2. Solve the system of equations $S_b = 0$ and $S_m = 0$:
$$12m + 6b - 440 = 0 \qquad (1)$$
$$28m + 12b - 886 = 0 \qquad (2)$$
The solution to this system is $b = \frac{211}{3}$ and $m = \frac{3}{2}$.
Thus, $\left(\frac{211}{3}, \frac{3}{2}\right)$ is a critical point, and $S\left(\frac{211}{3}, \frac{3}{2}\right)$ is a candidate for a maximum or minimum.

3. We must check to see if $S\left(\frac{211}{3}, \frac{3}{2}\right)$ is a maximum or minimum value:
$$D = S_{bb}\left(\frac{211}{3}, \frac{3}{2}\right) \cdot S_{mm}\left(\frac{211}{3}, \frac{3}{2}\right) - \left[S_{bm}\left(\frac{211}{3}, \frac{3}{2}\right)\right]^2$$
$$\begin{aligned} D &= 6 \cdot 28 - 12^2 \\ &= 168 - 144 \\ &= 24. \end{aligned}$$

4. Since $D > 0$ and $S_{bb}\left(\frac{211}{3}, \frac{3}{2}\right) = 6 > 0$, it follows that S has a relative minimum at $\left(\frac{211}{3}, \frac{3}{2}\right)$. The minimum value is found as follows:
$$\begin{aligned} S\left(\frac{211}{3}, \frac{3}{2}\right) &= \left(\frac{3}{2} + \frac{211}{3} - 72\right)^2 + \left(2 \cdot \frac{3}{2} + \frac{211}{3} - 73\right)^2 + \left(3 \cdot \frac{3}{2} + \frac{211}{3} - 75\right)^2 \\ &= \frac{1}{36} + \frac{1}{9} + \frac{1}{36} \\ &= \frac{1}{6}. \end{aligned}$$
The relative minimum value of S is $\frac{1}{6}$ at $\left(\frac{221}{3}, \frac{3}{2}\right)$.

27. $\boxed{tw}$ No, the cross-section of an anticlastic curve is not always a parabola. An example of one such curve is $f(x, y) = x^4 - y^4$.

28. $\boxed{tw}$ A function $f(x, y)$ has a relative minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points in a rectangular region containing (a, b). A function $f(x, y)$ has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of $f(x, y)$.

29. $f(x, y)$ has a relative minimum of -5 at $(0, 0)$.

30. $f(x, y)$ has a relative maximum of 1 at $(-1, -1)$. A saddle point occurs at $(0, 0)$.

31. $f(x, y)$ has no relative extrema.

32. $f(x, y)$ has a relative maximum of –6 at $(-0.5, 4)$.

Exercise Set 6.4

1. Find the regression line for the data set:

x	1	2	4	5
y	1	3	3	4

The data points are $(1,1), (2,3), (4,3),$ and $(5,4)$.
The points on the regression line are $(1, y_1), (2, y_2), (4, y_3),$ and $(5, y_4)$.
The y-deviations are $y_1 - 1,\ y_2 - 3,\ y_3 - 3,\ y_4 - 4.$
We want to minimize
$$S = (y_1 - 1)^2 + (y_2 - 3)^2 + (y_3 - 3)^2 + (y_4 - 4)^2$$
Where:
$$\begin{aligned} y_1 &= m \cdot 1 + b \\ y_2 &= m \cdot 2 + b \\ y_3 &= m \cdot 4 + b \\ y_4 &= m \cdot 5 + b \end{aligned}$$
Substituting we get:

$$S=(m+b-1)^2+(2m+b-3)^2+(4m+b-3)^2+(5m+b-4)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b}=2(m+b-1)+2(2m+b-3)+2(4m+b-3)+2(5m+b-4)$$
$$=2m+2b-2+4m+2b-6+8m+2b-6+10m+2b-8$$
$$=24m+8b-22$$
$$\frac{\partial S}{\partial m}=2(m+b-1)\cdot 1+2(2m+b-3)\cdot 2+2(4m+b-3)\cdot 4+2(5m+b-4)\cdot 5$$
$$=2m+2b-2+8m+4b-12+32m+8b-24+50m+10b-40$$
$$=92m+24b-78$$

We set these derivatives equal to 0 and solve the resulting system.

$$24m+8b-22=0$$
$$92m+24b-78=0$$

The solution to this system is $b=0.95,\ m=0.6.$

We use the D-test to verify that $S(0.95,0.6)$ is a relative minimum.

We first find the second-order partial derivatives.

$$S_{bb}=8, S_{bm}=24, S_{mm}=92$$
$$D=S_{bb}(0.95,0.6)\cdot S_{mm}(0.95,0.6)-\left[S_{bm}(0.6,0.95)\right]^2$$
$$D=8\cdot 92-[24]^2$$
$$=160$$

Since $D>0$ and $S_{bb}(0.95,0.6)=8>0$, S has a relative minimum at $(0.95,0.6)$. The regression line is $y=0.6x+0.95.$

2. Find the regression line for the data set:

x	1	3	5
y	2	4	7

The data points are $(1,2),(3,4),$ and $(5,7)$.

The points on the regression line are $(1,y_1),(3,y_2),$ and $(5,y_3)$.

The y-deviations are $y_1-2,\ y_2-4,$ and $y_3-7.$

We want to minimize

$$S=(y_1-2)^2+(y_2-4)^2+(y_3-7)^2$$

Where:

$$y_1=m\cdot 1+b$$
$$y_2=m\cdot 3+b$$
$$y_3=m\cdot 5+b$$

Substituting we get:

$$S=(m+b-2)^2+(3m+b-4)^2+(5m+b-7)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b}=2(m+b-2)+2(3m+b-4)+2(5m+b-7)$$
$$=2m+2b-4+6m+2b-8+10m+2b-14$$
$$=18m+6b-26$$
$$\frac{\partial S}{\partial m}=2(m+b-2)\cdot 1+2(3m+b-4)\cdot 3+2(5m+b-7)\cdot 5$$
$$=2m+2b-4+18m+6b-24+50m+10b-70$$
$$=70m+18b-98$$

We set these derivatives equal to 0 and solve the resulting system.

$$18m+6b-26=0$$
$$45m+18b-98=0$$

The solution to this system is $b=\frac{7}{12},\ m=1.25$

We use the D-test to verify that $S\left(\frac{7}{12},1.25\right)$ is a relative minimum.

We first find the second-order partial derivatives.

$$S_{bb}=6, S_{bm}=18, S_{mm}=70$$
$$D=S_{bb}\left(\tfrac{7}{12},1.25\right)\cdot S_{mm}\left(\tfrac{7}{12},1.25\right)-\left[S_{bm}\left(\tfrac{7}{12},1.25\right)\right]^2$$
$$D=6\cdot 70-[18]^2$$
$$=96$$

Since $D>0$ and $S_{bb}\left(\frac{7}{12},1.25\right)=6>0$, S has a relative minimum at $\left(\frac{7}{12},1.25\right)$. The regression line is $y=1.25x+\frac{7}{12}.$

3. Find the regression line for the data set:

x	1	2	3	5
y	0	1	3	4

The data points are $(1,0),(2,1),(3,3),$ and $(5,4)$.

The points on the regression line are $(1,y_1),(2,y_2),(3,y_3),$ and $(5,y_4)$.

The y-deviations are $y_1-0,\ y_2-1,\ y_3-3,$ and y_4-4.

We want to minimize

$$S=(y_1-0)^2+(y_2-1)^2+(y_3-3)^2+(y_4-4)^2$$

Where:

$$y_1=m\cdot 1+b$$
$$y_2=m\cdot 2+b$$
$$y_3=m\cdot 3+b$$
$$y_4=m\cdot 5+b$$

Substituting we get:

$$S=(m+b)^2+(2m+b-1)^2+(3m+b-3)^2+(5m+b-4)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\begin{aligned}\frac{\partial S}{\partial b}&=2(m+b)+2(2m+b-1)+2(3m+b-3)+2(5m+b-4)\\&=2m+2b+4m+2b-2+6m+2b-6+10m+2b-8\\&=22m+8b-16\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial m}&=2(m+b)\cdot 1+2(2m+b-1)\cdot 2+2(3m+b-3)\cdot 3+2(5m+b-4)\cdot 5\\&=2m+2b+8m+4b-4+18m+6b-18+50m+10b-40\\&=78m+22b-62\end{aligned}$$

We set these derivatives equal to 0 and solve the resulting system.

$$22m+8b-16=0$$
$$78m+22b-62=0$$

The solution to this system is $b=-\frac{29}{35},\ m=\frac{36}{35}$

We use the D-test to verify that $S\left(-\frac{29}{35},\frac{36}{35}\right)$ is a relative minimum.

We first find the second-order partial derivatives.

$S_{bb}=8, S_{bm}=22, S_{mm}=78$

$$D=S_{bb}\left(-\tfrac{29}{35},\tfrac{36}{35}\right)\cdot S_{mm}\left(-\tfrac{29}{35},\tfrac{36}{35}\right)-\left[S_{bm}\left(-\tfrac{29}{35},\tfrac{36}{35}\right)\right]^2$$

$$\begin{aligned}D&=8\cdot 78-[22]^2\\&=140\end{aligned}$$

Since $D>0$ and $S_{bb}\left(-\frac{29}{35},\frac{36}{35}\right)=8>0$, S has a relative minimum at $\left(-\frac{29}{35},\frac{36}{35}\right)$. The regression line is $y=\frac{36}{35}x-\frac{29}{35}$.

4. Find the regression line for the data set:

x	1	2	4
y	3	5	8

The data points are $(1,3),(2,5),$ and $(4,8)$.

The points on the regression line are $(1,y_1),(2,y_2),$ and $(4,y_3)$.

The y-deviations are $y_1-3,\ y_2-5,$ and y_3-8.

We want to minimize

$$S=(y_1-3)^2+(y_2-5)^2+(y_3-8)^2$$

Where:

$$y_1=m\cdot 1+b$$
$$y_2=m\cdot 2+b$$
$$y_3=m\cdot 4+b$$

Substituting we get:

$$S=(m+b-3)^2+(2m+b-5)^2+(4m+b-8)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\begin{aligned}\frac{\partial S}{\partial b}&=2(m+b-3)+2(2m+b-5)+2(4m+b-8)\\&=2m+2b-6+4m+2b-10+8m+2b-16\\&=14m+6b-32\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial m}&=2(m+b-3)\cdot 1+2(2m+b-5)\cdot 2+2(4m+b-8)\cdot 4\\&=2m+2b-6+8m+4b-20+32m+8b-64\\&=42m+14b-90\end{aligned}$$

We set the derivatives equal to 0 and solve the resulting system.

$$14m + 6b - 32 = 0$$
$$42m + 14b - 90 = 0$$

The solution to this system is $b = 1.5,\ m = \dfrac{23}{14}$

We use the D-test to verify that $S\left(1.5, \frac{23}{14}\right)$ is a relative minimum.

We first find the second-order partial derivatives.

$$S_{bb} = 6, S_{bm} = 14, S_{mm} = 42$$
$$D = S_{bb}\left(1.5, \tfrac{23}{14}\right) \cdot S_{mm}\left(1.5, \tfrac{23}{14}\right) - \left[S_{bm}\left(1.5, \tfrac{23}{14}\right)\right]^2$$
$$D = 6 \cdot 42 - [14]^2 = 56$$

Since $D > 0$ and $S_{bb}\left(1.5, \frac{23}{14}\right) = 6 > 0$, S has a relative minimum at $\left(1.5, \frac{23}{14}\right)$. The regression line is $y = \dfrac{23}{14}x + 1.5$.

5. a) The data points are $(0, 3.10), (1, 3.35), (10, 3.80), (11, 4.25),$ $(16, 4.75),$ and $(17, 5.15)$.

The points on the regression line are $(0, y_1), (1, y_2), (10, y_3), (11, y_4),$ $(16, y_5),$ and $(17, y_6)$.

The y-deviations are $y_1 - 3.10,\ y_2 - 3.35,\ y_3 - 3.80,$ $y_4 - 4.25, y_5 - 4.75,$ and $y_6 - 5.15$.

We want to minimize

$$S = (y_1 - 3.10)^2 + (y_2 - 3.35)^2 + (y_3 - 3.80)^2 + (y_4 - 4.25)^2 + (y_5 - 4.75)^2 + (y_6 - 5.15)^2$$

Where:

$$y_1 = m \cdot 0 + b$$
$$y_2 = m \cdot 1 + b$$
$$y_3 = m \cdot 10 + b$$
$$y_4 = m \cdot 11 + b$$
$$y_5 = m \cdot 16 + b$$
$$y_6 = m \cdot 17 + b$$

Substituting we get:

$$S = (b - 3.10)^2 + (m + b - 3.35)^2 + (10m + b - 3.80)^2 + (11m + b - 4.25)^2 + (16m + b - 4.75)^2 + (17m + b - 5.15)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b} = 2(b - 3.10) + 2(m + b - 3.35) + 2(10m + b - 3.80) + 2(11m + b - 4.25) + 2(16m + b - 4.75) + 2(17m + b - 5.15)$$
$$= 2b - 6.2 + 2m + 2b - 6.7 + 20m + 2b - 7.6 + 22m + 2b - 8.5 + 32m + 2b - 9.5 + 34m + 2b - 10.3$$
$$= 110m + 12b - 48.8$$

$$\frac{\partial S}{\partial m}$$
$$= 2(m + b - 3.35) \cdot 1 + 2(10m + b - 3.80) \cdot 10 + 2(11m + b - 4.25) \cdot 11 + 2(16m + b - 4.75) \cdot 16 + 2(17m + b - 5.15) \cdot 17$$
$$= 2m + 2b - 6.7 + 200m + 20b - 76 + 242m + 22b - 93.5 + 512m + 32b - 152 + 578m + 34b - 175.1$$
$$= 1534m + 110b - 503.3$$

We set these derivatives equal to 0 and solve the resulting system.

$$110m + 12b - 48.8 = 0$$
$$1534m + 110b - 503.3 = 0$$

The solution to this system is

$$b = 3.090710209$$
$$m = 0.1064679772$$

We use the D-test to verify that $S(b, m)$ is a relative minimum.

We first find the second-order partial derivatives.

$$S_{bb} = 12, S_{bm} = 110, S_{mm} = 1534$$
$$D = S_{bb} \cdot S_{mm} - [S_{bm}]^2$$
$$D = 12 \cdot 1534 - [110]^2 = 6308$$

Since $D > 0$ and $S_{bb} = 12 > 0$, S has a relative minimum at $(3.090710209, 0.1064679772)$. The regression line is $y = 0.1064679772x + 3.090710209$.

b) In 2010, $x = 2010 - 1980 = 30$

$y = 0.1064679772(30) + 3.090710209$

≈ 6.28

The minimum wage will be about \$6.28 in 2010.

In 2015, $x = 2015 - 1980 = 35$

$y = 0.1064679772(35) + 3.090710209$

≈ 6.82

The minimum wage will be about \$6.82 in 2015.

6. a) The data points are $(0,45.03),(1,49.35),(2,47.49),(3,50.02),(4,52.95),(5,54.75)$ and $(6,58.95)$.

The points on the regression line are $(0,y_1),(1,y_2),(2,y_3),(3,y_4),(4,y_5),(5,y_6)$, and $(6,y_7)$.

The y-deviations are $y_1 - 45.03,\ y_2 - 49.35,\ y_3 - 47.49,\ y_4 - 50.02, y_5 - 52.95, y_6 - 54.75$, and $y_7 - 58.95$.

We want to minimize

$$S = (y_1 - 45.03)^2 + (y_2 - 49.35)^2 + (y_3 - 47.49)^2 + (y_4 - 50.02)^2 + (y_5 - 52.95)^2 + (y_6 - 54.75)^2 + (y_7 - 58.95)^2.$$

Where:

$$y_1 = m \cdot 0 + b$$
$$y_2 = m \cdot 1 + b$$
$$y_3 = m \cdot 2 + b$$
$$y_4 = m \cdot 3 + b$$
$$y_5 = m \cdot 4 + b$$
$$y_6 = m \cdot 5 + b$$
$$y_7 = m \cdot 6 + b$$

Substituting we get:

$$S = (b - 45.03)^2 + (m + b - 49.35)^2 + (2m + b - 47.49)^2 + (3m + b - 50.02)^2 + (4m + b - 52.95)^2 + (5m + b - 54.75)^2 + (6m + b - 58.95)^2.$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b} = 2(b - 45.03) + 2(m + b - 49.35) + 2(2m + b - 47.49) + 2(3m + b - 50.02) + 2(4m + b - 52.95) + 2(5m + b - 54.75) + 2(6m + b - 58.95)$$
$$= 42m + 14b - 717.08$$

$$\frac{\partial S}{\partial m} = 0 + 2(m + b - 49.35) \cdot 1 + 2(2m + b - 47.49) \cdot 2 + 2(3m + b - 50.02) \cdot 3 + 2(4m + b - 52.95) \cdot 4 + 2(5m + b - 54.75) \cdot 5 + 2(6m + b - 58.95) \cdot 6.$$
$$= 182m + 42b - 2267.28$$

We set these derivatives equal to 0 and solve the resulting system.

$$42m + 14b - 717.08 = 0$$
$$182m + 42b - 2267.28 = 0$$

The solution to this system is

$b = 45.00357143$

$m = 2.072142857$

We use the D-test to verify that $S(b,m)$ is a relative minimum.

