all real numbers. Solve:

$Q'(x) = 0$
$70 - 2x = 0$
$x = 35$

$Q''(x) = -2 < 0$ for all values of x, so a maximum occurs at $x = 35$.

$Q(35) = 70(35) - (35)^2 = 1225$

Thus, the maximum product is 1225 when $x = 35$ and $y = 70 - 35 = 35$.

3. $\boxed{tw}$ Since $Q = x(50 - x)$ has no minimum, there is no minimum product.

4. $\boxed{tw}$ Since $Q = x(70 - x)$ has no minimum, there is no minimum product.

5. Let x be one number and y be the other number. Since the difference of the two numbers must be 4, we have $x - y = 4$.

The product, Q, of the two numbers is given by $Q = xy$, so our task is to minimize $Q = xy$, where $x - y = 4$.

First, we express $Q = xy$ as a function of one variable.

Solving $x - y = 4$ for y, we have:

$x - y = 4$
$-y = 4 - x$
$y = x - 4$

Next, we substitute $x - 4$ for y in $Q = xy$.

$Q = x(x - 4) = x^2 - 4x$
$Q(x) = x^2 - 4x$

Find $Q'(x) = 2x - 4$

The derivative exists for all values of x; thus, the only critical values are where $Q'(x) = 0$.

$2x - 4 = 0$
$2x = 4$
$x = 2$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.

$Q''(x) = 2 > 0$ for all values of x. Therefore, a minimum occurs at $x = 2$.

$Q(2) = (2)^2 - 4(2) = -4$

Thus, the minimum product is -4 when $x = 2$. Substitute 2 for x in $y = x - 4$ to find y.

$y = 2 - 4 = -2$.

The two numbers which have the minimum product are 2 and -2.

6. Let x be one number and y be the other number. The product, Q, of the two numbers is given by $Q = xy$, so our task is to minimize $Q = xy$, where $x - y = 6$.

First, we express $Q = xy$ as a function of one variable.

Solving $x - y = 6$ for *y*, we have $y = x - 6$, and
$Q(x) = x(x-6) = x^2 - 6x$
$Q'(x) = 2x - 6$
The derivative exists for all values of *x*; thus, the only critical values are where $Q'(x) = 0$.

$$2x - 6 = 0$$
$$2x = 6$$
$$x = 3$$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.
$Q''(x) = 2 > 0$ for all values of *x*. Therefore, a minimum occurs at $x = 3$.
$Q(3) = (3)^2 - 6(3) = -9$
Thus, the minimum product is -9 when $x = 3$, and $y = 3 - 6 = -3$.

7. Maximize $Q = xy^2$, where *x* and *y* are positive numbers such that $x + y^2 = 1$.
Express $Q = xy^2$ as a function of one variable.
First, we solve $x + y^2 = 1$ for y^2.

$$x + y^2 = 1$$
$$y^2 = 1 - x$$

Next, we substitute $1 - x$ for y^2 in $Q = xy^2$.

$Q = xy^2$
$Q = x(1-x)$ Substituting
$= x - x^2$

Now that *Q* is a function of one variable we can find the maximum. First, we find the critical values.
$Q'(x) = 1 - 2x$. Since $Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We solve;

$$1 - 2x = 0$$
$$1 = 2x$$
$$\frac{1}{2} = x$$

There is only one critical value. We use the second derivative to determine if the critical value is a maximum. Note that:
$Q''(x) = -2 < 0$. The second derivative is negative for all values of *x*. Therefore, a maximum occurs at $x = \frac{1}{2}$.

Now,
$$Q\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
Substitute $\frac{1}{2}$ in for *x* in $x + y^2 = 1$ and solve for *y*.

$$\frac{1}{2} + y^2 = 1$$
$$y^2 = \frac{1}{2}$$
$$y = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

$y = \frac{1}{\sqrt{2}}$, x and y must be positive

Therefore, the maximum value of Q is $\frac{1}{4}$ when $x = \frac{1}{2}$ and $y = \frac{1}{\sqrt{2}}$.

8. Maximize $Q = xy^2$, where *x* and *y* are positive numbers such that $x + y^2 = 4$.
Express $Q = xy^2$ as a function of one variable

$$x + y^2 = 4$$
$$y^2 = 4 - x$$
$$Q = xy^2$$
$$Q = x(4-x) = 4x - x^2$$
$$Q'(x) = 4 - 2x.$$

$Q'(x)$ exists for all real numbers, the only critical value will occur when $Q'(x) = 0$. We solve;

$$4 - 2x = 0$$
$$x = 2$$

There is only once critical value. We use the second derivative to determine if the critical value is a maximum. Note that:
$Q''(x) = -2 < 0$. The second derivative is negative for all values of *x*. Therefore, a maximum occurs at $x = 2$.
$Q(2) = (4)(2) - (2)^2 = 4$
When $x = 2$,

$$y^2 = 4 - 2$$
$$y^2 = 2$$

$y = \sqrt{2}$ x and y are positive.

Then *Q* is a maximum when $x = 2$ and $y = \sqrt{2}$.

9. Minimize $Q = 2x^2 + 3y^2$, where $x + y = 5$.
Express Q as a function of one variable. First, solve $x + y = 5$ for y.

$$x + y = 5$$
$$y = 5 - x$$

Then substitute $5 - x$ for y in $Q = 2x^2 + 3y^2$.

$$\begin{aligned} Q &= 2x^2 + 3(5-x)^2 \\ &= 2x^2 + 3(25 - 10x + x^2) \\ &= 2x^2 + 75 - 30x + 3x^2 \\ &= 5x^2 - 30x + 75 \end{aligned}$$

Find $Q'(x)$, where $Q(x) = 5x^2 - 30x + 75$.

$$Q'(x) = 10x - 30$$

This derivative exists for all values of x; thus the only critical values are where

$$\begin{aligned} Q'(x) &= 0 \\ 10x - 30 &= 0 \\ 10x &= 30 \\ x &= 3 \end{aligned}$$

Since there is only one critical value, we can use the second derivative to determine whether we have a minimum. Note that:
$Q''(x) = 10$, which is positive for all real numbers. Thus $Q''(3) > 0$, so a minimum occurs when $x = 3$. The value of Q is

$$\begin{aligned} Q(3) &= 2(3)^2 + 3(5-3)^2 \\ &= 18 + 12 \\ &= 30 \end{aligned}$$

Substitute 3 for x in $y = 5 - x$ to find y.

$$\begin{aligned} y &= 5 - x \\ y &= 5 - 3 \\ y &= 2 \end{aligned}$$

Thus, the minimum value of Q is 30 when $x = 3$ and $y = 2$.

10. $x + y = 3$, so $y = 3 - x$.

$$\begin{aligned} Q &= x^2 + 2y^2 = x^2 + 2(3-x)^2 \\ &= 3x^2 - 12x + 18 \end{aligned}$$

$$Q'(x) = 6x - 12$$

$Q'(x)$ exists for all real numbers. Solve:

$$\begin{aligned} Q'(x) &= 0 \\ 6x - 12 &= 0 \\ x &= 2 \end{aligned}$$

There is one critical value, we use the second derivative to determine if it is a minimum.
$Q''(x) = 6 > 0$ for all values of x, therefore, $Q''(2) > 0$ and a minimum occurs when $x = 2$.
When $x = 2$, $Q(2) = 2^2 + 2(3-2)^2 = 6$.
When $x = 2$, $y = 3 - 2 = 1$.
Therefore, the minimum value of Q is 6 when $x = 2$ and $y = 1$.

11. Maximize $Q = xy$, where x and y are positive numbers such that $\frac{4}{3}x^2 + y = 16$.
Express Q as a function of one variable. First, solve $\frac{4}{3}x^2 + y = 16$ for y.

$$\frac{4}{3}x^2 + y = 16$$
$$y = 16 - \frac{4}{3}x^2$$

Then substitute $16 - \frac{4}{3}x^2$ for y in $Q = xy$.

$$\begin{aligned} Q &= x\left(16 - \frac{4}{3}x^2\right) \\ &= 16x - \frac{4}{3}x^3 \end{aligned}$$

Find $Q'(x)$, where $Q(x) = 16x - \frac{4}{3}x^3$.

$$Q'(x) = 16 - 4x^2$$

This derivative exists for all values of x; thus the only critical values are where

$$\begin{aligned} Q'(x) &= 0 \\ 16 - 4x^2 &= 0 \\ -4x^2 &= -16 \\ x^2 &= 4 \\ x &= \pm 2 \\ x &= 2 \qquad x \text{ must be positive} \end{aligned}$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum. Note that:
$Q''(x) = -8x$
and
$Q''(2) = -8(2) = -16 < 0$.

Since $Q''(2)$ is negative, a maximum occurs at $x = 2$.

$$Q(2) = 16(2) - \frac{4}{3}(2)^3$$
$$= 32 - \frac{32}{3}$$
$$= \frac{64}{3}$$

Substitute 2 for *x* in $y = 16 - \frac{4}{3}x^2$ to find *y*.

$$y = 16 - \frac{4}{3}x^2$$
$$y = 16 - \frac{4}{3}(2)^2$$
$$y = 16 - \frac{16}{3}$$
$$y = \frac{32}{3}$$

Thus, the maximum value of *Q* is $\frac{64}{3}$ when $x = 2$ and $y = \frac{32}{3}$.

12. $x + \frac{4}{3}y^2 = 1$, so $x = 1 - \frac{4}{3}y^2$.

$$Q = xy = \left(1 - \frac{4}{3}y^2\right)y = y - \frac{4}{3}y^3$$
$$Q'(y) = 1 - 4y^2$$

$Q'(y)$ exists for all real numbers. Solve:

$$Q'(y) = 0$$
$$1 - 4y^2 = 0$$
$$y = \pm\frac{1}{2}$$

Only $\frac{1}{2}$ is in the domain of *Q*.

$$Q''(y) = -8y$$

When $y = \frac{1}{2}$, $Q''\left(\frac{1}{2}\right) = -8\left(\frac{1}{2}\right) = -4$, so a maximum occurs when $y = \frac{1}{2}$.

When $y = \frac{1}{2}$,

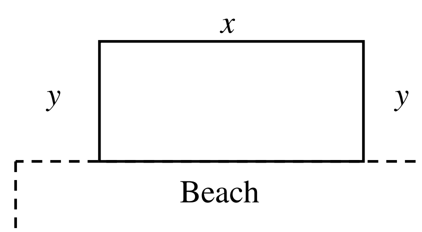
$$x = 1 - \frac{4}{3}\left(\frac{1}{2}\right)^2 = \frac{2}{3}$$

and

$$Q\left(\frac{1}{2}\right) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Therefore, the maximum value of *Q* is $\frac{1}{3}$ when $x = \frac{2}{3}$ and $y = \frac{1}{2}$.

13. Let *x* represent the length and *y* represent the width of the swimming area. It is helpful to draw a picture.



Since the life guard has 180 yd of rope and floats, the perimeter of the swimming area is $x + 2y = 180$. Solving this equation for *x*, we have $x = 180 - 2y$

The objective is to maximize area, which is given by

$$A = l \cdot w$$

Substituting $x = 180 - 2y$ for the length and *y* for the width, we have:

$$A = (180 - 2y)y = 180y - 2y^2.$$

We will maximize the area over the interval $0 < y < 90$, because *y* is the length of one side, and cannot be negative. Furthermore, since there is only 180 yards of rope, and we need two sides, *y* cannot be greater than 90 yards. If $y = 90$, then the length of the swimming area would be 0.

We now must find $A'(y)$, where

$$A(y) = 180y - 2y^2.$$
$$A'(y) = 180 - 4y$$

This derivative exists for all values of y in $(0,90)$. Thus the only critical values are where

$$A'(y)=0$$
$$180-4y=0$$
$$-4y=-180$$
$$y=45$$

Since there is only one critical value in the interval, we use the second derivative to determine whether we have a maximum. Note, $A''(y)=-4<0$ for all values of y, so there is a maximum at $y=45$.

Next, find the dimensions and the area.
When $y=45$,
we have $x=180-2(45)=90$
and
$A(45)=180(45)-2(45)^2=4050.$

Therefore, the maximum area is 4050 yd^2 when the overall dimensions are 45 yd by 90 yd.

14. Let x represent the width and y represent the length of the area.
The perimeter is $3x+y=240$, or $y=240-3x$. Since x and y must be positive, we are restricted to the interval $0<x<80$. Using the perimeter equation, we find the area as a function of x.

$$A(x)=xy=x(240-3x)=240x-3x^2$$
$$A'(x)=240-6x$$

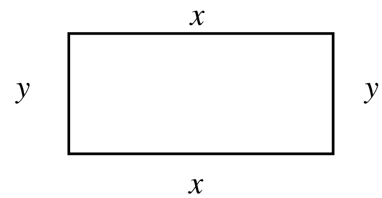
This derivative exists for all values of x. Solve:

$$A'(x)=0$$
$$240-6x=0$$
$$x=40$$

There is one critical value on the interval.
$A''(x)=-6<0$ for all values of x. So a maximum occurs at $x=40$.
When $x=40$,
$y=240-3x=240-3(40)=120$
and
$A(40)=240(40)-3(40)^2=4800.$

Therefore, the maximum area is 4800 yd^2 when the overall dimensions are 40 yd by 120 yd.

15. Let x represent the length and y represent the width. It is helpful to draw a picture.



The perimeter is found by adding up the length of the sides. Since it is fixed at 54 feet, the equation of the perimeter is $2x+2y=54$.
The area is given by $A=xy$.
First, we solve the perimeter equation for y.

$$2x+2y=54$$
$$2y=54-2x$$
$$y=27-x$$

Then we substitute for y into the area formula.

$$A=xy$$
$$=x(27-x)=27x-x^2$$

We want to maximize the area on the interval $(0,27)$. We consider this interval because x is the length of the shed and cannot be negative. Since the perimeter cannot exceed 54 feet, x cannot be greater than 27, also if x is 27 feet, the width of the shed would be 0 feet. We begin by finding $A'(x)$.

$A'(x)=27-2x$.

This derivative exists for all values of x in $(0,27)$. Thus, the only critical values occur where $A'(x)=0$. We solve the equation.

$$27-2x=0$$
$$-2x=-27$$
$$x=\frac{27}{2}=13.5$$

Since there is only one critical value in the interval, we can use the second derivative to determine whether we have a maximum. Note that
$A''(x)=-2<0$ for all values of x. Thus, $A''(13.5)<0$, so a maximum occurs at $x=13.5$.
Now,

$$A(x)=27x-x^2$$
$$A(13.5)=27(13.5)-(13.5)^2$$
$$=182.25$$

The maximum area is 182.25 ft^2.

Note: when $x = 13.5$, $y = 27 - 13.5 = 13.5$, so the overall dimensions that will achieve the maximum area are 13.5 ft by 13.5 ft.

16. The perimeter is $2l + 2w = 42$, so $l = 21 - w$
Since l and w must be positive, we are restricted to the interval $0 < w < 21$.

$A(w) = l \cdot w = (21 - w)w = 21w - w^2$

$A'(w) = 21 - 2w$

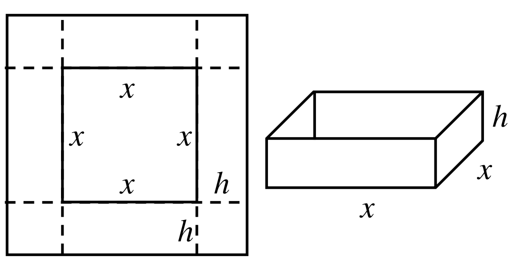
This derivative exists for all values of w. Solve:

$$A'(x) = 0$$
$$21 - 2w = 0$$
$$w = \frac{21}{2} = 10.5$$

There is one critical value on the interval.
$A''(w) = -2 < 0$ for all values of w. So a maximum occurs at $w = 10.5$.
When $w = 10.5$,
$l = 21 - 10.5 = 10.5$
and
$A(10.5) = 10.5(21 - 10.5) = 110.25.$

Therefore, the maximum area is 110.25 ft^2 when the overall dimensions are
10.5 ft by 10.5 ft.

17. When squares of length h on a side are cut out of the corners, we are left with a square base of length x. A picture will help.



The resulting volume of the box is
$V = lwh = x \cdot x \cdot h = x^2 h$.
We want to express V in terms of one variable. Note that the overall length of a side of the aluminum is 50 cm. We see from the drawing, that $h + x + h = 50, or\ x + 2h = 50.$ Solving for h we get:

$$2h = 50 - x$$
$$h = \frac{1}{2}(50 - x) = 25 - \frac{1}{2}x.$$

Substituting h into the volume equation, we have:

$V = x^2\left(25 - \frac{1}{2}x\right) = 25x^2 - \frac{1}{2}x^3$. The objective is to maximize $V(x)$ on the interval $(0, 50)$.
First, we find the derivative.

$$V'(x) = 50x - \frac{3}{2}x^2$$

This derivative exists for all x in the interval $(0, 50)$, so the critical values will occur when $V'(x) = 0$. Solving this equation, we have:

$$50x - \frac{3}{2}x^2 = 0$$
$$x\left(50 - \frac{3}{2}x\right) = 0$$
$$x = 0 \quad \text{or} \quad 50 - \frac{3}{2}x = 0$$
$$x = 0 \quad \text{or} \quad -\frac{3}{2}x = -50$$
$$x = 0 \quad \text{or} \quad x = \frac{100}{3} \approx 33\tfrac{1}{3}$$

The only critical value in $(0, 50)$ is $\frac{100}{3}$, or about 33.33. Therefore, we can use the second derivative $V''(x) = 50 - 3x$ to determine if we have a maximum. We have

$$V''\left(\frac{100}{3}\right) = 50 - 3\left(\frac{100}{3}\right) = -50 < 0.$$

Therefore, there is a maximum at $\frac{100}{3}$.

$$V\left(\frac{100}{3}\right) = 25\left(\frac{100}{3}\right)^2 - \frac{1}{2}\left(\frac{100}{3}\right)^3$$
$$= \frac{250{,}000}{27} = 9259\tfrac{7}{27}$$

Now, we find the height of the box.

$$h = 25 - \frac{1}{2}\left(\frac{100}{3}\right) = \frac{25}{3} = 8\tfrac{1}{3}.$$

Therefore, a box with dimensions $33\frac{1}{3}$ cm. by $33\frac{1}{3}$ cm. by $8\frac{1}{3}$ cm. will yield a maximum volume of $9259\frac{7}{27}\ \text{cm}^3$.

18. Using the drawing in Exercise 17, we see that
$V = lwh = x \cdot x \cdot h = x^2 h$.

$$h + x + h = 20, or\ h = 10 - \frac{1}{2}x.$$

Substituting h into the volume equation, we have:

$V = x^2\left(10 - \frac{1}{2}x\right) = 10x^2 - \frac{1}{2}x^3$. The objective is to maximize $V(x)$ on the interval $(0,20)$.

First, we find the derivative.

$V'(x) = 20x - \frac{3}{2}x^2$

This derivative exists for all x in the interval $(0,20)$, Solve:

$$\begin{aligned} V'(x) &= 0 \\ 20x - \frac{3}{2}x^2 &= 0 \\ x\left(20 - \frac{3}{2}x\right) &= 0 \end{aligned}$$

$x = 0 \quad \text{or} \quad x = \frac{40}{3} = 13\tfrac{1}{3}$

The only critical value in $(0,20)$ is $\frac{40}{3}$.

Therefore, we can use the second derivative $V''(x) = 20 - 3x$ to determine if we have a maximum. We have

$V''\left(\frac{40}{3}\right) = 20 - 3\left(\frac{40}{3}\right) = -20 < 0$.

Therefore, there is a maximum at $\frac{40}{3}$.

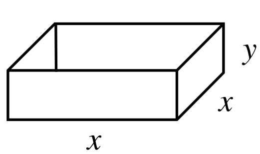
$$\begin{aligned} V\left(\frac{40}{3}\right) &= 10\left(\frac{40}{3}\right)^2 - \frac{1}{2}\left(\frac{40}{3}\right)^3 \\ &= \frac{16{,}000}{27} = 592\tfrac{16}{27} \end{aligned}$$

Now, we find the height of the box.

$h = 10 - \frac{1}{2}\left(\frac{40}{3}\right) = \frac{10}{3} = 3\tfrac{1}{3}$.

Therefore, a box with dimensions $13\tfrac{1}{3}$ in. by $13\tfrac{1}{3}$ in. by $3\tfrac{1}{3}$ in. will yield a maximum volume of $592\tfrac{16}{27}$ in^3.

19. First, we make a drawing.



The surface area of the open-top, square-based, rectangular box is found by adding the area of the base and the four sides. x^2 is the area of the base, xy is the area of one of the sides and there are four sides, therefore the surface area is given by $S = x^2 + 4xy$.

The volume must by 62.5 cubic inches, and is given by $V = l \cdot w \cdot h = x^2 y = 62.5$.

To express S in terms of one variable, we solve $x^2 y = 62.5$ for y:

$y = \frac{62.5}{x^2}$

Then

$$\begin{aligned} S(x) &= x^2 + 4x\left(\frac{62.5}{x^2}\right) \\ &= x^2 + \frac{250}{x} = x^2 + 250x^{-1} \end{aligned}$$

Now S is defined only for positive numbers, so we minimize S on the interval $(0,\infty)$.

First, we find $S'(x)$.

$$\begin{aligned} S'(x) &= 2x - 250x^{-2} \\ &= 2x - \frac{250}{x^2} \end{aligned}$$

Since $S'(x)$ exists for all x in $(0,\infty)$, the only critical values are where $S'(x) = 0$. We solve the following equation:

$$\begin{aligned} 2x - \frac{250}{x^2} &= 0 \\ 2x &= \frac{250}{x^2} \\ x^3 &= 125 \\ x &= 5 \end{aligned}$$

Since there is only one critical value, we use the second derivative to determine whether we have a minimum. Note that

$S''(x) = 2 + 500x^{-3} = 2 + \frac{500}{x^3}$.

$S''(5) = 2 + \frac{500}{5^3} = 6 > 0$. Since the second derivative is positive, we have a minimum at $x = 5$. We find y when $x = 5$.

$$\begin{aligned} y &= \frac{62.5}{x^2} \\ &= \frac{62.5}{5^2} \\ &= 2.5 \end{aligned}$$

The surface area is minimized when $x = 5$ in. and $y = 2.5$ in. We find the minimum surface area by substituting these values into the surface area equation.

$$\begin{aligned} S &= x^2 + 4xy \\ &= (5)^2 + 4(5)(2.5) \\ &= 25 + 50 \\ &= 75 \end{aligned}$$

The minimum surface area is 75 in^2 when the dimensions are 5 in. by 5 in. by 2.5 in.

20. Using the drawing in Exercise 19, we see that $S = x^2 + 4xy$ and $V = x^2 y = 32$.

Then $y = \dfrac{32}{x^2}$, and $S(x) = x^2 + 4x\left(\dfrac{32}{x^2}\right)$, or

$$S(x) = x^2 + \frac{128}{x} = x^2 + 128x^{-1}$$

We restrict the analysis to the interval $(0, \infty)$.

$$S'(x) = 2x - 128x^{-2} = 2x - \frac{128}{x^2}$$

$S'(x)$ exists for all values of x in $(0, \infty)$. Solve:

$$\begin{aligned} S'(x) &= 0 \\ 2x - \frac{128}{x^2} &= 0 \\ x^3 &= 64 \\ x &= 4 \end{aligned}$$

There is one critical value in the interval. We use the second derivative to determine if it is a minimum.

$$S''(x) = 2 + 256x^{-3} = 2 + \frac{256}{x^3}$$

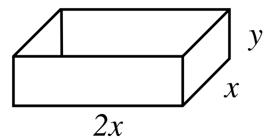
$S''(4) = 2 + \dfrac{256}{4^3} = 6 > 0$, so a minimum occurs when $x = 4$.

When $x = 4$, $y = \dfrac{32}{4^2} = 2$.

Therefore,
$S(4) = 4^2 + 4 \cdot 4 \cdot 2 = 48.$

Therefore, when the dimensions are 4 ft. by 4 ft. by 2 ft., the minimum surface area will be 48 ft^2.

21. First, we make a drawing.



The surface area of the open-top, square-based, rectangular dumpster is found by adding the area of the base and the four sides. $2x^2$ is the area of the base, xy is the area of two of the sides, while $2xy$ is the area of the other two sides. Therefore the surface area is given by $S = 2x^2 + 2xy + 2(2xy) = 2x^2 + 6xy$.

The volume must by 12 cubic yards, and is given by $V = l \cdot w \cdot h = 2x \cdot x \cdot y = 2x^2 y = 12$.

To express S in terms of one variable, we solve $2x^2 y = 12$ for y:

$$y = \frac{6}{x^2}$$

Then

$$\begin{aligned} S(x) &= 2x^2 + 6x\left(\frac{6}{x^2}\right) \\ &= 2x^2 + \frac{36}{x} = 2x^2 + 36x^{-1} \end{aligned}$$

Now S is defined only for positive numbers, so we minimize S on the interval $(0, \infty)$.

First, we find $S'(x)$.

$$\begin{aligned} S'(x) &= 4x - 36x^{-2} \\ &= 4x - \frac{36}{x^2} \end{aligned}$$

Since $S'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $S'(x) = 0$. We solve the following equation:

$$\begin{aligned} 4x - \frac{36}{x^2} &= 0 \\ 4x &= \frac{36}{x^2} \\ x^3 &= 9 \\ x &= \sqrt[3]{9} \approx 2.08 \end{aligned}$$

Since there is only one critical value, we use the second derivative to determine whether we have a minimum. Note that

$$S''(x) = 4 + 72x^{-3} = 4 + \frac{72}{x^3}.$$

$S''\left(\sqrt[3]{9}\right) = 4 + \frac{72}{\left(\sqrt[3]{9}\right)^3} = 12 > 0$. Since the second derivative is positive, we have a minimum at $x = \sqrt[3]{9} \approx 2.08$. The width is 2.08 yd.; therefore, the length is $2(2.08) \approx 4.16$. We find the height y

$$y = \frac{6}{x^2}$$
$$= \frac{6}{(2.08)^2}$$
$$\approx 1.387$$

The overall dimensions of the dumpster that will minimize surface area are 2.08 yd. by 4.16 yd. by 1.387 yd.

22. Let x be the width of the container, y be the length of the container and $2x$ be the height of the container.
Since we are including the top and the bottom of the container, the surface area is given by:
$S = 4x^2 + 6xy$ and the volume is given by
$V = y \cdot x \cdot 2x = 2x^2 y = 18$

Then, $y = \frac{9}{x^2}$, and

$S = 4x^2 + 6x\left(\frac{9}{x^2}\right) = 4x^2 + 54x^{-1}$.

We are restricted to the interval $(0, \infty)$.

$S'(x) = 8x - 54x^{-2} = 8x - \frac{54}{x^2}$

$S'(x)$ exists for all x in the interval. Solve;

$$S'(x) = 0$$
$$8x - \frac{54}{x^2} = 0$$
$$8x^3 = 54$$
$$x^3 = \frac{27}{4}$$
$$x \approx 1.89$$

Since there is only one critical value, we use the second derivative to determine if it is a minimum.

$S''(x) = 8 + \frac{108}{x^3}$

$S''(1.89) \approx 24 > 0$

Therefore, there is a minimum at $x = 1.89$.

The height is twice that of the width, therefore, the height is $2(1.89) \approx 3.78$. We solve for the length y using:

$$y = \frac{9}{(1.89)^2} \approx 2.52$$

Therefore, the dimensions of the compost container with minimal surface area are 1.89 ft. by 2.52 ft. by 3.78 ft.

23. $R(x) = 50x - 0.5x^2$; $C(x) = 4x + 10$

Profit is equal to revenue minus cost.

$$P(x) = R(x) - C(x)$$
$$= 50x - 0.5x^2 - (4x + 10)$$
$$= -0.5x^2 + 46x - 10$$

Because x is the number of units produced and sold, we are only concerned with the non-negative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0, \infty)$.

First, we find $P'(x)$.

$P'(x) = -x + 46$

The derivative exists for all values of x in $[0, \infty)$. Thus, we solve $P'(x) = 0$.

$$-x + 46 = 0$$
$$-x = -46$$
$$x = 46$$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.

$P''(x) = -1 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 46$.

$$P(46) = -0.5(46)^2 + 46(46) - 10$$
$$= -1058 + 2116 - 10$$
$$= 1048$$

The maximum profit is \$1048 when 46 units are produced and sold.

24. $R(x) = 50x - 0.5x^2$; $C(x) = 10x + 3$

$$P(x) = R(x) - C(x)$$
$$= 50x - 0.5x^2 - (10x + 3)$$
$$= -0.5x^2 + 40x - 3, \qquad 0 \le x < \infty$$

$P'(x) = -x + 40$

The derivative exists for all values of x in $[0, \infty)$. Thus, we solve $P'(x) = 0$.

$$-x + 40 = 0$$
$$x = 40$$

There is only one critical value.

$P''(x) = -1 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 40$.

$P(40) = -0.5(40)^2 + 40(40) - 3 = 797$

The maximum profit is \$797 when 40 units are produced and sold.

25. $R(x) = 2x;\;\; C(x) = 0.01x^2 + 0.6x + 30$

Profit is equal to revenue minus cost.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 2x - \left(0.01x^2 + 0.6x + 30\right) \\ &= -0.01x^2 + 1.4x - 30 \end{aligned}$$

Because x is the number of units produced and sold, we are only concerned with the non-negative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0, \infty)$.

First, we find $P'(x)$.

$P'(x) = -0.02x + 1.4$

The derivative exists for all values of x in $[0, \infty)$. Thus, we solve $P'(x) = 0$.

$$\begin{aligned} -0.02x + 1.4 &= 0 \\ -0.02x &= -1.4 \\ x &= 70 \end{aligned}$$

There is only one critical value. We can use the second derivative to determine whether we have a maximum.

$P''(x) = -0.02 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 70$.

$$\begin{aligned} P(70) &= -0.01(70)^2 + 1.4(70) - 30 \\ &= -49 + 98 - 30 \\ &= 19 \end{aligned}$$

The maximum profit is \$19 when 70 units are produced and sold.

26. $R(x) = 5x;\;\; C(x) = 0.001x^2 + 1.2x + 60$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 5x - \left(0.001x^2 + 1.2x + 60\right) \\ &= -0.001x^2 + 3.8x - 60, \qquad 0 \le x < \infty \end{aligned}$$

$P'(x) = -0.002x + 3.8$

The derivative exists for all values of x in $[0, \infty)$. Thus, we solve $P'(x) = 0$.

$$\begin{aligned} -0.002x + 3.8 &= 0 \\ x &= 1900 \end{aligned}$$

There is only one critical value.

$P''(x) = -0.002 < 0$

The second derivative is less than zero for all values of x. Thus, a maximum occurs at $x = 1900$.

$P(1900) = -0.002(1900)^2 + 3.8(1900) - 60 = 3550$

The maximum profit is \$3550 when 1900 units are produced and sold.

27. $R(x) = 9x - 2x^2$

$C(x) = x^3 - 3x^2 + 4x + 1$

$R(x)$ and $C(x)$ are in thousands of dollars and x is in thousands of units.

Profit is equal to revenue minus cost.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 9x - 2x^2 - \left(x^3 - 3x^2 + 4x + 1\right) \\ &= -x^3 + x^2 + 5x - 1 \end{aligned}$$

Because x is the number of units produced and sold, we are only concerned with the non-negative values of x. Therefore, we will find the maximum of $P(x)$ on the interval $[0, \infty)$.

First, we find $P'(x)$.

$P'(x) = -3x^2 + 2x + 5$

The derivative exists for all values of x in $[0, \infty)$. Thus, we solve $P'(x) = 0$.

$$\begin{aligned} -3x^2 + 2x + 5 &= 0 \\ 3x^2 - 2x - 5 &= 0 \\ (3x - 5)(x + 1) &= 0 \end{aligned}$$

$$\begin{aligned} 3x - 5 = 0 \quad &\text{or} \quad x + 1 = 0 \\ 3x = 5 \quad &\text{or} \quad x = -1 \\ x = \frac{5}{3} \quad &\text{or} \quad x = -1 \end{aligned}$$

There is only one critical value in the interval $[0, \infty)$. We can use the second derivative to determine whether we have a maximum.

$P''(x) = -6x + 2$

Therefore,

$$P''\left(\frac{5}{3}\right) = -6\left(\frac{5}{3}\right) + 2 = -10 + 2 = -8 < 0$$

The second derivative is less than zero for $x = \frac{5}{3}$. Thus, a maximum occurs at $x = \frac{5}{3}$.

$$P\left(\frac{5}{3}\right)=-\left(\frac{5}{3}\right)^3+\left(\frac{5}{3}\right)^2+5\left(\frac{5}{3}\right)-1$$
$$=-\frac{125}{27}+\frac{25}{9}+\frac{25}{3}-1$$
$$=-\frac{125}{27}+\frac{75}{27}+\frac{225}{27}-\frac{27}{27}$$
$$=\frac{148}{27}$$

Note that $x=\frac{5}{3}$ thousand is approximately 1.667 thousand or 1667 units, and that $\frac{148}{27}$ thousand is approximately 5.481 thousand or 5481.
Thus, the maximum profit is approximately \$5481 when approximately 1667 units are produced and sold.

28. $R(x)=100x-x^2$

$C(x)=\frac{1}{3}x^3-6x^2+89x+100$

$R(x)$ and $C(x)$ are in thousands of dollars and x is in thousands of units.

$$P(x)=R(x)-C(x)$$
$$=100x-x^2-\left(\frac{1}{3}x^3-6x^2+89x+100\right)$$
$$=\frac{1}{3}x^3+5x^2+11x-100, \qquad 0\le x<\infty$$

$P'(x)=-x^2+10x+11$

The derivative exists for all values of x in $[0,\infty)$. Thus, we solve $P'(x)=0$.

$$-x^2+10x+11=0$$
$$x^2-10x-11=0$$
$$(x-11)(x+1)=0$$
$$x-11=0 \quad \text{or} \quad x+1=0$$
$$x=11 \quad \text{or} \quad x=-1$$

There is only one critical value in the interval $[0,\infty)$. We can use the second derivative to determine whether we have a maximum.

$P''(x)=-2x+10$

Therefore,

$P''(11)=-2(11)+10=-12<0$

The second derivative is less than zero for $x=11$. Thus, a maximum occurs at $x=11$.

$$P(11)=-\frac{1}{3}(11)^3+5(11)^2+11(11)-100$$
$$\approx 182.333$$

Note that $x=11$ thousand is 11,000 units, and that 182.333 thousand is 182,333.
Thus, the maximum profit is approximately \$182,333 when 11,000 units are produced and sold.

29. $p=150-0.5x$ Price per suit.

$C(x)=4000+0.25x^2$ Cost per suit.

a) Revenue is price times quantity. Therefore, revenue can be found by multiplying the number of suits sold, x, by the price of the suit, p. Substituting $150-0.5x$ for p, we have:

$$R(x)=x\cdot p$$
$$=x(150-0.5x)$$
$$R(x)=150x-0.5x^2$$

b) Profit is revenue minus cost. Therefore,

$$P(x)=R(x)-C(x)$$
$$=150x-0.5x^2-\left(4000+0.25x^2\right)$$
$$=-0.75x^2+150x-4000$$

Since x is the number of suits produced and sold, we will restrict the domain to the interval $0\le x<\infty$.

c) To determine the number of suits required to maximize profit, we first find $P'(x)$.

$P'(x)=-1.5x+150$.

The derivative exists for all real numbers in the interval $[0,\infty)$. Thus, we solve

$$P'(x)=0$$
$$-1.5x+150=0$$
$$-1.5x=-150$$
$$x=100$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum.

$P''(x)=-1.5<0$

The second derivative is negative for all values of x; therefore, a maximum occurs at $x=100$.

Raggs, Ltd. must sell 100 suits to maximize profit.

d) The maximum profit is found by substituting 100 for x in the profit function.

$$P(100) = -0.75(100)^2 + 150(100) - 4000$$
$$= -7500 + 15,000 - 4000$$
$$= 3500$$

The maximum profit is \$3500.

e) The price per suit is given by:

$p = 150 - 0.5x$

Substituting 100 for x, we have:

$p = 150 - 0.5(100) = 150 - 50 = 100$.

The price per suit will be \$100.

30. $p = 280 - 0.4x$, $C(x) = 5000 + 0.6x^2$

a) $R(x) = x \cdot p = x(280 - 0.4x) = 280x - 0.4x^2$

b) $P(x) = R(x) - C(x)$

$$= 280x - 0.4x^2 - (5000 + 0.6x^2)$$
$$= -x^2 + 280x - 5000, \qquad 0 \le x < \infty$$

c) $P'(x) = -2x + 280$

$P'(x)$ exists for all real numbers in the interval $[0, \infty)$. Solve:

$$-2x + 280 = 0$$
$$x = 140$$

Since there is only one critical value, we can use the second derivative to determine whether we have a maximum.

$P''(x) = -2 < 0$

The second derivative is negative for all values of x; therefore, a maximum occurs at $x = 140$.

Riverside Appliances must sell 140 refrigerators to maximize profit.

d) Substitute 140 for x in the profit function.

$$P(140) = -(140)^2 + 280(140) - 5000$$
$$= 14,600$$

The maximum profit is \$14,600.

e) $p = 280 - 0.4(140) = 224$

The price per refrigerator will be \$224.

31. Let x be the amount by which the price of \$18 should be decreased (if x is negative, the price would be increased to maximize revenue). First, we express total revenue R as a function of x. There are two sources of revenue, revenue from tickets and revenue from concessions.

$$R(x) = \begin{pmatrix}\text{Revenue from}\\ \text{tickets}\end{pmatrix} + \begin{pmatrix}\text{Revenue from}\\ \text{concessions}\end{pmatrix}$$
$$= \begin{pmatrix}\text{Number of}\\ \text{People}\end{pmatrix} \cdot \begin{pmatrix}\text{Ticket}\\ \text{Price}\end{pmatrix} + \begin{pmatrix}\text{Number of}\\ \text{People}\end{pmatrix} \cdot 4.50$$

Note, the increase in ticket sales is 10,000 x, when price drops $3x$ dollars. Therefore, the increase in ticket sales is $\frac{10,000}{3}x$ when price drops x dollars.

$$R(x) = \left(40,000 + \frac{10,000}{3}x\right)(18 - x) + \left(40,000 + \frac{10,000}{3}x\right)(4.50)$$
$$= -\frac{10,000}{3}x^2 + 20,000x + 720,000 + 180,000 + 15,000x$$
$$= -\frac{10,000}{3}x^2 + 35,000x + 900,000$$

Therefore, the total revenue function is

$$R(x) = -\frac{10,000}{3}x^2 + 35,000x + 900,000$$

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$$R'(x) = -\frac{20,000}{3}x + 35,000$$

This derivative exists for all real numbers x. thus, the only critical values are where $R'(x) = 0$; so we solve that equation:

$$-\frac{20,000}{3}x + 35,000 = 0$$
$$-\frac{20,000}{3}x = -35,000$$
$$x = 5.25$$

Since this is the only critical value, we can use the second derivative,

$$R''(x) = -\frac{20,000}{3} < 0$$

to determine whether we have a maximum. since $R''(5.25)$ is negative, $R(5.25)$ is a maximum. Therefore, in order to maximize revenue, the university should charge $\$18 - \$5.25$, or \$12.75. Since \$12.75 is \$5.25 less than \$18, we can find the attendance using

$$40,000 + \frac{10,000}{3}(5.25) = 57,500$$

The average attendance when ticket price is \$12.75 is 57,500 people.

32. $R(x) = (300 - x)(80 + x) - 22(300 - x)$
$= -x^2 + 242x + 17,400$
$R'(x) = -2x + 242$
$R'(x)$ exists for all real numbers. Solve:
$R'(x) = 0$
$-2x + 242 = 0$
$x = 121$
There is only one critical value. The second derivative $R''(x) = -2 < 0$ is negative for all values of *x*, therefore a maximum occurs at $x = 121$.
The charge per unit should be \$80 + \$121, or \$201.

33. Let *x* equal the number of additional trees per acre which should be planted. Then the number of trees planted per acre is represented by $(20 + x)$ and the yield per tree by $(30 - x)$. The total yield per acre is equal to the yield per tree times the number of trees so, we have:
$Y(x) = (30 - x)(20 + x)$
$= 600 + 10x - x^2$
To find *x* such that $Y(x)$ is a maximum, we first find $Y'(x)$:
$Y'(x) = 10 - 2x.$
This derivative exists for all real numbers *x*. Thus, the only critical values are where $Y'(x) = 0$; so we solve that equation:
$10 - 2x = 0$
$-2x = -10$
$x = 5$
This corresponds to planting 5 trees.
Since this is the only critical value, we can use the second derivative,
$R''(x) = -2 < 0,$
to determine whether we have a maximum.
Since $R''(5)$ is negative, $R(5)$ is a maximum.
Therefore, in order to maximize yield, the apple farm should plant 20 + 5, or 25 trees per acre.

34. a) First find the slope of the line.
$$m = \frac{1.12 - 1}{0.59 - 1} = -\frac{12}{41}$$
$$y - 1 = -\frac{12}{41}(x - 1)$$
$$y = -\frac{12}{41}x + \frac{53}{41}$$
So the demand function is:
$$q(x) = -\frac{12}{41}x + \frac{53}{41}$$
b) $R(x) = x \cdot q(x)$
$$R(x) = x\left(-\frac{12}{41}x + \frac{53}{41}\right)$$
$$= -\frac{12}{41}x^2 + \frac{53}{41}x$$
Find the derivative:
$$R'(x) = -\frac{24}{41}x + \frac{53}{41}$$
$R'(x)$ exists for all real numbers. Solve:
$R'(x) = 0$
$$-\frac{24}{41}x + \frac{53}{41} = 0$$
$$x = \frac{53}{24}$$
$x \approx 2.21$
Since there is only once critical value, we can use the second derivative,
$$R''(x) = -\frac{24}{41} < 0,$$
to determine whether we have a maximum.
Since $R''(2.21) < 0$, a maximum occurs at $x \approx 2.21$.
To maximize revenue, the price of nitrogen should increase 221% from the January 2001 price.

35. a) When $x = 25$, $q = 2.13$. When $x = 25 + 1$, or 26, then $q = 2.13 - 0.04 = 2.09$. We use the points $(25, 2.13)$ and $(26, 2.09)$ to find the linear demand function $q(x)$. First, we find the slope:
$$m = \frac{2.13 - 2.09}{25 - 26} = \frac{0.04}{-1} = -0.04.$$
Next, we use the point-slope equation:
$q - 2.13 = -0.04(x - 25)$
$q - 2.13 = -0.04x + 1$
$q = -0.04x + 3.13$
Therefore, the linear demand function is:
$q(x) = -0.04x + 3.13.$

b) Revenue is price times quantity; therefore, the revenue function is:

$$R(x) = x \cdot q(x)$$
$$= x(-0.04x + 3.13)$$
$$= -0.04x^2 + 3.13x$$

To find x such that $R(x)$ is a maximum, we first find $R'(x)$:

$R'(x) = -0.08x + 3.13$.

This derivative exists for all values of x. So the only critical values occur when $R'(x) = 0$; so we solve that equation:

$$-0.08x + 3.13 = 0$$
$$-0.08x = -3.13$$
$$x = 39.125$$

Since this is the only critical value, we can use the second derivative,

$R''(x) = -0.08 < 0,$

to determine whether we have a maximum. Since $R''(39.125)$ is negative, $R(39.125)$ is a maximum.

In order to maximize revenue, the State of Maryland should charge \$39.125 or rounding up to \$39.13 per license plate.

36. Let x be the number of \$0.10 increase that should be made. Then,

$$R(x) = (5 + 0.1x)(180 - x)$$
$$= -0.1x^2 + 13x + 900$$
$$R'(x) = -0.2x + 13$$

$R'(x)$ exists for all real numbers. Solve:

$$R'(x) = 0$$
$$-0.2x + 13 = 0$$
$$x = 65$$

Since there is only one critical value, we can use the second derivative,

$R''(x) = -0.2 < 0,$

to determine whether we have a maximum. Since $R''(65)$ is negative, a maximum occurs at $x = 65$.

Therefore, the theater owner should charge $\$5 + 0.1(65)$ or \$11.50 per ticket.

37. The volume of the box is given by

$V = x \cdot x \cdot y = x^2 y = 320.$

The area of the base is x^2. The cost of the base is $15x^2$ cents.

The area of the top is x^2. The cost of the top is $10x^2$ cents.

The area of each side is xy. The total area for the four sides is $4xy$. The cost of the four sides is $2.5(4xy)$ cents.

The total costs in cents is given by

$C = 15x^2 + 10x^2 + 2.5(4xy) = 25x^2 + 10xy.$

To express C in terms of one variable, we solve $x^2 y = 320$ for y:

$$y = \frac{320}{x^2}.$$

Then,

$$C(x) = 25x^2 + 10x\left(\frac{320}{x^2}\right)$$
$$= 25x^2 + \frac{3200}{x}$$

The function is defined only for positive numbers, and the problem dictates that the length x must be positive, so we are minimizing C on the interval $(0, \infty)$.

First, we find $C'(x)$.

$$C'(x) = 50x - 3200x^{-2} = 50x - \frac{3200}{x^2}$$

Since $C'(x)$ exists for all x in $(0, \infty)$, the only critical values are where $C'(x) = 0$. Thus, we solve the following equation:

$$50x - \frac{3200}{x^2} = 0$$
$$50x = \frac{3200}{x^2}$$
$$50x^3 = 3200$$
$$x^3 = 64$$
$$x = 4$$

This is the only critical value, so we can use the second derivative to determine whether we have a minimum.

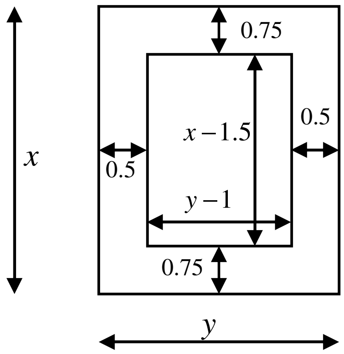
$$C''(x) = 50 + 6400x^{-3} = 50 + \frac{6400}{x^3}$$

Note that the second derivative is positive for all positive values of x, therefore we have a minimum at $x = 4$. We find y when $x = 4$.

$$y = \frac{320}{x^2} = \frac{320}{(4)^2} = \frac{320}{16} = 20$$

The cost is minimized when the dimensions are 4 ft by 4 ft by 20 ft.