We first find the second-order partial derivatives.

$S_{bb} = 14, S_{bm} = 42, S_{mm} = 182$

$D = S_{bb} \cdot S_{mm} - [S_{bm}]^2$

$D = 14 \cdot 182 - [42]^2$

$= 784$

Since $D > 0$ and $S_{bb} = 14 > 0$, S has a relative minimum at $(45.00357143, 2.072142857)$.

The regression line is $y = 2.072142857x + 45.00357143.$

b) In 2009, $x = 2009 - 1999 = 10$

$y = 2.072142857(10) + 45.00357143$

$\approx 65.73.$

The average ticket price for an NFL game in 2009 will be about \$65.73.

In 2015, $x = 2015 - 1999 = 16$

$y = 2.072142857(16) + 45.00357143$

$\approx 78.16.$

The average ticket price for an NFL game in 2015 will be about \$78.16.

7. a) The data points are $(1950,71.1),(1960,73.1),(1970,74.7),$ $(1980,77.4),(1990,78.8),(2000,79.5),$ and $(2003,80.1)$.

The points on the regression line are $(1950,y_1),(1960,y_2),(1970,y_3),$ $(1980,y_4),(1990,y_5),(2000,y_6),$ and $(2003,y_7)$.

The y-deviations are $y_1-71.1,\ y_2-73.1,\ y_3-74.7,$ $y_4-77.4, y_5-78.8, y_6-79.5,$ and $y_7-80.1$.

We want to minimize

$$S=(y_1-71.1)^2+(y_2-73.1)^2+(y_3-74.7)^2+(y_4-77.4)^2+(y_5-78.8)^2+(y_6-79.5)^2+(y_7-80.1)^2$$

Where:

$$y_1=m\cdot 1950+b$$
$$y_2=m\cdot 1960+b$$
$$y_3=m\cdot 1970+b$$
$$y_4=m\cdot 1980+b$$
$$y_5=m\cdot 1990+b$$
$$y_6=m\cdot 2000+b$$
$$y_7=m\cdot 2003+b$$

Substituting we get:

$$S=(1950m+b-71.1)^2+(1960m+b-73.1)^2+(1970m+b-74.7)^2+(1980m+b-77.4)^2+(1990m+b-78.8)^2+(2000m+b-79.5)^2+(2003m+b-80.1)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b}=2(1950m+b-71.1)+2(1960m+b-73.1)+2(1970m+b-74.7)+2(1980m+b-77.4)+2(1990m+b-78.8)+2(2000m+b-79.5)+2(2003m+b-80.1)$$
$$=3900m+2b-142.2+3920m+2b-146.2+3940m+2b-149.4+3960m+2b-154.8+3980m+2b-157.6+4000m+2b-159+4006m+2b-160.2$$
$$=27,706m+14b-1069.4$$

$$\frac{\partial S}{\partial m}=2(1950m+b-71.1)\cdot 1950+2(1960m+b-73.1)\cdot 1960+2(1970m+b-74.7)\cdot 1970+2(1980m+b-77.4)\cdot 1980+2(1990m+b-78.8)\cdot 1990+2(2000m+b-79.5)\cdot 2000+2(2003m+b-80.1)\cdot 2003$$
$$=7,605,000m+3900b-277,290+7,683,200m+3920b-286,552+7,761,800m+3940b-294,318+7,840,800m+3960b-306,504+7,920,200m+3980b-313,624+8,000,000m+4000b-318,000+8,024,018m+4006b-320,880.6$$
$$=54,835,018m+27,706b-2,117,168.6$$

We set these derivatives equal to 0 and solve the resulting system.

$$27,706m+14b-1069.4=0$$
$$54,835,018m+27,706b-2,117,168.6=0$$

The solution to this system is

$b=-261.0738233$

$m=0.1705202312$

We use the D-test to verify that $S(b,m)$ is a relative minimum.

We first find the second-order partial derivatives.

$S_{bb}=14, S_{bm}=27,706, S_{mm}=54,835,018$

$D=S_{bb}\cdot S_{mm}-[S_{bm}]^2$

$$D=14\cdot 54,835,018-[27,706]^2$$
$$=67,816$$

Since $D > 0$ and $S_{bb} = 14 > 0$, S has a relative minimum at $(-261.0738233, 0.1705202312)$. The regression line is
$y = 0.1705202312x - 261.0738233.$

b) In 2010,
$$y = 0.1705202312(2010) - 261.0738233 \approx 81.7.$$
In 2010, the average life expectancy of women will be about 81.7 years.
In 2015,
$$y = 0.1705202312(2015) - 261.0738233 \approx 82.5.$$
In 2015, the average life expectancy of women will be about 82.5 years.

8. a) The data points are $(1950, 65.6), (1960, 66.6), (1970, 67.1), (1980, 70.0), (1990, 71.8), (2000, 74.1),$ and $(2003, 74.8)$.
The points on the regression line are $(1950, y_1), (1960, y_2), (1970, y_3), (1980, y_4), (1990, y_5), (2000, y_6),$ and $(2003, y_7)$.
The y-deviations are
$y_1 - 65.6,\ y_2 - 66.6,\ y_3 - 67.1,$
$y_4 - 70.0, y_5 - 71.8, y_6 - 74.1,$
and $y_7 - 74.8$.
We want to minimize
$$S = (y_1 - 65.6)^2 + (y_2 - 66.6)^2 + (y_3 - 67.1)^2 + (y_4 - 70.0)^2 + (y_5 - 71.8)^2 + (y_6 - 74.1)^2 + (y_7 - 74.8)^2$$
Where:
$$y_1 = m \cdot 1950 + b$$
$$y_2 = m \cdot 1960 + b$$
$$y_3 = m \cdot 1970 + b$$
$$y_4 = m \cdot 1980 + b$$
$$y_5 = m \cdot 1990 + b$$
$$y_6 = m \cdot 2000 + b$$
$$y_7 = m \cdot 2003 + b$$
Substituting we get:
$$S = (1950m + b - 65.6)^2 + (1960m + b - 66.6)^2 + (1970m + b - 67.1)^2 + (1980m + b - 70.0)^2 + (1990m + b - 71.8)^2 + (2000m + b - 74.1)^2 + (2003m + b - 74.8)^2$$
In order to minimize S, we need to find the first partial derivatives.
$$\frac{\partial S}{\partial b} = 2(1950m + b - 65.6) + 2(1960m + b - 66.6) + 2(1970m + b - 67.1) + 2(1980m + b - 70.0) + 2(1990m + b - 71.8) + 2(2000m + b - 74.1) + 2(2003m + b - 74.8)$$
$$= 27706m + 14b - 980$$
$$\frac{\partial S}{\partial m} = 2(1950m + b - 65.6) \cdot 1950 + 2(1960m + b - 66.6) \cdot 1960 + 2(1970m + b - 67.1) \cdot 1970 + 2(1980m + b - 70.0) \cdot 1980 + 2(1990m + b - 71.8) \cdot 1990 + 2(2000m + b - 74.1) \cdot 2000 + 2(2003m + b - 74.8) \cdot 2003$$
$$= 54{,}835{,}018m + 27{,}706b - 1{,}940{,}298.8$$
We set these derivatives equal to 0 and solve the resulting system.
$$27{,}706m + 14b - 980 = 0$$
$$54{,}835{,}018m + 27{,}706b - 1{,}940{,}298.8 = 0$$
The solution to this system is
$b = -289.030801$
$m = 0.1814203138$
We use the D-test to verify that $S(b, m)$ is a relative minimum.
We first find the second-order partial derivatives.
$$S_{bb} = 14, S_{bm} = 27{,}706, S_{mm} = 54{,}835{,}018$$
$$D = S_{bb} \cdot S_{mm} - [S_{bm}]^2$$
$$D = 14 \cdot 54{,}835{,}018 - [27{,}706]^2 = 67{,}816$$
Since $D > 0$ and $S_{bb} = 14 > 0$, S has a relative minimum at $(-289.030801, 0.1814203138)$. The regression line is
$y = 0.1814203138x - 289.030801.$

b) In 2010,

$$y = 0.1814203138(2010) - 289.030801.$$
$$\approx 75.6.$$

In 2010, the average life expectancy of men will be about 75.6 years.

In 2015,

$$y = 0.1814203138(2015) - 289.030801.$$
$$\approx 76.5.$$

In 2015, the average life expectancy of men will be about 76.5 years.

9. a) The data points are $(70,75),(60,62),$ and $(85,89)$.

The points on the regression line are $(70,y_1),(60,y_2),$ and $(85,y_3)$.

The y-deviations are $y_1-75,\ y_2-62,$ and y_3-89.

We want to minimize

$$S=(y_1-75)^2+(y_2-62)^2+(y_3-89)^2$$

Where:

$$y_1 = m\cdot 70+b$$
$$y_2 = m\cdot 60+b$$
$$y_3 = m\cdot 85+b$$

Substituting we get:

$$S=(70m+b-75)^2+(60m+b-62)^2+(85m+b-89)^2$$

In order to minimize S, we need to find the first partial derivatives.

$$\frac{\partial S}{\partial b}$$
$$=2(70m+b-75)+2(60m+b-62)+2(85m+b-89)$$
$$=140m+2b-150+120m+2b-124+170m+2b-178$$
$$=430m+6b-452$$

$$\frac{\partial S}{\partial m}$$
$$=2(70m+b-75)\cdot 70+2(60m+b-62)\cdot 60+2(85m+b-89)\cdot 85$$
$$=9800m+140b-10,500+7200m+120b-7440+14,450m+170b-15,130$$
$$=31,450m+430b-33,070$$

We set these derivatives equal to 0 and solve the resulting system.

$$430m+6b-452=0$$
$$31,450m+430b-33,070=0$$

The solution to this system is

$$b=-1.236842105$$
$$m=1.068421053$$

We use the D-test to verify that $S(b,m)$ is a relative minimum.

We first find the second-order partial derivatives.

$$S_{bb}=6, S_{bm}=430, S_{mm}=31,450$$
$$D=S_{bb}\cdot S_{mm}-[S_{bm}]^2$$
$$D=6\cdot 31,450-[430]^2$$
$$=3800$$

Since $D>0$ and $S_{bb}=6>0$, S has a relative minimum at $(-1.236842105, 1.068421053)$.

The regression line is

$$y=1.068421053x-1.236842105$$

b) $x=81$

$$y=1.068421053(81)-1.236842105$$
$$\approx 85.$$

A student who scores 81% on the midterm will score about 85% on the final.

10. a) The data points are $(1912,78.0),(1956,84.5),(1973,90.5),$ $(1989,96.0),$ and $(1993,96.5)$.

The points on the regression line are $(1912,y_1),(1956,y_2),(1973,y_3),$ $(1989,y_4),$ and $(1993,y_5)$.

The y-deviations are $y_1-78.0,\ y_2-84.5,\ y_3-90.5,$ $y_4-96.0,$ and $y_5-96.5$.

We want to minimize

$$S=(y_1-78.0)^2+(y_2-84.5)^2+(y_3-90.5)^2+(y_4-96.0)^2+(y_5-96.5)^2$$

Where:

$$y_1=m\cdot 1912+b$$
$$y_2=m\cdot 1956+b$$
$$y_3=m\cdot 1973+b$$
$$y_4=m\cdot 1989+b$$
$$y_5=m\cdot 1993+b$$

Substituting we get:

$S = (1912m + b - 78.0)^2 + (1956m + b - 84.5)^2 +$
$(1973m + b - 90.5)^2 + (1989m + b - 96.0)^2 +$
$(1993m + b - 96.5)^2$

In order to minimize S, we need to find the first partial derivatives.

$$\begin{aligned}\frac{\partial S}{\partial b} &= 2(1912m + b - 78.0) + \\ &\quad 2(1956m + b - 84.5) + \\ &\quad 2(1973m + b - 90.5) + \\ &\quad 2(1989m + b - 96.0) + \\ &\quad 2(1993m + b - 96.5) \\ &= 19{,}646m + 10b - 891\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial m} &= 2(1912m + b - 78.0)\cdot 1912 + \\ &\quad 2(1956m + b - 84.5)\cdot 1956 + \\ &\quad 2(1973m + b - 90.5)\cdot 1973 + \\ &\quad 2(1989m + b - 96.0)\cdot 1989 + \\ &\quad 2(1993m + b - 96.5)\cdot 1993 \\ &= 38{,}605{,}158m + 19{,}646b - 1{,}752{,}486\end{aligned}$$

We set these derivatives equal to 0 and solve the resulting system.

$$19{,}646m + 10b - 891 = 0$$
$$38{,}605{,}158m + 19{,}646b - 1{,}752{,}486 = 0$$

The solution to this system is

$b = -372.6256376$

$m = 0.2350227209$

We use the D-test to verify that $S(b, m)$ is a relative minimum.

We first find the second-order partial derivatives.

$S_{bb} = 10, S_{bm} = 19{,}646, S_{mm} = 38{,}605{,}158$

$D = S_{bb} \cdot S_{mm} - [S_{bm}]^2$

$D = 10 \cdot 38{,}605{,}158 - [19{,}646]^2$
$= 86{,}264$

Since $D > 0$ and $S_{bb} = 10 > 0$, S has a relative minimum at $(-372.6256376, 0.2350227209)$.

The regression line is
$y = 0.2350227209x - 372.6256376.$

b) In 2010,
$y = 0.2350227209(2010) - 372.6256376$
$\approx 99.8.$
According to the model, the world record in the high jump will be 99.8 inches in 2010.
In 2050,
$y = 0.2350227209(2050) - 372.6256376$
$\approx 109.2.$
According to the model, the world record in the high jump will be 109.2 inches in 2050.

c) $\boxed{tw}$ It does not seem realistic that the record could be more than 9 feet. The linear function increases without limit, while human abilities eventually reach a limit.

11. $\boxed{tw}$ Answers will vary. The concept of linear regression involves fitting a linear equation to a set of data in such a way that the sum of the squares of the deviations of the actual y-values from those on the line is a minimum.

12. $\boxed{tw}$ Answers will vary.

13. a) Converting the times to decimal notation and using the STAT package on a calculator, we get the regression equation
$y = -0.0059379586x + 15.57191398$.

b) We predict that the world record in 2010 will be about 3.636617 minutes or 3:38.2.
We predict that the world record in 2015 will be about 3.60693 minutes or 3:36.4.

c) According to the regression model, we would predict the world record in 1999 to be 3.7019 minutes or 3:42.1. This is about a second faster than the actual world record.

Exercise Set 6.5

1. Find the maximum value of
$f(x,y) = xy$
subject to the constraint
$3x + y = 10$.
We first express $3x + y = 10$ as $3x + y - 10 = 0$.
We form the new function F , given by:
$F(x,y,\lambda) = xy - \lambda(3x + y - 10)$.
We find the first partial derivatives:
$F(x,y,\lambda) = \underline{x}y - \lambda(3\underline{x} + y - 10)$
$F_x = y - 3\lambda,$
$F(x,y,\lambda) = x\underline{y} - \lambda(3x + \underline{y} - 10)$
$F_y = x - \lambda,$
$F(x,y,\lambda) = xy - \underline{\lambda}(3x + y - 10)$
$F_\lambda = -(3x + y - 10).$
We set each derivative equal to 0 and solve the resulting system:

$$y - 3\lambda = 0 \qquad (1)$$
$$x - \lambda = 0 \qquad (2)$$
$$3x + y - 10 = 0 \qquad (3) \left[\begin{smallmatrix} -(3x+y-10)=0, \text{ or} \\ 3x+y-10=0 \end{smallmatrix}\right]$$

Solving Eq. (2) for λ , we get:
$\lambda = x.$
Substituting into Eq. (1) for λ , we get:
$y - 3(x) = 0, \text{ or } y = 3x. \qquad (4)$
Substituting $3x$ for y in Eq. (3) , we get:

$$3x + 3x - 10 = 0$$
$$6x = 10$$
$$x = \frac{10}{6} = \frac{5}{3}$$

Then, using Eq. (4) , we have:
$y = 3\left(\frac{5}{3}\right) = 5.$
The maximum value of f subject to the constraint occurs at $\left(\frac{5}{3}, 5\right)$ and is
$f\left(\frac{5}{3}, 5\right) = \frac{5}{3} \cdot 5 = \frac{25}{3}.$

2. Find the maximum value of
$f(x,y) = 2xy$
subject to the constraint
$4x + y = 16$.
We first express $4x + y = 16$ as $4x + y - 16 = 0$.
We form the new function F , given by:
$F(x,y,\lambda) = 2xy - \lambda(4x + y - 16)$.
We find the first partial derivatives:
$F_x = 2y - 4\lambda,$
$F_y = 2x - \lambda,$
$F_\lambda = -(4x + y - 16).$
We set each derivative equal to 0 and solve the resulting system:

$$2y - 4\lambda = 0 \qquad (1)$$
$$2x - \lambda = 0 \qquad (2)$$
$$4x + y - 16 = 0 \qquad (3)\left[\begin{smallmatrix} -(4x+y-16)=0, \text{ or} \\ 4x+y-16=0 \end{smallmatrix}\right]$$

Solving Eq. (2) for λ , we get:
$\lambda = 2x.$
Substituting into Eq. (1) for λ , we get:
$2y - 4(2x) = 0, \text{ or } y = 4x. \qquad (4)$
Substituting $4x$ for y in Eq. (3) , we get:

$$4x + (4x) - 16 = 0$$
$$8x = 16$$
$$x = 2$$

Then, using Eq. (4) , we have:
$y = 4(2) = 8.$
The maximum value of f subject to the constraint occurs at $(2,8)$ and is
$f(2,8) = 2 \cdot 2 \cdot 8 = 32.$

3. Find the maximum value of
$f(x,y) = 4 - x^2 - y^2$
subject to the constraint
$x + 2y = 10$.
We first express $x + 2y = 10$ as $x + 2y - 10 = 0$.
We form the new function F , given by:
$F(x,y,\lambda) = 4 - x^2 - y^2 - \lambda(x + 2y - 10)$.
We find the first partial derivatives:

$$F(x,y,\lambda)=4-\underline{x}^2-y^2-\lambda(\underline{x}+2y-10)$$
$$F_x=-2x-\lambda,$$
$$F(x,y,\lambda)=4-x^2-\underline{y}^2-\lambda(x+2\underline{y}-10)$$
$$F_y=-2y-2\lambda,$$
$$F(x,y,\lambda)=4-x^2-y^2-\underline{\lambda}(x+2y-10)$$
$$F_\lambda=-(x+2y-10).$$

We set each derivative equal to 0 and solve the resulting system:

$$-2x-\lambda=0 \qquad (1)$$
$$-2y-2\lambda=0 \qquad (2)$$
$$x+2y-10=0 \qquad (3)\left[\begin{smallmatrix}-(x+2y-10)=0,\text{ or}\\ x+2y-10=0\end{smallmatrix}\right]$$

Solving Eq. (1) for λ, we get:

$\lambda=-2x.$

Substituting into Eq. (2) for λ, we get:

$-2y-2(-2x)=0,\ \text{or } y=2x. \qquad (4)$

Substituting $2x$ for y in Eq. (3), we get:

$$x+2(2x)-10=0$$
$$5x=10$$
$$x=2.$$

Then, using Eq. (4), we have:

$y=2(2)=4.$

The maximum value of f subject to the constraint occurs at $(2,4)$ and is

$$f(2,4)=4-(2)^2-(4)^2$$
$$=4-4-16$$
$$=-16.$$

4. Find the maximum value of
$f(x,y)=3-x^2-y^2$
subject to the constraint
$x+6y=37$.

We first express $x+6y=37$ as $x+6y-37=0$.