38. Let x and y represent the outside length and width, respectively.



We know that $xy = 73.125$, so $y = \dfrac{73.125}{x}$.

We want to maximize the print area:

$$A = (x-1.5)(y-1)$$
$$= xy - x - 1.5y + 1.5.$$

Substituting for y, we get:

$$A(x) = x\left(\frac{73.125}{x}\right) - x - 1.5\left(\frac{73.125}{x}\right) + 1.5$$
$$= 73.125 - x - \frac{109.6875}{x} + 1.5$$
$$= 74.625 - x - \frac{109.6875}{x}$$

$$A'(x) = -1 + 109.6875x^{-2} = -1 + \frac{109.6875}{x^2}$$

$A'(x)$ exists for all values in the domain of A.

Solve:

$$A'(x) = 0$$
$$-1 + \frac{109.6875}{x^2} = 0$$
$$x^2 = 109.6875$$
$$x = \pm\sqrt{109.6875}$$

The only critical value in the domain of A is $x = \sqrt{109.6875}$; therefore, we can use the second derivative to determine if it is a maximum.

$$A''(x) = -219.375x^{-3} = -\frac{219.375}{x^3}.$$

$A''\left(\sqrt{109.6875}\right) < 0$, so $A\left(\sqrt{109.6875}\right)$ is a maximum.

When

$x = \sqrt{109.6875} \approx 10.47$,

$y = \dfrac{73.125}{\sqrt{109.6875}} \approx 6.98$.

Therefore, the outside dimensions should be approximately 10.47 in. by 6.98 in. to maximize the print area.

39. Let x equal the lot size. Now the inventory costs are given by:

$$C(x) = \underset{\text{Cost}}{\text{Yearly Carrying}} + \underset{\text{Cost}}{\text{Yearly reorder}}$$

We consider each cost separately.

Yearly carrying costs, $C_c(x)$*:* Can be found by multiplying the cost to store the items by the number of items in storage. The average amount held in stock is $\dfrac{x}{2}$, and it cost \$20 per pool table for storage. Thus:

$$C_c(x) = 20 \cdot \frac{x}{2}$$
$$= 10x$$

Yearly reorder costs, $C_r(x)$*:* Can be found by multiplying the cost of each order by the number of reorders. The cost of each order is $40 + 16x$, and the number of orders per year is $\dfrac{100}{x}$. Therefore,

$$C_r(x) = (40+16x)\left(\frac{100}{x}\right)$$
$$= \frac{4000}{x} + 1600$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x)$$
$$= 10x + \frac{4000}{x} + 1600.$$

We want to find the minimum value of C on the interval $[1,100]$. First, we find $C'(x)$:

$$C'(x) = 10 - 4000x^{-2} = 10 - \frac{4000}{x^2}.$$

The derivative exists for all x in $[1,100]$, so the only critical values are where $C'(x) = 0$. We solve that equation:

$$C'(x)=0$$
$$10-\frac{4000}{x^2}=0$$
$$10=\frac{4000}{x^2}$$
$$10x^2=4000$$
$$x^2=400$$
$$x=\pm 20$$

The only critical value in the interval is $x=20$, so we can use the second derivative to determine whether we have a minimum.

$$C''(x)=8000x^{-3}=\frac{8000}{x^3}$$

Notice that $C''(x)$ is positive for all values of x in $[1,100]$, we have a minimum at $x=20$. Thus, to minimize inventory costs, the store should order pool tables $\frac{100}{20}=5$ times per year. The lot size will be 20 tables.

40. Let x be the lot size.

Yearly carrying cost: $C_c(x)=4\cdot\frac{x}{2}=2x$

Yearly reorder cost:
$$C_r(x)=(1+0.5x)\left(\frac{200}{x}\right)=\frac{200}{x}+100$$

Then,
$$C(x)=C_c(x)+C_r(x)$$
$$=2x+\frac{200}{x}+100,\quad 1\le x\le 200$$
$$C'(x)=2-200x^{-2}=2-\frac{200}{x^2}$$

$C'(x)$ exists for all x in $[1,200]$. Solve:

$$C'(x)=0$$
$$2-\frac{200}{x^2}=0$$
$$x=\pm 10$$

The only critical value in the domain is $x=10$. Therefore, we use the second derivative,

$$C''(x)=400x^{-3}=\frac{400}{x^3}$$

to determine whether we have a minimum.
$C''(10)=0.4>0$, so $C(10)$ is a minimum.

In order to minimize inventory costs. The store should order $\frac{200}{10}=20$ times per year. The lot size will be 10 bowling balls.

41. Let x equal the lot size. Now the inventory costs are given by:

$$C(x)=\underset{\text{Cost}}{\text{Yearly Carrying}}+\underset{\text{Cost}}{\text{Yearly reorder}}$$

We consider each cost separately.
Yearly carrying costs, $C_c(x)$*:* Can be found by multiplying the cost to store the items by the number of items in storage. The average amount held in stock is $\frac{x}{2}$, and it cost \$2 per calculator for storage. Thus:

$$C_c(x)=2\cdot\frac{x}{2}$$
$$=x$$

Yearly reorder costs, $C_r(x)$*:* Can be found by multiplying the cost of each order by the number of reorders. The cost of each order is $5+2.50x$, and the number of orders per year is $\frac{720}{x}$. Therefore,

$$C_r(x)=(5+2.50x)\left(\frac{720}{x}\right)$$
$$=\frac{3600}{x}+1800$$

Hence, the total inventory cost is:

$$C(x)=C_c(x)+C_r(x)$$
$$=x+\frac{3600}{x}+1800.$$

We want to find the minimum value of C on the interval $[1,720]$. First, we find $C'(x)$:

$$C'(x)=1-3600x^{-2}=1-\frac{3600}{x^2}.$$

The derivative exists for all x in $[1,720]$, so the only critical values are where $C'(x)=0$. We solve that equation:

$$1-\frac{3600}{x^2}=0$$
$$1=\frac{3600}{x^2}$$
$$x^2=3600$$
$$x=\pm 60$$

The only critical value in the interval is $x=60$, so we can use the second derivative to determine whether we have a minimum.

$C''(x) = 7200x^{-3} = \frac{7200}{x^3}$

Notice that $C''(x)$ is positive for all values of x in $[1, 720]$, we have a minimum at $x = 60$. Thus, to minimize inventory costs, the store should order calculators $\frac{720}{60} = 12$ times per year. The lot size will be 60 calculators.

42. Let x be the lot size.

Yearly carrying cost: $C_c(x) = 8 \cdot \frac{x}{2} = 4x$

Yearly reorder cost:
$$C_r(x) = (10 + 5x)\left(\frac{360}{x}\right) = \frac{3600}{x} + 1800$$

Then,

$$C(x) = C_c(x) + C_r(x) = 4x + \frac{3600}{x} + 1800, \quad 1 \le x \le 360$$

$$C'(x) = 4 - 3600x^{-2} = 4 - \frac{3600}{x^2}$$

$C'(x)$ exists for all x in $[1, 360]$. Solve:

$$C'(x) = 0$$
$$4 - \frac{3600}{x^2} = 0$$
$$x = \pm 30$$

The only critical value in the domain is $x = 30$. Therefore, we use the second derivative,

$$C''(x) = 7200x^{-3} = \frac{7200}{x^3}$$

to determine whether we have a minimum.

$C''(30) = 0.27 > 0$, so $C(30)$ is a minimum.

In order to minimize inventory costs. The store should order $\frac{360}{30} = 12$ times per year. The lot size will be 30 surf boards.

43. Let x equal the lot size.

Yearly carrying costs:

$$C_c(x) = 2 \cdot \frac{x}{2} = x$$

Yearly reorder costs:

$$C_r(x) = (4 + 2.50x)\left(\frac{256}{x}\right) = \frac{1024}{x} + 640$$

Hence, the total inventory cost is:

$$C(x) = C_c(x) + C_r(x) = x + \frac{1024}{x} + 640.$$

We want to find the minimum value of C on the interval $[1, 256]$. First, we find $C'(x)$:

$$C'(x) = 1 - 1024x^{-2} = 1 - \frac{1024}{x^2}.$$

The derivative exists for all x in $[1, 256]$, so the only critical values are where $C'(x) = 0$. We solve that equation:

$$1 - \frac{1024}{x^2} = 0$$
$$1 = \frac{1024}{x^2}$$
$$x^2 = 1024$$
$$x = \pm 32$$

The only critical value in the interval is $x = 32$, so we can use the second derivative to determine whether we have a minimum.

$$C''(x) = 2048x^{-3} = \frac{2048}{x^3}$$

Notice that $C''(x)$ is positive for all values of x in $[1, 256]$, we have a minimum at $x = 32$. Thus, to minimize inventory costs, the store should order calculators $\frac{256}{32} = 8$ times per year. The lot size will be 32 calculators.

44. Let x be the lot size.

Yearly carrying cost: $C_c(x) = 8 \cdot \frac{x}{2} = 4x$

Yearly reorder cost:
$$C_r(x) = (10 + 6x)\left(\frac{360}{x}\right) = \frac{3600}{x} + 2160$$

Then,

$$C(x) = C_c(x) + C_r(x) = 4x + \frac{3600}{x} + 2160, \quad 1 \le x \le 360$$

$$C'(x) = 4 - 3600x^{-2} = 4 - \frac{3600}{x^2}$$

$C'(x)$ exists for all x in $[1,360]$. Solve:

$$C'(x)=0$$

$$4-\frac{3600}{x^2}=0$$

$$x=\pm 30$$

The only critical value in the domain is $x=30$. Therefore, we use the second derivative,

$$C''(x)=7200x^{-3}=\frac{7200}{x^3}$$

to determine whether we have a minimum.

$C''(30)=0.27>0$, so $C(30)$ is a minimum.

In order to minimize inventory costs. The store should order $\frac{360}{30}=12$ times per year. The lot size will be 30 surf boards.

45. Case I.
If y is the length, the girth is $x+x+x+x$, or $4x$.
Case II.
If x is the length, the girth is $x+y+x+y$, or $2x+2y$.

Case I.
The combine length and girth is $y+4x=84$.

The volume is $V=x\cdot x\cdot y=x^2y$.

We want express V in terms of one variable. We solve $y+4x=84$, for y.

$$y+4x=84$$
$$y=84-4x$$

Thus,
$$V=x^2(84-4x)=84x^2-4x^3.$$

To maximize $V(x)$ we first find $V'(x)$.

$$V'(x)=168x-12x^2$$

This derivative exists for all x, so the critical values will occur when $V'(x)=0$; therefore, we solve that equation.

$$168x-12x^2=0$$
$$12x(14-x)=0$$
$$12x=0 \quad\text{or}\quad 14-x=0$$
$$x=0 \quad\text{or}\quad x=14$$

Since $x\neq 0$, the only critical value is $x=14$.
We can use the second derivative,
$$V'(x)=168-24x,$$
to determine whether we have a maximum.
$$V'(14)=168-24(14)=-168<0$$
Therefore, we have a maximum at $x=14$.

If $x=14$, then $y=84-4(14)=28$.

Therefore, the dimensions that will maximize the volume of the package are 14 in. by 14 in. by 28 in. The volume is
$14\times 14\times 28=5488\text{ in}^3$.

Case II.
The combine length and girth is $x+2x+2y=3x+2y=84$.

The volume is $V=x\cdot x\cdot y=x^2y$.

We want express V in terms of one variable. We solve $3x+2y=84$, for y.

$$3x+2y=84$$
$$2y=84-3x$$
$$y=42-\frac{3}{2}x$$

Thus,
$$V=x^2\left(42-\frac{3}{2}x\right)=42x^2-\frac{3}{2}x^3.$$

To maximize $V(x)$ we first find $V'(x)$.

$$V'(x)=84x-\frac{9}{2}x^2$$

This derivative exists for all x, so the critical values will occur when $V'(x)=0$; therefore, we solve that equation.

$$84x-\frac{9}{2}x^2=0$$
$$3x\left(28-\frac{3}{2}x\right)=0$$
$$3x=0 \quad\text{or}\quad 28-\frac{3}{2}x=0$$
$$x=0 \quad\text{or}\quad -\frac{3}{2}x=-28$$
$$x=0 \quad\text{or}\quad x=\frac{56}{3}$$

Since $x\neq 0$, the only critical value is $x=\frac{56}{3}$

We can use the second derivative,
$$V''(x)=84-9x,$$
to determine whether we have a maximum.
$$V''\left(\frac{56}{3}\right)=84-9\left(\frac{56}{3}\right)=-84<0$$

Therefore, we have a maximum at $x=\frac{56}{3}$.

If $x=\frac{56}{3}\approx 18.67$, then $y=42-\frac{3}{2}\left(\frac{56}{3}\right)=14$.

Therefore, the dimensions that will maximize the volume of the package are 18.67 in. by 18.67 in. by 14 in. The volume is $\frac{56}{3}\times\frac{56}{3}\times 14 \approx 4878.2 \text{ in}^3$.

Comparing Case I and Case II, we see that the maximum volume is 5488 in^3 when the dimensions are 14 in. by 14 in. by 28 in.

46. Let y represent the dimension on the lot line and let x represent the other dimension. Then the length of fencing that the person must pay for is $\frac{1}{2}y + x + y + x = 2x + \frac{3}{2}y$.

We know $xy = 48$; therefore, $y = \frac{48}{x}$.

The length of fencing as a function of x is:

$$F(x) = 2x + \frac{3}{2}\left(\frac{48}{x}\right) = 2x + \frac{72}{x}, \quad 0 < x < \infty.$$

$$F'(x) = 2 - 72x^{-2} = 2 - \frac{72}{x^2}$$

$F'(x)$ exists for all x in $(0,\infty)$. Solve:

$$F'(x) = 0$$
$$2 - \frac{72}{x^2} = 0$$
$$x = \pm 6$$

Only the critical value $x = 6$ is in $(0,\infty)$. Since there is only one critical value, we use the second derivative to determine whether we have a minimum.

$$F''(x) = 144x^{-3} = \frac{144}{x^3}$$

$F''(6) = \frac{2}{3} > 0$, so $F(6)$ is a minimum.

When $x = 6$, $y = \frac{48}{6} = 8$. The dimensions that minimize the cost are 6 yd. by 8 yd. Where the longer side of the lot is adjacent to the neighbors yard.

47. Use the figure in the text book. Since the radius of the window is x, the diameter of the window is $2x$, which is also the length of the base of the window.
The circumference of a circle whose radius is x is given by:

$$C = 2\pi x. \qquad (C = 2\pi r)$$

Therefore, the perimeter of the semicircle is $\frac{1}{2}C = \frac{2\pi x}{2} = \pi x.$
The perimeter of the three sides of the rectangle which form the remaining part of the total perimeter of the window is given by:
$2x + y + y = 2x + 2y.$
The total perimeter of the window is:
$\pi x + 2x + 2y = 24$.
Maximizing the amount of light is the same as maximizing the area of the window. The area of the circle with radius x is:

$$A = \pi x^2, \qquad \left(A = \pi r^2\right).$$

Therefore, the area of the semicircle is:

$$\frac{1}{2}A = \frac{1}{2}\pi x^2.$$

The area of the rectangle is $2x \cdot y$.
The total area of the Norman window is

$$A = \frac{1}{2}\pi x^2 + 2xy.$$

To express A in terms of one variable, we solve $\pi x + 2x + 2y = 24$ for y:

$$\pi x + 2x + 2y = 24$$
$$2y = 24 - 2x - \pi x$$
$$y = 12 - x - \frac{\pi}{2}x$$

Then,

$$A(x) = \frac{1}{2}\pi x^2 + 2x\left(12 - x - \frac{\pi}{2}x\right)$$
$$= \frac{1}{2}\pi x^2 + 24x - 2x^2 - \pi x^2$$
$$= \left(-\frac{1}{2}\pi - 2\right)x^2 + 24x$$

We maximize A on the interval $(0,24)$. We first find $A'(x)$.

$$A'(x) = (-\pi - 4)x + 24.$$

Since $A'(x)$ exists for all x in $(0,24)$, the only critical points are where $A'(x) = 0$. Thus, we solve the following equation:

$$A'(x) = 0$$
$$(-\pi - 4)x + 24 = 0$$
$$(-\pi - 4)x = -24$$
$$x = \frac{-24}{(-\pi - 4)}$$
$$x = \frac{-24}{-(\pi + 4)}$$
$$x = \frac{24}{\pi + 4} \approx 3.36$$

This is the only critical value, so we can use the second derivative to determine whether we have a maximum.

$$A''(x) = -\pi - 4 < 0$$

Since $A''(x)$ is negative for all values of x, we have a maximum at $x = \frac{24}{\pi + 4}$. We find y when $x = \frac{24}{\pi + 4}$:

$$y = 12 - \frac{\pi}{2}x - x$$
$$= 12 - \frac{\pi}{2}\left(\frac{24}{\pi + 4}\right) - \frac{24}{\pi + 4}$$
$$= 12\left(\frac{\pi + 4}{\pi + 4}\right) - \frac{12\pi}{\pi + 4} - \frac{24}{\pi + 4}$$
$$= \frac{12\pi + 48 - 12\pi - 24}{\pi + 4}$$
$$= \frac{24}{\pi + 4} \approx 3.36$$

To maximize the amount of light through the window, the dimensions must be $x = \frac{24}{\pi + 4}$ ft and $y = \frac{24}{\pi + 4}$ ft, or approximately $x \approx 3.36$ ft and $y \approx 3.36$ ft.

48. Since the stained glass transmits only half as much light as the semicircle in Exercise 47, we express the function A as:

$$A = \frac{1}{2} \cdot \frac{1}{2}\pi x^2 + 2xy$$
$$= \frac{1}{4}\pi x^2 + 2xy$$

The perimeter is still the same, so we can substitute $12 - x - \frac{\pi}{2}x$ for y to get:

$$A(x) = \frac{1}{4}\pi x^2 + 2x\left(12 - x - \frac{\pi}{2}x\right)$$
$$= \frac{1}{4}\pi x^2 + 24x - 2x^2 - \pi x^2$$
$$= \left(-\frac{3}{4}\pi - 2\right)x^2 + 24x, \quad 0 < x < 24.$$

Find $A'(x)$.

$$A'(x) = \left(-\frac{3}{2}\pi - 4\right)x + 24$$

Since $A'(x)$ exists for all x in $(0, 24)$, the only critical points are where $A'(x) = 0$. Thus, we solve the following equation:

$$\left(-\frac{3}{2}\pi - 4\right)x + 24 = 0$$
$$(3\pi + 8)x - 48 = 0$$
$$x = \frac{48}{3\pi + 8} \approx 2.75$$

This is the only critical value, so we can use the second derivative to determine whether we have a maximum.

$$A''(x) = -\frac{3}{2}\pi - 4 < 0$$

Since $A''(x)$ is negative for all values of x, we have a maximum at $x = \frac{48}{3\pi + 8}$. We find y when $x = \frac{48}{3\pi + 8}$:

$$y = 12 - \frac{\pi}{2}x - x$$
$$= 12 - \frac{\pi}{2}\left(\frac{48}{3\pi + 8}\right) - \frac{48}{3\pi + 8}$$
$$= \frac{12\pi + 48}{3\pi + 8} \approx 4.92$$

To maximize the amount of light through the window, the dimensions must be $x = \frac{48}{3\pi + 8}$ ft and $y = \frac{12\pi + 48}{3\pi + 8}$ ft or approximately $x \approx 2.75$ ft and $y \approx 4.92$ ft.

49. Let x represent a positive number. Then, $\frac{1}{x}$ is the reciprocal of the number, and x^2 is the square of the number. The sum, S, of the reciprocal and five times the square is given by:

$$S(x) = \frac{1}{x} + 5x^2.$$

We want to minimize $S(x)$ on the interval $(0,\infty)$. First, we find $S'(x)$

$S'(x) = -x^{-2} + 10x = -\frac{1}{x^2} + 10x$

Since $S'(x)$ exists for all values of x in $(0,\infty)$, the only critical values occur when $S'(x) = 0$. We solve the following equation:

$$-\frac{1}{x^2} + 10x = 0$$
$$10x = \frac{1}{x^2}$$
$$10x^3 = 1$$
$$x^3 = \frac{1}{10}$$
$$x = \sqrt[3]{\frac{1}{10}} = \frac{1}{\sqrt[3]{10}}$$

Since there is only one critical value, we use the second derivative,

$S''(x) = 2x^{-3} + 10 = \frac{2}{x^3} + 10$,

to determine whether it is a minimum. The second derivative is positive for all x in $(0,\infty)$; therefore, the sum is a minimum when

$x = \frac{1}{\sqrt[3]{10}}$.

50. Let x represent a positive number. The sum, S, of the reciprocal and four times the square is given by:

$S(x) = \frac{1}{x} + 4x^2$.

We want to minimize $S(x)$ on the interval $(0,\infty)$. First, we find $S'(x)$

$S'(x) = -x^{-2} + 8x = -\frac{1}{x^2} + 8x$

$S'(x)$ exists for all values of x in $(0,\infty)$. Solve:

$$S'(x) = 0$$
$$-\frac{1}{x^2} + 8x = 0$$
$$x = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$$

Since there is only one critical value, we use the second derivative,

$S''(x) = 2x^{-3} + 8 = \frac{2}{x^3} + 8$,

to determine whether it is a minimum. The second derivative is positive for all x in $(0,\infty)$; therefore, the sum is a minimum when $x = \frac{1}{2}$.

51. Let A represent the amount deposited in savings accounts and i represent the interest rate paid on the money deposited. If A is directly proportional to i, then there is some positive constant k such that $A = ki$. The interest earned by the bank is represented by $18\%A$, or $0.18A$. The interest paid by the bank is represented by iA. Thus the profit received by the bank is given by

$P = 0.18A - iA.$

We express P as a function of the interest the bank pays on the money deposited, i, by substituting ki for A.

$$P = 0.18(ki) - i(ki)$$
$$= 0.18ki - ki^2$$

We maximize P on the interval $(0,\infty)$. First, we find $P'(i)$.

$P'(i) = 0.18k - 2ki$

Since $P'(i)$ exists for all i in $(0,\infty)$, the only critical values are where $P'(i) = 0$. We solve the following equation:

$$0.18k - 2ki = 0$$
$$-2ki = -0.18k$$
$$i = \frac{-0.18k}{-2k}$$
$$i = 0.09$$

Since there is only one critical point, we can use the second derivative to determine whether we have a minimum. Notice that $P''(i) = -2k$, which is a negative constant $(k > 0)$. Thus, $P''(0.09)$ is negative, so $P(0.09)$ is a maximum. To maximize profit, the bank should pay 9% on its savings accounts.

52. The circumference of the circle is $x = 2\pi r$.

Solving the equation for *r* we get, $r = \frac{x}{2\pi}$.

The area of the circle is $A = \pi r^2$, thus,

substituting for *r* we have $A = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$.

The perimeter of the square is $24 - x$.

The length of a side of the square is $\frac{24-x}{4}$.

Therefore, the area of the square is:

$$A_s = \left(\frac{24-x}{4}\right)^2 = \frac{x^2 - 48x + 576}{16}.$$

The total area is:

$$\begin{aligned} A &= A_c + A_s \\ &= \frac{x^2}{4\pi} + \frac{x^2 - 48x + 576}{16} \\ &= \frac{x^2}{4\pi} + \frac{1}{16}x^2 - 3x + 36 \\ &= \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - 3x + 36 \\ &= \left(\frac{4+\pi}{16\pi}\right)x^2 - 3x + 36 \end{aligned}$$

We minimize the area on the interval $(0,24)$.

First, we find $A'(x)$.

$$\begin{aligned} A'(x) &= 2\left(\frac{4+\pi}{16\pi}\right)x - 3 \\ &= \left(\frac{4+\pi}{8\pi}\right)x - 3 \end{aligned}$$

$A'(x)$ exists for all *x* in $(0,24)$; therefore, the only critical value occurs when $A'(x) = 0$.

Solve:

$$\left(\frac{4+\pi}{8\pi}\right)x - 3 = 0$$

$$x = 3\left(\frac{8\pi}{4+\pi}\right) = \frac{24\pi}{4+\pi} \approx 10.56$$

Since there is only one critical value, we use the second derivative,

$$A''(x) = \frac{4+\pi}{8\pi} > 0$$

to determine whether we have a minimum. The second derivative is positive for all values of *x* in the interval; therefore a minimum occurs at

$$x = \frac{24}{4+\pi} \approx 10.56.$$

The wire should be cut to $x \approx 10.56$ in. in order to form the circle and $24 - 10.56 \approx 13.44$ in. in order to form the square.

There is no maximum if the string is to be cut. One would interpret the maximum to be at the endpoint, with the string uncut and used to form a circle.

53. Using the drawing in the text, we write a function that gives the cost of the power line. The length of the power line on the land is given by $4 - x$, so the cost of laying the power line underground is given by:

$$C_L(x) = 3000(4-x) = 12{,}000 - 3000x.$$

The length of the power line that will be under water is $\sqrt{1+x^2}$, so the cost of laying the power line underwater is given by:

$$C_W(x) = 5000\sqrt{1+x^2}$$

Therefore, the total cost of laying the power line is:

$$\begin{aligned} C(x) &= C_L(x) + C_W(x) \\ &= 12{,}000 - 3000x + 5000\sqrt{1+x^2}. \end{aligned}$$

We want to minimize $C(x)$ over the interval $0 \le x \le 4$. First, we find the derivative.

$$\begin{aligned} C'(x) &= -3000 + 5000\left(\frac{1}{2}\right)\left(1+x^2\right)^{-1/2}(2x) \\ &= -3000 + 5000x\left(1+x^2\right)^{-1/2} \\ &= -3000 + \frac{5000x}{\sqrt{1+x^2}} \end{aligned}$$

Since the derivative exists for all *x*, we find the critical values by solving the equation:

$$C'(x)=0$$
$$-3000+\frac{5000x}{\sqrt{1+x^2}}=0$$
$$-3000\sqrt{1+x^2}+5000x=0$$
$$5000x=3000\sqrt{1+x^2}$$
$$\frac{5}{3}x=\sqrt{1+x^2}$$
$$\left(\frac{5}{3}x\right)^2=\left(\sqrt{1+x^2}\right)^2$$
$$\frac{25}{9}x^2=1+x^2$$
$$\frac{16}{9}x^2=1$$
$$x^2=\frac{9}{16}$$
$$x=\pm\sqrt{\frac{9}{16}}$$
$$x=\pm\frac{3}{4}$$

The only critical value in the interval $[0,4]$ is $x=\frac{3}{4}$, so we can use the second derivative to determine if we have a minimum.

$$C''(x)=\frac{(1+x^2)^{1/2}(5000)-5000x\left[\frac{1}{2}(1+x^2)^{-1/2}(2x)\right]}{\left[(1+x^2)^{1/2}\right]^2}$$
$$=\frac{5000\sqrt{1+x^2}-\frac{5000x^2}{\sqrt{1+x^2}}}{(1+x^2)}$$
$$=\frac{5000}{\sqrt{1+x^2}}-\frac{5000x^2}{(1+x^2)^{3/2}}$$
$$=\frac{5000(1+x^2)-5000x^2}{(1+x^2)^{3/2}}$$
$$=\frac{5000}{(1+x^2)^{3/2}}$$

$C''(x)$ is positive for all x in $[0,4]$; therefore, a minimum occurs at $x=\frac{3}{4}$. When $x=\frac{3}{4}$,

$4-\frac{3}{4}=\frac{13}{4}=3.25$.

Therefore, S should be 3.25 miles down shore from the power station.

Note: since we are minimizing cost over a closed interval, we could have used Max-Min Principle 1 to determine the minimum, and avoided finding the second derivative. The critical value and the endpoints are $0, \frac{3}{4},$ and 4. The function values at these three points are:

$$C(0)=12{,}000-3000(0)+5000\left(\sqrt{1+(0)^2}\right)$$
$$=17{,}000$$
$$C\left(\frac{3}{4}\right)=12{,}000-3000\left(\frac{3}{4}\right)+5000\left(\sqrt{1+\left(\frac{3}{4}\right)^2}\right)$$
$$=16{,}000$$
$$C(4)=12{,}000-3000(4)+5000\left(\sqrt{1+(4)^2}\right)$$
$$\approx 20{,}615.53$$

Therefore, the minimum occurs when $x=\frac{3}{4}$, or when S is 3.25 miles down shore from the power station.

54. The distance from point C to point S is given by $\sqrt{9+x^2}$, and the distance from point S to point A is $8-x$. Therefore, the total energy expended by the pigeon is:

$E(x)=1.28r\sqrt{9+x^2}+r(8-x)$, where r is a positive constant measuring the rate of energy the pigeon uses.

We want to minimize $E(x)$ on the interval $[0,8]$.

$$E'(x)=\frac{1.28rx}{(9+x^2)^{1/2}}-r$$

$E'(x)$ exists for all x in $[0,8]$. Solve:

$$E'(x)=0$$
$$\frac{1.28rx}{(9+x^2)^{1/2}}-r=0$$
$$x\approx\pm3.755$$

The only critical value in $[0,8]$ is $x\approx3.755$, so we can use the second derivative to determine whether we have a maximum.

Note that $E''(x)=\dfrac{11.52r}{\left(9+x^2\right)^{3/2}}$ and that

$E''(x)>0$ for all x in the interval $[0,8]$.

Therefore, a minimum occurs when $x\approx 3.755$. The pigeon should reach land about $8-3.755$ or 4.245 miles down shore from A.

55. Using the drawing in the text, we write a function which gives the total distance between the cities.

The distance from C_1 to the bridge can be given by $\sqrt{a^2+(p-x)^2}$. The distance over the bridge is r. The distance from the bridge to C_2 can be given by $\sqrt{b^2+x^2}$. Therefore, the total distance between the two cities is given by:

$$D(x)=\sqrt{a^2+(p-x)^2}+r+\sqrt{x^2+b^2}$$

To minimize the distance, we find the derivative of the function first.

$$D'(x)=\frac{1}{2}\left[a^2+(p-x)^2\right]^{-1/2}\cdot 2(p-x)(-1)+\frac{1}{2}\left[b^2+x^2\right]^{-1/2}(2x)$$

$$=\frac{x-p}{\sqrt{a^2+(p-x)^2}}+\frac{x}{\sqrt{b^2+x^2}}$$

The derivative exists for all values of x in the interval $[0,p]$. Therefore, the only critical values occur when $D'(x)=0$. We solve this equation.

$$\frac{x-p}{\sqrt{a^2+(p-x)^2}}+\frac{x}{\sqrt{b^2+x^2}}=0.$$

The solution to this equation is

$x=\dfrac{bp}{b-a}$ or $x=\dfrac{bp}{b+a}$.

Only $x=\dfrac{bp}{b+a}$ is in $[0,p]$.

Since there is only one critical value, we can use the second derivative to determine if there is a minimum.

$$D''(x)=\frac{a^2}{\left[a^2+(p-x)^2\right]^{3/2}}+\frac{b^2}{\left[x^2+b^2\right]^{3/2}}.$$

$D''(x)>0$ for all values of x; therefore, a minimum occurs at $x=\dfrac{bp}{b+a}$.

The bridge should be located such that the distance x is $\dfrac{bp}{b+a}$ units.

56. $C(x)=8x+20+\dfrac{x^3}{100}$

a) $A(x)=\dfrac{C(x)}{x}$.

$$A(x)=\frac{8x+20+\frac{x^3}{100}}{x}$$

$$=8+\frac{20}{x}+\frac{x^2}{100}$$

b) $C'(x)=8+\dfrac{3}{100}x^2$

$$A'(x)=-20x^{-2}+\frac{1}{100}(2x)$$

$$=-\frac{20}{x^2}+\frac{1}{50}x$$

c) The derivative exists for all x in $(0,\infty)$; therefore, the critical values occur when $A'(x)=0$. Solve:

$$-\frac{20}{x^2}+\frac{1}{50}x=0$$

$$\frac{1}{50}x=\frac{20}{x^2}$$

$$x^3=1000$$

$$x=10$$

There is only one critical value, so we use the second derivative to determine whether we have a minimum.

$$A''(x)=\frac{40}{x^3}+\frac{1}{50}$$

$A''(10)=\dfrac{3}{50}>0.$ Thus $A(10)$ is a minimum.

$$A(10)=8+\frac{20}{10}+\frac{10^2}{100}=11.$$

The minimum average cost is \$11 when 10 units are produced.

$$C'(10)=8+\frac{3}{100}\left(10^2\right)=11.$$

The marginal cost is \$11 when 10 units are produced.

d) $A(10)=C'(10)=11$.

57. $A(x)=\dfrac{C(x)}{x}$

a) Taking the derivative of $A(x)$ we have:

$$A'(x)=\frac{d}{dx}\left[\frac{C(x)}{x}\right]$$
$$=\frac{x\cdot C'(x)-C(x)\cdot 1}{x^2} \quad \text{Quotient Rule}$$
$$=\frac{x\cdot C'(x)-C(x)}{x^2}$$

b) The derivative exists for all x in $(0,\infty)$, therefore, the critical values will occur when $A'(x_0)=0$. We solve the equation:

$$\frac{x_0\cdot C'(x_0)-C(x_0)}{x_0^{\ 2}}=0$$
$$x_0\cdot C'(x_0)-C(x_0)=0 \quad \text{Multiplying by } x_0^{\ 2}\neq 0.$$
$$x_0\cdot C'(x_0)=C(x_0)$$
$$C'(x_0)=\frac{C(x_0)}{x_0}=A(x_0)$$

58. Express Q as a function of one variable. First, solve $x+y=1$ for y. We have:
$y=1-x$.
Substituting we have:
$$Q=x^3+2(1-x)^3$$
$$=x^3+2(1-3x+3x^2-x^3)$$
$$=-x^3+6x^2-6x+2$$
Next, we find $Q'(x)$.
$$Q'(x)=-3x^2+12x-6$$
The derivative exists for all values of x in the interval $(0,\infty)$; thus, the only critical values are where $Q'(x)=0$. We solve the equation:
$$-3x^2+12x-6=0$$
$$x^2-4x+2=0$$
Using the quadratic formula, we have:
$x=2\pm\sqrt{2}$.
When $x=2+\sqrt{2}$, $y=1-(2+\sqrt{2})=-1-\sqrt{2}$.
When $x=2-\sqrt{2}$, $y=1-(2-\sqrt{2})=-1+\sqrt{2}$.
Since x and y must be positive, we only consider $x=2-\sqrt{2}$ and $y=-1+\sqrt{2}$.

Note, $Q''(x)=-6x+12$ and
$Q''(2-\sqrt{2})=-6(2-\sqrt{2})+12\approx 15.51>0$, so we have a minimum at $x=2-\sqrt{2}$.
The minimum value of Q is found by substituting.
$$Q=x^3+2y^3$$
$$=(2-\sqrt{2})^3+2(-1+\sqrt{2})^3$$
$$=6-4\sqrt{2}$$

59. Express Q as a function of one variable. First, solve $x^2+y^2=2$ for y. We have:
$$y^2=2-x^2$$
$$y=\pm\sqrt{2-x^2}$$
y is a real number of x in the interval $\left[-\sqrt{2},\sqrt{2}\right]$.
If $y=-\sqrt{2-x^2}$, we substitute for y to get:
$$Q=3x+y^3$$
$$Q=3x+\left(-\sqrt{2-x^2}\right)^3$$
$$=3x-(2-x^2)^{3/2}$$
Next, we find $Q'(x)$.
$$Q'(x)=3-\left(\frac{3}{2}\right)(2-x^2)^{1/2}(-2x)$$
$$=3+3x(2-x^2)^{1/2}$$
$$=3+3x\sqrt{2-x^2}$$
The derivative exists for all values of x in the interval $\left[-\sqrt{2},\sqrt{2}\right]$; thus, the only critical values are where $Q'(x)=0$. We solve the equation:

$$Q'(x) = 0$$
$$3 + 3x\sqrt{2-x^2} = 0$$
$$3 = -3x\sqrt{2-x^2}$$
$$1 = -x\sqrt{2-x^2}$$
$$1^2 = \left(-x\sqrt{2-x^2}\right)^2$$
$$1 = x^2\left(2-x^2\right)$$
$$1 = 2x^2 - x^4$$
$$x^4 - 2x^2 + 1 = 0$$
$$\left(x^2-1\right)^2 = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

We notice that $x = 1$ is an extraneous solution which does not work.

$3 + 3(1)\sqrt{2-(1)^2} = 3 + 3 = 6 \neq 0$.

Therefore, the only critical value is $x = -1$. The critical point and the endpoints are $-\sqrt{2},\ -1$, and $\sqrt{2}$.

$$Q\left(-\sqrt{2}\right) = 3\left(-\sqrt{2}\right) - \left(2-\left(-\sqrt{2}\right)^2\right)^{3/2} = -3\sqrt{2}$$

$$Q(-1) = 3(-1) - \left(2-(-1)^2\right)^{3/2} = -4$$

$$Q\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right) - \left(2-\left(\sqrt{2}\right)^2\right)^{3/2} = 3\sqrt{2}$$

The minimum value of Q is $-3\sqrt{2}$ and occurs when $x = -\sqrt{2}$ and $y = -\sqrt{2-\left(-\sqrt{2}\right)^2} = 0$

Next, we repeat the process for $y = \sqrt{2-x^2}$. We notice that:

$$Q = 3x + y^3$$
$$Q = 3x + \left(\sqrt{2-x^2}\right)^3$$
$$= 3x + \left(2-x^2\right)^{3/2}$$

Next, we find $Q'(x)$.

$$Q'(x) = 3 + \left(\frac{3}{2}\right)\left(2-x^2\right)^{1/2}(-2x)$$
$$= 3 - 3x\left(2-x^2\right)^{1/2}$$
$$= 3 - 3x\sqrt{2-x^2}$$

The derivative exists for all values of x in the interval $\left[-\sqrt{2},\sqrt{2}\right]$; thus, the only critical values are where $Q'(x) = 0$. We solve the equation:

$$3 - 3x\sqrt{2-x^2} = 0$$
$$3 = 3x\sqrt{2-x^2}$$
$$1 = x\sqrt{2-x^2}$$
$$1^2 = \left(x\sqrt{2-x^2}\right)^2$$
$$1 = x^2\left(2-x^2\right)$$
$$1 = 2x^2 - x^4$$
$$x^4 - 2x^2 + 1 = 0$$
$$\left(x^2-1\right)^2 = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

We notice that $x = -1$ is an extraneous solution which does not work.

$3 - 3(-1)\sqrt{2-(-1)^2} = 3 + 3 = 6 \neq 0$.

Therefore, the only critical value is $x = 1$. The critical point and the endpoints are $-\sqrt{2},\ 1$, and $\sqrt{2}$.

$$Q\left(-\sqrt{2}\right) = 3\left(-\sqrt{2}\right) + \left(2-\left(-\sqrt{2}\right)^2\right)^{3/2} = -3\sqrt{2}$$

$$Q(1) = 3(1) + \left(2-(1)^2\right)^{3/2} = 4$$

$$Q\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right) + \left(2-\left(\sqrt{2}\right)^2\right)^{3/2} = 3\sqrt{2}$$

The minimum value of Q is $-3\sqrt{2}$ occurs when $x = -\sqrt{2}$ and $y = -\sqrt{2-\left(-\sqrt{2}\right)^2} = 0$.

Regardless of what value of y we chose, we see that the minimum of Q, is $-3\sqrt{2}$, when $x = -\sqrt{2}$ and $y = 0$.

60. Let x be the lot size.

Yearly carrying cost: $C_c(x) = a \cdot \dfrac{x}{2} = \dfrac{ax}{2}$

Yearly reorder cost:
$$C_r(x) = (b + cx)\left(\frac{Q}{x}\right)$$
$$= \frac{bQ}{x} + cQ$$

Then,

$$C(x) = C_c(x) + C_r(x)$$
$$= \frac{ax}{2} + \frac{bQ}{x} + cQ, \quad 1 \le x \le Q$$

$$C'(x) = \frac{a}{2} - bQx^{-2} = \frac{a}{2} - \frac{bQ}{x^2}$$

$C'(x)$ exists for all x in $[1, Q]$. Solve:

$$C'(x) = 0$$
$$\frac{a}{2} - \frac{bQ}{x^2} = 0$$
$$x = \pm\sqrt{\frac{2bQ}{a}}$$

The only critical value in the domain is

$x = \sqrt{\frac{2bQ}{a}}$.

Therefore, we use the second derivative,

$$C''(x) = 2bQx^{-3} = \frac{2bQ}{x^3}$$

to determine whether we have a minimum.

$C''(x) > 0$ for all x in $[1, Q]$, so a minimum occurs at $x = \sqrt{\frac{2bQ}{a}}$

In order to minimize inventory costs. The store should order $\frac{Q}{\sqrt{\frac{2bQ}{a}}} = \sqrt{\frac{aQ}{2b}}$ times per year. The lot size will be $\sqrt{\frac{2bQ}{a}}$ units.

61. From Exercise 60, we know that the store should order a lot size of $\sqrt{\frac{2bQ}{a}}$ units, $\sqrt{\frac{aQ}{2b}}$ times per year.

When $Q = 2500,\ a = 10,\ b = 20,\ c = 9$, the store should order:

$$\sqrt{\frac{aQ}{2b}} = \sqrt{\frac{10(2500)}{2(20)}} = 25 \text{ times per year.}$$

The lot size of each order should be:

$$\sqrt{\frac{2(20)(2500)}{10}} = 100 \text{ units.}$$

Exercise Set 2.6

1. $R(x) = 5x;\ C(x) = 0.001x^2 + 1.2x + 60$

a) Total profit is revenue minus cost.

$$P(x) = R(x) - C(x)$$
$$= 5x - (0.001x^2 + 1.2x + 60)$$
$$= 5x - 0.001x^2 - 1.2x - 60$$
$$= -0.001x^2 + 3.8x - 60$$

b) Substituting 100 for x into the three functions, we have:

$$R(100) = 5(100) = 500$$

The total revenue from the sale of the first 50 units is \$500.

$$C(100) = 0.001(100)^2 + 1.2(100) + 60 = 190$$

The total cost of producing the first 100 units is \$190.

$$P(100) = R(100) - C(100)$$
$$= 500 - 190$$
$$= 310$$

The total profit is \$310 when the first 100 units are produced and sold.

Note, we could have also used the profit function, $P(x)$, from part (a) to find the profit.

$$P(100) = -0.001(100)^2 + 3.8(100) - 60 = 310$$

c) Finding the derivative for each of the functions, we have:

$$R'(x) = 5$$
$$C'(x) = 0.002x + 1.2$$
$$P'(x) = -0.002x + 3.8$$

d) Substituting 100 for x in each of the three marginal functions, we have:

$$R'(100) = 5$$

Once 100 units have been sold, the approximate revenue for the 51st unit is \$5.

$$C'(100) = 1.4$$

Once 100 units have been produced, the approximate cost for the 101st unit is \$1.40.

$$P'(100) = -0.002(100) + 3.8 = 3.6$$

Once 100 units have been produced and sold, the approximate profit from the sale of the 101st unit is \$3.60.

e) $\boxed{tw}$ In part (b), we are observing the total revenue, cost and profit from the production and sale of the first 100 items. In part (d), we are observing the approximate revenue, cost and profit from the production and sale of the 101st unit only. These quantities are also known as the marginal revenue, marginal cost and marginal profit.

2. $R(x) = 50x - 0.5x^2;\ C(x) = 4x + 10$

a) $P(x) = R(x) - C(x)$

$P(x) = 50x - 0.5x^2 - (4x + 10)$

$= -0.5x^2 + 46x - 10$

b) $R(20) = 50(20) - 0.5(20)^2 = 800$

$C(20) = 4(20) + 10 = 90$

$P(20) = -0.5(20)^2 + 46(20) - 10 = 710$

c) $R'(x) = 50 - x$

$C'(x) = 4$

$P'(x) = -x + 46$

d) $R'(20) = 50 - 20 = 30$

$C'(20) = 4$

$P'(20) = -20 + 46 = 26$

3. $C(x) = 0.001x^3 + 0.07x^2 + 19x + 700$

a) Substituting 25 for x into the cost function, we have:

$C(25) = 0.001(25)^3 + 0.07(25)^2 + 19(25) + 700$

$= 1234.375$

The current monthly cost of producing 25 chairs is \$1234.38.

b) In order to find the additional cost of producing 26 chairs monthly, we first find the total cost of producing 26 chairs in a month.

$C(26) = 0.001(26)^3 + 0.07(26)^2 + 19(26) + 700$

$= 1258.896$

Next, we subtract the cost of producing 25 chairs monthly found in part (a) from the cost of producing 26 chairs monthly.

$C(26) - C(25) = 1258.896 - 1234.375$

$= 24.521$

The additional cost of increasing production to 26 chairs monthly is \$24.52.

c) First, we find the marginal cost function,

$C'(x) = 0.003x^2 + 0.14x + 19.$

Next, we substituting 25 for x, we have:

$C'(25) = 0.003(25)^2 + 0.14(25) + 19$

$= 24.375$

The marginal cost when 25 chairs have been produced is \$24.38.

d) Using the marginal cost from part (c), the additional cost required to produce 2 additional chairs monthly is:

$2(24.375) = 48.75.$

Therefore, the difference in cost between producing 25 and 27 chairs per month is approximately \$48.75.

e) In part (a) we found that it cost 1234.38 to produce 25 chairs per month. In part (d) we found that the difference in cost between 25 chairs and 27 chairs per month was \$48.75. Therefore, the approximate total cost of producing 27 chairs per month is

$C(27) \approx 1234.38 + 48.75 = 1283.13.$

We predict the cost of producing 27 chairs monthly will be \$1283.13.

4. $C(x) = 0.002x^3 + 0.1x^2 + 42x + 300$

a) $C(40) = 2268$

The current daily cost of producing 40 radios is \$2268.

b) $C(41) - C(40) = 2327.94 - 2268 = 59.94$

The additional daily cost of increasing production to 41 radios daily is \$59.94.

c) $C'(x) = 0.006x^2 + 0.2x + 42$

$C'(40) = 59.6$

The marginal cost when 40 radios are produced daily is \$59.60.

d) The additional cost of producing 2 additional radios is $2(\$59.60) = \$119.20$.