We form the new function F, given by:
$F(x,y,\lambda)=3-x^2-y^2-\lambda(x+6y-37).$

We find the first partial derivatives:

$$F_x=-2x-\lambda,$$
$$F_y=-2y-6\lambda,$$
$$F_\lambda=-(x+6y-37).$$

We set each derivative equal to 0 and solve the resulting system:

$$-2x-\lambda=0 \qquad (1)$$
$$-2y-6\lambda=0 \qquad (2)$$
$$x+6y-37=0 \qquad (3)\left[\begin{smallmatrix}-(x+6y-37)=0,\text{ or}\\ x+6y-37=0\end{smallmatrix}\right]$$

Solving Eq. (1) for λ, we get:

$\lambda=-2x.$

Substituting into Eq. (2) for λ, we get:

$-2y-6(-2x)=0,\ \text{or } y=6x. \qquad (4)$

Substituting $6x$ for y in Eq. (3), we get:

$$x+6(6x)-37=0$$
$$37x=37$$
$$x=1.$$

Then, using Eq. (4), we have:

$y=6(1)=6.$

The maximum value of f subject to the constraint occurs at $(1,6)$ and is

$$f(1,6)=3-(1)^2-(6)^2$$
$$=3-1-36$$
$$=-34.$$

5. Find the minimum value of
$f(x,y)=x^2+y^2$
subject to the constraint
$2x+y=10$.

We first express $2x+y=10$ as $2x+y-10=0$.

We form the new function F, given by:
$F(x,y,\lambda)=x^2+y^2-\lambda(2x+y-10).$

We find the first partial derivatives:

$$F(x,y,\lambda)=\underline{x}^2+y^2-\lambda(2\underline{x}+y-10).$$
$$F_x=2x-2\lambda,$$
$$F(x,y,\lambda)=x^2+\underline{y}^2-\lambda(2x+\underline{y}-10).$$
$$F_y=2y-\lambda,$$
$$F(x,y,\lambda)=x^2+y^2-\underline{\lambda}(2x+y-10)$$
$$F_\lambda=-(2x+y-10).$$

We set each derivative equal to 0 and solve the resulting system:

$$2x-2\lambda=0 \qquad (1)$$
$$2y-\lambda=0 \qquad (2)$$
$$2x+y-10=0 \qquad (3)\left[\begin{smallmatrix}-(2x+y-10)=0,\text{ or}\\ 2x+y-10=0\end{smallmatrix}\right]$$

Solving Eq. (2) for λ, we get:
$\lambda = 2y.$
Substituting into Eq. (1) for λ, we get:
$2x - 2(2y) = 0,$ or $x = 2y.$ $\qquad (4)$
Substituting $2y$ for x in Eq. (3), we get:

$$\begin{aligned}2(2y) + y - 10 &= 0\\ 5y &= 10\\ y &= 2.\end{aligned}$$

Then, using Eq. (4), we have:
$x = 2(2) = 4.$
The minimum value of f subject to the constraint occurs at $(4,2)$ and is

$$\begin{aligned}f(4,2) &= 4^2 + 2^2\\ &= 16 + 4\\ &= 20.\end{aligned}$$

6. Find the minimum value of
$f(x,y) = x^2 + y^2$
subject to the constraint
$x + 4y = 17$.
We first express $x + 4y = 17$ as $x + 4y - 17 = 0$.
We form the new function F, given by:
$F(x,y,\lambda) = x^2 + y^2 - \lambda(x + 4y - 17).$
We find the first partial derivatives:
$F_x = 2x - \lambda,$
$F_y = 2y - 4\lambda,$
$F_\lambda = -(x + 4y - 17).$
We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned}2x - \lambda &= 0 && (1)\\ 2y - 4\lambda &= 0 && (2)\\ x + 4y - 17 &= 0 && (3)\ \left[\begin{smallmatrix}-(x+4y-17)=0, \text{ or}\\ x+4y-17=0\end{smallmatrix}\right]\end{aligned}$$

Solving Eq. (1) for λ, we get:
$\lambda = 2x.$
Substituting into Eq. (2) for λ, we get:
$2y - 4(2x) = 0,$ or $y = 4x.$ $\qquad (4)$
Substituting $4x$ for y in Eq. (3), we get:

$$\begin{aligned}x + 4(4x) - 17 &= 0\\ 17x &= 17\\ x &= 1.\end{aligned}$$

Then, using Eq. (4), we have:
$y = 4(1) = 4.$

The minimum value of f subject to the constraint occurs at $(1,4)$ and is

$$\begin{aligned}f(1,4) &= 1^2 + 4^2\\ &= 1 + 16\\ &= 17.\end{aligned}$$

7. Find the minimum value of
$f(x,y) = 2y^2 - 6x^2$
subject to the constraint
$2x + y = 4$.
We first express $2x + y = 4$ as $2x + y - 4 = 0$.
We form the new function F, given by:
$F(x,y,\lambda) = 2y^2 - 6x^2 - \lambda(2x + y - 4).$
We find the first partial derivatives:
$F(x,y,\lambda) = 2y^2 - 6\underline{x}^2 - \lambda(2\underline{x} + y - 4)$
$F_x = -12x - 2\lambda,$
$F(x,y,\lambda) = 2\underline{y}^2 - 6x^2 - \lambda(2x + \underline{y} - 4)$
$F_y = 4y - \lambda,$
$F(x,y,\lambda) = 2y^2 - 6x^2 - \underline{\lambda}(2x + y - 4)$
$F_\lambda = -(2x + y - 4).$
We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned}-12x - 2\lambda &= 0 && (1)\\ 4y - \lambda &= 0 && (2)\\ 2x + y - 4 &= 0 && (3)\left[\begin{smallmatrix}-(2x+y-4)=0, \text{ or}\\ 2x+y-4=0\end{smallmatrix}\right]\end{aligned}$$

Solving Eq. (2) for λ, we get:
$\lambda = 4y.$
Substituting into Eq. (1) for λ, we get:

$$\begin{aligned}-12x - 2(4y) &= 0\\ 8y &= -12x\\ y &= -\frac{3}{2}x. \qquad (4)\end{aligned}$$

Substituting $-\frac{3}{2}x$ for y in Eq. (3), we get:

$$\begin{aligned}2x + \left(-\frac{3}{2}x\right) - 4 &= 0\\ \frac{1}{2}x &= 4\\ x &= 8.\end{aligned}$$

Then, using Eq. (4), we have:
$y = -\frac{3}{2}(8) = -12.$

The minimum value of *f* subject to the constraint occurs at $(8,-12)$ and is

$$\begin{aligned} f(8,-12) &= 2(-12)^2 - 6(8)^2 \\ &= 2(144) - 6(64) \\ &= 288 - 384 \\ &= -96. \end{aligned}$$

8. Find the minimum value of
$f(x,y) = 2x^2 + y^2 - xy$
subject to the constraint
$x + y = 8$.
We first express $x + y = 8$ as $x + y - 8 = 0$.
We form the new function F, given by:
$F(x,y,\lambda) = 2x^2 + y^2 - xy - \lambda(x + y - 8)$.
We find the first partial derivatives:
$F_x = 4x - y - \lambda,$
$F_y = 2y - x - \lambda,$
$F_\lambda = -(x + y - 8).$
We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} 4x - y - \lambda &= 0 && (1) \\ 2y - x - \lambda &= 0 && (2) \\ x + y - 8 &= 0 && (3)\ \left[\begin{smallmatrix} -(x+y-8)=0, \text{ or} \\ x+y-8=0 \end{smallmatrix}\right] \end{aligned}$$

Subtracting Eq. (2) from Eq. (1) we get:

$5x - 3y = 0$, or $x = \frac{3}{5}y$. (4).

Substituting $\frac{3}{5}y$ for x in Eq. (3), we get:

$$\begin{aligned} \frac{3}{5}y + y - 8 &= 0 \\ \frac{8}{5}y &= 8 \\ y &= 5. \end{aligned}$$

Then, using Eq. (4), we have:

$x = \frac{3}{5}(5) = 3.$

The minimum value of *f* subject to the constraint occurs at $(3,5)$ and is

$$\begin{aligned} f(3,5) &= 2(3)^2 + (5)^2 - (3)(5) \\ &= 18 + 25 - 15 \\ &= 28. \end{aligned}$$

9. Find the minimum value of
$f(x,y,z) = x^2 + y^2 + z^2$
subject to the constraint
$y + 2x - z = 3$.
We first express $y + 2x - z = 3$
as $y + 2x - z - 3 = 0$.
We form the new function F, given by:
$F(x,y,z,\lambda)$
$= x^2 + y^2 + z^2 - \lambda(y + 2x - z - 3)$
We find the first partial derivatives:
$F_x = 2x - 2\lambda,$
$F_y = 2y - \lambda,$
$F_z = 2z + \lambda,$
$F_\lambda = -(y + 2x - z - 3).$
We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} 2x - 2\lambda &= 0 && (1) \\ 2y - \lambda &= 0 && (2) \\ 2z + \lambda &= 0 && (3) \\ y + 2x - z - 3 &= 0 && (4)\left[\begin{smallmatrix} -(y+2x-z-3)=0, \text{ or} \\ y+2x-z-3=0 \end{smallmatrix}\right] \end{aligned}$$

Solving Eq. (1) for *x*, we get:
$x = \lambda.$
Solving Eq. (2) for *y*, we get:
$y = \frac{1}{2}\lambda.$
Solving Eq. (3) for *z*, we get:
$z = -\frac{1}{2}\lambda.$
Substituting λ for *x*, $\frac{1}{2}\lambda$ for *y*, and $-\frac{1}{2}\lambda$ for *z* into Eq. (4), we get:

$$\begin{aligned} y + 2x - z - 3 &= 0 \\ \frac{1}{2}\lambda + 2\lambda - \left(-\frac{1}{2}\lambda\right) - 3 &= 0 \\ 3\lambda &= 3 \\ \lambda &= 1. \end{aligned}$$

Then,
$x = \lambda = 1$
$y = \frac{1}{2}\lambda = \frac{1}{2}$
$z = -\frac{1}{2}\lambda = -\frac{1}{2}$

The minimum value of f subject to the constraint occurs at $\left(1,\frac{1}{2},-\frac{1}{2}\right)$ and is

$$f\left(1,\frac{1}{2},-\frac{1}{2}\right)=1^2+\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2$$
$$=1+\frac{1}{4}+\frac{1}{4}$$
$$=\frac{3}{2}.$$

10. Find the minimum value of
$f(x,y,z)=x^2+y^2+z^2$
subject to the constraint
$x+y+z=2$.
We first express $x+y+z=2$ as $x+y+z-2=0$.
We form the new function F, given by:
$F(x,y,z,\lambda)$
$=x^2+y^2+z^2-\lambda(x+y+z-2)$
We find the first partial derivatives:
$F_x=2x-\lambda,$
$F_y=2y-\lambda,$
$F_z=2z-\lambda,$
$F_\lambda=-(x+y+z-2).$
We set each derivative equal to 0 and solve the resulting system:

$$2x-\lambda=0 \qquad (1)$$
$$2y-\lambda=0 \qquad (2)$$
$$2z-\lambda=0 \qquad (3)$$
$$x+y+z-2=0 \qquad (4)\left[\begin{smallmatrix}-(x+y+z-2)=0, \text{ or}\\ x+y+z-2=0\end{smallmatrix}\right]$$

Solving Eq. (1) for x, we get:
$$x=\frac{1}{2}\lambda.$$
Solving into Eq. (2) for y, we get:
$$y=\frac{1}{2}\lambda.$$
Solving into Eq. (3) for z, we get:
$$z=\frac{1}{2}\lambda.$$
Substituting $\frac{1}{2}\lambda$ for x, $\frac{1}{2}\lambda$ for y, and $\frac{1}{2}\lambda$ for z into Eq. (3), we get:
$$\frac{1}{2}\lambda+\frac{1}{2}\lambda+\frac{1}{2}\lambda-2=0$$
$$\frac{3}{2}\lambda=2$$
$$\lambda=\frac{4}{3}.$$
Then,
$$x=\frac{1}{2}\lambda=\frac{2}{3}$$
$$y=\frac{1}{2}\lambda=\frac{2}{3}$$
$$z=\frac{1}{2}\lambda=\frac{2}{3}$$
The minimum value of f subject to the constraint occurs at $\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$ and is
$$f\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)=\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2$$
$$=\frac{4}{9}+\frac{4}{9}+\frac{4}{9}$$
$$=\frac{12}{9}=\frac{4}{3}.$$

11. Find the maximum value of
$f(x,y)=xy \qquad$ (Product is $x\cdot y$)
subject to the constraint
$x+y=50. \qquad$ (Sum is 50.).
We first express $x+y=50$ as $x+y-50=0$.
We form the new function F, given by:
$F(x,y,\lambda)=xy-\lambda(x+y-50).$
We find the first partial derivatives:
$F_x=y-\lambda,$
$F_y=x-\lambda,$
$F_\lambda=-(x+y-50).$
We set each derivative equal to 0 and solve the resulting system:

$$y-\lambda=0 \qquad (1)$$
$$x-\lambda=0 \qquad (2)$$
$$x+y-50=0 \qquad (3)\left[\begin{smallmatrix}-(x+y-50)=0, \text{ or}\\ x+y-50=0\end{smallmatrix}\right]$$

Solving Eq. (2) for λ, we get:
$\lambda = x.$
Substituting into Eq. (1) for λ, we get:
$y-(x)=0, \text{ or } y=x. \qquad (4)$
Substituting x for y in Eq. (3), we get:
$$x+x-50=0$$
$$2x=50$$
$$x=25.$$
Then, using Eq. (4), we have:
$y=25.$
The maximum value of f subject to the constraint occurs at $(25,25)$. Thus, the two numbers whose sum is 50 that have the maximum product are 25 and 25.

12. Find the maximum value of
$f(x,y)=xy \qquad (\text{Product is } x\cdot y)$
subject to the constraint
$x+y=70. \qquad (\text{Sum is 70.}).$
We first express $x+y=70$ as $x+y-70=0$.
We form the new function F, given by:
$F(x,y,\lambda)=xy-\lambda(x+y-70).$
We find the first partial derivatives:
$F_x = y-\lambda,$
$F_y = x-\lambda,$
$F_\lambda = -(x+y-70).$
We set each derivative equal to 0 and solve the resulting system:
$$y-\lambda=0 \qquad (1)$$
$$x-\lambda=0 \qquad (2)$$
$$x+y-70=0 \qquad (3)\ \left[\begin{matrix}-(x+y-70)=0, \text{ or}\\ x+y-70=0\end{matrix}\right]$$
Solving Eq. (2) for λ, we get:
$\lambda = x.$
Substituting into Eq. (1) for λ, we get:
$y-(x)=0, \text{ or } y=x. \qquad (4)$
Substituting x for y in Eq. (3), we get:
$$x+x-70=0$$
$$2x=70$$
$$x=35.$$
Then, using Eq. (4), we have:
$y=35.$
The maximum value of f subject to the constraint occurs at $(35,35)$. Thus, the two numbers whose sum is 70 that have the maximum product are 35 and 35.