Therefore, the estimated daily cost of producing 42 radios per day is

$C(42) \approx \$2268 + \$119.20 = \$2387.20.$

5. $R(x) = 0.005x^3 + 0.01x^2 + 0.5x$

a) Substituting 70 for x, we have:

$R(70) = 0.005(70)^3 + 0.01(70)^2 + 0.5(70)$

$= 1715 + 49 + 35$

$= 1799$

The currently daily revenue from selling 70 lawn chairs per day is \$1799.

b) Substituting 73 for x, we have:
$$R(73) = 0.005(73)^3 + 0.01(73)^2 + 0.5(73)$$
$$= 2034.875$$
$$= 2034.88$$
Therefore, the increase in revenue from increasing sales to 73 chairs per day is:
$$R(73) - R(70) = 2034.88 - 1799$$
$$= 235.88$$
Revenue will increase \$235.88 per day if the number of chairs sold increases to 73 per day.

c) First we find the marginal revenue function by finding the derivative of the revenue function.
$$R'(x) = 0.015x^2 + 0.02x + 0.5$$
Substituting 70 for x, we have:
$$R'(70) = 0.015(70)^2 + 0.02(70) + 0.5$$
$$= 75.40$$
The marginal revenue when 70 lawn chairs are sold daily is \$75.40.

d) In part (a) we found that selling 70 lawn chairs per day resulted in a revenue of \$1799. In part (c) we found that the marginal revenue when 70 chairs were sold is \$75.40. Using these two numbers, we estimate the daily revenue generated by selling 71 chairs is
$$R(71) \approx R(70) + R'(70)$$
$$= \$1799 + \$75.40 = \$1874.40.$$
Similarly, the daily revenue generated by selling 72 chairs, or 2 additional chairs, daily is approximately
$$R(72) \approx R(70) + 2 \cdot R'(70)$$
$$\approx \$1799 + 2(\$75.40) \approx \$1949.80.$$
The daily revenue generated by selling 73 chairs, or 3 additional chairs, daily is approximately
$$R(73) \approx R(70) + 3 \cdot R'(70)$$
$$\approx \$1799 + 3(\$75.40) \approx \$2025.20.$$

6. $P(x) = -0.006x^3 - 0.2x^2 + 900x - 1200$

a) $P(60) = \$50,784$

b) $P(60) - P(59) = 50,784 - 49,971.53 = 812.47$
The dealership would lose \$812.47 per week if it were only able to sell 59 cars weekly.

c) $P'(x) = -0.018x^2 - 0.4x + 900$
$$P'(60) = -0.018(60)^2 - 0.4(60) + 900$$
$$= 811.20$$
The marginal profit is \$811.20 when 60 cars are sold each week.

d) $P(61) \approx P(60) + P'(60) = \$51,595.20$
The estimated profit of selling 61 cars per week is \$51,595.20.

7. $P(x) = -0.004x^3 - 0.3x^2 + 600x - 800$

a) Substituting 9 for x, we have:
$$P(9) = -0.004(9)^3 - 0.3(9)^2 + 600(9) - 800$$
$$= -2.916 - 24.30 + 5400 - 800$$
$$= 4572.784$$
The currently weekly profit is \$4572.78.

b) First, we find the total weekly profit of selling 8 laptops per week.
$$P(8) = -0.004(8)^3 - 0.3(8)^2 + 600(8) - 800$$
$$= 3978.752$$
The difference in weekly profit from selling 8 laptops and 9 laptops per week is
$$P(9) - P(8) = 4572.78 - 3978.75$$
$$= 594.03$$
Therefore, Crawford Computing would lose \$594.03 each week if 8 laptops were sold each week instead of 9.

c) First, we find the marginal profit function by taking the derivative of the profit function.
$$P'(x) = -0.012x^2 - 0.6x + 600$$
Substituting 9 for x, we have:
$$P'(9) = -0.012(9)^2 - 0.6(9) + 600$$
$$= 593.628$$
The marginal profit is \$593.63 when 9 laptops are sold weekly.

d) From part (a), we know that when 9 laptops are built and sold, total weekly profit is \$4572.78. From part (c), we know that when 9 laptops are built and sold, marginal profit is \$593.63. Therefore, we estimate:
$$P(10) \approx P(9) + P'(9)$$
$$\approx 4572.78 + 593.63$$
$$\approx 5166.41$$
The total weekly profit is approximately \$5166.41 when 10 laptops are built and sold weekly.

8. $R(x)=0.007x^3-0.5x^2+150x$

a) $R(26)=\$3685.03$

b) If sales increased from 26 to 28 suitcase, revenue would increase:

$R(28)-R(26)=\$3961.66-\$3685.03$

$=\$276.63.$

c) $R'(x)=0.021x^2-x+150$

$R'(26)=0.021(26)^2-(26)+150=138.196$

Marginal revenue is 138.20 when 26 suitcases are sold.

d) $R(27)\approx R(26)+R'(26)$

$R(27)\approx \$3685.03+\$138.20\approx \$3823.23$

We estimate the revenue from selling 27 suitcases per month to be \$3823.23.

9. $N(1000)=500{,}000$ means that 500,000 computers will be sold annually when the price of the computer is \$1000.

$N'(1000)=-100$ means that when the price is increased \$1 to \$1001, sales will decrease by 100 computers per year.

10. $N(1025)\approx N(1000)+25\cdot N'(1000)$

$N(1025)\approx 500{,}000+25(-100)\approx 497{,}500$

We estimate that 497,500 computers will be sold annually if the price is increased to \$1025.

11. $C(x)=0.01x^2+0.6x+30$

$\Delta C=C(x+\Delta x)-C(x)$

Substituting $x=70,$ and $\Delta x=1$ we have

$\Delta C=C(70+1)-C(70)$

$=C(71)-C(70)$

$=0.01(71)^2+0.6(71)+30-$

$\left[0.01(70)^2+0.6(70)+30\right]$

$=2.01$

The additional cost of producing the 71st unit is \$2.01.

Finding the derivative of $C(x)$ we have:

$C'(x)=0.02x+0.6$

Substituting 70 for x, we have:

$C'(70)=0.02(70)+0.6=2.00$

The marginal cost when 70 units are produced is \$2.00.

12. $C(x)=0.01x^2+1.6x+100$

$\Delta C=C(x+\Delta x)-C(x)$

$\Delta C=C(80+1)-C(80)$

$=0.01(81)^2+1.6(81)+100-$

$\left[0.01(80)^2+1.6(80)+100\right]$

$=3.21$

The additional cost of producing the 81st unit is \$3.21.

$C'(x)=0.02x+1.6$

$C'(80)=0.02(80)+1.6=3.20$

The marginal cost when 80 units are produced is \$3.20.

13. $R(x)=2x$

$\Delta R=R(x+\Delta x)-R(x)$

Substituting $x=70,$ and $\Delta x=1$ we have

$\Delta R=R(70+1)-R(70)$

$=R(71)-R(70)$

$=2(71)-\left[2(70)\right]$

$=2$

The additional cost of producing the 71st unit is \$2.00.

Finding the derivative of $R(x)$ we have:

$R'(x)=2$.

The derivative is constant; therefore,

$R'(70)=2$

The marginal cost when 70 units are produced is \$2.00.

14. $R(x)=3x$

$\Delta R=R(x+\Delta x)-R(x)$

$\Delta R=R(80+1)-R(80)$

$=3(81)-\left[3(80)\right]$

$=3$

The additional cost of producing the 81st unit is \$3.00.

$R'(x)=3$.

$R'(80)=3$

The marginal cost when 80 units are produced is \$3.00.

15. $C(x) = 0.01x^2 + 0.6x + 30;\ R(x) = 2x$

a) Finding the profit function we have:

$$P(x) = R(x) - C(x)$$
$$= 2x - (0.01x^2 + 0.6x + 30)$$
$$= -0.01x^2 + 1.4x - 30$$

b) $\Delta P = P(x + \Delta x) - P(x)$

Substituting $x = 70,$ and $\Delta x = 1$ we have

$$\Delta P = P(70 + 1) - P(70)$$
$$= P(71) - P(70)$$
$$= -0.01(71)^2 + 1.4(71) - 30 - \left[-0.01(70)^2 + 1.4(70) - 30\right]$$
$$= -0.01$$

The additional profit of producing and selling the 71st unit is $-\$0.01$.

Finding the derivative of $P(x)$ we have:

$$P'(x) = -0.02x + 1.4$$

Substituting 70 for x, we have:

$$P'(70) = -0.02(70) + 1.4 = 0.00$$

The marginal profit when 70 units are produced and sold is \$0.00.

16. $C(x) = 0.01x^2 + 1.6x + 100;\ R(x) = 3x$

a) Finding the profit function we have:

$$P(x) = R(x) - C(x)$$
$$= 3x - (0.01x^2 + 1.6x + 100)$$
$$= -0.01x^2 + 1.4x - 100$$

b) $\Delta P = P(x + \Delta x) - P(x)$

$$\Delta P = P(80 + 1) - P(80)$$
$$= -0.01(81)^2 + 1.4(81) - 100 - \left[-0.01(80)^2 + 1.4(80) - 100\right]$$
$$= -0.21$$

he additional profit of producing and selling the 81st unit is $-\$0.21$.

$$P'(x) = -0.02x + 1.4$$
$$P'(80) = -0.02(80) + 1.4 = -0.20$$

The marginal profit when 80 units are produced and sold is $-\$0.20$.

Note: We notice that $\Delta P = \Delta R - \Delta C$ and $P'(x) = R'(x) - C'(x)$. We could have used this knowledge and our work from Exercises 12 and 14 to simplify our work. We have:

$$\Delta P = \Delta R - \Delta C$$
$$= 3 - 3.21 = -0.21$$
$$P'(80) = R'(80) - C'(80)$$
$$= 3 - 3.20$$
$$= -0.20$$

17. $D = 0.007p^3 - 0.5p^2 + 150p$

a) We take the derivative of the demand function with respect to price.

$$\frac{dD}{dp} = 0.021p^2 - p + 150$$

b) Substituting 25 for p in the demand function we have:

$$D = 0.007(25)^3 - 0.5(25)^2 + 150(25)$$
$$= 109.375 - 312.50 + 3750$$
$$= 3546.875$$

Consumers will want to buy 3547 units when price is \$25 per unit.

c) $\boxed{tw}$ Substituting 25 for p into the answer from part (a) we have:

$$\left.\frac{dD}{dp}\right|_{p=25} = 0.021(25)^2 - 25 + 150 = 138.125$$

This result implies when the price is \$25, a \$1 increase in price will lead to an increase in demand of approximately 138 pens.

d) $\boxed{tw}$ We would expect the rate of change of quantity with respect to price to be negative. All things being equal, it is reasonable to assume as the price of a good or service increases, the demand for that good or service will fall.

18. $M(t) = -2t^2 + 100t + 180$

a) Substituting for t, we have:

$$M(5) = -2(5)^2 + 100(5) + 180 = 630$$
$$M(10) = -2(10)^2 + 100(10) + 180 = 980$$
$$M(25) = -2(25)^2 + 100(25) + 180 = 1430$$
$$M(45) = -2(45)^2 + 100(45) + 180 = 630$$

b) $M'(t) = -4t + 100$

c) $\boxed{tw}$ Substituting for t, we have:

$$M'(5) = -4(5) + 100 = 80$$
$$M'(10) = -4(10) + 100 = 60$$
$$M'(25) = -4(25) + 100 = 0$$
$$M'(45) = -4(45) + 100 = -80$$

We see that the additional monthly output per years of service decreases each year the employee is with the company.

d) $\boxed{tw}$ The employees *marginal productivity* is at it's highest point when the employee is new to the company. The employee is still learning how to do the job and will make the greatest gains. As the employee gains experience, the *marginal productivity* begins to decrease. The employee is still being more productive each month, but just doesn't increase total output as much as the previous month's increase. Eventually, age catches up to the employee and they can not produce the output they once did. *Marginal productivity* becomes negative as total output starts to fall.

19. $A(x)=\dfrac{13x+100}{x}$

To estimate the change in average cost as production goes from 100 to 101 units, we establish that $x=100$ and $\Delta x=1$. Next, we find the derivative of $A(x)$.

$$A'(x)=\frac{x(13)-(13x+100)(1)}{x^2}$$
$$=\frac{13x-13x-100}{x^2}$$
$$=-\frac{100}{x^2}$$

Therefore,

$$\Delta A\approx A'(x)\Delta x$$
$$\approx A'(100)\Delta x \qquad (x=100)$$
$$\approx -\frac{100}{(100)^2}\Delta x$$
$$\approx -\frac{100}{(100)^2}(1) \qquad (\Delta x=1)$$
$$\approx -0.01$$

The average cost changes by about $-\$0.01$. (We see an approximate decrease in average cost of one cent.)

20. $S(p)=0.08p^3+2p^2+10p+11$

$p=18.00,\ \Delta p=18.20-18.00=0.20$

$S'(p)=0.24p^2+4p+10$

$$\Delta S\approx S'(p)\Delta p$$
$$\approx S'(18.00)\Delta p$$
$$\approx\left(0.24(18)^2+4(18)+10\right)(0.2)$$
$$\approx 31.952$$

The supplier will supply approximately 32 more units.

21. $P(x)=567+x\left(36x^{0.6}-104\right)$

$=567+36x^{1.6}-104x$

x is the number of years since 1960; therefore, the year 2009 corresponds to $x=2009-1960=49$, and the year 2010 corresponds to $x=2010-1960=50$. To estimate the increase in gross domestic product from 2009 to 2010, we establish that $x=49$ and $\Delta x=1$. Next, we find the derivative of $P(x)$:

$P'(x)=36(1.6)x^{0.6}-104=57.6x^{0.6}-104$.

Therefore,

$$\Delta P\approx P'(x)\Delta x$$
$$\approx P'(49)\Delta x \qquad [x=49]$$
$$\approx\left(57.6(49)^{0.6}-104\right)\Delta x$$
$$\approx(491.03173875)(1) \qquad [\Delta x=1]$$
$$\approx 491.03$$

The gross domestic product should increase about \$491.03 billion between 2009 and 2010.

22. $N(x)=-x^2+300x+6$

x is in thousands of dollars so, $x=100,\ \Delta x=1$.

$N'(x)=-2x+300$

$$\Delta N\approx N'(x)\Delta x$$
$$\approx N'(100)\Delta x$$
$$\approx\left[-2(100)+300\right](1)$$
$$\approx 100$$

Norris will sell approximately 100 more units by increasing its advertising expenditure from \$100,000 to \$101,000.

23. $\boxed{tw}$ No, the taxation in 2005 was not progressive. The 25,001st dollar is taxed at a rate of 21%, the 80,001st dollar is taxed at a rate of 15%, and the 140,001st dollar is taxed at a rate of 28%.

24. Marcy's marginal tax rate is about 42.5%, while Tyrone marginal tax rate is about 36%. Therefore, since Tyrone is in a lower marginal tax bracket, he will keep more of the \$5000 after taxes.

25. Alan's marginal tax rate is 21%, therefore for each additional dollar he earns, he will have to pay \$0.21 in taxes. If he earns another \$2000, dollars, we will pay an additional $\$2000(0.21) = \$420$ in taxes.

26. The marginal tax rate at \$50,000 is 0%. Therefore, her tax liability will not grow if she takes the extra work.

27. $y = f(x) = x^2,\ x = 2,$ and $\Delta x = 0.01$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&= f(2+0.01) - f(2) \quad \text{Substituting 2 for } x \text{ and } 0.01 \text{ for } \Delta x. \\
&= f(2.01) - f(2) \\
&= (2.01)^2 - (2)^2 \\
&= 0.0401
\end{aligned}$$

$$f'(x)\Delta x = 2x \cdot \Delta x \qquad \left[f(x) = x^2; f'(x) = 2x\right]$$

$$\begin{aligned}
f'(2)\Delta x &= 2(2)\cdot(0.01) \quad \text{Substituting 2 for } x \text{ and } 0.01 \text{ for } \Delta x. \\
&= 4(0.01) \\
&= 0.04
\end{aligned}$$

28. $y = f(x) = x^3,\ x = 2,$ and $\Delta x = 0.01$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&= f(2+0.01) - f(2) \\
&= (2.01)^3 - (2)^3 \\
&= 0.1206
\end{aligned}$$

$$f'(x)\Delta x = 3x^2 \cdot \Delta x$$

$$\begin{aligned}
f'(2)\Delta x &= 3(2)^2(0.01) \\
&= 0.12
\end{aligned}$$

29. $y = f(x) = x + x^2,\ x = 3,$ and $\Delta x = 0.04$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&= f(3+0.04) - f(3) \quad \text{Substituting 3 for } x \text{ and } 0.04 \text{ for } \Delta x. \\
&= f(3.04) - f(3) \\
&= \left[(3.04) + (3.04)^2\right] - \left[(3) + (3)^2\right] \\
&= [12.2816] - [12] \\
&= 0.2816
\end{aligned}$$

$$f'(x)\Delta x = (1+2x)\cdot\Delta x \qquad \begin{bmatrix} f(x) = x + x^2; \\ f'(x) = 1 + 2x \end{bmatrix}$$

$$\begin{aligned}
f'(3)\Delta x &= \left[1 + 2(3)\right]\cdot(0.04) \quad \text{Substituting 3 for } x \text{ and } 0.04 \text{ for } \Delta x. \\
&= [7](0.04) \\
&= 0.28
\end{aligned}$$

30. $y = f(x) = x - x^2,\ x = 3,$ and $\Delta x = 0.02$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&= f(3+0.02) - f(3) \\
&= \left[(3.02) - (3.02)^2\right] - \left[(3) - (3)^2\right] \\
&= -0.1004
\end{aligned}$$

$$f'(x)\Delta x = (1-2x)\cdot\Delta x$$

$$\begin{aligned}
f'(3)\Delta x &= (1 - 2(3))(0.02) \\
&= -0.10
\end{aligned}$$

31. $y = f(x) = \dfrac{1}{x^2} = x^{-2},\ x = 1,$ and $\Delta x = 0.5$

$$\begin{aligned}
\Delta y &= f(x+\Delta x) - f(x) \\
&= f(1+0.5) - f(1) \quad \text{Substituting 1 for } x \text{ and } 0.5 \text{ for } \Delta x. \\
&= f(1.5) - f(1) \\
&= \left[\frac{1}{(1.5)^2}\right] - \left[\frac{1}{(1)^2}\right] \\
&= \left[\frac{1}{2.25}\right] - [1] \\
&= -0.5556
\end{aligned}$$

$$f'(x)\Delta x = -2x^{-3}\cdot\Delta x \quad \begin{bmatrix} f(x)=x^{-2}; \\ f'(x)=-2x^{-3} \end{bmatrix}$$

$$f'(1)\Delta x = \left[-2(1)^{-3}\right]\cdot(0.5) \quad \text{Substituting 1 for } x \text{ and 0.5 for } \Delta x.$$

$$= [-2](0.5)$$
$$= -1$$

32. $y = f(x) = \frac{1}{x} = x^{-1},\ x = 1,\ \text{and } \Delta x = 0.2$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= f(1+0.2) - f(1)$$
$$= \left[\frac{1}{1.2}\right] - \left[\frac{1}{1}\right]$$
$$= -0.1667$$
$$f'(x)\Delta x = \left(-x^{-2}\right)\cdot\Delta x$$
$$f'(1)\Delta x = \left(-(1)^{-2}\right)(0.2)$$
$$= -0.2$$

33. $y = f(x) = 3x-1,\ x = 4,\ \text{and } \Delta x = 2$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= f(4+2) - f(4) \quad \text{Substituting 4 for } x \text{ and 2 for } \Delta x.$$
$$= f(6) - f(4)$$
$$= [3(6)-1] - [3(4)-1]$$
$$= [17] - [11]$$
$$= 6$$
$$f'(x)\Delta x = (3)\cdot\Delta x \quad [f(x)=3x-1; f'(x)=3]$$
$$f'(4)\Delta x = (3)\cdot(2) \quad \text{Substituting 4 for } x \text{ and 2 for } \Delta x.$$
$$= 6$$

34. $y = f(x) = 2x-3,\ x = 8,\ \text{and } \Delta x = 0.5$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= f(8+0.5) - f(8)$$
$$= [2(8.5)-3] - [2(8)-3]$$
$$= 1$$
$$f'(x)\Delta x = (2)\cdot\Delta x$$
$$f'(8)\Delta x = (2)\cdot(0.5)$$
$$= 1$$

35. We first think of the number closest to 26 that is a perfect square. This is 25. What we will do is approximate how $y = \sqrt{x}$, changes when x changes from 25 to 26. Let

$$y = f(x) = \sqrt{x} = x^{1/2}$$

Then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

Using, $\Delta y \approx f'(x)\Delta x$, we have

$$\Delta y \approx f'(x)\Delta x$$
$$\approx \frac{1}{2\sqrt{x}}\cdot\Delta x$$

We are interested in Δy as x changes from 25 to 26, so

$$\Delta y \approx \frac{1}{2\sqrt{x}}\cdot\Delta x$$
$$\approx \frac{1}{2\sqrt{25}}\cdot 1 \quad \text{Replacing } x \text{ with 25 and } \Delta x \text{ with 1}$$
$$\approx \frac{1}{2\cdot 5}$$
$$\approx \frac{1}{10} = 0.1$$

We can now approximate $\sqrt{26}$;

$$\sqrt{26} = \sqrt{25} + \Delta y$$
$$= 5 + \Delta y$$
$$\approx 5 + 0.1$$
$$\approx 5.1$$

To five decimal places $\sqrt{26} = 5.09902$. Thus, our approximation is fairly accurate.

36. Let $y = f(x) = \sqrt{x},\ x = 9,\ \Delta x = 1$

$$\Delta y \approx f'(x)\Delta x$$
$$\approx \frac{1}{2\sqrt{x}}\cdot\Delta x$$
$$\approx \frac{1}{2\sqrt{9}}\cdot(1)$$
$$\approx \frac{1}{6} = 0.167$$

We can now approximate $\sqrt{10}$;

$$\sqrt{10} = \sqrt{9} + \Delta y$$
$$= 3 + \Delta y$$
$$\approx 3 + 0.167$$
$$\approx 3.167$$

To five decimal places $\sqrt{10} = 3.16228$. Thus, our approximation is fairly accurate.

37. We first think of the number closest to 102 that is a perfect square. This is 100. What we will do is approximate how $y=\sqrt{x}$, changes when x changes from 100 to 102. Let

$y=f(x)=\sqrt{x}=x^{1/2}$

Then $f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

Using, $\Delta y\approx f'(x)\Delta x$, we have

$$\begin{aligned}\Delta y&\approx f'(x)\Delta x\\&\approx\frac{1}{2\sqrt{x}}\cdot\Delta x\end{aligned}$$

We are interested in Δy as x changes from 100 to 102, so

$$\begin{aligned}\Delta y&\approx\frac{1}{2\sqrt{x}}\cdot\Delta x\\&\approx\frac{1}{2\sqrt{100}}\cdot 2 \quad \text{Replacing } x \text{ with 100 and } \Delta x \text{ with 2}\\&\approx\frac{1}{2\cdot 10}\cdot 2\\&\approx\frac{1}{10}=0.1\end{aligned}$$

We can now approximate $\sqrt{102}$;

$$\begin{aligned}\sqrt{102}&=\sqrt{100}+\Delta y\\&=10+\Delta y\\&\approx 10+0.1\\&\approx 10.1\end{aligned}$$

To five decimal places $\sqrt{26}=10.09950$. Thus, our approximation is fairly accurate.

38. Let $y=f(x)=\sqrt{x},\ x=100,\ \Delta x=3$

$$\begin{aligned}\Delta y&\approx f'(x)\Delta x\\&\approx\frac{1}{2\sqrt{x}}\cdot\Delta x\\&\approx\frac{1}{2\sqrt{100}}\cdot(3)\\&\approx\frac{3}{20}=0.15\end{aligned}$$

We can now approximate $\sqrt{103}$;

$$\begin{aligned}\sqrt{103}&=\sqrt{100}+\Delta y\\&\approx 10+0.15\\&\approx 10.15\end{aligned}$$

To five decimal places $\sqrt{103}=10.14889$. Thus, our approximation is fairly accurate.

39. We first think of the number closest to 1005 that is a perfect cube. This is 1000. What we will do is approximate how $y=\sqrt[3]{x}$, changes when x changes from 1000 to 1005. Let

$y=f(x)=\sqrt[3]{x}=x^{1/3}$

Then $f'(x)=\frac{1}{3}x^{-2/3}=\frac{1}{2\sqrt[3]{x^2}}$

Using, $\Delta y\approx f'(x)\Delta x$, we have

$$\begin{aligned}\Delta y&\approx f'(x)\Delta x\\&\approx\frac{1}{2\sqrt[3]{x^2}}\cdot\Delta x\end{aligned}$$

We are interested in Δy as x changes from 1000 to 1005, so

$$\Delta y\approx\frac{1}{3\sqrt[3]{x^2}}\cdot\Delta x$$

Replacing x with 1000 and Δx with 5, we have

$$\begin{aligned}&\approx\frac{1}{3\cdot\sqrt[3]{(1000)^2}}\cdot 5\\&\approx\frac{1}{3\cdot 100}\cdot 5\\&\approx\frac{1}{60}=0.017\end{aligned}$$

We can now approximate $\sqrt[3]{1005}$;

$$\begin{aligned}\sqrt[3]{1005}&=\sqrt[3]{1000}+\Delta y\\&=10+\Delta y\\&\approx 10+0.017\\&\approx 10.017\end{aligned}$$

To five decimal places $\sqrt[3]{1000}=10.01664$ Thus, our approximation is fairly accurate.

40. Let $y=f(x)=\sqrt[3]{x},\ x=27,\ \Delta x=1$

$$\begin{aligned}\Delta y&\approx f'(x)\Delta x\\&\approx\frac{1}{3\cdot\sqrt[3]{x^2}}\cdot\Delta x\\&\approx\frac{1}{3\cdot\sqrt[3]{(27)^2}}\cdot(1)\\&\approx\frac{1}{27}=0.037\end{aligned}$$

We can now approximate $\sqrt[3]{28}$;

$$\sqrt[3]{28} = \sqrt[3]{27} + \Delta y \\ \approx 3 + 0.037 \\ \approx 3.037$$

To five decimal places $\sqrt[3]{28} = 3.03659$ Thus, our approximation is fairly accurate.

41. $y = \sqrt{x+1} = (x+1)^{1/2}$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2}(1) = \frac{1}{2\sqrt{x+1}}.$$

Then

$$dy = \frac{1}{2\sqrt{x+1}}\,dx.$$

Note that the expression for dy contains two variables x and dx.

42. $y = \sqrt{3x-2} = (3x-2)^{1/2}$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{2}(3x-2)^{-1/2}(3) = \frac{3}{2\sqrt{3x-2}}.$$

Then

$$dy = \frac{3}{2\sqrt{3x-2}}\,dx.$$

43. $y = (2x^3+1)^{3/2}$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{3}{2}(2x^3+1)^{1/2}(6x^2) \quad \text{By the extended power rule}$$
$$= 9x^2(2x^3+1)^{1/2}$$
$$= 9x^2\sqrt{2x^3+1}.$$

Then

$$dy = 9x^2\sqrt{2x^3+1}\,dx.$$

Note that the expression for dy contains two variables x and dx.

44. $y = x^3(2x+5)^2$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = x^3\left[2(2x+5)(2)\right] + 3x^2(2x+5)^2$$
$$= 4x^3(2x+5) + 3x^2(2x+5)^2$$
$$= \left(4x^3 + 3x^2(2x+5)\right)(2x+5)$$
$$= \left(4x^3 + 6x^3 + 15x^2\right)(2x+5)$$
$$= \left(10x^3 + 15x^2\right)(2x+5)$$
$$= 5x^2(2x+3)(2x+5).$$

Then

$$dy = 5x^2(2x+3)(2x+5)\,dx.$$

45. $y = \sqrt[5]{x+27} = (x+27)^{1/5}$

First, we find $\frac{dy}{dx}$. By the extended power rule we have:

$$\frac{dy}{dx} = \frac{1}{5}(x+27)^{-4/5}(1) = \frac{1}{5\cdot\sqrt[5]{(x+27)^4}}.$$

Then

$$dy = \frac{1}{5\cdot\sqrt[5]{(x+27)^4}}\,dx.$$

Note that the expression for dy contains two variables x and dx.

46. $y = \frac{x^3+x+2}{x^2+3}$

First, we find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{(x^2+3)(3x^2+1) - (x^3+x+2)(2x)}{(x^2+3)^2}$$
$$= \frac{(3x^4+10x^2+3) - (2x^4+2x^2+4x)}{(x^2+3)^2}.$$
$$= \frac{x^4+8x^2-4x+3}{(x^2+3)^2}.$$

Then

$$dy = \frac{x^4+8x^2-4x+3}{(x^2+3)^2}\,dx.$$

47. $y = x^4 - 2x^3 + 5x^2 + 3x - 4$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx} = 4x^3 - 6x^2 + 10x + 3$.

Then

$dy = \left(4x^3 - 6x^2 + 10x + 3\right)dx$.

Note that the expression for dy contains two variables x and dx.

48. $y = (7 - x)^8$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx} = 8(7 - x)^7(-1) = -8(7 - x)^7$.

Then

$dy = -8(7 - x)^7\, dx$.

49. From Exercise 47, we know:

$dy = \left(4x^3 - 6x^2 + 10x + 3\right)dx$.

When $x = 2$ and $dx = 0.1$ we have:

$$\begin{aligned} dy &= \left(4(2)^3 - 6(2)^2 + 10(2) + 3\right)(0.1) \\ &= (32 - 24 + 20 + 3)(0.1) \\ &= (31)(0.1) \\ &= 3.1 \end{aligned}$$

50. From Exercise 48, we know:

$dy = -8(7 - x)^7\, dx$,

when $x = 1$ and $dx = 0.01$, we have:

$$\begin{aligned} dy &= -8(7 - 1)^7(0.01) \\ &= -8(6)^7(0.01) \\ &= -22{,}394.88. \end{aligned}$$

51. $y = (3x - 10)^5$

First, we find $\frac{dy}{dx}$:

$$\begin{aligned} \frac{dy}{dx} &= 5(3x - 10)^4(3) \quad \text{By the extended power rule} \\ &= 15(3x - 10)^4 \end{aligned}$$

Then

$dy = 15(3x - 10)^4\, dx$.

When $x = 4$ and $dx = 0.03$ we have:

$$\begin{aligned} dy &= 15(3(4) - 10)^4(0.03) \\ &= 15(2)^4(0.03) \\ &= 7.2 \end{aligned}$$

52. $y = x^5 - 2x^3 - 7x$

First, we find $\frac{dy}{dx}$:

$\frac{dy}{dx} = 5x^4 - 6x^2 - 7$.

Then

$dy = \left(5x^4 - 6x^2 - 7\right)dx$.

When $x = 3$ and $dx = 0.02$ we have:

$$\begin{aligned} dy &= \left(5(3)^4 - 6(3)^2 - 7\right)(0.02) \\ &= 6.88. \end{aligned}$$

53. Let $y = f(x) = x^4 - x^2 + 8$

First we find $f'(x)$:

$f'(x) = 4x^3 - 2x$.

Then

$$\begin{aligned} dy &= f'(x)dx \\ &= \left(4x^3 - 2x\right)dx \end{aligned}$$

To approximate $f(5.1)$, we will use $x = 5$ and $dx = 0.1$ to determine the differential dy.

Substituting 5 for x and 0.1 for dx we have:

$$\begin{aligned} dy &= f'(5)dx \\ &= \left(4(5)^3 - 2(5)\right)(0.1) \\ &= \left(4(125) - 10\right)(0.1) \\ &= (500 - 10)(0.1) \\ &= (490)(0.1) \\ &= 49 \end{aligned}$$

Next, we find

$$\begin{aligned} f(5) &= (5)^4 - (5)^2 + 8 \\ &= 625 - 25 + 8 \\ &= 608 \end{aligned}$$

Now,

$$\begin{aligned} f(5.1) &\approx f(5) + f'(5)dx \\ &\approx 608 + 49 \\ &\approx 657 \end{aligned}$$

54. Let $y = f(x) = x^3 - 5x + 9$

First we find $f'(x)$:

$f'(x) = 3x^2 - 5$.

Then

$$dy = f'(x)dx$$
$$= (3x^2 - 5)dx$$

To approximate $f(3.2)$, we will use $x = 3$ and $dx = 0.2$ to determine the differential dy.

$$dy = f'(3)dx$$
$$= (3(3)^2 - 5)(0.2)$$
$$= (27 - 5)(0.2)$$
$$= (22)(0.2)$$
$$= 4.4$$

Next, we find

$$f(3) = (3)^3 - 5(3) + 9$$
$$= 27 - 15 + 9$$
$$= 21$$

Now,

$$f(3.2) \approx f(3) + f'(3)dx$$
$$\approx 21 + 4.4$$
$$\approx 25.4$$

55. $S = 0.02235h^{0.42246}w^{0.51456}$

We begin by noticing that we are wanting to estimate the change in surface area due to a change in weight w; therefore, we will first find $\frac{dS}{dw}$. Since $h = 160$, we have:

$$S = 0.02235(160)^{0.42246}\ w^{0.51456}$$
$$= 0.02235(8.53399783)w^{0.51456}$$
$$= 0.19073485w^{0.51456}$$

Now we can take the derivative of S with respect to w.

$$\frac{dS}{dw} = 0.19073485(0.51456)w^{-0.48544}$$
$$= 0.09814452w^{-0.48544}$$

Therefore,

$$dS = (0.09814452w^{-0.48544})dw$$

Now that we have the differential, we can use her weight of 60 kg to approximate how much her surface area changes when her weight drops 1 kg. We substitute 60 for w and -1 for dw to get:

$$dS \approx (0.09814452(60)^{-0.48544})(-1)$$
$$\approx -0.01345$$

The patient's surface area will change by $-0.01345\ \text{m}^2$.

56. $A = \pi r^2$

$$dA = A'(r)dr$$
$$= (2\pi r)dr$$

Using 3.14 for π, 2 for r, and -0.1 for dr. We have:

$$dA = [2(3.14)(2)](-0.1)$$
$$= -1.256$$

The area will change by $-1.256\ \text{cm}^2$ when the radius decreases from 2 cm to 1.9 cm.

57. $N(t) = \dfrac{0.8t + 1000}{5t + 4}$

First we find $N'(t)$ by the quotient rule.

$$N'(t) = \frac{(5t+4)(0.8) - (0.8t+1000)(5)}{(5t+4)^2}$$
$$= \frac{4t + 3.2 - 4t - 5000}{(5t+4)^2}$$
$$= -\frac{4996.8}{(5t+4)^2}$$

The differential is:

$$dN = N'(t)dt$$
$$= -\frac{4996.8}{(5t+4)^2} \cdot dt$$

We approximate the change in bodily concentration from 1.0 hr to 1.1 hr by using 1.0 for t and 0.1 for dt.

$$dN = -\frac{4996.8}{(5(1.0)+4)^2} \cdot (0.1)$$
$$= -\frac{4996.8}{(9)^2}(0.1)$$
$$= -\frac{4996.8}{81}(0.1)$$
$$\approx -61.6889(0.1)$$
$$\approx -6.16889$$

Next, we approximate the change in bodily concentration from 2.8 hr to 2.9 hr by using 2.8 for *t* and 0.1 for *dt*.

$$\begin{aligned} dN &= -\frac{4996.8}{\left(5(2.8)+4\right)^2}\cdot(0.1) \\ &= -\frac{4996.8}{(18)^2}(0.1) \\ &= -\frac{4996.8}{324}(0.1) \\ &\approx -15.4222(0.1) \\ &\approx -1.54222 \end{aligned}$$

The concentration changes more from 1.0 hr to 1.1 hr.

58. $p(x)=0.09x^2-0.19x+9.41$

$$\begin{aligned} dp &= p'(x)\,dx \\ &= (0.18x-0.19)\,dx \end{aligned}$$

Since *x* is the number of years since1990, we have 2010 implies $x=20$ and 2012 implies $x=22$. To estimate the change in ticket prices from 2010 and 2012, we substitute 20 for *x* and 2 for *dx*.

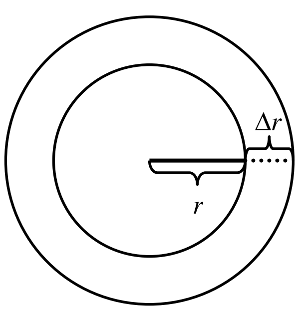
$$dp=(0.18(20)-0.19)(2)=6.82$$

To estimate the change in ticket prices from 2030 and 2031, we substitute 40 for *x* and 1 for *dx*.

$$dp=(0.18(40)-0.19)(1)=7.01$$

Ticket prices will increase more between 2030 and 2031.

59. The circumference of the earth, which is the original length of the rope, is given by $C(r)=2\pi r$, where *r* is the radius of the earth. We need to find the change in the length of the radius, Δr, when the length of the rope is increased 10 feet.



Using differentials, $\Delta C\approx C'(r)\Delta r$ represents the change in the length of the rope. Therefore, $\Delta C=10$, and we have:

$$10=C'(r)\Delta r$$

Noticing that $C'(r)=2\pi$, we have:

$$10=2\pi\cdot\Delta r$$

$$\frac{10}{2\pi}=\Delta r$$

Therefore, the rope is raised approximately $\Delta r=\frac{5}{\pi}\approx 1.59$ feet above the earth.

60. $A(x)=\frac{C(x)}{x}$

By the quotient rule we have:

$$\begin{aligned} A'(x) &= \frac{x\cdot C'(x)-C(x)(1)}{x^2} \\ &= \frac{x\cdot C'(x)-C(x)}{x^2}. \end{aligned}$$

61. $p=100-\sqrt{x}$

Since revenue is price times quantity, the revenue function is given by:

$$\begin{aligned} R(x) &= p\cdot x \\ &= \left(100-\sqrt{x}\right)x \\ &= 100x-x^{3/2}. \end{aligned}$$

To find the marginal revenue, we take the derivative of the revenue function. Thus:

$$\begin{aligned} R'(x) &= 100-\frac{3}{2}x^{1/2} \\ &= 100-\frac{3\sqrt{x}}{2}. \end{aligned}$$

62. $p=400-x$

$$\begin{aligned} R(x) &= p\cdot x \\ &= (400-x)x \\ &= 400x-x^2 \end{aligned}$$

$$R'(x)=400-2x$$

63. $p = 500 - x$

Since revenue is price times quantity, the revenue function is given by:

$$\begin{aligned} R(x) &= p \cdot x \\ &= (500 - x)x \\ &= 500x - x^2 \end{aligned}$$

To find the marginal revenue, we take the derivative of the revenue function. Thus:

$R'(x) = 500 - 2x$.

64. $p = \dfrac{4000}{x} + 3$

$$\begin{aligned} R(x) &= p \cdot x \\ &= \left(\frac{4000}{x} + 3\right)x \\ &= 4000 + 3x \end{aligned}$$

$R'(x) = 3$

65. $p = \dfrac{3000}{x} + 5$

Since revenue is price times quantity, the revenue function is given by:

$$\begin{aligned} R(x) &= p \cdot x \\ &= \left(\frac{3000}{x} + 5\right)x \\ &= 3000 + 5x. \end{aligned}$$

To find the marginal revenue, we take the derivative of the revenue function. Thus:

$R'(x) = 5$.

66. $\boxed{tw}$ Answers will vary. Calculus in its present form was essentially developed independently in the 17th century by Isaac Newton and Gottfried Wilhelm von Leibniz. During the 18th century calculus was challenged by some philosophers and religious leader who argued that the infinitely small quantities represented by differentials were meaningless. These critics were silenced when the concept of "limit" was introduced. In it, a differential was not thought of as an infinitely small quantity; rather, a derivative was considered to be the limit approached by two differentials as each becomes infinitely small.

67. $\boxed{tw}$ For a function $y = f(x)$, the differential, *dy,* can be used to approximate the true change in the value of $f(x)$ when a small change is made in the value of *x*.

Exercise Set 2.7

1. Differentiate implicitly to find $\dfrac{dy}{dx}$.

We have

$x^3 + 2y^3 = 6$

Differentiating both sides with respect to *x* yields:

$$\frac{d}{dx}\left(x^3 + 2y^3\right) = \frac{d}{dx}(6)$$

$$\frac{d}{dx}x^3 + 2\frac{d}{dx}y^3 = \frac{d}{dx}6$$

$$3x^2 + 2 \cdot 3y^2 \cdot \frac{dy}{dx} = 0$$

Next, we isolate $\dfrac{dy}{dx}$

$$6y^2 \cdot \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{6y^2}$$

$$\frac{dy}{dx} = \frac{-x^2}{2y^2}$$

Find the slope of the tangent line to the curve at $(2, -1)$.

$$\frac{dy}{dx} = \frac{-x^2}{2y^2}$$

Replacing *x* with 2 and *y* with -1, we have:

$$\frac{dy}{dx} = \frac{-(2)^2}{2(-1)^2} = \frac{-4}{2} = -2\ .$$

The slope of the tangent line to the curve at $(2, -1)$ is -2.

2. $3x^3 - y^2 = 8$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(3x^3 - y^2\right) = \frac{d}{dx}(8)$$

$$9x^2 - 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{9x^2}{2y}$$

Find the slope of the tangent line to the curve at $(2,4)$.

$$\frac{dy}{dx} = \frac{9x^2}{2y}$$

$$\frac{dy}{dx} = \frac{9(2)^2}{2(4)} = \frac{36}{8} = \frac{9}{2}.$$

The slope of the tangent line to the curve at $(2,4)$ is $\frac{9}{2}$.

3. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$2x^2 - 3y^3 = 5$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^2 - 3y^3\right) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}2x^2 - 3\frac{d}{dx}y^3 = \frac{d}{dx}5$$

$$2\cdot 2x - 3\cdot 3y^2\cdot\frac{dy}{dx} = 0$$

Next, we isolate $\frac{dy}{dx}$

$$-9y^2\cdot\frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{-4x}{-9y^2}$$

$$\frac{dy}{dx} = \frac{4x}{9y^2}$$

Find the slope of the tangent line to the curve at $(-2,1)$.

$$\frac{dy}{dx} = \frac{4x}{9y^2}$$

Replacing x with -2 and y with 1, we have:

$$\frac{dy}{dx} = \frac{4(-2)}{9(1)^2} = \frac{-8}{9} = -\frac{8}{9}.$$

The slope of the tangent line to the curve at $(-2,1)$ is $-\frac{8}{9}$.

4. $2x^3 + 4y^2 = -12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^3 + 4y^2\right) = \frac{d}{dx}(-12)$$

$$6x^2 + 8y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-6x^2}{8y} = -\frac{3x^2}{4y}$$

Find the slope of the tangent line to the curve at $(-2,-1)$.

$$\frac{dy}{dx} = \frac{-3(-2)^2}{4(-1)} = \frac{-12}{-4} = 3.$$

The slope of the tangent line to the curve at $(-2,-1)$ is 3.

5. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$x^2 - y^2 = 1$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}x^2 - \frac{d}{dx}y^2 = \frac{d}{dx}1$$

$$2x - 2y\cdot\frac{dy}{dx} = 0$$

Next, we isolate $\frac{dy}{dx}$

$$-2y\cdot\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-2y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Find the slope of the tangent line to the curve at $\left(\sqrt{3},\sqrt{2}\right)$.

$$\frac{dy}{dx} = \frac{x}{y}$$

Replacing x with $\sqrt{3}$ and y with $\sqrt{2}$, we have:

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}.$$

The slope of the tangent line to the curve at $\left(\sqrt{3}, \sqrt{2}\right)$ is $\sqrt{\frac{3}{2}}$.

6. $x^2 + y^2 = 1$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Find the slope of the tangent line to the curve at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$\frac{dy}{dx} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}.$$

The slope of the tangent line to the curve at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $-\frac{1}{\sqrt{3}}$.

7. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$3x^2y^4 = 12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(3x^2y^4\right) = \frac{d}{dx}(12)$$

$$3x^2\frac{d}{dx}y^4 + y^4\frac{d}{dx}3x^2 = \frac{d}{dx}12 \quad \text{Product Rule}$$

$$3x^2\left(4y^3 \cdot \frac{dy}{dx}\right) + y^4 \cdot (3 \cdot 2x) = 0$$

$$12x^2y^3 \cdot \frac{dy}{dx} + 6xy^4 = 0$$

$$12x^2y^3 \cdot \frac{dy}{dx} = -6xy^4$$

$$\frac{dy}{dx} = \frac{-6xy^4}{12x^2y^3}$$

$$\frac{dy}{dx} = -\frac{y}{2x}$$

Find the slope of the tangent line to the curve at $(2, -1)$.

$$\frac{dy}{dx} = -\frac{y}{2x}$$

Replacing x with 2 and y with -1, we have:

$$\frac{dy}{dx} = -\frac{(-1)}{2(2)} = \frac{1}{4}.$$

The slope of the tangent line to the curve at $(2, -1)$ is $\frac{1}{4}$.

8. $2x^3y^2 = -18$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(2x^3y^2\right) = \frac{d}{dx}(-18)$$

$$2x^3\left(2y \cdot \frac{dy}{dx}\right) + y^2\left(6x^2\right) = 0$$

$$4x^3y \cdot \frac{dy}{dx} + 6x^2y^2 = 0$$

$$\frac{dy}{dx} = \frac{-6x^2y^2}{4x^3y} = -\frac{3y}{2x}$$

Find the slope of the tangent line to the curve at $(-1, 3)$.

$$\frac{dy}{dx} = -\frac{3(3)}{2(-1)} = \frac{9}{2}$$

The slope of the tangent line to the curve at $(-1, 3)$ is $\frac{9}{2}$.

9. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$x^3 - x^2y^2 = -9$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3 - x^2y^2\right) = \frac{d}{dx}(-9)$$

$$\frac{d}{dx}x^3 - \frac{d}{dx}x^2y^2 = \frac{d}{dx}(-9)$$

$$3x^2 - \left[x^2\left(2y\cdot\frac{dy}{dx}\right) + y^2(2x)\right] = 0$$

$$3x^2 - 2x^2y\cdot\frac{dy}{dx} - 2xy^2 = 0$$

$$-2x^2y\cdot\frac{dy}{dx} = 2xy^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{2xy^2 - 3x^2}{-2x^2y}$$

$$\frac{dy}{dx} = \frac{-x\left(3x - 2y^2\right)}{-x(2xy)}$$

$$\frac{dy}{dx} = \frac{3x - 2y^2}{2xy}$$

Find the slope of the tangent line to the curve at $(3,-2)$.

$$\frac{dy}{dx} = \frac{3x - 2y^2}{2xy}$$

Replacing x with 3 and y with -2, we have:

$$\frac{dy}{dx} = \frac{3(3) - 2(-2)^2}{2(3)(-2)} = \frac{9-8}{-12} = -\frac{1}{12}.$$

The slope of the tangent line to the curve at $(3,-2)$ is $-\frac{1}{12}$.