13. Find the minimum value of
$f(x,y)=xy \qquad (\text{Product is } x\cdot y)$
subject to the constraint
$x-y=6. \qquad (\text{Difference is 6.}).$
We first express $x-y=6$ as $x-y-6=0$.
We form the new function F, given by:
$F(x,y,\lambda)=xy-\lambda(x-y-6).$
We find the first partial derivatives:
$F_x = y-\lambda,$
$F_y = x+\lambda,$
$F_\lambda = -(x-y-6).$
We set each derivative equal to 0 and solve the resulting system:
$$y-\lambda=0 \qquad (1)$$
$$x+\lambda=0 \qquad (2)$$
$$x-y-6=0 \qquad (3)\ \left[\begin{matrix}-(x-y-6)=0, \text{ or}\\ x-y-6=0\end{matrix}\right]$$
Solving Eq. (1) for λ, we get:
$\lambda = y.$
Substituting into Eq. (2) for λ, we get:
$x+(y)=0, \text{ or } y=-x. \qquad (4)$
Substituting $-x$ for y in Eq. (3), we get:
$$x-(-x)-6=0$$
$$2x=6$$
$$x=3.$$
Then, using Eq. (4), we have:
$y=-3.$
The minimum value of f subject to the constraint occurs at $(3,-3)$. Thus, the two numbers whose difference is 6 that have the minimum product are 3 and -3.

14. Find the minimum value of
$f(x,y)=xy \qquad (\text{Product is } x\cdot y)$
subject to the constraint
$x-y=4. \qquad (\text{Difference is 4.}).$
We first express $x-y=4$ as $x-y-4=0$.
We form the new function F, given by:
$F(x,y,\lambda)=xy-\lambda(x-y-4).$

We find the first partial derivatives:

$F_x = y - \lambda,$

$F_y = x + \lambda,$

$F_\lambda = -(x - y - 4).$

We set each derivative equal to 0 and solve the resulting system:

$$y - \lambda = 0 \qquad (1)$$

$$x + \lambda = 0 \qquad (2)$$

$$x - y - 4 = 0 \qquad (3) \quad \left[\begin{smallmatrix} -(x-y-4)=0, \text{ or} \\ x-y-4=0 \end{smallmatrix}\right]$$

Solving Eq. (1) for λ, we get:

$\lambda = y.$

Substituting into Eq. (1) for λ, we get:

$x + (y) = 0,$ or $y = -x. \qquad (4)$

Substituting $-x$ for y in Eq. (3), we get:

$$x - (-x) - 4 = 0$$
$$2x = 4$$
$$x = 2.$$

Then, using Eq. (4), we have:

$y = -2.$

The minimum value of f subject to the constraint occurs at $(2,-2)$. Thus, the two numbers whose difference is 4 that have the minimum product are 2 and -2.

15. Find the minimum value of

$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$

subject to the constraint

$x + 2y + 3z = 13$.

We first express $x + 2y + 3z = 13$ as $x + 2y + 3z - 13 = 0$.

We form the new function F, given by:

$$F(x,y,z,\lambda)$$
$$= (x-1)^2 + (y-1)^2 + (z-1)^2 - \lambda(x + 2y + 3z - 13)$$

We find the first partial derivatives:

$F_x = 2(x-1) - \lambda,$

$F_y = 2(y-1) - 2\lambda,$

$F_z = 2(z-1) - 3\lambda,$

$F_\lambda = -(x + 2y + 3z - 13).$

We set each derivative equal to 0 and solve the resulting system:

$$2(x-1) - \lambda = 0 \qquad (1)$$

$$2(y-1) - 2\lambda = 0 \qquad (2)$$

$$2(z-1) - 3\lambda = 0 \qquad (3)$$

$$x + 2y + 3z - 13 = 0 \qquad (4)\left[\begin{smallmatrix} -(x+2y+3z-13)=0, \text{ or} \\ x+2y+3z-13=0 \end{smallmatrix}\right]$$

Solving Eq. (1) for x, we get:

$$2x - 2 - \lambda = 0$$
$$2x = 2 + \lambda$$
$$x = 1 + \frac{1}{2}\lambda.$$

Solving Eq. (2) for y, we get:

$$2y - 2 - 2\lambda = 0$$
$$2y = 2 + 2\lambda$$
$$y = 1 + \lambda.$$

Solving Eq. (3) for z, we get:

$$2z - 2 - 3\lambda = 0$$
$$2z = 2 + 3\lambda$$
$$z = 1 + \frac{3}{2}\lambda.$$

Substituting $1 + \frac{1}{2}\lambda$ for x, $1 + \lambda$ for y, and $1 + \frac{3}{2}\lambda$ for z into Eq. (4), we get:

$$x + 2y + 3z - 13 = 0$$
$$\left(1 + \frac{1}{2}\lambda\right) + 2(1 + \lambda) + 3\left(1 + \frac{3}{2}\lambda\right) - 13 = 0$$
$$1 + \frac{1}{2}\lambda + 2 + 2\lambda + 3 + \frac{9}{2}\lambda = 13$$
$$6 + 7\lambda = 13$$
$$7\lambda = 7$$
$$\lambda = 1.$$

Then,

$$x = 1 + \frac{1}{2}\lambda = 1 + \frac{1}{2} = \frac{3}{2}$$
$$y = 1 + \lambda = 1 + 1 = 2$$
$$z = 1 + \frac{3}{2}\lambda = 1 + \frac{3}{2} = \frac{5}{2}.$$

The minimum value of f subject to the constraint occurs at $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$.

16. Find the minimum value of
$f(x,y,z)=(x-1)^2+(y-4)^2+(z-2)^2$
subject to the constraint
$3x+4y+2z=52$.
We first express $3x+4y+2z=52$
as $3x+4y+2z-52=0$.
We form the new function F, given by:
$F(x,y,z,\lambda)$
$=(x-1)^2+(y-4)^2+(z-2)^2-$
$\lambda(3x+4y+2z-52)$
We find the first partial derivatives:
$F_x=2(x-1)-3\lambda,$
$F_y=2(y-4)-4\lambda,$
$F_z=2(z-2)-2\lambda,$
$F_\lambda=-(3x+4y+2z-52).$
We set each derivative equal to 0 and solve the resulting system:

$$2(x-1)-3\lambda=0 \quad (1)$$
$$2(y-4)-4\lambda=0 \quad (2)$$
$$2(z-2)-2\lambda=0 \quad (3)$$
$$3x+4y+2z-52=0 \quad (4)\left[\begin{array}{l}-(3x+4y+2z-52)=0,\text{ or}\\ 3x+4y+2z-52=0\end{array}\right]$$

Solving Eq. (1) for x, we get:

$$2x-2-3\lambda=0$$
$$2x=2+3\lambda$$
$$x=1+\frac{3}{2}\lambda.$$

Solving Eq. (2) for y, we get:

$$2y-8-4\lambda=0$$
$$2y=8+4\lambda$$
$$y=4+2\lambda.$$

Solving Eq. (3) for z, we get:

$$2z-4-2\lambda=0$$
$$2z=4+2\lambda$$
$$z=2+\lambda.$$

Substituting $1+\frac{3}{2}\lambda$ for x, $4+2\lambda$ for y, and $2+\lambda$ for z into Eq. (3), we get:

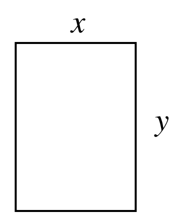
$$3x+4y+2z-52=0$$
$$3\left(1+\frac{3}{2}\lambda\right)+4(4+2\lambda)+2(2+\lambda)-52=0$$
$$\frac{29}{2}\lambda=29$$
$$\lambda=2.$$

Then,

$$x=1+\frac{3}{2}\lambda=1+\frac{3}{2}\cdot 2=4$$
$$y=4+2\lambda=4+2\cdot 2=8$$
$$z=2+\lambda=2+2=4.$$

The minimum value of f subject to the constraint occurs at $(4,8,4)$.

17. The area of the page is given by $A=xy$ and the perimeter of the page is given by $P=2x+2y$. See the figure.

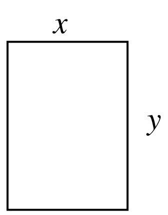


We want to maximize the area
$A=xy$
Subject to the constraint
$2x+2y=39$.
We first express $2x+2y=39$ as
$2x+2y-39=0$.
We form the new function F, given by:
$F(x,y,\lambda)=xy-\lambda(2x+2y-39).$
We find the first partial derivatives:
$F_x=y-2\lambda,$
$F_y=x-2\lambda,$
$F_\lambda=-(2x+2y-39).$
We set each derivative equal to 0 and solve the resulting system:

$$y-2\lambda=0 \quad (1)$$
$$x-2\lambda=0 \quad (2)$$
$$2x+2y-39=0 \quad (3)\left[\begin{array}{l}-(2x+2y-39)=0,\text{ or}\\ 2x+2y-39=0\end{array}\right]$$

From Eqs. (1) and (2) we see:
$y = 2\lambda = x.$
Substituting x for y in Eq. (3), we get:
$$2x + 2x - 39 = 0$$
$$4x = 39$$
$$x = \frac{39}{4} = 9\tfrac{3}{4}.$$
Then,
$y = x = 9\tfrac{3}{4}.$
The maximum area subject to the constraint occurs at $\left(9\tfrac{3}{4}, 9\tfrac{3}{4}\right)$. The maximum area is $A = 9\tfrac{3}{4} \cdot 9\tfrac{3}{4} = 95\tfrac{1}{16}$ in^2. The area of the standard $8\tfrac{1}{2} \times 11$ paper is not the maximum area of paper that has a perimeter of 39 in.

18. The area of the room is given by $A = xy$ and the perimeter of the room is given by $P = 2x + 2y$. See the figure.



We want to maximize the area
$A = xy$
Subject to the constraint
$2x + 2y = 80.$
We first express $2x + 2y = 80$ as
$2x + 2y - 80 = 0$.
We form the new function F, given by:
$F(x, y, \lambda) = xy - \lambda(2x + 2y - 80).$
We find the first partial derivatives:
$F_x = y - 2\lambda,$
$F_y = x - 2\lambda,$
$F_\lambda = -(2x + 2y - 80).$
We set each derivative equal to 0 and solve the resulting system:
$$y - 2\lambda = 0 \quad (1)$$
$$x - 2\lambda = 0 \quad (2)$$
$$2x + 2y - 80 = 0 \quad (3) \left[\begin{array}{l} -(2x+2y-80)=0, \text{ or} \\ 2x+2y-80=0 \end{array}\right]$$

From Eqs. (1) and (2) we see:
$y = 2\lambda = x.$
Substituting x for y in Eq. (3), we get:
$$2x + 2x - 80 = 0$$
$$4x = 80$$
$$x = \frac{80}{4} = 20.$$
Then,
$y = x = 20.$
The maximum area subject to the constraint occurs at $(20, 20)$. The dimensions of the largest room that can be built are 20 ft by 20 ft. The maximum area the room is
$A = 20 \cdot 20 = 400$ ft^2.

19. We want to minimize the function s given by
$s(h, r) = 2\pi rh + 2\pi r^2$
subject to the volume constraint
$\pi r^2 h = 27,$ or $\pi r^2 h - 27 = 0.$
We form the new function S given by
$S(h, r, \lambda) = 2\pi rh + 2\pi r^2 - \lambda\left(\pi r^2 h - 27\right).$
We find the first partial derivatives.
$S_h = 2\pi r - \lambda \pi r^2,$
$S_r = 2\pi h + 4\pi r - 2\lambda \pi rh,$
$S_\lambda = -\left(\pi r^2 h - 27\right).$
We set these derivatives equal to 0 and solve the resulting system.
$$2\pi r - \lambda \pi r^2 = 0 \quad (1)$$
$$2\pi h + 4\pi r - 2\lambda \pi rh = 0 \quad (2)$$
$$\pi r^2 h - 27 = 0. \quad (3) \left[\begin{array}{l} -\left(\pi r^2 h - 27\right)=0, \text{ or} \\ \pi r^2 h - 27 = 0 \end{array}\right]$$
We solve Eq. (1) for r:
$$\pi r(2 - \lambda r) = 0$$
$$\pi r = 0 \quad \text{or} \quad 2 - \lambda r = 0$$
$$r = 0 \quad \text{or} \quad r = \frac{2}{\lambda}$$
Note, $r = 0$ can not be a solution to the original problem, so we continue by substituting $\frac{2}{\lambda}$ for r in Eq. (2).

$$2\pi h+4\pi\left(\frac{2}{\lambda}\right)-2\lambda\pi\left(\frac{2}{\lambda}\right)h=0$$
$$2\pi h+\frac{8\pi}{\lambda}-4\pi h=0$$
$$\frac{8\pi}{\lambda}-2\pi h=0$$
$$h=\frac{4}{\lambda}$$

Since $h=\frac{4}{\lambda}$ and $r=\frac{2}{\lambda}$, it follows that $h=2r$.

substituting $2r$ for h in Eq. (3) yields:

$$\pi r^2(2r)-27=0$$
$$2\pi r^3=27$$
$$r^3=\frac{27}{2\pi}$$
$$r=\sqrt[3]{\frac{27}{2\pi}}\approx 1.6$$

So when $r\approx 1.6$ ft and $h=2(1.6)\approx 3.2$ ft, the surface area of the oil drum is a minimum. The minimum area is about

$2\pi(1.6)(3.2)+2\pi(1.6)^2\approx 48.3\text{ ft}^2$.

(Answers will vary due to rounding differences.)

20. We want to minimize the function s given by

$s(h,r)=2\pi rh+2\pi r^2$

subject to the volume constraint

$\pi r^2h=99$, or $\pi r^2h-99=0$.

We form the new function S given by

$S(h,r,\lambda)=2\pi rh+2\pi r^2-\lambda(\pi r^2h-99)$.

We find the first partial derivatives.

$S_h=2\pi r-\lambda\pi r^2$,

$S_r=2\pi h+4\pi r-2\lambda\pi rh$,

$S_\lambda=-(\pi r^2h-99)$.

We set these derivatives equal to 0 and solve the resulting system.

$$2\pi r-\lambda\pi r^2=0 \quad (1)$$
$$2\pi h+4\pi r-2\lambda\pi rh=0 \quad (2)$$
$$\pi r^2h-99=0. \quad (3)\left[\begin{array}{l}-(\pi r^2h-99)=0,\text{ or}\\ \pi r^2h-99=0\end{array}\right]$$

We solve Eq. (1) for r:

$\pi r(2-\lambda r)=0$

$\pi r=0$ or $2-\lambda r=0$

$r=0$ or $r=\frac{2}{\lambda}$

Note, $r=0$ can not be a solution to the original problem, so we continue by substituting $\frac{2}{\lambda}$ for r in Eq. (2).

$$2\pi h+4\pi\left(\frac{2}{\lambda}\right)-2\lambda\pi\left(\frac{2}{\lambda}\right)h=0$$
$$2\pi h+\frac{8\pi}{\lambda}-4\pi h=0$$
$$\frac{8\pi}{\lambda}-2\pi h=0$$
$$h=\frac{4}{\lambda}$$

Since $h=\frac{4}{\lambda}$ and $r=\frac{2}{\lambda}$, it follows that $h=2r$.

substituting $2r$ for h in Eq. (3) yields:

$$\pi r^2(2r)-99=0$$
$$2\pi r^3=99$$
$$r^3=\frac{99}{2\pi}$$
$$r=\sqrt[3]{\frac{99}{2\pi}}\approx 2.5$$

So when $r\approx 2.5$ in. and $h=2(2.5)\approx 5$ in., the surface area of the juice can is a minimum. The minimum area is about

$2\pi(2.5)(5.0)+2\pi(2.5)^2\approx 117.8\text{ in}^2$.

(Answers will vary due to rounding differences.)

21. We want maximize

$S(L,M)=ML-L^2$

subject to the constraint

$M+L=90$.

We first express $M+L=90$ as $M+L-90=0$.

We form the new function F, given by:

$F(L,M,\lambda)=ML-L^2-\lambda(M+L-90)$.

We find the first partial derivatives:

$F_L=M-2L-\lambda$,

$F_M=L-\lambda$,

$F_\lambda=-(M+L-90)$.

We set each derivative equal to 0 and solve the resulting system:

$$M-2L-\lambda=0 \qquad (1)$$
$$L-\lambda=0 \qquad (2)$$
$$M+L-90=0 \qquad (3)\ \left[\begin{smallmatrix}-(M+L-90)=0,\text{ or}\\ M+L-90=0\end{smallmatrix}\right]$$

Solving Eq. (2) for λ, we get:

$\lambda=L.$

Substituting into Eq. (1) for λ, we get:

$$M-2L-L=0$$
$$M-3L=0$$
$$M=3L. \qquad (4)$$

Substituting $3L$ for M in Eq. (3), we get:

$$3L+L-90=0$$
$$4L=90$$
$$L=22.5.$$

Then, using Eq. (4), we have:

$M=3(22.5)=67.5.$

The maximum value of S subject to the constraint occurs at $(22.5,67.5)$ and is

$$\begin{aligned}S(22.5,67.5)&=(67.5)(22.5)-(22.5)^2\\&=1518.75-506.25\\&=1012.5\end{aligned}$$

22. We want maximize

$S(L,M)=ML-L^2$

subject to the constraint

$M+L=70.$

We first express $M+L=70$ as $M+L-70=0$.

We form the new function F, given by:

$F(x,y,\lambda)=ML-L^2-\lambda(M+L-70).$

We find the first partial derivatives:

$F_L=M-2L-\lambda,$

$F_M=L-\lambda,$

$F_\lambda=-(M+L-70).$

We set each derivative equal to 0 and solve the resulting system:

$$M-2L-\lambda=0 \qquad (1)$$
$$L-\lambda=0 \qquad (2)$$
$$M+L-70=0 \qquad (3)\ \left[\begin{smallmatrix}-(M+L-70)=0,\text{ or}\\ M+L-70=0\end{smallmatrix}\right]$$

Solving Eq. (2) for λ, we get:

$\lambda=L.$

Substituting into Eq. (1) for λ, we get:

$$M-2L-L=0$$
$$M-3L=0$$
$$M=3L. \qquad (4)$$

Substituting $3L$ for M in Eq. (3), we get:

$$3L+L-70=0$$
$$4L=70$$
$$L=17.5.$$

Then, using Eq. (4), we have:

$M=3(17.5)=52.5.$

The maximum value of S subject to the constraint occurs at $(17.5,52.5)$ and is

$$\begin{aligned}S(17.5,52.5)&=(52.5)(17.5)-(17.5)^2\\&=612.5\end{aligned}$$

23. a) The area of the floor is xy.

The cost of the floor is $4xy$.

The area of the walls is $2xz+2yz$.

The cost of the walls is $3(2xz+2yz)$.

The area of the ceiling is xy.

The cost of the ceiling is $3xy$.

Therefore, the total cost function is

$$\begin{aligned}C(x,y,z)&=4xy+3(2xz+2yz)+3xy\\&=7xy+6xz+6yz.\end{aligned}$$

b) We want to minimize the value of

$C(x,y,z)=7xy+6xz+6yz$

subject to the constraint of

$x\cdot y\cdot z=252{,}000 \qquad (\text{Volume}=l\cdot w\cdot h)$

We first express $x\cdot y\cdot z=252{,}000$ as $x\cdot y\cdot z-252{,}000=0$.

We form the new function F, given by:

$F(x,y,z,\lambda)$

$=7xy+6xz+6yz-\lambda(x\cdot y\cdot z-252{,}000)$

We find the first partial derivatives:

$F_x=7y+6z-\lambda yz,$

$F_y=7x+6z-\lambda xz,$

$F_z=6x+6y-\lambda xy,$

$F_\lambda=-(x\cdot y\cdot z-252{,}000).$

We set each derivative equal to 0 and solve the resulting system:

$$7y+6z-\lambda yz=0 \qquad (1)$$
$$7x+6z-\lambda xz=0 \qquad (2)$$
$$6x+6y-\lambda xy=0 \qquad (3)$$
$$xyz-252{,}000=0 \quad (4)\left[\begin{smallmatrix}-(x\cdot y\cdot z-252{,}000)=0,\text{ or}\\ x\cdot y\cdot z-252{,}000=0\end{smallmatrix}\right]$$

Solving Eq. (2) for x and Eq. (1) for y, we get:

$x=\dfrac{6z}{\lambda z-7}$ and $y=\dfrac{6z}{\lambda z-7}$.

Thus, $x=y$.

Substituting x for y we get the following system:

$$7x+6z-\lambda xz=0$$
$$6x+6x-\lambda xx=0$$
$$xxz-252{,}000=0$$

Which simplifies to:

$$7x+6z-\lambda xz=0 \qquad (5)$$
$$12x-\lambda x^2=0 \qquad (6)$$
$$x^2z-252{,}000=0 \qquad (7)$$

Solving Eq. (6) for x, we get

$$12x-\lambda x^2=0$$
$$x(12-\lambda x)=0$$
$$x=0 \quad\text{or}\quad 12-\lambda x=0$$
$$x=0 \quad\text{or}\quad x=\frac{12}{\lambda}$$

We only consider $x=\dfrac{12}{\lambda}$ since x cannot be 0 in the original problem. We continue by substituting $\dfrac{12}{\lambda}$ for x into Eq. (7) and solving for z.

$$\left(\frac{12}{\lambda}\right)^2 z-252{,}000=0$$
$$\frac{144}{\lambda^2}\cdot z=252{,}000$$
$$z=\frac{252{,}000}{144}\lambda^2$$
$$z=1750\lambda^2$$

Next we substitute $\dfrac{12}{\lambda}$ for x and $1750\lambda^2$ for z in Eq. (5) and solve for λ.

$$7\left(\frac{12}{\lambda}\right)+6\cdot 1750\lambda^2-\lambda\left(\frac{12}{\lambda}\right)1750\lambda^2=0$$
$$\frac{84}{\lambda}+10{,}500\lambda^2-21{,}000\lambda^2=0$$
$$10{,}500\lambda^2=\frac{84}{\lambda}$$
$$\lambda^3=\frac{84}{10{,}500}$$
$$\lambda^3=\frac{1}{125}$$
$$\lambda=\frac{1}{5}$$

Thus,

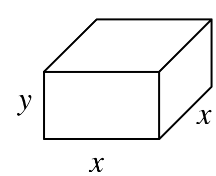
$$x=\frac{12}{\lambda}=\frac{12}{\frac{1}{5}}=12\cdot\frac{5}{1}=60$$
$$y=\frac{12}{\lambda}=\frac{12}{\frac{1}{5}}=12\cdot\frac{5}{1}=60$$
$$z=1750\lambda^2=1750\left(\frac{1}{5}\right)^2=70$$

The minimum total cost subject to the constraint occurs when the dimensions are 60 ft by 60 ft by 70 ft. The minimum cost is found as follows:

$$C(60,60,70)=7\cdot 60\cdot 60+6\cdot 60\cdot 70+6\cdot 60\cdot 70$$
$$=25{,}200+25{,}200+25{,}200$$
$$=75{,}600.$$

The minimum total cost of the building is \$75,600.

24.



Area of top: x^2.
Cost of top: $2x^2$.
Area of sides: $4xy$.
Cost of sides: $2\cdot 4xy=8xy$.
Area of bottom: x^2.
Cost of bottom: $3x^2$.
Total cost:
$C(x,y)=2x^2+8xy+3x^2=5x^2+8xy$
Volume: $x^2y=12$.

We want to minimize
$C(x,y) = 5x^2 + 8xy$
subject to the constraint
$x^2 y - 12 = 0.$
We form the new function F, given by:
$F(x,y,\lambda) = 5x^2 + 8xy - \lambda(x^2 y - 12).$
We find the first partial derivatives:
$F_x = 10x + 8y - 2\lambda xy,$
$F_y = 8x - \lambda x^2,$
$F_\lambda = -(x^2 y - 12).$
We set each derivative equal to 0 and solve the resulting system:

$$10x + 8y - 2\lambda xy = 0 \qquad (1)$$
$$8x - \lambda x^2 = 0 \qquad (2)$$
$$x^2 y - 12 = 0 \qquad (3) \quad \begin{bmatrix} -(x^2y-12)=0, \text{ or} \\ x^2y-12=0 \end{bmatrix}$$

We solve Eq. (2) for x:

$$8x - \lambda x^2 = 0$$
$$x(8 - \lambda x) = 0$$
$$x = 0 \quad \text{or} \quad 8 - \lambda x = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{8}{\lambda}$$

We only consider $x = \frac{8}{\lambda}$, since x cannot be 0 in the original problem.
Substituting $\frac{8}{\lambda}$ for x in Eq. (3), we get:

$$\left(\frac{8}{\lambda}\right)^2 y - 12 = 0$$
$$\frac{64}{\lambda^2} y = 12$$
$$y = \frac{12}{64}\lambda^2$$
$$y = \frac{3}{16}\lambda^2.$$

Substituting $\frac{8}{\lambda}$ for x and $\frac{3}{16}\lambda^2$ for y in Eq. (1) and solving for λ, we get:

$$10\left(\frac{8}{\lambda}\right) + 8\left(\frac{3\lambda^2}{16}\right) - 2\lambda\left(\frac{8}{\lambda}\right)\left(\frac{3\lambda^2}{16}\right) = 0$$
$$\frac{80}{\lambda} - \frac{3\lambda^2}{2} = 0$$
$$160 - 3\lambda^3 = 0$$
$$\lambda^3 = \frac{160}{3}$$
$$\lambda = \sqrt[3]{\frac{160}{3}}$$
$$\lambda \approx 3.764$$

Then,

$$x \approx \frac{8}{3.764} \approx 2.13$$
$$y \approx \frac{3(3.764)^2}{16} \approx 2.66.$$

The dimensions that will minimize the costs are 2.13 ft by 2.13 ft by 2.66 ft.

25. $C(x,y) = C(x) + C(y)$

$$C(x,y) = 10 + \frac{x^2}{6} + 200 + \frac{y^3}{9}$$
$$= 210 + \frac{x^2}{6} + \frac{y^3}{9}$$

We need to minimize
$C(x,y) = 210 + \frac{x^2}{6} + \frac{y^3}{9}$
subject to the constraint
$x + y = 10{,}100.$
We first
express $x + y = 10{,}100$ as $x + y - 10{,}100 = 0.$
We form the new function F, given by:
$F(x,y,\lambda)$
$= 210 + \frac{x^2}{6} + \frac{y^3}{9} - \lambda(x + y - 10{,}100).$
We find the first partial derivatives:
$F_x = \frac{x}{3} - \lambda,$
$F_y = \frac{1}{3}y^2 - \lambda,$
$F_\lambda = -(x + y - 10{,}100).$

We set each derivative equal to 0 and solve the resulting system:

$$\frac{x}{3}-\lambda=0 \qquad (1)$$

$$\frac{1}{3}y^2-\lambda=0 \qquad (2)$$

$$x+y-10{,}100=0 \qquad (3)\ \left[\begin{smallmatrix}-(x+y-10{,}100)=0,\text{ or}\\ x+y-10{,}100=0\end{smallmatrix}\right]$$

From Eq. (1) and Eq. (2) we see:

$x=3\lambda=y^2.$

Thus, $x=y^2$.

Substituting y^2 for x in Eq. (3), we get:

$$y^2+y-10{,}100=0$$
$$(y+101)(y-100)=0$$
$$y+101=0 \quad\text{or}\quad y-100=0$$
$$y=-101 \quad\text{or}\quad y=100.$$

Since y cannot be -101 in the original problem, we only consider $y=100$. If $y=100$, then $x=100^2=10{,}000.$ To minimize total costs, 10,000 units should be made on machine *A* and 100 units should be made on machine *B*.

26. Find the minimum value of

$f(x,y)=xy$

subject to the constraint

$x^2+y^2=9$.

We first express $x^2+y^2=9$ as $x^2+y^2-9=0$.

We form the new function F, given by:

$F(x,y,\lambda)=xy-\lambda(x^2+y^2-9).$

We find the first partial derivatives:

$F_x=y-2x\lambda,$

$F_y=x-2y\lambda,$

$F_\lambda=-(x^2+y^2-9).$

We set each derivative equal to 0 and solve the resulting system:

$$y-2x\lambda=0 \qquad (1)$$

$$x-2y\lambda=0 \qquad (2)$$

$$x^2+y^2-9=0 \qquad (3)\left[\begin{smallmatrix}-(x^2+y^2-9)=0,\text{ or}\\ x^2+y^2-9=0\end{smallmatrix}\right]$$

Solving Eq. (1) for λ, we get:

$$\lambda=\frac{y}{2x}.$$

Substituting into Eq. (2) for λ, we get:

$$x-2\left(\frac{y}{2x}\right)\cdot y=0$$
$$x-\frac{y^2}{x}=0$$
$$x^2=y^2.$$

Substituting x^2 for y^2 in Eq. (3), we get:

$$x^2+(x^2)-9=0$$
$$2x^2=9$$
$$x^2=\frac{9}{2}$$
$$x=\pm\sqrt{\frac{9}{2}}$$

Then,

$$y^2=\left(\pm\sqrt{\frac{9}{2}}\right)^2$$
$$y=\pm\sqrt{\frac{9}{2}}.$$

Thus, there are four pairs that satisfy the constraint:

$\left(\sqrt{\frac{9}{2}},\sqrt{\frac{9}{2}}\right),\left(\sqrt{\frac{9}{2}},-\sqrt{\frac{9}{2}}\right),\left(-\sqrt{\frac{9}{2}},\sqrt{\frac{9}{2}}\right),$

and $\left(-\sqrt{\frac{9}{2}},-\sqrt{\frac{9}{2}}\right)$

Checking each point we see:

$$f\left(\sqrt{\frac{9}{2}},\sqrt{\frac{9}{2}}\right)=\sqrt{\frac{9}{2}}\cdot\sqrt{\frac{9}{2}}=\frac{9}{2}$$

$$f\left(\sqrt{\frac{9}{2}},-\sqrt{\frac{9}{2}}\right)=\sqrt{\frac{9}{2}}\cdot\left(-\sqrt{\frac{9}{2}}\right)=-\frac{9}{2}$$

$$f\left(-\sqrt{\frac{9}{2}},\sqrt{\frac{9}{2}}\right)=\left(-\sqrt{\frac{9}{2}}\right)\cdot\sqrt{\frac{9}{2}}=-\frac{9}{2}$$

$$f\left(-\sqrt{\frac{9}{2}},-\sqrt{\frac{9}{2}}\right)=\left(-\sqrt{\frac{9}{2}}\right)\cdot\left(-\sqrt{\frac{9}{2}}\right)=\frac{9}{2}$$

The minimum value of *f* subject to the constraint occurs at

$\left(\sqrt{\frac{9}{2}},-\sqrt{\frac{9}{2}}\right)$ or $\left(-\sqrt{\frac{9}{2}},\sqrt{\frac{9}{2}}\right)$ The minimum value is $-\frac{9}{2}$.

27. Find the minimum value of
$f(x,y)=2x^2+y^2+2xy+3x+2y$
subject to the constraint
$y^2=x+1.$
We first express $y^2=x+1$ as $y^2-x-1=0$.
We form the new function F, given by:
$F(x,y,\lambda)$
$=2x^2+y^2+2xy+3x+2y-\lambda\left(y^2-x-1\right).$
We find the first partial derivatives:
$F_x=4x+2y+3+\lambda,$
$F_y=2y+2x+2-2\lambda y,$
$F_\lambda=-\left(y^2-x-1\right).$
We set each derivative equal to 0 and solve the resulting system:
$$4x+2y+3+\lambda=0 \qquad (1)$$
$$2y+2x+2-2\lambda y=0 \qquad (2)$$
$$y^2-x-1=0 \qquad (3)\left[\begin{array}{l}-\left(y^2-x-1\right)=0,\text{ or}\\ y^2-x-1=0\end{array}\right]$$
Solving Eq. (1) for λ, we get:
$$4x+2y+3+\lambda=0$$
$$\lambda=-4x-2y-3. \qquad (4)$$
Solving Eq. (2) for λ, we get:
$$2y+2x+2-2\lambda y=0$$
$$2\lambda y=2y+2x+2$$
$$\lambda=\frac{2y+2x+2}{2y}$$
$$\lambda=\frac{y+x+1}{y}. \qquad (5)$$
Setting Eq. (4) equal to Eq. (5) and solving for x we have:
$$-4x-2y-3=\frac{y+x+1}{y}$$
$$-4xy-2y^2-3y=y+x+1$$
$$-2y^2-3y-y-1=x+4xy$$
$$-2y^2-4y-1=x(1+4y)$$
$$\frac{-2y^2-4y-1}{1+4y}=x. \qquad (6)$$
Solving Eq. (3) for x we have:
$$y^2-x-1=0$$
$$y^2-1=x \qquad (7)$$
Substituting Eq. (7) into Eq. (6), we have:
$$\frac{-2y^2-4y-1}{1+4y}=y^2-1$$
$$-2y^2-4y-1=\left(y^2-1\right)(1+4y)$$
$$-2y^2-4y-1=y^2+4y^3-1-4y$$
$$0=4y^3+3y^2$$
$$0=y^2(4y+3)$$
$$y^2=0 \quad\text{or}\quad 4y+3=0$$
$$y=0 \quad\text{or}\quad y=-\frac{3}{4}$$
Using equation (7) When $y=0$,
$$x=(0)^2-1$$
$$=-1$$
$$f(-1,0)=2(-1)^2+(0)^2+2(-1)(0)+3(-1)+2(0)$$
$$=2-3$$
$$=-1.$$
Using Eq. (7), when $y=-\frac{3}{4}$,
$$x=\left(\frac{-3}{4}\right)^2-1$$
$$=\frac{9}{16}-1$$
$$=-\frac{7}{16}.$$
$$f\left(-\frac{7}{16},-\frac{3}{4}\right)=2\left(-\frac{7}{16}\right)^2+\left(-\frac{3}{4}\right)^2+2\left(-\frac{7}{16}\right)\left(-\frac{3}{4}\right)+3\left(-\frac{7}{16}\right)+2\left(-\frac{3}{4}\right)$$
$$=\frac{49}{128}+\frac{9}{16}+\frac{21}{32}-\frac{21}{16}-\frac{3}{2}$$
$$=-\frac{155}{128}.$$
The minimum value of f subject to the constraint occurs at $\left(-\frac{7}{16},-\frac{3}{4}\right)$ and is $f\left(-\frac{7}{16},-\frac{3}{4}\right)=-\frac{155}{128}$.

28. Find the maximum value of
$f(x,y,z) = x+y+z$
subject to the constraint
$x^2+y^2+z^2=1$.
We first express $x^2+y^2+z^2=1$
as $x^2+y^2+z^2-1=0$.
We form the new function F, given by:
$F(x,y,z,\lambda)$
$= x+y+z-\lambda(x^2+y^2+z^2-1)$
We find the first partial derivatives:
$F_x = 1-2\lambda x,$
$F_y = 1-2\lambda y,$
$F_z = 1-2\lambda z,$
$F_\lambda = -(x^2+y^2+z^2-1).$
We set each derivative equal to 0 and solve the resulting system:

$$1-2\lambda x = 0 \quad (1)$$
$$1-2\lambda y = 0 \quad (2)$$
$$1-2\lambda z = 0 \quad (3)$$
$$x^2+y^2+z^2-1=0 \quad (4)\left[\begin{array}{l}-(x^2+y^2+z^2-1)=0, \text{ or}\\ x^2+y^2+z^2-1=0\end{array}\right]$$

From Eq. (1), Eq. (2), and Eq. (3) we see:
$$\frac{1}{2\lambda} = x = y = z.$$
Substituting x for y and z in Eq. (4), we have:
$$x^2+(x)^2+(x)^2-1=0$$
$$3x^2=1$$
$$x^2=\frac{1}{3}$$
$$x=\pm\sqrt{\frac{1}{3}}=\pm\frac{1}{\sqrt{3}}$$
Since $x=y=z$ it follows that
When $x=\frac{1}{\sqrt{3}}$
$y=\frac{1}{\sqrt{3}}$ and $z=\frac{1}{\sqrt{3}}$.
When $x=-\frac{1}{\sqrt{3}}$
$y=-\frac{1}{\sqrt{3}}$ and $z=-\frac{1}{\sqrt{3}}$.

However, it is clear looking at the function that the point that will yield a maximum value is:
$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.
The maximum value is found as follows:
$$f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{3}.$$

29. Find the maximum value of
$f(x,y,z)=x^2y^2z^2$
subject to the constraint
$x^2+y^2+z^2=2$.
We first express $x^2+y^2+z^2=2$
as $x^2+y^2+z^2-2=0$.
We form the new function F, given by:
$F(x,y,z,\lambda)$
$= x^2y^2z^2-\lambda(x^2+y^2+z^2-2)$
We find the first partial derivatives:
$F_x = 2xy^2z^2-2\lambda x$
$F_y = 2x^2yz^2-2\lambda y,$
$F_z = 2x^2y^2z-2\lambda z,$
$F_\lambda = -(x^2+y^2+z^2-2).$
We set each derivative equal to 0 and solve the resulting system:

$$2xy^2z^2-2\lambda x=0$$
$$2x^2yz^2-2\lambda y=0$$
$$2x^2y^2z-2\lambda z=0$$
$$x^2+y^2+z^2-2=0 \quad \left[\begin{array}{l}-(x^2+y^2+z^2-2)=0, \text{ or}\\ x^2+y^2+z^2-2=0\end{array}\right]$$

Rewriting the system we get:
$$x(2y^2z^2-2\lambda)=0 \quad (1)$$
$$y(2x^2z^2-2\lambda)=0 \quad (2)$$
$$z(2x^2y^2-2\lambda)=0 \quad (3)$$
$$x^2+y^2+z^2-2=0 \quad (4)\left[\begin{array}{l}-(x^2+y^2+z^2-2)=0, \text{ or}\\ x^2+y^2+z^2-2=0\end{array}\right]$$

Note that for
$x=0, y=0,$ or $z=0, f(x,y,z)=0.$ For all values of $x, y,$ and $z\neq 0, f(x,y,z)>0.$ Thus the maximum value of f cannot occur when any or all of the variables is 0. Thus we will only consider nonzero values of x, y, and z.

Using the Principle of Zero Products, we get:

From Eq. (1)	From Eq. (2)	From Eq. (3)
$y^2z^2-\lambda=0$	$x^2z^2-\lambda=0$	$x^2y^2-\lambda=0$
$y^2z^2=\lambda$	$x^2z^2=\lambda$	$x^2y^2=\lambda$

Thus, $y^2z^2=x^2z^2=x^2y^2$ and $x^2=y^2=z^2$.

Substituting x^2 for y^2 and z^2 in Eq. (4), we have:

$$x^2+x^2+x^2-2=0$$
$$3x^2=2$$
$$x^2=\frac{2}{3}$$
$$x=\pm\sqrt{\frac{2}{3}}$$

Since $x^2=y^2=z^2$ it follows that

$$y^2=\frac{2}{3} \quad\text{and}\quad z^2=\frac{2}{3}$$

$$y=\pm\sqrt{\frac{2}{3}} \quad\text{and}\quad z=\pm\sqrt{\frac{2}{3}}.$$

For $\left(\pm\sqrt{\frac{2}{3}},\pm\sqrt{\frac{2}{3}},\pm\sqrt{\frac{2}{3}}\right)$:

$$f(x,y,z)=\left(\pm\sqrt{\frac{2}{3}}\right)^2\left(\pm\sqrt{\frac{2}{3}}\right)^2\left(\pm\sqrt{\frac{2}{3}}\right)^2$$
$$=\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}$$
$$=\frac{8}{27}$$

Thus $f(x,y,z)$ has a maximum value of $\frac{8}{27}$ at $\left(\pm\sqrt{\frac{2}{3}},\pm\sqrt{\frac{2}{3}},\pm\sqrt{\frac{2}{3}}\right)$.

30. Find the maximum value of

$f(x,y,z)=x+2y-2z$

subject to the constraint

$x^2+y^2+z^2=4$.

We first express $x^2+y^2+z^2=4$ as $x^2+y^2+z^2-4=0$.

We form the new function F, given by:

$$F(x,y,z,\lambda)$$
$$=x+2y-2z-\lambda\left(x^2+y^2+z^2-4\right)$$

We find the first partial derivatives:

$$F_x=1-2\lambda x,$$
$$F_y=2-2\lambda y,$$
$$F_z=-2-2\lambda z,$$
$$F_\lambda=-\left(x^2+y^2+z^2-4\right).$$

We set each derivative equal to 0 and solve the resulting system:

$$1-2\lambda x=0 \quad (1)$$
$$2-2\lambda y=0 \quad (2)$$
$$-2-2\lambda z=0 \quad (3)$$
$$x^2+y^2+z^2-4=0 \quad (4)\left[\begin{array}{l}-\left(x^2+y^2+z^2-4\right)=0,\text{ or}\\ x^2+y^2+z^2-4=0\end{array}\right]$$

From Eq. (1), Eq. (2), and Eq. (3) we see:

$x=\frac{1}{2\lambda}, y=\frac{1}{\lambda},$ and $z=-\frac{1}{\lambda}$. Then $y=2x$ and $z=-2x$. Substitute these values into Eq. (4) and solve for x:

$$x^2+(2x)^2+(-2x)^2-4=0$$
$$x^2+4x^2+4x^2=4$$
$$9x^2=4$$
$$x^2=\frac{4}{9}$$
$$x=\pm\sqrt{\frac{4}{9}}=\pm\frac{2}{3}.$$

When $x=\frac{2}{3}$

$y=2\cdot\frac{2}{3}=\frac{4}{3}$ and $z=-2\cdot\frac{2}{3}=-\frac{4}{3}$.

When $x=-\frac{2}{3}$

$y=2\cdot\left(-\frac{2}{3}\right)=-\frac{4}{3}$ and $z=-2\cdot\left(-\frac{2}{3}\right)=\frac{4}{3}$.

We evaluate the function at each of the possibilities to find the maximum.

$$f\left(\frac{2}{3},\frac{4}{3},-\frac{4}{3}\right)=\frac{2}{3}+2\left(\frac{4}{3}\right)-2\left(-\frac{4}{3}\right)$$
$$=\frac{2}{3}+\frac{8}{3}+\frac{8}{3}$$
$$=6 \qquad \text{Maximum.}$$

$$f\left(-\frac{2}{3},-\frac{4}{3},\frac{4}{3}\right)=-\frac{2}{3}+2\left(-\frac{4}{3}\right)-2\left(\frac{4}{3}\right)$$
$$=-\frac{2}{3}-\frac{8}{3}-\frac{8}{3}$$
$$=-6 \qquad \text{Minimum.}$$

Therefore, $f(x,y,z)$ has a maximum value of 6 at $\left(\frac{2}{3},\frac{4}{3},-\frac{4}{3}\right)$.

31. Find the maximum value of
$f(x,y,z,t)=x+y+z+t$
subject to the constraint
$x^2+y^2+z^2+t^2=1$.

We first express $x^2+y^2+z^2+t^2=1$ as $x^2+y^2+z^2+t^2-1=0$.

We form the new function F, given by:
$F(x,y,z,t,\lambda)$
$=x+y+z+t-\lambda\left(x^2+y^2+z^2+t^2-1\right)$

We find the first partial derivatives:
$F_x=1-2\lambda x,$
$F_y=1-2\lambda y,$
$F_z=1-2\lambda z,$
$F_t=1-2\lambda t,$
$F_\lambda=-\left(x^2+y^2+z^2+t^2-1\right).$

We set each derivative equal to 0 and solve the resulting system:

$$1-2\lambda x=0 \quad (1)$$
$$1-2\lambda y=0 \quad (2)$$
$$1-2\lambda z=0 \quad (3)$$
$$1-2\lambda t=0 \quad (4)$$
$$x^2+y^2+z^2+t^2-1=0 \quad (5)$$
$$\left[\begin{array}{l}-\left(x^2+y^2+z^2+t^2-1\right)=0, \text{ or}\\ x^2+y^2+z^2+t^2-1=0\end{array}\right]$$

From Eq. (1), Eq. (2), Eq. (3), and Eq. (4), we see:
$$\frac{1}{2\lambda}=x=y=z=t.$$

Substituting x for y, z, and t in Eq. (5), we have:

$$x^2+(x)^2+(x)^2+(x)^2-1=0$$
$$4x^2=1$$
$$x^2=\frac{1}{4}$$
$$x=\pm\frac{1}{2}$$

Since $x=y=z=t$ it follows that

When $x=\frac{1}{2}$

$y=\frac{1}{2}$ and $z=\frac{1}{2}$ and $t=\frac{1}{2}$.

When $x=-\frac{1}{2}$

$y=-\frac{1}{2}$ and $z=-\frac{1}{2}$ and $t=-\frac{1}{2}$.

However, it is clear looking at the function that the point that will yield a maximum value is:
$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

The maximum value is found as follows:
$$f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=2.$$

32. Find the minimum value of
$f(x,y,z)=x^2+y^2+z^2$
subject to the constraint
$x-2y+5z=1$.

We first express $x-2y+5z=1$ as $x-2y+5z-1=0$.

We form the new function F, given by:
$F(x,y,z,\lambda)$
$=x^2+y^2+z^2-\lambda(x-2y+5z-1)$

We find the first partial derivatives:
$F_x=2x-\lambda,$
$F_y=2y+2\lambda,$
$F_z=2z-5\lambda,$
$F_\lambda=-(x-2y+5z-1).$

We set each derivative equal to 0 and solve the resulting system:

$$2x-\lambda=0 \quad (1)$$
$$2y+2\lambda=0 \quad (2)$$
$$2z-5\lambda=0 \quad (3)$$
$$x-2y+5z-1=0 \quad (4)\left[\begin{array}{l}-(x-2y+5z-1)=0, \text{ or}\\ x-2y+5z-1=0\end{array}\right]$$

From Eq. (1), Eq. (2) and Eq. (3), we see that:

$x=\frac{\lambda}{2}, y=-\lambda,$ and $z=\frac{5\lambda}{2}$. Then,

$y=-2x$ and $z=5x$.

Substituting $-2x$ for y, and $5x$ for z into Eq. (4), we get:

$$x-2(-2x)+5(5x)-1=0$$
$$x+4x+25x=1$$
$$30x=1$$
$$x=\frac{1}{30}.$$

Then,

$$x=\frac{1}{30}$$
$$y=-2\left(\frac{1}{30}\right)=-\frac{1}{15}$$
$$z=5\left(\frac{1}{30}\right)=\frac{1}{6}.$$

The minimum value of f subject to the constraint occurs at $\left(\frac{1}{30},-\frac{1}{15},\frac{1}{6}\right)$ and is

$$f\left(\frac{1}{30},-\frac{1}{15},\frac{1}{6}\right)=\left(\frac{1}{30}\right)^2+\left(-\frac{1}{15}\right)^2+\left(\frac{1}{6}\right)^2$$
$$=\frac{1}{900}+\frac{1}{225}+\frac{1}{36}$$
$$=\frac{1}{30}.$$

33. We want to maximize

$p(x,y)$

subject to the constraint.

$B=c_1x+c_2y.$

We first express $B=c_1x+c_2y$ as

$c_1x+c_2y-B=0$

Then we form the new function P, given by:

$P(x,y,\lambda)=p(x,y)-\lambda(c_1x+c_2y-B)$.

We find the first partial derivatives.

$P_x=p_x-\lambda c_1$

$P_y=p_y-\lambda c_2$.

We set these derivatives equal to 0 and solve for λ.

$$p_x-\lambda c_1=0 \qquad p_y-\lambda c_2=0$$
$$p_x=\lambda c_1 \qquad p_y=\lambda c_2$$
$$\frac{p_x}{c_1}=\lambda \qquad \frac{p_y}{c_2}=\lambda$$

Thus, $\lambda=\frac{p_x}{c_1}=\frac{p_y}{c_2}$.

34. Find the maximum value of

$p(x,y)=800x^{3/4}y^{1/4}$

subject to the constraint

$x+y=1{,}000{,}000.$

We first express $x+y=1{,}000{,}000$ as

$x+y-1{,}000{,}000=0.$

We form the new function F, given by:

$F(x,y,\lambda)$

$=800x^{3/4}y^{1/4}-\lambda(x+y-1{,}000{,}000.).$

We find the first partial derivatives:

$$F_x=800\cdot\frac{3}{4}x^{-1/4}y^{1/4}-\lambda$$
$$=600x^{-1/4}y^{1/4}-\lambda,$$
$$F_y=800\cdot\frac{1}{4}x^{3/4}y^{-3/4}-\lambda$$
$$=200x^{3/4}y^{-3/4}-\lambda,$$
$$F_\lambda=-(x+y-1{,}000{,}000).$$

We set each derivative equal to 0 and solve the resulting system:

$$600x^{-1/4}y^{1/4}-\lambda=0 \qquad (1)$$
$$200x^{3/4}y^{-3/4}-\lambda=0 \qquad (2)$$
$$x+y-1{,}000{,}000=0 \qquad (3)$$
$$\left[\begin{array}{l}-(x+y-1{,}000{,}000)=0,\text{ or}\\ x+y-1{,}000{,}000=0\end{array}\right]$$

From Eq. (1) we see:

$$\lambda=600\left(\frac{y}{x}\right)^{1/4}$$

Substituting for λ in Eq. (2),

$$200\left(\frac{x}{y}\right)^{3/4}-600\left(\frac{y}{x}\right)^{1/4}=0$$
$$200\left(\frac{x}{y}\right)^{3/4}=600\left(\frac{y}{x}\right)^{1/4}$$
$$200x=600y$$
$$x=3y.$$

Substituting $3y$ for x in Eq. (3), we get:

$$3y + y - 1{,}000{,}000 = 0$$
$$4y = 1{,}000{,}000$$
$$y = 250{,}000.$$

Then $x = 2(250{,}000) = 750{,}000.$

In order to maximize production, the company should allocate \$750,000 for labor and \$250,000 for capital.

35. $\boxed{tw}$ The method in Section 6.3 is used to find relative maximum and minimum values of a function of two variables whereas the method of Lagrange multipliers is used to find the maximum and minimum values of a function of two or more variables subject to a constraint on the variables. The difference is seen graphically in the figures on pages 557,558, and 574 in the text book.

36. $\boxed{tw}$ The French mathematician and astronomer Joseph Louis Lagrange (1736-1813) was born and educated in Turin, Italy. He was appointed professor of geometry at the Turin military academy at the age of nineteen. In 1766 he was appointed director of the Berlin Academy of Sciences and twenty years later went to Paris at the invitation of Louis XVI. During the French Revolution he was in charge of the commission for establishing a new system of weights and measures, the metric system. Lagrange was one of the greatest mathematicians of the eighteenth century. He created the calculus of variations, systematized the field of differential equations, and worked on the theory of numbers. Among his accomplishments in astronomy were calculations of the libration of the moon and motions of the planets.

37 – 44. Left to the student.

Exercise Set 6.6

1. $\int_0^3 \int_0^1 2y\,dx\,dy$

$$= \int_0^3 \left(\int_0^1 2y\,dx \right) dy$$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_0^1 2y\,dx = 2y[x]_0^1$$
$$= 2y[1-0]$$
$$= 2y.$$

Then we evaluate the outside y-integral:

$$\int_0^3 \left(\int_0^1 2y\,dx \right) dy = \int_0^3 2y\,dy \qquad \left(\int_0^1 2y\,dx = 2y \right)$$
$$= \left[y^2 \right]_0^3$$
$$= 3^2 - 0^2$$
$$= 9.$$

2. $\int_0^1 \int_0^4 3x\,dx\,dy$

$$= \int_0^1 \left(\int_0^4 3x\,dx \right) dy$$

We first evaluate the inside x-integral:

$$\int_0^4 3x\,dx = \left[\frac{3x^2}{2} \right]_0^4$$
$$= \frac{3}{2}\left[4^2 - 0^2 \right]$$
$$= \frac{3}{2}[16] = 24.$$

Then we evaluate the outside y-integral:

$$\int_0^1 \left(\int_0^4 3x\,dx \right) dy = \int_0^1 24\,dy \qquad \left(\int_0^4 3x\,dx = 24 \right)$$
$$= [24y]_0^1$$
$$= 24 \cdot 1 - 24 \cdot 0$$
$$= 24.$$

3. $\int_{-1}^{3}\int_{1}^{2} x^2 y\,dy\,dx$

$= \int_{-1}^{3}\left(\int_{1}^{2} x^2 y\,dy\right)dx$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_{1}^{2} x^2 y\,dy = x^2\left[\frac{1}{2}y^2\right]_1^2$$
$$= x^2\left[\frac{1}{2}(2)^2 - \frac{1}{2}(1)^2\right]$$
$$= x^2\left[2 - \frac{1}{2}\right]$$
$$= \frac{3}{2}x^2.$$

Then we evaluate the outside x-integral:

$$\int_{-1}^{3}\left(\int_{1}^{2} x^2 y\,dy\right)dx = \int_{-1}^{3}\frac{3}{2}x^2\,dx \quad \left(\int_{1}^{2} x^2 y\,dy = \tfrac{3}{2}x^2\right)$$
$$= \left[\frac{1}{2}x^3\right]_{-1}^{3}$$
$$= \frac{1}{2}\left[3^3 - (-1)^3\right]$$
$$= \frac{1}{2}\left[27 - (-1)\right]$$
$$= \frac{1}{2}[28]$$
$$= 14.$$

4. $\int_{1}^{4}\int_{-2}^{1} x^3 y\,dy\,dx$

$= \int_{1}^{4}\left(\int_{-2}^{1} x^3 y\,dy\right)dx$

We first evaluate the inside y-integral:

$$\int_{-2}^{1} x^3 y\,dy = x^3\left[\frac{y^2}{2}\right]_{-2}^{1}$$
$$= x^3\left[\frac{1^2}{2} - \frac{(-2)^2}{2}\right]$$
$$= x^3\left[\frac{1}{2} - 2\right]$$
$$= -\frac{3}{2}x^3.$$

Then we evaluate the outside y-integral:

$$\int_{1}^{4}\left(\int_{-2}^{1} x^3 y\,dy\right)dx$$
$$= \int_{1}^{4} -\frac{3}{2}x^3\,dx \qquad \left(\int_{-2}^{1} x^3 y\,dy = -\frac{3}{2}x^3\right)$$
$$= \left[-\frac{3}{8}x^4\right]_1^4$$
$$= -\frac{3}{8}\left[4^4 - 1^4\right]$$
$$= -\frac{3}{8}(256 - 1)$$
$$= -\frac{765}{8}.$$

5. $\int_{0}^{5}\int_{-2}^{-1}(3x + y)\,dx\,dy$

$= \int_{0}^{5}\left(\int_{-2}^{-1}(3x + y)\,dx\right)dy$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_{-2}^{-1}(3x + y)\,dx$$
$$= \left[\frac{3}{2}x^2 + yx\right]_{-2}^{-1}$$
$$= \left[\frac{3}{2}(-1)^2 + y(-1)\right] - \left[\frac{3}{2}(-2)^2 + y(-2)\right]$$
$$= \left[\frac{3}{2} - y\right] - \left[\frac{3}{2}\cdot 4 - 2y\right]$$
$$= y - \frac{9}{2}.$$

Then we evaluate the outside y-integral:

$$\int_{0}^{5}\left(\int_{-2}^{-1}(3x + y)\,dx\right)dy$$
$$= \int_{0}^{5}\left(y - \frac{9}{2}\right)dy \qquad \left(\int_{-2}^{-1}(3x + y)\,dx = y - \tfrac{9}{2}\right)$$
$$= \left[\frac{1}{2}y^2 - \frac{9}{2}y\right]_0^5$$
$$= \left[\frac{1}{2}(5)^2 - \frac{9}{2}(5)\right] - \left[\frac{1}{2}(0)^2 - \frac{9}{2}(0)\right]$$
$$= \frac{25}{2} - \frac{45}{2} - 0$$
$$= \frac{-20}{2}$$
$$= -10.$$

6. $\int_{-4}^{-1}\int_{1}^{3}(x+5y)\,dx\,dy$

$=\int_{-4}^{-1}\left(\int_{1}^{3}(x+5y)\,dx\right)dy$

We first evaluate the inside x-integral, treating y as a constant:

$\int_{1}^{3}(x+5y)\,dx$

$=\left[\frac{1}{2}x^2+5yx\right]_1^3$

$=\left[\frac{1}{2}(3)^2+5y(3)\right]-\left[\frac{1}{2}(1)^2+5y(1)\right]$

$=\frac{9}{2}+15y-\frac{1}{2}-5y$

$=10y+4.$

Then we evaluate the outside y-integral:

$\int_{-4}^{-1}\left(\int_{1}^{3}(x+5y)\,dx\right)dy$

$=\int_{-4}^{-1}(10y+4)\,dy \qquad \left(\int_1^3(x+5y)\,dx=10y+4\right)$

$=\left[5y^2+4y\right]_{-4}^{-1}$

$=\left[5(-1)^2+4(-1)\right]-\left[5(-4)^2+4(-4)\right]$

$=(5-4)-\left[5(16)-16\right]$

$=1-80+16$

$=-63.$

7. $\int_{-1}^{1}\int_{x}^{1}xy\,dy\,dx$

$=\int_{-1}^{1}\left(\int_{x}^{1}xy\,dy\right)dx$

We first evaluate the inside y-integral, treating x as a constant:

$\int_x^1 xy\,dy = x\left[\frac{1}{2}y^2\right]_x^1$

$=x\left[\frac{1}{2}(1)^2-\frac{1}{2}(x)^2\right]$

$=x\left[\frac{1}{2}-\frac{1}{2}x^2\right]$

$=\frac{1}{2}x-\frac{1}{2}x^3.$

Then we evaluate the outside x-integral:

$\int_{-1}^{1}\left(\int_{x}^{1}xy\,dy\right)dx$

$=\int_{-1}^{1}\left(\frac{1}{2}x-\frac{1}{2}x^3\right)dx \qquad \left(\int_x^1 xy\,dy=\frac{1}{2}x-\frac{1}{2}x^3\right)$

$=\left[\frac{1}{4}x^2-\frac{1}{8}x^4\right]_{-1}^{1}$

$=\left[\frac{1}{4}(1)^2-\frac{1}{8}(1)^4\right]-\left[\frac{1}{4}(-1)^2-\frac{1}{8}(-1)^4\right]$

$=\left[\frac{1}{4}-\frac{1}{8}\right]-\left[\frac{1}{4}-\frac{1}{8}\right]$

$=0.$

8. $\int_{-1}^{1}\int_{x}^{2}(x+y)\,dy\,dx$

$=\int_{-1}^{1}\left(\int_{x}^{2}(x+y)\,dy\right)dx$

We first evaluate the inside y-integral, treating x as a constant:

$\int_x^2(x+y)\,dy$

$=\left[xy+\frac{1}{2}y^2\right]_x^2$

$=\left[x(2)+\frac{1}{2}(2)^2\right]-\left[x(x)+\frac{1}{2}(x)^2\right]$

$=\left[2x+2\right]-\left[x^2+\frac{1}{2}x^2\right]$

$=-\frac{3}{2}x^2+2x+2.$

Then we evaluate the outside x-integral:

$\int_{-1}^{1}\left(\int_{x}^{2}(x+y)\,dy\right)dx$

$=\int_{-1}^{1}\left(-\frac{3}{2}x^2+2x+2\right)dx$

$\left(\int_x^2(x+y)\,dy=-\frac{3}{2}x^2+2x+2\right)$

$=\left[-\frac{1}{2}x^3+x^2+2x\right]_{-1}^{1}$

$=\left[-\frac{1}{2}(1)^3+(1)^2+2(1)\right]-$

$\left[-\frac{1}{2}(-1)^3+(-1)^2+2(-1)\right]$

$=\left[-\frac{1}{2}+1+2\right]-\left[\frac{1}{2}+1-2\right]$

$=-1+4$

$=3.$

9. $\int_0^1 \int_{x^2}^x (x+y)\,dy\,dx$

$= \int_0^1 \left(\int_{x^2}^x (x+y)\,dy \right) dx$

We first evaluate the inside y-integral, treating x as a constant:

$\int_{x^2}^x (x+y)\,dy$

$= \left[xy + \frac{1}{2}y^2 \right]_{x^2}^x$

$= \left[x(x) + \frac{1}{2}(x)^2 \right] - \left[x(x^2) + \frac{1}{2}(x^2)^2 \right]$

$= \left[x^2 + \frac{1}{2}x^2 \right] - \left[x^3 + \frac{1}{2}x^4 \right]$

$= -\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2.$

Then we evaluate the outside x-integral:

$\int_0^1 \left(\int_{x^2}^x (x+y)\,dy \right) dx$

$= \int_0^1 \left(-\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 \right) dx$

$\left(\int_{x^2}^x (x+y)\,dy = -\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 \right)$

$= \left[-\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{1}{2}x^3 \right]_0^1$

$= \left[-\frac{1}{10}(1)^5 - \frac{1}{4}(1)^4 + \frac{1}{2}(1)^3 \right] - \left[-\frac{1}{10}(0)^5 - \frac{1}{4}(0)^4 + \frac{1}{2}(0)^3 \right]$

$= \left[-\frac{1}{10} - \frac{1}{4} + \frac{1}{2} \right] - [0]$

$= \frac{3}{20}.$

10. $\int_0^2 \int_0^x e^{x+y}\,dy\,dx$

$= \int_0^2 \left(\int_0^x e^{x+y}\,dy \right) dx$

We first evaluate the inside y-integral, treating x as a constant:

$\int_0^x e^{x+y}\,dy$

$= \left[e^{x+y} \right]_0^x$

$= e^{x+x} - e^{x+0}$

$= e^{2x} - e^x$

Then we evaluate the outside x-integral:

$\int_0^2 \left(\int_0^x e^{x+y}\,dy \right) dx$

$= \int_0^2 \left(e^{2x} - e^x \right) dx \qquad \left(\int_0^x e^{x+y}\,dy = e^{2x} - e^x \right)$

$= \left[\frac{1}{2}e^{2x} - e^x \right]_0^2$

$= \left[\frac{1}{2}e^{2(2)} - e^2 \right] - \left[\frac{1}{2}e^{2(0)} - e^{(0)} \right]$

$= \left[\frac{1}{2}e^4 - e^2 \right] - \left[\frac{1}{2} - 1 \right]$

$= \frac{1}{2}e^4 - e^2 + \frac{1}{2}.$

11. $\int_0^1 \int_1^{e^x} \frac{1}{y}\,dy\,dx$

$= \int_0^1 \left(\int_1^{e^x} \frac{1}{y}\,dy \right) dx$

We first evaluate the inside y-integral, treating x as a constant:

$\int_1^{e^x} \frac{1}{y}\,dy = [\ln y]_1^{e^x}$

$= \ln e^x - \ln 1$

$= x.$

Then we evaluate the outside x-integral:

$\int_0^1 \left(\int_1^{e^x} \frac{1}{y}\,dy \right) dx$

$= \int_0^1 x\,dx \qquad \left(\int_1^{e^x} \frac{1}{y}\,dy = x \right)$

$= \left[\frac{1}{2}x^2 \right]_0^1$

$= \left[\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 \right]$

$= \frac{1}{2}.$

12. $\int_0^1 \int_{-1}^x \left(x^2 + y^2 \right) dy\,dx$

$= \int_0^1 \left(\int_{-1}^x \left(x^2 + y^2 \right) dy \right) dx$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_{-1}^{x}\left(x^2+y^2\right)dy$$
$$=\left[x^2y+\frac{1}{3}y^3\right]_{-1}^{x}$$
$$=\left[x^2(x)+\frac{1}{3}(x)^3\right]-\left[x^2(-1)+\frac{1}{3}(-1)^3\right]$$
$$=\left[x^3+\frac{1}{3}x^3\right]-\left[-x^2-\frac{1}{3}\right]$$
$$=\frac{4}{3}x^3+x^2+\frac{1}{3}.$$

Then we evaluate the outside x-integral:

$$\int_0^1\left(\int_{-1}^{x}\left(x^2+y^2\right)dy\right)dx$$
$$=\int_0^1\left(\frac{4}{3}x^3+x^2+\frac{1}{3}\right)dx$$
$$\left(\int_{-1}^{x}\left(x^2+y^2\right)dy=\frac{4}{3}x^3+x^2+\frac{1}{3}\right)$$
$$=\left[\frac{1}{3}x^4+\frac{1}{3}x^3+\frac{1}{3}x\right]_0^1$$
$$=\left[\frac{1}{3}(1)^4+\frac{1}{3}(1)^3+\frac{1}{3}(1)\right]-\left[\frac{1}{3}(0)^4+\frac{1}{3}(0)^3+\frac{1}{3}(0)\right]$$
$$=\left[\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right]-[0]$$
$$=1.$$

13. $\int_0^2\int_0^x\left(x+y^2\right)dy\,dx$

$$=\int_0^2\left(\int_0^x\left(x+y^2\right)dy\right)dx$$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_0^x\left(x+y^2\right)dy$$
$$=\left[xy+\frac{1}{3}y^3\right]_0^x$$
$$=\left[x(x)+\frac{1}{3}(x)^3\right]-\left[x(0)+\frac{1}{3}(0)^3\right]$$
$$=\left[x^2+\frac{1}{3}x^3\right]-[0]$$
$$=\frac{1}{3}x^3+x^2$$

Then we evaluate the outside x-integral:

$$\int_0^2\left(\int_0^x\left(x+y^2\right)dy\right)dx$$
$$=\int_0^2\left(\frac{1}{3}x^3+x^2\right)dx$$
$$\left(\int_0^x\left(x+y^2\right)dy=\frac{1}{3}x^3+x^2\right)$$
$$=\left[\frac{1}{12}x^4+\frac{1}{3}x^3\right]_0^2$$
$$=\left[\frac{1}{12}(2)^4+\frac{1}{3}(2)^3\right]-\left[\frac{1}{12}(0)^4+\frac{1}{3}(0)^3\right]$$
$$=\left[\frac{16}{12}+\frac{8}{3}\right]-[0]$$
$$=\frac{4}{3}+\frac{8}{3}$$
$$=4.$$

14. $\int_1^3\int_0^x 2e^{x^2}\,dy\,dx$

$$=\int_1^3\left(\int_0^x 2e^{x^2}\,dy\right)dx$$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_0^x 2e^{x^2}\,dy$$
$$=2e^{x^2}\left[y\right]_0^x$$
$$=2e^{x^2}\left[x-0\right]$$
$$=2xe^{x^2}.$$

Then we evaluate the outside x-integral:

$$\int_1^3\left(\int_0^x 2e^{x^2}\,dy\right)dx$$
$$=\int_1^3\left(2xe^{x^2}\right)dx \qquad \left(\int_0^x 2e^{x^2}\,dy=2xe^{x^2}\right)$$
$$=\left[e^{x^2}\right]_1^3$$
$$=\left[e^{(3)^2}-e^{(1)^2}\right]$$
$$=e^9-e.$$

15. $\int_0^1\int_0^{1-x^2}\left(1-y-x^2\right)dy\,dx$

$$=\int_0^1\left(\int_0^{1-x^2}\left(1-y-x^2\right)dy\right)dx$$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_0^{1-x^2}\left(1-y-x^2\right)dy$$
$$=\left[y-\frac{1}{2}y^2-x^2y\right]_0^{1-x^2}$$
$$=\left[\left(1-x^2\right)-\frac{1}{2}\left(1-x^2\right)^2-x^2\left(1-x^2\right)\right]-\left[(0)-\frac{1}{2}(0)^2-x^2(0)\right]$$
$$=1-x^2-\frac{1}{2}\left(1-2x^2+x^4\right)-x^2+x^4$$
$$=1-x^2-\frac{1}{2}+x^2-\frac{1}{2}x^4-x^2+x^4$$
$$=\frac{1}{2}x^4-x^2+\frac{1}{2}$$

Then we evaluate the outside x-integral:

$$=\int_0^1\left(\int_0^{1-x^2}\left(1-y-x^2\right)dy\right)dx$$
$$=\int_0^1\left(\frac{1}{2}x^4-x^2+\frac{1}{2}\right)dx$$
$$=\left[\frac{1}{10}x^5-\frac{1}{3}x^3+\frac{1}{2}x\right]_0^1$$
$$=\left[\frac{1}{10}(1)^5-\frac{1}{3}(1)^3+\frac{1}{2}(1)\right]-\left[\frac{1}{10}(0)^5-\frac{1}{3}(0)^3+\frac{1}{2}(0)\right]$$
$$=\frac{1}{10}-\frac{1}{3}+\frac{1}{2}$$
$$=\frac{4}{15}.$$

The volume of the solid is $\frac{4}{15}$ units^3.

16. $\int_0^1\int_0^{1-x}(x+y)\,dy\,dx$

$$=\int_0^1\left(\int_0^{1-x}(x+y)dy\right)dx$$

We first evaluate the inside y-integral, treating x as a constant:

$$\int_0^{1-x}(x+y)dy$$
$$=\left[xy+\frac{1}{2}y^2\right]_0^{1-x}$$
$$=\left[x(1-x)+\frac{1}{2}(1-x)^2\right]-\left[x(0)-\frac{1}{2}(0)^2\right]$$
$$=1-x^2+\frac{1}{2}\left(1-2x+x^2\right)$$
$$=x-x^2+\frac{1}{2}-x+\frac{1}{2}x^2$$
$$=-\frac{1}{2}x^2+\frac{1}{2}$$

Then we evaluate the outside x-integral:

$$=\int_0^1\left(\int_0^{1-x}(x+y)dy\right)dx$$
$$=\int_0^1\left(-\frac{1}{2}x^2+\frac{1}{2}\right)dx$$
$$=\left[-\frac{1}{6}x^3+\frac{1}{2}x\right]_0^1$$
$$=\left[-\frac{1}{6}(1)^3+\frac{1}{2}(1)\right]-\left[-\frac{1}{6}(0)^3+\frac{1}{2}(0)\right]$$
$$=-\frac{1}{6}+\frac{1}{2}-0$$
$$=\frac{1}{3}.$$

The volume of the solid is $\frac{1}{3}$ units^3.

17. $f(x,y)=x^2+\frac{1}{3}xy$

$0\le x\le 1$

$0\le y\le 2$

Find

$$\int_0^2\int_0^1 f(x,y)\,dx\,dy$$
$$=\int_0^2\left(\int_0^1\left(x^2+\frac{1}{3}xy\right)dx\right)dy$$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_0^1 \left(x^2 + \frac{1}{3}xy \right) dx$$
$$= \left[\frac{1}{3}x^3 + \frac{1}{6}x^2 y \right]_0^1$$
$$= \left[\frac{1}{3}(1)^3 + \frac{1}{6}(1)^2 y \right] - \left[\frac{1}{3}(0)^3 + \frac{1}{6}(0)^2 y \right]$$
$$= \frac{1}{3} + \frac{1}{6}y.$$

Then we evaluate the outside y-integral:

$$\int_0^2 \left(\int_0^1 \left(x^2 + \frac{1}{3}xy \right) dx \right) dy$$
$$= \int_0^2 \left(\frac{1}{3} + \frac{1}{6}y \right) dy$$
$$= \left[\frac{1}{3}y + \frac{1}{12}y^2 \right]_0^2$$
$$= \left[\frac{1}{3}(2) + \frac{1}{12}(2)^2 \right] - \left[\frac{1}{3}(0) + \frac{1}{12}(0)^2 \right]$$
$$= \frac{2}{3} + \frac{4}{12}$$
$$= 1.$$

18. $f(x,y) = x^2 + \frac{1}{3}xy$

$0 \le x \le 1$

$0 \le y \le 2$

Find

$$\int_1^2 \int_0^{1/2} f(x,y)\,dx\,dy$$
$$= \int_1^2 \left(\int_0^{1/2} \left(x^2 + \frac{1}{3}xy \right) dx \right) dy$$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_0^{1/2} \left(x^2 + \frac{1}{3}xy \right) dx$$
$$= \left[\frac{1}{3}x^3 + \frac{1}{6}x^2 y \right]_0^{1/2}$$
$$= \left[\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{6}\left(\frac{1}{2}\right)^2 y \right] - \left[\frac{1}{3}(0)^3 + \frac{1}{6}(0)^2 y \right]$$
$$= \frac{1}{24} + \frac{1}{24}y.$$

Then we evaluate the outside y-integral:

$$\int_1^2 \left(\int_0^{1/2} \left(x^2 + \frac{1}{3}xy \right) dx \right) dy$$
$$= \int_1^2 \left(\frac{1}{24} + \frac{1}{24}y \right) dy$$
$$= \left[\frac{1}{24}y + \frac{1}{48}y^2 \right]_1^2$$
$$= \left[\frac{1}{24}(2) + \frac{1}{48}(2)^2 \right] - \left[\frac{1}{24}(1) + \frac{1}{48}(1)^2 \right]$$
$$= \left[\frac{1}{12} + \frac{1}{12} \right] - \left[\frac{1}{24} + \frac{1}{48} \right]$$
$$= \frac{2}{12} - \frac{1}{16}$$
$$= \frac{5}{48}.$$

19. $f(x,y) = x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2$

$1 \le x \le 2$

$3 \le y \le 5$

Find

$$\int_3^4 \int_1^2 f(x,y)\,dx\,dy$$
$$= \int_3^4 \left(\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx \right) dy$$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx$$
$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{1}{6}x^2 y - \frac{1}{3}xy + 2x \right]_1^2$$
$$= \left[\frac{1}{3}(2)^3 - \frac{3}{2}(2)^2 + \frac{1}{6}(2)^2 y - \frac{1}{3}(2)y + 2(2) \right] -$$
$$\left[\frac{1}{3}(1)^3 - \frac{3}{2}(1)^2 + \frac{1}{6}(1)^2 y - \frac{1}{3}(1)y + 2(1) \right]$$
$$= \left[\frac{8}{3} - 6 + \frac{2}{3}y - \frac{2}{3}y + 4 \right] -$$
$$\left[\frac{1}{3} - \frac{3}{2} + \frac{1}{6}y - \frac{1}{3}y + 2 \right]$$
$$= \frac{2}{3} - \left[\frac{5}{6} - \frac{1}{6}y \right]$$
$$= \frac{1}{6}y - \frac{1}{6}.$$

Then we evaluate the outside y-integral:

$$= \int_3^4 \left(\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx \right) dy$$
$$= \int_3^4 \left(\frac{1}{6}y - \frac{1}{6} \right) dy$$
$$= \left[\frac{1}{12}y^2 - \frac{1}{6}y \right]_3^4$$
$$= \left[\frac{1}{12}(4)^2 - \frac{1}{6}(4) \right] - \left[\frac{1}{12}(3)^2 - \frac{1}{6}(3) \right]$$
$$= \left[\frac{4}{3} - \frac{2}{3} \right] - \left[\frac{3}{4} - \frac{1}{2} \right]$$
$$= \frac{2}{3} - \frac{1}{4}$$
$$= \frac{5}{12}.$$

20. $f(x,y) = x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2$

$1 \le x \le 2$

$3 \le y \le 5$

Find

$$\int_4^5 \int_1^2 f(x,y)\,dx\,dy$$
$$= \int_4^5 \left(\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx \right) dy$$

We first evaluate the inside x-integral, treating y as a constant:

$$\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx$$
$$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{1}{6}x^2 y - \frac{1}{3}xy + 2x \right]_1^2$$
$$= \left[\frac{1}{3}(2)^3 - \frac{3}{2}(2)^2 + \frac{1}{6}(2)^2 y - \frac{1}{3}(2)y + 2(2) \right] -$$
$$\left[\frac{1}{3}(1)^3 - \frac{3}{2}(1)^2 + \frac{1}{6}(1)^2 y - \frac{1}{3}(1)y + 2(1) \right]$$
$$= \left[\frac{8}{3} - 6 + \frac{2}{3}y - \frac{2}{3}y + 4 \right] -$$
$$\left[\frac{1}{3} - \frac{3}{2} + \frac{1}{6}y - \frac{1}{3}y + 2 \right]$$
$$= \frac{2}{3} - \left[\frac{5}{6} - \frac{1}{6}y \right]$$
$$= \frac{1}{6}y - \frac{1}{6}.$$

Then we evaluate the outside y-integral:

$$= \int_4^5 \left(\int_1^2 \left(x^2 - 3x + \frac{1}{3}xy - \frac{1}{3}y + 2 \right) dx \right) dy$$
$$= \int_4^5 \left(\frac{1}{6}y - \frac{1}{6} \right) dy$$
$$= \left[\frac{1}{12}y^2 - \frac{1}{6}y \right]_4^5$$
$$= \left[\frac{1}{12}(5)^2 - \frac{1}{6}(5) \right] - \left[\frac{1}{12}(4)^2 - \frac{1}{6}(4) \right]$$
$$= \left[\frac{25}{12} - \frac{5}{6} \right] - \left[\frac{4}{3} - \frac{2}{3} \right]$$
$$= \frac{5}{4} - \frac{2}{3}$$
$$= \frac{7}{12}.$$

21. $\int_0^1 \int_1^3 \int_{-1}^2 (2x + 3y - z)\,dx\,dy\,dz$

$$= \int_0^1 \int_1^3 \left(\int_{-1}^2 (2x + 3y - z)\,dx \right) dy\,dz$$

We first evaluate the inside x-integral, treating y and z as constants:

$$\int_{-1}^2 (2x + 3y - z)\,dx$$
$$= \left[x^2 + 3yx - zx \right]_{-1}^2$$
$$= \left[(2)^2 + 3y(2) - z(2) \right] -$$
$$\left[(-1)^2 + 3y(-1) - z(-1) \right]$$
$$= [4 + 6y - 2z] - [1 - 3y + z]$$
$$= 3 + 9y - 3z$$

Then we evaluate the middle y-integral, treating z as a constant:

$$\int_1^3 \left(\int_{-1}^2 (2x + 3y - z)\,dx \right) dy$$
$$= \int_1^3 (3 + 9y - 3z)\,dy$$
$$= \left[3y + \frac{9}{2}y^2 - 3zy \right]_1^3$$
$$= \left[3(3) + \frac{9}{2}(3)^2 - 3z(3) \right] -$$
$$\left[3(1) + \frac{9}{2}(1)^2 - 3z(1) \right]$$
$$= \left[9 + \frac{81}{2} - 9z \right] - \left[3 + \frac{9}{2} - 3z \right]$$
$$= 42 - 6z.$$

Finally, we evaluate the outside z integral:

$$\int_0^1\left(\int_1^3\left(\int_{-1}^2(2x+3y-z)\,dx\right)dy\right)dz$$
$$=\int_0^1(42-6z)\,dz$$
$$=\left[42z-3z^2\right]_0^1$$
$$=\left[42(1)-3(1)^2\right]-\left[42(0)-3(0)^2\right]$$
$$=42-3$$
$$=39.$$

22. $$\int_0^2\int_1^4\int_{-1}^2(8x-2y+z)\,dx\,dy\,dz$$
$$=\int_0^2\int_1^4\left(\int_{-1}^2(8x-2y+z)\,dx\right)dy\,dz$$

We first evaluate the inside x-integral, treating y and z as constants:

$$\int_{-1}^2(8x-2y+z)\,dx$$
$$=\left[4x^2-2yx+zx\right]_{-1}^2$$
$$=\left[4(2)^2-2y(2)+z(2)\right]-\left[4(-1)^2-2y(-1)+z(-1)\right]$$
$$=\left[16-4y+2z\right]-\left[4+2y-z\right]$$
$$=12-6y+3z$$

Then we evaluate the middle y-integral, treating z as a constant:

$$\int_1^4\left(\int_{-1}^2(8x-2y+z)\,dx\right)dy$$
$$=\int_1^4(12-6y+3z)\,dy$$
$$=\left[12y-3y^2+3zy\right]_1^4$$
$$=\left[12(4)-3(4)^2+3z(4)\right]-\left[12(1)-3(1)^2+3z(1)\right]$$
$$=\left[48-48+12z\right]-\left[12-3+3z\right]$$
$$=9z-9.$$

Finally, we evaluate the outside z integral:

$$\int_0^2\left(\int_1^4\left(\int_{-1}^2(8x-2y+z)\,dx\right)dy\right)dz$$
$$=\int_0^2(9z-9)\,dz$$
$$=\left[\frac{9}{2}z^2-9z\right]_0^2$$
$$=\left[\frac{9}{2}(2)^2-9(2)\right]-\left[\frac{9}{2}(0)^2-9(0)\right]$$
$$=18-18$$
$$=0.$$

23. $$\int_0^1\int_0^{1-x}\int_0^{2-x}(xyz)\,dz\,dy\,dx$$
$$=\int_0^1\int_0^{1-x}\left(\int_0^{2-x}(xyz)\,dz\right)dy\,dx$$

We first evaluate the inside z-integral, treating x and y as constants:

$$\int_0^{2-x}(xyz)\,dz$$
$$=\left[\frac{1}{2}xyz^2\right]_0^{2-x}$$
$$=\left[\frac{1}{2}xy(2-x)^2\right]-\left[\frac{1}{2}xy(0)^2\right]$$
$$=\left[\frac{1}{2}xy(2-x)^2\right]-[0]$$
$$=\frac{1}{2}x(2-x)^2\,y$$

Then we evaluate the middle y-integral, treating x as a constant:

$$\int_0^{1-x}\left(\int_0^{2-x}(xyz)\,dz\right)dy$$
$$=\int_0^{1-x}\left(\frac{1}{2}x(2-x)^2\,y\right)dy$$
$$=\frac{1}{2}x(2-x)^2\left[\frac{1}{2}y^2\right]_0^{1-x}$$
$$=\frac{1}{2}x(2-x)^2\left[\frac{1}{2}(1-x)^2-\frac{1}{2}(0)^2\right]$$
$$=\frac{1}{4}x(2-x)^2\left[(1-x)^2\right]$$
$$=\frac{1}{4}x\left(4-4x+x^2\right)\left[1-2x+x^2\right]$$
$$=\left[x-x^2+\frac{1}{4}x^3\right]\left[1-2x+x^2\right]$$
$$=x-2x^2+x^3-x^2+2x^3-x^4+\frac{1}{4}x^3-\frac{1}{2}x^4+\frac{1}{4}x^5$$
$$=\frac{1}{4}x^5-\frac{3}{2}x^4+\frac{13}{4}x^3-3x^2+x$$

Finally, we evaluate the outside x integral:

$$=\int_0^1\int_0^{1-x}\left(\int_0^{2-x}(xyz)\,dz\right)dy\,dx$$
$$=\int_0^1\left(\frac{1}{4}x^5-\frac{3}{2}x^4+\frac{13}{4}x^3-3x^2+x\right)dx$$
$$=\left[\frac{1}{24}x^6-\frac{3}{10}x^5+\frac{13}{16}x^4-x^3+\frac{1}{2}x^2\right]_0^1$$
$$=\left[\frac{1}{24}(1)^6-\frac{3}{10}(1)^5+\frac{13}{16}(1)^4-(1)^3+\frac{1}{2}(1)^2\right]-\left[\frac{1}{24}(0)^6-\frac{3}{10}(0)^5+\frac{13}{16}(0)^4-(0)^3+\frac{1}{2}(0)^2\right]$$
$$=\left(\frac{1}{24}-\frac{3}{10}+\frac{13}{16}-1+\frac{1}{2}\right)-(0)$$
$$=\frac{10}{240}-\frac{72}{240}+\frac{195}{240}-\frac{240}{240}+\frac{120}{240}$$
$$=\frac{13}{240}.$$

24. $$\int_0^2\int_{2-y}^{6-2y}\int_0^{\sqrt{4-y^2}}z\,dz\,dx\,dy$$
$$=\int_0^2\int_{2-y}^{6-2y}\left(\int_0^{\sqrt{4-y^2}}z\,dz\right)dx\,dy$$

We first evaluate the inside z-integral, treating x and y as constants:

$$\int_0^{\sqrt{4-y^2}}z\,dz$$
$$=\left[\frac{1}{2}z^2\right]_0^{\sqrt{4-y^2}}$$
$$=\left[\frac{1}{2}\left(\sqrt{4-y^2}\right)^2\right]-\left[\frac{1}{2}(0)^2\right]$$
$$=\frac{1}{2}\left(4-y^2\right)$$
$$=2-\frac{1}{2}y^2$$

Then we evaluate the middle x-integral, treating y as a constant:

$$\int_{2-y}^{6-2y}\left(\int_0^{\sqrt{4-y^2}}z\,dz\right)dx$$
$$=\int_{2-y}^{6-2y}\left(2-\frac{1}{2}y^2\right)dx$$
$$=\left(2-\frac{1}{2}y^2\right)[x]_{2-y}^{6-2y}$$
$$=\left(2-\frac{1}{2}y^2\right)\left[(6-2y)-(2-y)\right]$$
$$=\left(2-\frac{1}{2}y^2\right)[4-y]$$
$$=8-2y-2y^2+\frac{1}{2}y^3.$$

Finally, we evaluate the outside y integral:

$$=\int_0^2\left(\int_{2-y}^{6-2y}\left(\int_0^{\sqrt{4-y^2}}z\,dz\right)dx\right)dy$$
$$=\int_0^2\left(8-2y-2y^2+\frac{1}{2}y^3\right)dy$$
$$=\left[8y-y^2-\frac{2}{3}y^3+\frac{1}{8}y^4\right]_0^2$$
$$=\left[8(2)-(2)^2-\frac{2}{3}(2)^3+\frac{1}{8}(2)^4\right]-\left[8(0)-(0)^2-\frac{2}{3}(0)^3+\frac{1}{8}(0)^4\right]$$
$$=\left(16-4-\frac{16}{3}+2\right)-(0)$$
$$=14-\frac{16}{3}$$
$$=\frac{26}{3}.$$

25. $\boxed{tw}$ The multiple integral of a function of two variables $f(x,y)$ represents the volume of a solid based in some region of the *xy*-plane and capped by the surface $z = f(x,y)$.

26. $\boxed{tw}$ $f(x,y)$ represents a joint probability density function, on a region R. By definition $\iint_R f(x,y)\,dxdy = 1$.

27. Left to the student.