10. $x^4 - x^2y^3 = 12$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^4 - x^2y^3\right) = \frac{d}{dx}(12)$$

$$4x^3 - \left[x^2\left(3y^2\cdot\frac{dy}{dx}\right) + y^3(2x)\right] = 0$$

$$4x^3 - 3x^2y^2\cdot\frac{dy}{dx} - 2xy^3 = 0$$

$$\frac{dy}{dx} = \frac{2xy^3 - 4x^3}{-3x^2y^2}$$

$$\frac{dy}{dx} = \frac{4x^2 - 2y^3}{3xy^2}$$

Find the slope of the tangent line to the curve at $(-2,1)$.

$$\frac{dy}{dx} = \frac{4(-2)^2 - 2(1)^3}{3(-2)(1)^2} = -\frac{7}{3}$$

The slope of the tangent line to the curve at $(-2,1)$ is $-\frac{7}{3}$.

11. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$xy - x + 2y = 3$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(xy - x + 2y) = \frac{d}{dx}(3)$$

$$\frac{d}{dx}xy - \frac{d}{dx}x + 2\frac{d}{dx}y = \frac{d}{dx}(3)$$

$$\left[x\left(\frac{dy}{dx}\right) + y(1)\right] - 1 + 2\cdot\frac{dy}{dx} = 0$$

$$x\cdot\frac{dy}{dx} + y - 1 + 2\cdot\frac{dy}{dx} = 0$$

$$x\cdot\frac{dy}{dx} + 2\cdot\frac{dy}{dx} = 1 - y$$

$$(x+2)\cdot\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{dx} = \frac{1-y}{x+2}$$

Find the slope of the tangent line to the curve at $\left(-5,\frac{2}{3}\right)$.

$$\frac{dy}{dx} = \frac{1-y}{x+2}$$

Replacing x with -5 and y with $\frac{2}{3}$, we have:

$$\frac{dy}{dx}=\frac{1-\left(\frac{2}{3}\right)}{(-5)+2}=\frac{\frac{1}{3}}{-3}=-\frac{1}{9}.$$

The slope of the tangent line to the curve at $\left(-5,\frac{2}{3}\right)$ is $-\frac{1}{9}$.

12. $xy+y^2-2x=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(xy+y^2-2x\right)=\frac{d}{dx}(0)$$

$$\left[x\left(\frac{dy}{dx}\right)+y(1)\right]+2y\cdot\frac{dy}{dx}-2(1)=0$$

$$x\cdot\frac{dy}{dx}+y+2y\cdot\frac{dy}{dx}-2=0$$

$$(x+2y)\cdot\frac{dy}{dx}=2-y$$

$$\frac{dy}{dx}=\frac{2-y}{x+2y}$$

Find the slope of the tangent line to the curve at $(1,-2)$.

$$\frac{dy}{dx}=\frac{2-(-2)}{(1)+2(-2)}=\frac{4}{-3}=-\frac{4}{3}$$

The slope of the tangent line to the curve at $(1,-2)$ is $-\frac{4}{3}$.

13. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$x^2y-2x^3-y^3+1=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2y-2x^3-y^3+1\right)=\frac{d}{dx}(0)$$

$$\frac{d}{dx}x^2y-\frac{d}{dx}2x^3-\frac{d}{dx}y^3+\frac{d}{dx}1=0$$

$$\left[x^2\left(\frac{dy}{dx}\right)+y(2x)\right]-2\left(3x^2\right)-\left(3y^2\right)\cdot\frac{dy}{dx}=0$$

$$x^2\cdot\frac{dy}{dx}+2xy-6x^2-3y^2\cdot\frac{dy}{dx}=0$$

$$x^2\cdot\frac{dy}{dx}-3y^2\cdot\frac{dy}{dx}=6x^2-2xy$$

$$\left(x^2-3y^2\right)\cdot\frac{dy}{dx}=6x^2-2xy$$

$$\frac{dy}{dx}=\frac{6x^2-2xy}{x^2-3y^2}$$

Find the slope of the tangent line to the curve at $(2,-3)$.

$$\frac{dy}{dx}=\frac{6x^2-2xy}{x^2-3y^2}$$

Replacing x with 2 and y with -3, we have:

$$\frac{dy}{dx}=\frac{6(2)^2-2(2)(-3)}{(2)^2-3(-3)^2}=\frac{24+12}{4-27}=\frac{36}{-23}=-\frac{36}{23}$$

The slope of the tangent line to the curve at $(2,-3)$ is $-\frac{36}{23}$.

14. $4x^3-y^4-3y+5x+1=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(4x^3-y^4-3y+5x+1\right)=\frac{d}{dx}(0)$$

$$12x^2-4y^3\cdot\frac{dy}{dx}-3\cdot\frac{dy}{dx}+5+0=0$$

$$-4y^3\cdot\frac{dy}{dx}-3\cdot\frac{dy}{dx}=-12x^2-5$$

$$\left(-4y^3-3\right)\cdot\frac{dy}{dx}=-12x^2-5$$

$$\frac{dy}{dx}=\frac{-12x^2-5}{-4y^3-3}$$

$$\frac{dy}{dx}=\frac{12x^2+5}{4y^3+3}$$

Find the slope of the tangent line to the curve at $(1,-2)$.

$$\frac{dy}{dx}=\frac{12(1)^2+5}{4(-2)^3+3}=-\frac{17}{29}$$

The slope of the tangent line to the curve at $(1,-2)$ is $-\frac{17}{29}$.

15. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$2xy+3=0$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(2xy+3)=\frac{d}{dx}(0)$$
$$2\cdot\frac{d}{dx}xy+\frac{d}{dx}3=\frac{d}{dx}(0)$$
$$2\cdot\left[x\left(\frac{dy}{dx}\right)+y(1)\right]+0=0$$
$$2x\cdot\frac{dy}{dx}+2y=0$$
$$2x\cdot\frac{dy}{dx}=-2y$$
$$\frac{dy}{dx}=\frac{-2y}{2x}$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

16. $x^2+2xy=3y^2$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(x^2+2xy)=\frac{d}{dx}(3y^2)$$
$$2x+2\left[x\left(\frac{dy}{dx}\right)+y(1)\right]=3\left(2y\cdot\frac{dy}{dx}\right)$$
$$2x+2x\cdot\frac{dy}{dx}+2y=6y\cdot\frac{dy}{dx}$$
$$2x\cdot\frac{dy}{dx}-6y\cdot\frac{dy}{dx}=-2x-2y$$
$$(2x-6y)\cdot\frac{dy}{dx}=-2(x+y)$$
$$\frac{dy}{dx}=\frac{-2(x+y)}{2x-6y}$$
$$\frac{dy}{dx}=\frac{-2(x+y)}{-2(-x+3y)}$$
$$\frac{dy}{dx}=\frac{x+y}{3y-x}$$

17. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$x^2-y^2=16$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(16)$$
$$\frac{d}{dx}x^2-\frac{d}{dx}y^2=\frac{d}{dx}(16)$$
$$2x-2y\cdot\frac{dy}{dx}=0$$
$$-2y\cdot\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=\frac{-2x}{-2y}$$
$$\frac{dy}{dx}=\frac{x}{y}$$

18. $x^2+y^2=25$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25)$$
$$2x+2y\cdot\frac{dy}{dx}=0$$
$$2y\cdot\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=\frac{-2x}{2y}$$
$$\frac{dy}{dx}=-\frac{x}{y}$$

19. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$y^5=x^3$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(y^5)=\frac{d}{dx}(x^3)$$
$$5y^4\cdot\frac{dy}{dx}=3x^2$$
$$\frac{dy}{dx}=\frac{3x^2}{5y^4}$$

20. $y^3 = x^5$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{dx}\left(x^5\right)$$

$$3y^2 \cdot \frac{dy}{dx} = 5x^4$$

$$\frac{dy}{dx} = \frac{5x^4}{3y^2}$$

21. Differentiate implicitly to find $\frac{dy}{dx}$.

We have

$$x^2y^3 + x^3y^4 = 11$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^2y^3 + x^3y^4\right) = \frac{d}{dx}(11)$$

$$\frac{d}{dx}\left(x^2y^3\right) + \frac{d}{dx}\left(x^3y^4\right) = 0$$

Notice:

$$\frac{d}{dx}\left(x^2y^3\right) = x^2\left(3y^2 \cdot \frac{dy}{dx}\right) + y^3(2x)$$

$$= 3x^2y^2 \cdot \frac{dy}{dx} + 2xy^3$$

and

$$\frac{d}{dx}\left(x^3y^4\right) = x^3\left(4y^3 \cdot \frac{dy}{dx}\right) + y^4\left(3x^2\right)$$

$$= 4x^3y^3 \cdot \frac{dy}{dx} + 3x^2y^4$$

Therefore,

$$\frac{d}{dx}\left(x^2y^3\right) + \frac{d}{dx}\left(x^3y^4\right) = 0$$

$$3x^2y^2 \cdot \frac{dy}{dx} + 2xy^3 + 4x^3y^3 \cdot \frac{dy}{dx} + 3x^2y^4 = 0$$

Isolating $\frac{dy}{dx}$, we have:

$$\left(4x^3y^3 + 3x^2y^2\right) \cdot \frac{dy}{dx} = -3x^2y^4 - 2xy^3$$

$$\frac{dy}{dx} = \frac{-3x^2y^4 - 2xy^3}{4x^3y^3 + 3x^2y^2}$$

$$\frac{dy}{dx} = \frac{xy^2\left(-3xy^2 - 2y\right)}{xy^2\left(4x^2y + 3x\right)}$$

$$\frac{dy}{dx} = -\frac{2y + 3xy^2}{4x^2y + 3x}$$

22. $x^3y^2 - x^5y^3 = -19$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(x^3y^2 - x^5y^3\right) = \frac{d}{dx}(-19)$$

$$2x^3y \cdot \frac{dy}{dx} + 3x^2y^2 - 3x^5y^2 \cdot \frac{dy}{dx} - 5x^4y^3 = 0$$

$$\left(2x^3y - 3x^5y^2\right)\frac{dy}{dx} = 5x^4y^3 - 3x^2y^2$$

$$\frac{dy}{dx} = -\frac{5x^4y^3 - 3x^2y^2}{3x^5y^2 - 2x^3y}$$

$$\frac{dy}{dx} = \frac{-5x^2y^2 + 3y}{3x^3y - 2x}$$

23. Differentiate implicitly to find $\frac{dp}{dx}$.

$$p^3 + p - 3x = 50$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(p^3 + p - 3x\right) = \frac{d}{dx}(50)$$

$$3p^2 \cdot \frac{dp}{dx} + \frac{dp}{dx} - 3 \cdot 1 = 0$$

$$\left(3p^2 + 1\right) \cdot \frac{dp}{dx} = 3$$

$$\frac{dp}{dx} = \frac{3}{3p^2 + 1}$$

24. $p^2 + p + 2x = 40$

$$\frac{d}{dx}\left(p^2 + p + 2x\right) = \frac{d}{dx}(40)$$

$$2p \cdot \frac{dp}{dx} + \frac{dp}{dx} + 2 \cdot 1 = 0$$

$$(2p + 1) \cdot \frac{dp}{dx} = -2$$

$$\frac{dp}{dx} = \frac{-2}{2p + 1}$$

25. Differentiate implicitly to find $\frac{dp}{dx}$.

$$xp^3 = 24$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}\left(xp^3\right)=\frac{d}{dx}(24)$$
$$x\left(3p^2\cdot\frac{dp}{dx}\right)+p^3(1)=0 \qquad \text{Product Rule}$$
$$3xp^2\cdot\frac{dp}{dx}=-p^3$$
$$\frac{dp}{dx}=\frac{-p^3}{3xp^2}$$
$$\frac{dp}{dx}=-\frac{p}{3x}$$

26. $x^3p^2=108$

$$\frac{d}{dx}\left(x^3p^2\right)=\frac{d}{dx}(108)$$
$$x^3\left(2p\cdot\frac{dp}{dx}\right)+p^2\left(3x^2\right)=0$$
$$2x^3p\cdot\frac{dp}{dx}=-3x^2p^2$$
$$\frac{dp}{dx}=\frac{-3x^2p^2}{2x^3p}$$
$$\frac{dp}{dx}=-\frac{3p}{2x}$$

27. Differentiate implicitly to find $\frac{dp}{dx}$.

$$\frac{xp}{x+p}=2$$

First, we multiply both sides by $x+p$ to clear the fraction.

$$(x+p)\left(\frac{xp}{x+p}\right)=(2)(x+p)$$
$$xp=2x+2p$$

Next, differentiating both sides with respect to x yields:

$$\frac{d}{dx}(xp)=\frac{d}{dx}(2x+2p)$$
$$x\cdot\frac{dp}{dx}+p\cdot1=2\cdot1+2\cdot\frac{dp}{dx}$$
$$x\cdot\frac{dp}{dx}-2\cdot\frac{dp}{dx}=2-p$$
$$(x-2)\cdot\frac{dp}{dx}=2-p$$
$$\frac{dp}{dx}=\frac{2-p}{x-2}$$

28. $\frac{x^2p+xp+1}{2x+p}=1$

Multiplying by $2x+p$ we have:

$$x^2p+xp+1=2x+p$$
$$\frac{d}{dx}\left(x^2p+xp+1\right)=\frac{d}{dx}(2x+p)$$
$$x^2\cdot\frac{dp}{dx}+p\cdot2x+x\cdot\frac{dp}{dx}+p\cdot1+0=2+\frac{dp}{dx}$$
$$x^2\cdot\frac{dp}{dx}+x\cdot\frac{dp}{dx}-1\cdot\frac{dp}{dx}=2-2xp-p$$
$$\left(x^2+x-1\right)\frac{dp}{dx}=2-2xp-p$$
$$\frac{dp}{dx}=\frac{2-2xp-p}{x^2+x-1}$$

29. Differentiate implicitly to find $\frac{dp}{dx}$.

$$(p+4)(x+3)=48$$

Expanding the left hand side of the equation we have:

$$px+3p+4x+12=48$$
$$px+3p+4x=36$$

Differentiating both sides with respect to x yields:

$$\frac{d}{dx}(px+3p+4x)=\frac{d}{dx}(36)$$
$$p\cdot1+x\cdot\frac{dp}{dx}+3\cdot\frac{dp}{dx}+4\cdot1=0$$
$$(x+3)\cdot\frac{dp}{dx}=-p-4$$
$$\frac{dp}{dx}=\frac{-p-4}{x+3}$$

30. $1000-300p+25p^2=x$

$$\frac{d}{dx}\left(1000-300p+25p^2\right)=\frac{d}{dx}(x)$$
$$-300\cdot\frac{dp}{dx}+25\cdot2p\cdot\frac{dp}{dx}=1$$
$$(50p-300)\cdot\frac{dp}{dx}=1$$
$$\frac{dp}{dx}=\frac{1}{50p-300}$$

31. $A^3 + B^3 = 9$

We differentiate both sides with respect to *t*.

$$\frac{d}{dt}\left(A^3 + B^3\right) = \frac{d}{dt}(9)$$

$$3A^2 \cdot \frac{dA}{dt} + 3B^2 \cdot \frac{dB}{dt} = 0$$

$$3A^2 \cdot \frac{dA}{dt} = -3B^2 \cdot \frac{dB}{dt}$$

$$\frac{dA}{dt} = \frac{-3B^2}{3A^2} \cdot \frac{dB}{dt}$$

We find *B* when $A = 2$:

$$A^3 + B^3 = 9$$

$$(2)^3 + B^3 = 9$$

$$8 + B^3 = 9$$

$$B^3 = 1$$

$$B = 1$$

Next, we substitute 2 for *A*, 1 for *B*, and 3 for $\frac{dB}{dt}$ into the formula for $\frac{dA}{dt}$:

$$\frac{dA}{dt} = \frac{-3B^2}{3A^2} \cdot \frac{dB}{dt}$$

$$= \frac{-3(1)^2}{3(2)^2} \cdot (3)$$

$$= \frac{-3}{12} \cdot 3$$

$$= -\frac{3}{4}$$

32. $G^2 + H^2 = 25$

$$\frac{d}{dt}\left(G^2 + H^2\right) = \frac{d}{dt}(25)$$

$$2G \cdot \frac{dG}{dt} + 2H\frac{dH}{dt} = 0$$

$$2H \cdot \frac{dH}{dt} = -2G \cdot \frac{dG}{dt}$$

$$\frac{dH}{dt} = -\frac{G}{H} \cdot \frac{dG}{dt}$$

When $\frac{dG}{dt} = 3$ and $G = 0$:

$$(0)^2 + H^2 = 25$$

$$H^2 = 25$$

$$H = 5, \qquad H \text{ is nonnegative}$$

$$\frac{dH}{dt} = -\frac{0}{5} \cdot 3 = 0$$

When $\frac{dG}{dt} = 3$ and $G = 1$:

$$(1)^2 + H^2 = 25$$

$$H^2 = 24$$

$$H = \sqrt{24} = 2\sqrt{6}, \qquad H \text{ is nonnegative}$$

$$\frac{dH}{dt} = -\frac{1}{2\sqrt{6}} \cdot 3 = -\frac{3}{2\sqrt{6}}$$

When $\frac{dG}{dt} = 3$ and $G = 3$:

$$(3)^2 + H^2 = 25$$

$$H^2 = 16$$

$$H = 4, \qquad H \text{ is nonnegative}$$

$$\frac{dH}{dt} = -\frac{3}{4} \cdot 3 = -\frac{9}{4}$$

33. $R(x) = 50x - 0.5x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x) = \frac{d}{dt}\left(50x - 0.5x^2\right)$$

$$\frac{dR}{dt} = 50 \cdot \frac{dx}{dt} - x \cdot \frac{dx}{dt}$$

$$\frac{dR}{dt} = (50 - x) \cdot \frac{dx}{dt}$$

Next, we substitute 30 for *x* and 20 for *dx/dt.*

$$\frac{dR}{dt} = (50 - 30) \cdot 20 = (20) \cdot 20 = 400$$

The rate of change of total revenue with respect to time is \$400 per day.

$$C(x) = 4x + 10$$

Differentiating with respect to time we have:

$$\frac{d}{dt}C(x) = \frac{d}{dt}(4x + 10)$$

$$\frac{dC}{dt} = 4 \cdot \frac{dx}{dt}$$

Next, we substitute 30 for *x* and 20 for *dx/dt.*

$$\frac{dC}{dt} = 4 \cdot (20) = 80$$

The rate of change of total cost with respect to time is \$80 per day.

Profit is revenue minus cost. Therefore;

$$P(x) = R(x) - C(x)$$

$$= 50x - 0.5x^2 - (4x + 10)$$

$$= -0.5x^2 + 46x - 10$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x)=\frac{d}{dt}\left(-0.5x^2+46x-10\right)$$

$$\frac{dP}{dt}=-x\cdot\frac{dx}{dt}+46\frac{dx}{dt}$$

$$\frac{dP}{dt}=(46-x)\cdot\frac{dx}{dt}$$

Next, we substitute 30 for x and 20 for dx/dt.

$$\frac{dP}{dt}=\left(46-(30)\right)\cdot(20)=(16)(20)=320$$

The rate of change of total profit with respect to time is \$320 per day.

34. $R(x)=50x-0.5x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x)=\frac{d}{dt}\left(50x-0.5x^2\right)$$

$$\frac{dR}{dt}=50\cdot\frac{dx}{dt}-x\cdot\frac{dx}{dt}$$

$$\frac{dR}{dt}=(50-x)\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dR}{dt}=(50-10)\cdot 5=(40)\cdot 5=200$$

The rate of change of total revenue with respect to time is \$200 per day.

$C(x)=10x+3$

Differentiating with respect to time we have:

$$\frac{d}{dt}C(x)=\frac{d}{dt}(10x+3)$$

$$\frac{dC}{dt}=10\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dC}{dt}=10\cdot(5)=50$$

The rate of change of total cost with respect to time is \$50 per day.

Profit is revenue minus cost. Therefore;

$$\begin{aligned}P(x)&=R(x)-C(x)\\&=50x-0.5x^2-(10x+3)\\&=-0.5x^2+40x-3\end{aligned}$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x)=\frac{d}{dt}\left(-0.5x^2+40x-3\right)$$

$$\frac{dP}{dt}=-x\cdot\frac{dx}{dt}+40\frac{dx}{dt}$$

$$\frac{dP}{dt}=(40-x)\cdot\frac{dx}{dt}$$

Next, we substitute 10 for x and 5 for dx/dt.

$$\frac{dP}{dt}=\left(40-(10)\right)\cdot(5)=(30)(5)=150$$

The rate of change of total profit with respect to time is \$150 per day.

35. $R(x)=2x$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x)=\frac{d}{dt}(2x)$$

$$\frac{dR}{dt}=2\cdot\frac{dx}{dt}$$

Next, we substitute 20 for x and 8 for dx/dt.

$$\frac{dR}{dt}=2\cdot 8=16$$

The rate of change of total revenue with respect to time is \$16 per day.

$C(x)=0.01x^2+0.6x+30$

Differentiating with respect to time we have:

$$\frac{d}{dt}C(x)=\frac{d}{dt}\left(0.01x^2+0.6x+30\right)$$

$$\frac{dC}{dt}=0.02x\cdot\frac{dx}{dt}+0.6\cdot\frac{dx}{dt}$$

$$\frac{dC}{dt}=(0.02x+0.6)\cdot\frac{dx}{dt}$$

Next, we substitute 20 for x and 8 for dx/dt.

$$\frac{dC}{dt}=\left(0.02(20)+0.6\right)\cdot 8=8$$

The rate of change of total cost with respect to time is \$8 per day.

Profit is revenue minus cost. Therefore;

$$\begin{aligned}P(x)&=R(x)-C(x)\\&=2x-\left(0.01x^2+0.6x+30\right)\\&=-0.01x^2+1.4x-30\end{aligned}$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x)=\frac{d}{dt}\left(-0.01x^2+1.4x-30\right)$$

$$\frac{dP}{dt}=-0.02x\cdot\frac{dx}{dt}+1.4\frac{dx}{dt}$$

$$\frac{dP}{dt}=(1.4-0.02x)\cdot\frac{dx}{dt}$$

Next, we substitute 20 for x and 8 for dx/dt.

$$\frac{dP}{dt}=\left(1.4-0.02(20)\right)\cdot(8)=8$$

The rate of change of total profit with respect to time is \$8 per day.

36. $R(x) = 280x - 0.4x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}R(x) = \frac{d}{dt}\left(280x - 0.4x^2\right)$$

$$\frac{dR}{dt} = 280 \cdot \frac{dx}{dt} - 0.8x \cdot \frac{dx}{dt}$$

$$\frac{dR}{dt} = (280 - 0.8x) \cdot \frac{dx}{dt}$$

Next, we substitute 200 for *x* and 300 for *dx/dt.*

$$\frac{dR}{dt} = (280 - 0.8(200)) \cdot 300 = 36{,}000$$

The rate of change of total revenue with respect to time is \$36,000 per day.

$C(x) = 5000 + 0.6x^2$

Differentiating with respect to time we have:

$$\frac{d}{dt}C(x) = \frac{d}{dt}\left(5000 + 0.6x^2\right)$$

$$\frac{dC}{dt} = 1.2x \cdot \frac{dx}{dt}$$

Next, we substitute 200 for *x* and 300 for *dx/dt.*

$$\frac{dC}{dt} = 1.2(200) \cdot 300 = 72{,}000$$

The rate of change of total cost with respect to time is \$72,000 per day.

Profit is revenue minus cost. Therefore;

$$P(x) = R(x) - C(x)$$
$$= 280x - 0.4x^2 - \left(5000 + 0.6x^2\right)$$
$$= -x^2 + 280x - 5000$$

Differentiating with respect to time we have:

$$\frac{d}{dt}P(x) = \frac{d}{dt}\left(-x^2 + 280x - 5000\right)$$

$$\frac{dP}{dt} = -2x \cdot \frac{dx}{dt} + 280\frac{dx}{dt}$$

$$\frac{dP}{dt} = (280 - 2x) \cdot \frac{dx}{dt}$$

Next, we substitute 200 for *x* and 300 for *dx/dt.*

$$\frac{dP}{dt} = (280 - 2(200)) \cdot (300) = -36{,}000$$

The rate of change of total profit with respect to time is $-\$36{,}000$ per day.

37. $5p + 4x + 2px = 60$

First, we take the derivative of both sides of the equation with respect to *t*.

$$\frac{d}{dt}[5p + 4x + 2px] = \frac{d}{dt}[60]$$

$$5\frac{dp}{dt} + 4\frac{dx}{dt} + 2\left(\underbrace{p \cdot \frac{dx}{dt} + \frac{dp}{dt} \cdot x}_{\text{Product Rule}}\right) = 0$$

$$5\frac{dp}{dt} + 4\frac{dx}{dt} + 2p \cdot \frac{dx}{dt} + 2x \cdot \frac{dp}{dt} = 0$$

Next, we solve for $\frac{dx}{dt}$.

$$4\frac{dx}{dt} + 2p \cdot \frac{dx}{dt} = -5\frac{dp}{dt} - 2x \cdot \frac{dp}{dt}$$

$$(4 + 2p)\frac{dx}{dt} = -(5 + 2x) \cdot \frac{dp}{dt}$$

$$\frac{dx}{dt} = \frac{-(5 + 2x)}{(4 + 2p)} \cdot \frac{dp}{dt}$$

Substituting 3 for *x*, 5 for *p*, and 1.5 for $\frac{dp}{dt}$, we have:

$$\frac{dx}{dt} = \frac{-(5 + 2(3))}{(4 + 2(5))} \cdot (1.5)$$
$$= \frac{-(11)}{14} \cdot (1.5)$$
$$= \frac{-16.5}{14}$$
$$\approx -1.18$$

Sales are changing at a rate of -1.18 sales per day.

38. $R = xp$

From Exercise 37, we know that $\frac{dx}{dt} = \frac{-16.5}{14}$

$$\frac{dR}{dt} = x \cdot \frac{dp}{dt} + p \cdot \frac{dx}{dt}$$

Substituting the appropriate values, we have:

$$\frac{dR}{dt} = (3)(1.5) + (5)\left(\frac{-16.5}{14}\right) \approx -1.39$$

Total revenue is changing at a rate of $-\$1.39$ per day.

39. $A = \pi r^2$

To find the rate of change of the area of the Arctic ice cap with respect to time, we take the derivative of both sides of the equation with respect to *t*.

$$\frac{d}{dt}A = \frac{d}{dt}\left[\pi r^2\right]$$

$$\frac{dA}{dt} = \pi\frac{d}{dt}\left[r^2\right] \qquad \text{Constant Multiple Rule}$$

$$\frac{dA}{dt} = \pi\left[2r\cdot\frac{dr}{dt}\right] \qquad \text{Chain Rule}$$

$$\frac{dA}{dt} = 2\pi r\cdot\frac{dr}{dt}$$

In 2005, the r was 808 miles, and $\frac{dr}{dt} = -4.3$ miles per year. Substituting these values into the derivative, we have:

$$\frac{dA}{dt} = 2\pi(808)(-4.3)$$
$$\approx -21{,}830.2990313$$
$$\approx -21{,}830$$

Therefore, in 2005 the Arctic ice cap was changing at a rate of $-21{,}830\ \text{mi}^2$ per year. Another way of stating this is to say that the Arctic ice cap was *shrinking* at a rate of 21,830 mi^2/yr.

40. $A = \pi r^2$

$$\frac{d}{dt}A = \frac{d}{dt}\left[\pi r^2\right]$$

$$\frac{dA}{dt} = 2\pi r\cdot\frac{dr}{dt}$$

Substituting, we have:

$$\frac{dA}{dt} = 2\pi(25)(-1) = -50\pi \approx -157.0796$$

The area of the wound is decreasing at a rate of 157.08 mm^2/day.

41. $S = \frac{\sqrt{hw}}{60}$

First, we substitute 165 for h, and then we take the derivative of both sides with respect to t.

$$S = \frac{\sqrt{165w}}{60} = \frac{\sqrt{165}}{60}\cdot w^{1/2}$$

$$\frac{d}{dt}[S] = \frac{d}{dt}\left[\frac{\sqrt{165}}{60}\cdot w^{1/2}\right]$$

$$\frac{dS}{dt} = \frac{\sqrt{165}}{60}\cdot\frac{1}{2}w^{-1/2}\cdot\frac{dw}{dt}$$

$$= \frac{\sqrt{165}}{120}\frac{1}{w^{1/2}}\cdot\frac{dw}{dt}$$

$$= \frac{\sqrt{165}}{120\sqrt{w}}\cdot\frac{dw}{dt}$$

Now, we will substitute 70 for w and -2 for $\frac{dw}{dt}$.

$$\frac{dS}{dt} = \frac{\sqrt{165}}{120\sqrt{70}}\cdot(-2)$$
$$\approx -0.0256$$

Therefore, Tom's surface area is changing at a rate of $-0.0256\ \text{m}^2$/month. We could also say that Tom's surface area is *decreasing* by 0.0256 m^2/month.

42. $V = \frac{p}{4Lv}\left(R^2 - r^2\right)$

We assume that r, p, L and v are constants

a) Taking the derivative of both sides with respect to t, we have:

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{p}{4Lv}\left(R^2 - r^2\right)\right]$$
$$= \frac{p}{4Lv}\left[\frac{d}{dt}R^2 - \frac{d}{dt}r^2\right]$$
$$= \frac{p}{4Lv}\left[2R\cdot\frac{dR}{dt}\right]$$
$$= \frac{pR}{2Lv}\cdot\frac{dR}{dt}$$

Substituting, we have:

$$\frac{dV}{dt} = \frac{500R}{2(80)(0.003)}\cdot\frac{dR}{dt}$$
$$= \frac{500R}{0.48}\cdot\frac{dR}{dt}$$

b) Using the derivative found in part (a), and substituting the values for R and dR/dt, we have:

$$\frac{dV}{dt} = \frac{500(0.075)}{0.48}\cdot(-0.0002)$$
$$\approx -0.0156$$

The speed of the blood is changing at a rate of $-0.0156\ \text{mm}/\text{sec}^2$.

43. $V = \frac{p}{4Lv}\left(R^2 - r^2\right)$

We assume that r, p, L and v are constants

a) Taking the derivative of both sides with respect to t, we have:

$$\frac{dV}{dt}=\frac{d}{dt}\left[\frac{p}{4Lv}\left(R^2-r^2\right)\right]$$
$$=\frac{p}{4Lv}\left[\frac{d}{dt}R^2-\frac{d}{dt}r^2\right]$$
$$=\frac{p}{4Lv}\left[2R\cdot\frac{dR}{dt}-0\right]$$
$$=\frac{pR}{2Lv}\cdot\frac{dR}{dt}$$

Substituting 70 for L, 400 for p and 0.003 for v, we have:

$$\frac{dV}{dt}=\frac{400R}{2(70)(0.003)}\cdot\frac{dR}{dt}$$
$$=\frac{400R}{0.42}\cdot\frac{dR}{dt}$$
$$\approx 952.38R\cdot\frac{dR}{dt}$$

b) Using the derivative in part (a), we substitute 0.00015 for *dR/dt* and 0.1 for *R* to get:

$$\frac{dV}{dt}=952.38(0.1)\cdot(0.00015)$$
$$\approx 0.0143$$

The speed of the person's blood will be increasing at a rate of 0.0143 mm/sec^2.

44. $D^2=x^2+y^2$

After 1 hour,

$$D^2=25^2+60^2$$
$$D^2=4225$$
$$D=65$$

$\frac{dx}{dt}=25$ and $\frac{dy}{dt}=60$

Differentiating both sides of the distance equation with respect to t, we have:

$$\frac{d}{dt}D^2=\frac{d}{dt}\left[x^2+y^2\right]$$
$$2D\cdot\frac{dD}{dt}=2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}$$
$$\frac{dD}{dt}=\frac{x\cdot\frac{dx}{dt}+y\cdot\frac{dy}{dt}}{D}$$

Substituting the appropriate information, we have:

$$\frac{dD}{dt}=\frac{(25)\cdot(25)+(60)\cdot(60)}{(65)}=65$$

One hour after the cars leave, the distance between the two cars is increasing at a rate of 65 mph.

45. Since the ladder forms a right triangle with the wall and the ground, we know that:

$$x^2+y^2=26^2$$
$$x^2+y^2=676$$

We are looking for $\frac{dy}{dt}$. Differentiating both sides with respect to t, we have:

$$\frac{d}{dt}\left[x^2+y^2\right]=\frac{d}{dt}[676]$$
$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$
$$2y\frac{dy}{dt}=-2x\frac{dx}{dt} \qquad \text{Subtracting}$$
$$\frac{dy}{dt}=\frac{-2x}{2y}\cdot\frac{dx}{dt} \qquad \text{Dividing by } 2y$$
$$\frac{dy}{dt}=\frac{-x}{y}\cdot\frac{dx}{dt}$$

The lower end of the wall is being pulled away from the wall at a rate of 5 feet per second; therefore, $\frac{dx}{dt}=5$. When the lower end is 10 feet away from the wall, $x=10$, and

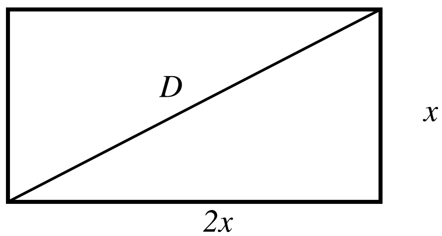
$$(10)^2+y^2=676$$
$$100+y^2=676$$
$$y^2=676-100$$
$$y^2=576$$
$$y=\pm\sqrt{576}$$
$$y=\pm 24$$
$$y=24 \qquad \text{Since } y \text{ must be positive.}$$

We substitute 10 for x, 24 for y and 5 for $\frac{dx}{dt}$ into the derivative to get:

$$\frac{dy}{dt}=\frac{-x}{y}\cdot\frac{dx}{dt}$$
$$=-\frac{(10)}{(24)}\cdot(5)$$
$$=-\frac{25}{12}$$
$$=-2\tfrac{1}{12}$$

When the lower end of the ladder is 10 feet from the wall, the top of the ladder is moving down the wall at a rate of $-2\frac{1}{12}$ feet per second.

46.



First, from the picture we know that:

$$x^2 + (2x)^2 = D^2$$
$$5x^2 = D^2$$

When x is 440 m wide, we have

$$D^2 = 5(440)^2$$
$$D^2 = 968,000$$
$$D = 440\sqrt{5}$$

Differentiating both sides of $5x^2 = D^2$ with respect to t, we have:

$$10x\frac{dx}{dt} = 2D\frac{dD}{dt}$$
$$\frac{dx}{dt} = \frac{D}{5x}\frac{dD}{dt}$$

Now, when $\frac{dD}{dt} = 90$, $x = 440$, and $D = 440\sqrt{5}$ we have:

$$\frac{dx}{dt} = \frac{(440\sqrt{5})}{5(440)}(90)$$
$$= \frac{\sqrt{5}}{5}(90)$$
$$= 18\sqrt{5}$$

We are looking to find how fast the area is changing.

$$A = 2x^2$$
$$\frac{dA}{dt} = 4x\frac{dx}{dt}$$

Substituting, we have:

$$\frac{dA}{dt} = 4(440)(18\sqrt{5})$$
$$= 31,680\sqrt{5}$$
$$\approx 70,838.63$$

The area is changing at a rate of 70,838.63 m^2/yr.

47. $V = \frac{4}{3}\pi r^3$

Differentiating both sides with respect to t, we have:

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$
$$= \frac{4}{3}\pi \cdot \frac{d}{dt}\left[r^3\right]$$
$$= \frac{4}{3}\pi\left[3r^2\frac{dr}{dt}\right]$$
$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

Next, substituting 0.7 for *dr/dt* and 7.5 for r, we have:

$$\frac{dV}{dt} = 4\pi(7.5)^2(0.7)$$
$$= 4\pi(56.25)(0.7)$$
$$= 157.5\pi$$
$$\approx 494.8$$

The cantaloupe's volume is changing at the rate of 494.8 cm^3/week.

48. $\sqrt{x} + \sqrt{y} = 1$

$$\frac{d}{dx}\left(x^{1/2} + y^{1/2}\right) = \frac{d}{dx}[1]$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\cdot\frac{dy}{dx} = 0$$
$$\frac{1}{2}y^{-1/2}\cdot\frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$
$$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}}$$
$$\frac{dy}{dx} = \frac{-x^{-1/2}}{y^{-1/2}}$$
$$\frac{dy}{dx} = \frac{-y^{1/2}}{x^{1/2}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

49. $\frac{1}{x^2} + \frac{1}{y^2} = 5$

$$x^{-2} + y^{-2} = 5$$

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\left[x^{-2}\right]+\frac{d}{dx}\left[y^{-2}\right]=\frac{d}{dx}[5]$$
$$-2x^{-3}-2y^{-3}\frac{dy}{dx}=0$$
$$-2y^{-3}\frac{dy}{dx}=2x^{-3}$$
$$\frac{dy}{dx}=-\frac{x^{-3}}{y^{-3}}$$
$$\frac{dy}{dx}=-\frac{y^3}{x^3}$$

50. $y^3=\dfrac{x-1}{x+1}$

$$\frac{d}{dx}\left[y^3\right]=\frac{d}{dx}\left[\frac{x-1}{x+1}\right]$$
$$3y^2\frac{dy}{dx}=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$
$$3y^2\frac{dy}{dx}=\frac{2}{(x+1)^2}$$
$$\frac{dy}{dx}=\frac{2}{3y^2(x+1)^2}$$

51. $y^2=\dfrac{x^2-1}{x^2+1}$

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\left[y^2\right]=\frac{d}{dx}\left[\frac{x^2-1}{x^2+1}\right]$$
$$2y\frac{dy}{dx}=\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$
$$2y\frac{dy}{dx}=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$
$$2y\frac{dy}{dx}=\frac{4x}{(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{4x}{2y(x^2+1)^2}$$
$$\frac{dy}{dx}=\frac{2x}{y(x^2+1)^2}$$

52. $x^{3/2}+y^{2/3}=1$

$$\frac{d}{dx}\left[x^{3/2}+y^{2/3}\right]=\frac{d}{dx}(1)$$
$$\frac{3}{2}x^{1/2}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$
$$\frac{2}{3}y^{-1/3}\frac{dy}{dx}=-\frac{3}{2}x^{1/2}$$
$$\frac{dy}{dx}=\frac{-\frac{3}{2}x^{1/2}}{\frac{2}{3}y^{-1/3}}$$
$$\frac{dy}{dx}=\frac{-9}{4}\cdot\frac{x^{1/2}}{y^{-1/3}}$$
$$\frac{dy}{dx}=\frac{-9}{4}\cdot x^{1/2}y^{1/3}$$

53. $(x-y)^3+(x+y)^3=x^5+y^5$

Differentiating both sides with respect to x, we have:

$$\frac{d}{dx}\left[(x-y)^3+(x+y)^3\right]=\frac{d}{dx}\left[x^5+y^5\right]$$
$$\frac{d}{dx}(x-y)^3+\frac{d}{dx}(x+y)^3=\frac{d}{dx}x^5+\frac{d}{dx}y^5$$
$$3(x-y)^2\cdot\frac{d}{dx}(x-y)+3(x+y)^2\frac{d}{dx}(x+y)=5x^4+5y^4\frac{dy}{dx}$$
$$3(x-y)^2\left(1-\frac{dy}{dx}\right)+3(x+y)^2\left(1+\frac{dy}{dx}\right)=5x^4+5y^4\frac{dy}{dx}$$
$$3(x-y)^2-3(x-y)^2\frac{dy}{dx}+3(x+y)^2+3(x+y)^2\frac{dy}{dx}=5x^4+5y^4\frac{dy}{dx}$$
$$\left[3(x+y)^2-3(x-y)^2-5y^4\right]\frac{dy}{dx}=5x^4-3(x-y)^2-3(x+y)^2$$
$$\frac{dy}{dx}=\frac{5x^4-3(x-y)^2-3(x+y)^2}{3(x+y)^2-3(x-y)^2-5y^4}$$

Simplification will yield:

$$\frac{dy}{dx}=\frac{5x^4-6x^2-6y^2}{12xy-5y^4}$$

54. $xy + x - 2y = 4$

Differentiate implicitly to find $\dfrac{dy}{dx}$

$$\frac{d}{dx}[xy + x - 2y] = \frac{d}{dx}[4]$$

$$x\frac{dy}{dx} + y\cdot 1 + 1 - 2\frac{dy}{dx} = 0$$

$$(x-2)\frac{dy}{dx} = -1 - y$$

$$\frac{dy}{dx} = \frac{-1-y}{x-2}$$

$$\frac{dy}{dx} = \frac{1+y}{2-x}$$

Differentiate $\dfrac{dy}{dx}$ implicitly to find $\dfrac{d^2y}{dx^2}$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1+y}{2-x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(2-x)\left(\frac{dy}{dx}\right) - (1+y)(-1)}{(2-x)^2}$$

$$= \frac{(2-x)\left(\frac{1+y}{2-x}\right) + (1+y)}{(2-x)^2}$$

Substituting $\dfrac{1+y}{2-x}$ for $\dfrac{dy}{dx}$

$$= \frac{1+y+1+y}{(2-x)^2}$$

$$= \frac{2+2y}{(2-x)^2}$$

55. $y^2 - xy + x^2 = 5$

Differentiate implicitly to find $\dfrac{dy}{dx}$

$$\frac{d}{dx}\left[y^2 - xy + x^2\right] = \frac{d}{dx}[5]$$

$$2y\frac{dy}{dx} - \left[x\frac{dy}{dx} + y\cdot 1\right] + 2x = 0$$

$$(2y - x)\frac{dy}{dx} - y + 2x = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

Differentiate $\dfrac{dy}{dx}$ implicitly to find $\dfrac{d^2y}{dx^2}$

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{y-2x}{2y-x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(2y-x)\left(\frac{dy}{dx} - 2\right) - (y-2x)\left(2\frac{dy}{dx} - 1\right)}{(2y-x)^2}$$

Simplifying the numerator we have:

$$\frac{d^2y}{dx^2} = \left[\left(2y\frac{dy}{dx} - 4y - x\frac{dy}{dx} + 2x\right) - \left(2y\frac{dy}{dx} - y - 4x\frac{dy}{dx} + 2x\right)\right] \div (2y-x)^2$$

$$\frac{d^2y}{dx^2} = \frac{-3y + 3x\frac{dy}{dx}}{(2y-x)^2}$$

Substituting $\dfrac{y-2x}{2y-x}$ for $\dfrac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{-3y + 3x\cdot\frac{y-2x}{2y-x}}{(2y-x)^2}$$

$$= \frac{-3y\frac{2y-x}{2y-x} + 3x\cdot\frac{y-2x}{2y-x}}{(2y-x)^2}$$

$$= \frac{-6y^2 + 3xy + 3xy - 6x^2}{(2y-x)^3}$$

$$= \frac{-6y^2 + 6xy - 6x^2}{(2y-x)^3}$$

$$= \frac{-6\left(y^2 - xy + x^2\right)}{(2y-x)^3}$$

56. $x^2 - y^2 = 5$

Differentiate implicitly to find $\dfrac{dy}{dx}$

$$\frac{d}{dx}\left[x^2 - y^2\right] = \frac{d}{dx}[5]$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Differentiate $\dfrac{dy}{dx}$ implicitly to find $\dfrac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x}{y}\right]$$

$$= \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\left[\frac{x}{y}\right]}{y^2} \quad \text{Substituting for } \frac{dy}{dx}$$

$$= \frac{\frac{y^2}{y} - \frac{x^2}{y}}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

57. $x^3 - y^3 = 8$

Differentiate implicitly to find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[x^3 - y^3\right] = \frac{d}{dx}[8]$$

$$3x^2 - 3y^2\frac{dy}{dx} = 0$$

$$-3y^2\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{-3y^2}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

Differentiate $\frac{dy}{dx}$ implicitly to find $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x^2}{y^2}\right]$$

$$= \frac{y^2 \cdot (2x) - x^2 \cdot 2y\frac{dy}{dx}}{\left(y^2\right)^2}$$

$$= \frac{2xy^2 - 2x^2y\left[\frac{x^2}{y^2}\right]}{y^4} \quad \text{Substituting for } \frac{dy}{dx}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$= \frac{\frac{2xy^2}{1}\cdot\frac{y}{y} - \frac{2x^4}{y}}{y^4}$$

$$= \frac{\frac{2xy^3 - 2x^4}{y}}{y^4}$$

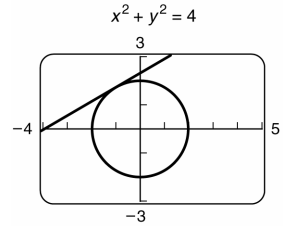
$$= \frac{2xy^3 - 2x^4}{y^5}$$

$$= \frac{2x\left(y^3 - x^3\right)}{y^5}$$

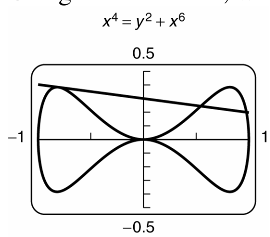
58. $\boxed{tw}$ Given an equation in x and y where y is a function of x but where it is difficult or impossible to express y in terms of x, implicit differentiation allows us to find the derivative of y with respect to x.

59. $\boxed{tw}$ One dictionary defines "implicit" as "capable of being understood from something else though unexpressed" or "involved in the nature or essence of something though not revealed, expressed, or developed." When a function y of x is defined implicitly it is written as an equation in x and y where y is not expressed in terms of x but where it is understood or implied that y is indeed a function of x.

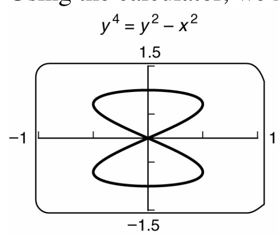
60. Using the calculator, we have:



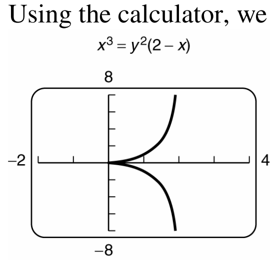
61. Using the calculator, we have:



62. Using the calculator, we have:



63. Using the calculator, we have:



64. Using the calculator, we have: