Exercise Set 2.2

1. $f(x)=5-x^2$

First, find $f'(x)$ and $f''(x)$.

$f'(x)=-2x$

$f''(x)=-2$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$f'(x)=0$

$-2x=0$

$x=0$

We find the function value at $x=0$.

$f(0)=5-(0)^2=5$.

The critical point is $(0,5)$.

Next, we apply the Second Derivative test.

$f''(x)=-2$

$f''(0)=-2<0$

Therefore, $f(0)=5$ is a relative maximum.

2. $f(x)=4-x^2$

$f'(x)=-2x$

$f''(x)=-2$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x)=0$.

$f'(x)=0$

$-2x=0$

$x=0$

We find the function value at $x=0$.

$f(0)=4-(0)^2=4$.

The critical point is $(0,4)$.

Next, we apply the Second Derivative test.

$f''(x)=-2$

$f''(0)=-2<0$

Therefore, $f(0)=4$ is a relative maximum.

3. $f(x)=x^2-x$

First, find $f'(x)$ and $f''(x)$.

$f'(x)=2x-1$

$f''(x)=2$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$f'(x)=0$

$2x-1=0$

$2x=1$

$x=\frac{1}{2}$

We find the function value at $x=\frac{1}{2}$.

$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$.

The critical point is $\left(\frac{1}{2},-\frac{1}{4}\right)$.

Next, we apply the Second Derivative test.

$f''(x)=2$

$f''\left(\frac{1}{2}\right)=2>0$

Therefore, $f\left(\frac{1}{2}\right)=-\frac{1}{4}$ is a relative minimum.

4. $f(x)=x^2+x-1$

$f'(x)=2x+1$

$f''(x)=2$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x)=0$.

$2x+1=0$

$x=-\frac{1}{2}$

We find the function value at $x=-\frac{1}{2}$.

$f\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)-1=\frac{1}{4}-\frac{1}{2}-1=-\frac{5}{4}$.

The critical point is $\left(-\frac{1}{2},-\frac{5}{4}\right)$.

Next, we apply the Second Derivative test.

$f''(x)=2$

$f''\left(-\frac{1}{2}\right)=2>0$

Therefore, $f\left(-\frac{1}{2}\right)=-\frac{5}{4}$ is a relative minimum.

5. $f(x) = -5x^2 + 8x - 7$

First, find $f'(x)$ and $f''(x)$.

$f'(x) = -10x + 8$

$f''(x) = -10$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers *x*, the only critical points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ -10x + 8 &= 0 \\ -10x &= -8 \\ x &= \frac{-8}{-10} \\ x &= \frac{4}{5} \end{aligned}$$

We find the function value at $x = \frac{4}{5}$.

$$\begin{aligned} f\left(\frac{4}{5}\right) &= -5\left(\frac{4}{5}\right)^2 + 8\left(\frac{4}{5}\right) - 7 \\ &= -5\left(\frac{16}{25}\right) + \frac{32}{5} - \frac{7}{1} \\ &= -\frac{16}{5} + \frac{32}{5} - \frac{35}{5} \\ &= -\frac{19}{5} \end{aligned}.$$

The critical point is $\left(\frac{4}{5}, -\frac{19}{5}\right)$.

Next, we apply the Second Derivative test.

$f''(x) = -10$

$f''\left(\frac{4}{5}\right) = -10 < 0$

Therefore, $f\left(\frac{4}{5}\right) = -\frac{19}{5}$ is a relative maximum.

6. $f(x) = -4x^2 + 3x - 1$

$f'(x) = -8x + 3$

$f''(x) = -8$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ -8x + 3 &= 0 \\ x &= \frac{3}{8} \end{aligned}$$

We find the function value at $x = \frac{3}{8}$.

$f\left(\frac{3}{8}\right) = -4\left(\frac{3}{8}\right)^2 + 3\left(\frac{3}{8}\right) - 1 = -\frac{7}{16}$.

The critical point is $\left(\frac{3}{8}, -\frac{7}{16}\right)$.

Next, we apply the Second Derivative test.

$f''(x) = -8$

$f''\left(\frac{3}{8}\right) = -8 < 0$

Therefore, $f\left(\frac{3}{8}\right) = -\frac{7}{16}$ is a relative maximum.

7. $f(x) = 8x^3 - 6x + 1$

First, find $f'(x)$ and $f''(x)$.

$f'(x) = 24x^2 - 6$

$f''(x) = 48x$

Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers *x*, the only critical points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 24x^2 - 6 &= 0 \\ 4x^2 - 1 &= 0 \qquad \text{Dividing by 6} \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

There are two critical values $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

We find the function value at $x = -\frac{1}{2}$

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= 8\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right) + 1 \\ &= 8\left(-\frac{1}{8}\right) + 3 + 1 \\ &= -1 + 3 + 1 \\ &= 3 \end{aligned}.$$

The critical point is $\left(-\frac{1}{2}, 3\right)$.

Next, we apply the Second Derivative test.

$f''(x) = 48x$

$f''\left(-\frac{1}{2}\right) = 48\left(-\frac{1}{2}\right) = -24 < 0$

Therefore, $f\left(-\frac{1}{2}\right) = 3$ is a relative maximum.

Now, we find the function value at $x = \frac{1}{2}$.

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 8\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right) + 1 \\ &= 8\left(\frac{1}{8}\right) - 3 + 1 \\ &= 1 - 3 + 1 \\ &= -1 \end{aligned}$$

The critical point is $\left(\frac{1}{2}, -1\right)$.

Next, we apply the Second Derivative test.

$f''(x) = 48x$

$f''\left(\frac{1}{2}\right) = 48\left(\frac{1}{2}\right) = 24 > 0$

Therefore, $f\left(\frac{1}{2}\right) = -1$ is a relative minimum.

8. $f(x) = x^3 - 12x - 1$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

$f'(x)$ exists for all real numbers. The only critical points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

There are two critical values.

First, we find the function value at $x = -2$.

$f(-2) = (-2)^3 - 12(-2) - 1 = 15$.

The critical point is $(-2, 15)$.

Next, we apply the Second Derivative test.

$f''(x) = 6x$

$f''(-2) = 6(-2) = -12 < 0$

Therefore, $f(-2) = 15$ is a relative maximum.

Next, we find the function value at $x = 2$.

$f(2) = (2)^3 - 12(2) - 1 = -17$.

The critical point is $(2, -17)$.

Next, we apply the Second Derivative test.

$f''(x) = 6x$

$f''(2) = 6(2) = 12 > 0$

Therefore, $f(2) = -17$ is a relative minimum.

9. $f(x) = x^3 - 12x$

a) Find $f'(x)$ and $f''(x)$.

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

There are two critical values $x = -2$ and $x = 2$.

We find the function value at $x = -2$

$f(-2) = (-2)^3 - 12(-2) = 16$.

The critical point on the graph is $(-2, 16)$.

Next, we find the function value at $x = 2$.

$f(2) = (2)^3 - 12(2) = -16$.

The critical point on the graph is $(2, -16)$.

c) Apply the Second Derivative test to the critical points.

For $x = -2$

$f''(x) = 6x$

$f''(-2) = 6(-2) = -12 < 0$

The critical point $(-2, 16)$ is a relative maximum.

For $x = 2$

$f''(x) = 6x$

$f''(2) = 6(2) = 12 > 0$

The critical point $(2,-16)$ is a relative minimum.

If we use the critical values $x = -2$ and $x = 2$ to divide the real line into three intervals, $(-\infty,-2),(-2,2)$, and $(2,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-2)$, decreasing over the interval $(-2,2)$ and then increasing again over the interval $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$f''(x) = 0$

$6x = 0$

$x = 0$

Therefore, a possible inflection point occurs at $x = 0$.

$f(0) = (0)^3 - 12(0) = 0$.

This gives the point $(0,0)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B:$(0,\infty)$. We test a point in each interval

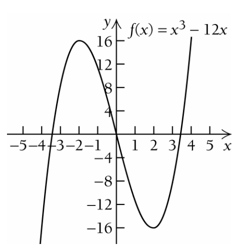
A: Test $-1: f''(-1) = 6(-1) = -6 < 0$

B: Test $1: \quad f''(1) = 6(1) = 6 > 0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,0)$ is an inflection point.

f) Finally, we use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	9
-1	11
1	-11
3	-9



10. $f(x) = x^3 - 27x$

a) $f'(x) = 3x^2 - 27$

$f''(x) = 6x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$f'(x) = 0$

$3x^2 - 27 = 0$

$x^2 = 9$

$x = \pm 3$

There are two critical values $x = -3$ and $x = 3$.

$f(-3) = (-3)^3 - 27(-3) = 54$.

The critical point on the graph is $(-3,54)$.

$f(3) = (3)^3 - 27(3) = -54$.

The critical point on the graph is $(3,-54)$.

c) We apply the Second Derivative test to the critical points.

For $x = -3$

$f''(x) = 6x$

$f''(-3) = 6(-3) = -18 < 0$

The critical point $(-3,54)$ is a relative maximum.

For $x = 3$

$f''(x) = 6x$

$f''(3) = 6(3) = 18 > 0$

The critical point $(3,-54)$ is a relative minimum.

Therefore, $f(x)$ is increasing over the interval $(-\infty,-3)$, decreasing over the interval $(-3,3)$ and then increasing again over the interval $(3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$f''(x) = 0$

$6x = 0$

$x = 0$

$f(0) = (0)^3 - 27(0) = 0$.

A possible inflection point on the graph is the point $(0,0)$.

e) We use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B:$(0,\infty)$. We test a point in each interval

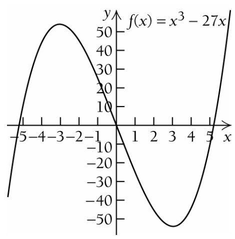
A: Test $-1: f''(-1)=6(-1)=-6<0$

B: Test $1: \quad f''(1)=6(1)=6>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,0)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–4	44
–1	26
1	–26
4	–44



11. $f(x)=3x^3-36x-3$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x)=9x^2-36$

$f''(x)=18x$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$$f'(x)=0$$
$$9x^2-36=0$$
$$9x^2=36$$
$$x^2=4$$
$$x=\pm 2$$

There are two critical values $x=-2$ and $x=2$.

We find the function value at $x=-2$

$f(-2)=3(-2)^3-36(-2)-3=45$.

The critical point on the graph is $(-2,45)$.

Next, we find the function value at $x=2$.

$f(2)=3(2)^3-36(2)-3=-51$.

The critical point on the graph is $(2,-51)$.

c) Apply the Second Derivative test to the critical points.

For $x=-2$

$f''(x)=18x$

$f''(-2)=18(-2)=-36<0$

The critical point $(-2,45)$ is a relative maximum.

For $x=2$

$f''(x)=18x$

$f''(2)=18(2)=36>0$

The critical point $(2,-51)$ is a relative minimum.

c) We use the critical values $x=-2$ and $x=2$ to divide the real line into three intervals, A:$(-\infty,-2)$, B: $(-2,2)$, and C: $(2,\infty)$, we know from the extrema above, that $f(x)$ is increasing over the interval $(-\infty,-2)$, decreasing over the interval $(-2,2)$ and then increasing again over the interval $(2,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$18x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$f(0)=3(0)^3-36(0)-3=-3$.

This gives the point $(0,-3)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B:$(0,\infty)$. We test a point in each interval

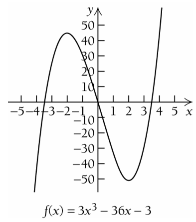
A: Test $-1: f''(-1)=18(-1)=-18<0$

B: Test $1: \quad f''(1)=18(1)=18>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $(0,-3)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–3	24
–1	30
1	–36
3	–30



12. $f(x) = 2x^3 - 3x^2 - 36x + 28$

a) $f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x) = 0$$
$$6x^2 - 6x - 36 = 0$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x - 3 = 0 \text{ or } x + 2 = 0$$
$$x = 3 \text{ or } x = -2$$

There are two critical values $x = -2$ and $x = 3$.

$f(-2)$

$= 2(-2)^3 - 3(-2)^2 - 36(-2) + 28 = 72$

The critical point on the graph is $(-2, 72)$.

$f(3)$

$= 2(3)^3 - 3(3)^2 - 36(3) + 28 = -53$.

The critical point on the graph is $(3, -53)$.

c) Apply the Second Derivative test to the critical points.

For $x = -2$

$f''(x) = 12x - 6$

$f''(-2) = 12(-2) - 6 = -30 < 0$

The critical point $(-2, 72)$ is a relative maximum.

For $x = 3$

$f''(x) = 12x - 6$

$f''(3) = 12(3) - 6 = 30 > 0$

The critical point $(3, -53)$ is a relative minimum.

Therefore, $f(x)$ is increasing over the interval $(-\infty, -2)$, decreasing over the interval $(-2, 3)$ and then increasing again over the interval $(3, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$
$$12x - 6 = 0$$
$$x = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 36\left(\frac{1}{2}\right) + 28$$
$$= \frac{19}{2}.$$

A possible inflection point on the graph is the point $\left(\frac{1}{2}, \frac{19}{2}\right)$.

e) We use the possible inflection point to divide the real number line into two intervals $A: \left(-\infty, \frac{1}{2}\right)$ and B: $\left(\frac{1}{2}, \infty\right)$. We test a point in each interval

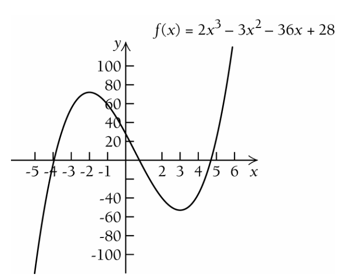
A: Test 0: $f''(0) = 12(0) - 6 = -6 < 0$

B: Test 1: $f''(1) = 12(1) - 6 = 6 > 0$

Then, $f(x)$ is concave down on the interval $\left(-\infty, \frac{1}{2}\right)$ and concave up on the interval $\left(\frac{1}{2}, \infty\right)$, so $\left(\frac{1}{2}, \frac{19}{2}\right)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
–3	55
–1	59
0	28
1	–9
4	–36



13. $f(x)=\frac{8}{3}x^3-2x+\frac{1}{3}$

a) Find $f'(x)$ and $f''(x)$.

$f'(x)=8x^2-2$

$f''(x)=16x$

The domain of f is $\mathbb{R}$.

b) Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x)=0$.

$$f'(x)=0$$
$$8x^2-2=0$$
$$8x^2=2$$
$$x^2=\frac{1}{4}$$
$$x=\pm\frac{1}{2}$$

There are two critical values $x=-\frac{1}{2}$ and $x=\frac{1}{2}$.

We find the function value at $x=-\frac{1}{2}$

$$\begin{aligned} f\left(-\frac{1}{2}\right)&=\frac{8}{3}\left(-\frac{1}{2}\right)^3-2\left(-\frac{1}{2}\right)+\frac{1}{3}\\ &=\frac{8}{3}\left(-\frac{1}{8}\right)+1+\frac{1}{3}\\ &=-\frac{1}{3}+1+\frac{1}{3}\\ &=1 \end{aligned}$$

The critical point on the graph is $\left(-\frac{1}{2},1\right)$.

Next, we find the function value at $x=\frac{1}{2}$.

$$\begin{aligned} f\left(\frac{1}{2}\right)&=\frac{8}{3}\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)+\frac{1}{3}\\ &=\frac{8}{3}\left(\frac{1}{8}\right)-1+\frac{1}{3}\\ &=\frac{1}{3}-1+\frac{1}{3}\\ &=-\frac{1}{3} \end{aligned}$$

The critical point on the graph is $\left(\frac{1}{2},-\frac{1}{3}\right)$.

c) Apply the Second Derivative test to the critical points.

For $x=-\frac{1}{2}$

$$f''(x)=16x$$
$$f''\left(-\frac{1}{2}\right)=16\left(-\frac{1}{2}\right)=-8<0$$

The critical point $\left(-\frac{1}{2},1\right)$ is a relative maximum.

For $x=\frac{1}{2}$

$$f''(x)=16x$$
$$f''\left(\frac{1}{2}\right)=16\left(\frac{1}{2}\right)=8>0$$

The critical point $\left(\frac{1}{2},-\frac{1}{3}\right)$ is a relative minimum.

We use the critical values $x=-\frac{1}{2}$, and $x=\frac{1}{2}$ to divide the real line into three intervals,

A : $\left(-\infty,-\frac{1}{2}\right)$, B: $\left(-\frac{1}{2},\frac{1}{2}\right)$, and C: $\left(\frac{1}{2},\infty\right)$,

we know from the extrema above, that $f(x)$ is increasing over the interval $\left(-\infty,-\frac{1}{2}\right)$, decreasing over the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ and then increasing again over the interval $\left(\frac{1}{2},\infty\right)$.

d) We find the points of inflection.

$f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$16x=0$$
$$x=0$$

Therefore, a possible inflection point occurs at $x=0$.

$$f(0)=\frac{8}{3}(0)^3-2(0)+\frac{1}{3}=\frac{1}{3}.$$

This gives the point $\left(0,\frac{1}{3}\right)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A:(-\infty,0)$ and B:$(0,\infty)$. We test a point in each interval

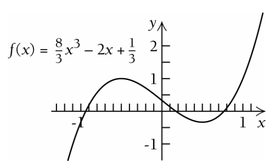
A: Test $-1: f''(-1)=16(-1)=-16<0$

B: Test $1:\quad f''(1)=16(1)=16>0$

Then, $f(x)$ is concave down on the interval $(-\infty,0)$ and concave up on the interval $(0,\infty)$, so $\left(0,\frac{1}{3}\right)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	-17
-1	$-\frac{1}{3}$
1	1
2	$\frac{53}{3}$



14. $f(x)=80-9x^2-x^3$

a) $f'(x)=-18x-3x^2$

$f''(x)=-18-6x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$-18x-3x^2=0$$
$$-3x(x+6)=0$$
$$-3x=0 \text{ or } x+6=0$$
$$x=0 \text{ or } \quad x=-6$$

There are two critical values $x=-6$ and $x=0$.

$f(-6)=80-9(-6)^2-(-6)^3=-28$.

The critical point on the graph is $(-6,-28)$.

$f(0)=80-9(0)^2-(0)^3=80$.

The critical point on the graph is $(0,80)$.

c) We Apply the Second Derivative test to the critical points.

For $x=-6$

$f''(x)=-18-6x$

$f''(-6)=-18-6(-6)=18>0$

The critical point $(-6,-28)$ is a relative minimum.

For $x=0$

$f''(x)=-18-6x$

$f''(0)=-18-6(0)=-18<0$

The critical point $(0,80)$ is a relative maximum.

Therefore, $f(x)$ is decreasing over the interval $(-\infty,-6)$, increasing over the interval $(-6,0)$ and then decreasing again over the interval $(0,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$-18-6x=0$$
$$x=-3$$

$f(-3)=80-9(-3)^2-(-3)^3=26$.

A possible inflection point on the graph is the point $(-3,26)$.

e) We use the possible inflection point to divide the real number line into two intervals $A:(-\infty,-3)$ and B:$(-3,\infty)$. We test a point in each interval

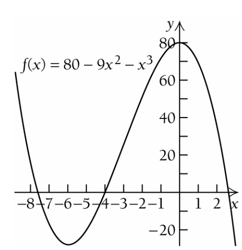
A: Test $-4: f''(-4)=-18-6(-4)=6>0$

B: Test $0: f''(0)=-18-6(0)=-18<0$

Then, $f(x)$ is concave up on the interval $(-\infty,-3)$ and concave down on the interval $(-3,\infty,)$, so $(-3,26)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-9	80
-4	0
-2	52
0	80
2	36
3	-28



15. $f(x) = -x^3 + 3x^2 - 4$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = -3x^2 + 6x$

$f''(x) = -6x + 6$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers *x*, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$-3x^2 + 6x = 0$$
$$-3x(x-2) = 0$$
$$-3x = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 0 \quad \text{or} \quad x = 2$$

There are two critical values $x = 0$ and $x = 2$.

We find the function value at $x = 0$

$f(0) = -(0)^3 + 3(0)^2 - 4 = -4$

The critical point on the graph is $(0, -4)$.

Next, we find the function value at $x = 6$.

$$\begin{aligned} f(2) &= -(2)^3 + 3(2)^2 - 4 \\ &= -8 + 12 - 4 \\ &= 0 \end{aligned}$$

The critical point on the graph is $(2, 0)$.

c) Apply the Second Derivative test to the critical points.

For $x = 0$

$f''(x) = -6x + 6$

$f''(0) = -6(0) + 6 = 6 > 0$

The critical point $(0, -4)$ is a relative minimum.

For $x = 2$

$f''(x) = -6x + 6$

$f''(2) = -6(2) + 6 = -6 < 0$

The critical point $(2, 0)$ is a relative maximum.

We use the critical values $x = 0$ and $x = 2$ to divide the real line into three intervals, A: $(-\infty, 0)$, B: $(0, 2)$, and C: $(2, \infty)$, we know from the extrema above, that $f(x)$ is decreasing over the interval $(-\infty, 0)$, increasing over the interval $(0, 2)$ and then decreasing again over the interval $(2, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$
$$-6x + 6 = 0$$
$$-6x = -6$$
$$x = 1$$

Therefore, a possible inflection point occurs at $x = 1$.

$$\begin{aligned} f(1) &= -(1)^3 + 3(1)^2 - 4 \\ &= -1 + 3 - 4 \\ &= -2 \end{aligned}$$

This gives the point $(1, -2)$ on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into two intervals $A: (-\infty, 1)$ and B: $(1, \infty)$. We test a point in each interval

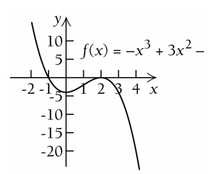
A: Test $0: f''(0) = -6(0) + 6 = 6 > 0$

B: Test $2: f''(2) = -6(2) + 6 = -2 < 0$

Then, $f(x)$ is concave up on the interval $(-\infty, 1)$ and concave down on the interval $(1, \infty)$, so $(1, -2)$ is an inflection point.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	16
-1	0
3	-4
4	-20



16. $f(x) = -x^3 + 3x - 2$

a) $f'(x) = -3x^2 + 3$

$f''(x) = -6x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x) = 0$$
$$-3x^2 + 3 = 0$$
$$-3(x^2 - 1) = 0$$
$$x^2 - 1 = 0$$
$$x = \pm 1$$

There are two critical values
$x = -1$ and $x = 1$.

$f(-1) = -(-1)^3 + 3(-1) - 2 = -4$.

The critical point on the graph is $(-1, -4)$.

$f(1) = -(1)^3 + 3(1) - 2 = 0$.

The critical point on the graph is $(1, 0)$.

c) We apply the Second Derivative test to the critical points.
For $x = -1$

$f''(x) = -6x$

$f''(-1) = -6(-1) = 6 > 0$

The critical point $(-1, -4)$ is a relative minimum.
For $x = 1$

$f''(x) = -6x$

$f''(1) = -6(1) = -6 < 0$

The critical point $(1, 0)$ is a relative maximum.

Therefore, $f(x)$ is decreasing over the interval $(-\infty, -1)$, increasing over the interval $(-1, 1)$ and then decreasing again over the interval $(1, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$f''(x) = 0$

$-6x = 0$

$x = 0$

$f(0) = -(0)^3 + 3(0) - 2 = -2$.

A possible inflection point on the graph is the point $(0, -2)$.

f) We use the possible inflection point to divide the real number line into two intervals $A: (-\infty, 0)$ and B:$(0, \infty)$. We test a point in each interval

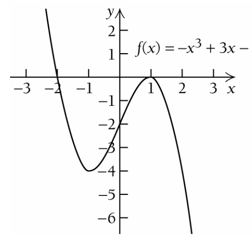
A: Test $-1: f''(-1) = -6(-1) = 6 > 0$

B: Test $1: f''(1) = -6(1) = -6 < 0$

Then, $f(x)$ is concave up on the interval $(-\infty, 0)$ and concave down on the interval $(0, \infty,)$, so $(0, -2)$ is an inflection point.

e) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	16
-2	0
2	-4
3	-20



17. $f(x) = 3x^4 - 16x^3 + 18x^2$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = 12x^3 - 48x^2 + 36x$

$f''(x) = 36x^2 - 96x + 36$

The domain of f is $\mathbb{R}$.

b) Next, find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$12x^3 - 48x^2 + 36x = 0$$
$$12x(x^2 - 4x + 3) = 0$$
$$12x(x-1)(x-3) = 0$$
$$12x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = 3$$

There are three critical values $x = 0$, $x = 1$, and $x = 3$.
Then

$f(0) = 3(0)^4 - 16(0)^3 + 18(0)^2 = 0$

$f(1) = 3(1)^4 - 16(1)^3 + 18(1)^2 = 5$

$f(3) = 3(3)^4 - 16(3)^3 + 18(3)^2 = -27$

Thus, the critical points $(0, 0)$, $(1, 5)$, and $(3, -27)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(0) = 36(0)^2 - 96(0) + 36 = 36 > 0$

The critical point $(0, 0)$ is a relative minimum.

$f''(1) = 36(1)^2 - 96(1) + 36 = -24 < 0$

The critical point $(1,5)$ is a relative maximum.

$f''(3) = 36(3)^2 - 96(3) + 36 = 72 > 0$

The critical point $(3,-27)$ is a relative minimum.

We use the critical values $0, 1,$ and 3 to divide the real line into four intervals, $A:(-\infty,0)$, B:$(0,1)$, C:$(1,3)$ and D:$(3,\infty)$, we know from the extrema above, that $f(x)$ is decreasing over the intervals $(-\infty,0]$ and $[1,3]$ and $f(x)$ increasing over the intervals $[0,1]$ and $[3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ 36x^2 - 96x + 36 &= 0 \\ 12(3x^2 - 8x + 3) &= 0 \\ 3x^2 - 8x + 3 &= 0 \end{aligned}$$

Using the quadratic formula, we find that $x = \dfrac{4 \pm \sqrt{7}}{3}$, so $x \approx 0.451$ or $x \approx 2.215$ are possible inflection points.

$f(0.451) \approx 2.321$

$f(2.215) \approx -13.358$.

So, $(0.451, 2.321)$ and $(2.215, -13.358)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A:(-\infty, 0.451)$, B:$(0.451, 2.215)$, and C:$(2.215,\infty)$. We test a point in each interval

A: Test 0:

$f''(0) = 36(0)^2 - 96(0) + 36 = 36 > 0$

B: Test 1:

$f''(1) = 36(1)^2 - 96(1) + 36 = -24 < 0$

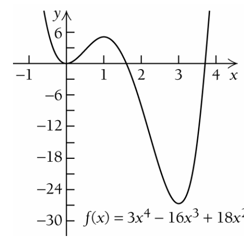
C: Test 3:

$f''(3) = 36(3)^2 - 96(3) + 36 = 72 > 0$

Then, $f(x)$ is concave up on the interval $(-\infty, 0.451)$ and concave down on the interval $(0.451, 2.215)$ and concave up on the interval $(2.215,\infty)$, so $(0.451, 2.321)$ and $(2.215, -13.358)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-1	37
2	-8
4	32



18. $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$

a) $f'(x) = 12x^3 + 12x^2 - 24x$

$f''(x) = 36x^2 + 24x - 24$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$\begin{aligned} f'(x) &= 0 \\ 12x^3 + 12x^2 - 24x &= 0 \\ 12x(x^2 + x - 2) &= 0 \\ 12x(x+2)(x-1) &= 0 \end{aligned}$$

$12x = 0 \quad$ or $\quad x + 2 = 0 \quad$ or $\quad x - 1 = 0$

$x = 0 \quad$ or $\quad x = -2 \quad$ or $\quad x = 1$

There are three critical values $x = -2$, $x = 0$, and $x = 1$.

Then

$$\begin{aligned} f(-2) &= 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 5 \\ &= -27 \end{aligned}$$

$f(0) = 3(0)^4 + 4(0)^3 - 12(0)^2 + 5 = 5$

$f(1) = 3(1)^4 + 4(1)^3 - 12(1)^2 + 5 = 0$

Thus, the critical points $(-2,-27)$, $(0,5)$, and $(1,0)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(-2) = 36(-2)^2 + 24(-2) - 24 = 72 > 0$

The critical point $(-2,-27)$ is a relative minimum.

$f''(0) = 36(0)^2 + 24(0) - 24 = -24 < 0$

The critical point $(0,5)$ is a relative maximum.

$f''(1) = 36(1)^2 + 24(1) - 24 = 36 > 0$

The critical point $(1,0)$ is a relative minimum.

Then $f(x)$ is decreasing over the intervals $(-\infty,-2)$ and $(0,1)$ and $f(x)$ increasing over the intervals $(-2,0)$ and $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$
$$36x^2 + 24x - 24 = 0$$

Using the quadratic formula, we find that

$x = \dfrac{-1 \pm \sqrt{7}}{3}$, so

$x \approx -1.215$ or $x \approx 0.549$ are possible inflection points.

$f(-1.215) \approx -13.358$

$f(0.549) \approx 2.321$

So, $(-1.215,-13.358)$ and $(0.549,2.321)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A: (-\infty,-1.215)$,

B: $(-1.215,0.549)$, and C: $(0.549,\infty)$.

We test a point in each interval

A: Test $-2: f''(-2) = 72 > 0$

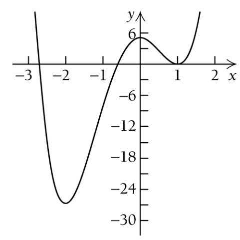
B: Test $0: f''(0) = -24 < 0$

C: Test $1: f''(1) = 36 > 0$

Then, $f(x)$ is concave up on the interval $(-\infty,-1.215)$ and concave down on the interval $(-1.215,0.549)$ and concave up on the interval $(0.549,\infty)$, so $(-1.215,-13.358)$ and $(0.549,2.321)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	32
-1	-8
2	37



$f(x) = 3x^4 + 4x^3 - 12x^2 + 5$

19. $f(x) = x^4 - 6x^2$

a) First, find $f'(x)$ and $f''(x)$.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

The domain of f is $\mathbb{R}$.

b) Find the critical points of $f(x)$. Since $f'(x)$ exists for all real numbers x, the only critical points occur when $f'(x) = 0$.

$$f'(x) = 0$$
$$4x^3 - 12x = 0$$
$$4x(x^2 - 3) = 0$$
$$4x = 0 \quad \text{or} \quad x^2 - 3 = 0$$
$$x = 0 \quad \text{or} \quad x = \pm\sqrt{3}$$

There are three critical values $-\sqrt{3}, 0,$ and $\sqrt{3}$.

Then

$$f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 = 9 - 6(3) = -9$$

$$f(0) = (0)^4 - 6(0)^2 = 0$$

$$f(\sqrt{3}) = (\sqrt{3})^4 - 6(\sqrt{3})^2 = 9 - 6(3) = -9$$

Thus, the critical points $(-\sqrt{3},-9)$, $(0,0)$, $(\sqrt{3},-9)$ and are on the graph.

c) Apply the Second Derivative test to the critical points.

$$f''(-\sqrt{3}) = 12(-\sqrt{3})^2 - 12 = 12(3) - 12 = 24 > 0$$

The critical point $(-\sqrt{3},-9)$ is a relative minimum.

$f''(0)=12(0)^2-12=-12<0$

The critical point $(0,0)$ is a relative maximum.

$f''(\sqrt{3})=12(\sqrt{3})^2-12$
$=12(3)-12=24>0$

The critical point $(\sqrt{3},-9)$ is a relative minimum.

If we use the critical values $-\sqrt{3}$, 0, and $\sqrt{3}$ to divide the real line into four intervals, A: $(-\infty,-\sqrt{3})$, B: $(-\sqrt{3},0)$, C: $(0,\sqrt{3})$, and D: $(\sqrt{3},\infty)$

Then $f(x)$ is decreasing over the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$, and $f(x)$ increasing over the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x)=0$$
$$12x^2-12=0$$
$$x^2-1=0$$
$$x^2=1$$
$$x=\pm1$$

So $x=-1$ or $x=1$ are possible inflection points.

$f(-1)=(-1)^4-6(-1)^2=1-6=-5$

$f(1)=(1)^4-6(1)^2=1-6=-5$.

So, $(-1,-5)$ and $(1,-5)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A:(-\infty,-1)$, B: $(-1,1)$, and C: $(1,\infty)$. We test a point in each interval

A: Test -2:

$f''(-2)=12(-2)^2-12=36>0$

B: Test 0:

$f''(0)=12(0)^2-12=-12<0$

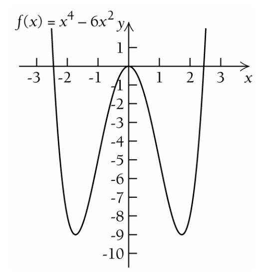
C: Test 2:

$f''(2)=12(2)^2-12=36>0$

Then, $f(x)$ is concave up on the intervals $(-\infty,-1)$ and $(1,\infty)$ and concave down on the interval $(-1,1)$, so $(-1,-5)$ and $(1,-5)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-3	27
-2	-8
2	-8
3	27



20. $f(x)=2x^2-x^4$

a) $f'(x)=4x-4x^3$

$f''(x)=4-12x^2$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$4x-4x^3=0$$
$$4x(1-x^2)=0$$
$$4x(1-x)(1+x)=0$$
$$4x=0 \quad\text{or}\quad 1-x=0 \quad\text{or}\quad 1+x=0$$
$$x=0 \quad\text{or}\quad x=1 \quad\text{or}\quad x=-1$$

There are three critical values $x=-1$, $x=0$, and $x=1$.

Then

$f(-1)=2(-1)^2-(-1)^4=1$

$f(0)=2(0)^2-(0)^4=0$

$f(1)=2(1)^2-(1)^4=1$

Thus, the critical points $(-1,1)$, $(0,0)$, and $(1,1)$ are on the graph.

c) Apply the Second Derivative test to the critical points.

$f''(-1)=4-12(-1)^2=-8<0$

The critical point $(-1,1)$ is a relative maximum.

$f''(0)=4-12(0)^2=4>0$

The critical point $(0,0)$ is a relative minimum.

$f''(1) = 4 - 12(1)^2 = -8 < 0$

The critical point $(1,1)$ is a relative maximum.

Then $f(x)$ is increasing over the intervals $(-\infty,-1)$ and $(0,1)$, and $f(x)$ is decreasing over the intervals $(-1,0)$ and $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers, so we solve the equation

$$f''(x) = 0$$
$$4 - 12x^2 = 0$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

So at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ are possible inflection points.

$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{5}{9}$$
$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{9}.$$

So, $\left(-\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ are two more points on the graph.

e) To determine concavity, we use the possible inflection point to divide the real number line into three intervals $A: \left(-\infty, -\frac{1}{\sqrt{3}}\right)$, B: $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and C: $\left(\frac{1}{\sqrt{3}}, \infty\right)$.

We test a point in each interval

A: Test $-1: f''(-1) = -8 < 0$

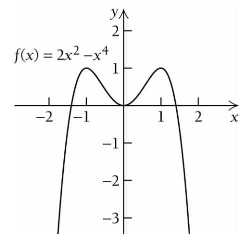
B: Test $0: f''(0) = 4 > 0$

C: Test $1: f''(1) = -8 < 0$

Therefore, $f(x)$ is concave down on the intervals $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$ and $f(x)$ is concave up on the interval $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, so $\left(-\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{5}{9}\right)$ are inflection points.

f) We use the preceding information to sketch the graph of the function. Additional function values can also be calculated as needed.

x	$f(x)$
-2	-8
-1	1
0	0
1	1
2	-8



21. $f(x) = x^3 - 2x^2 - 4x + 3$

a) $f'(x) = 3x^2 - 4x - 4$

$f''(x) = 6x - 4$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x) = 0$.

$$3x^2 - 4x - 4 = 0$$
$$(3x+2)(x-2) = 0$$
$$3x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -\frac{2}{3} \quad \text{or} \quad x = 2$$

The critical values are $-\frac{2}{3}$ and 2.

$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 3$$
$$= -\frac{8}{27} - \frac{8}{9} + \frac{8}{3} + 3$$
$$= -\frac{8}{27} - \frac{24}{27} + \frac{72}{27} + \frac{81}{27}$$
$$= \frac{121}{27}$$

$$f(2) = (2)^3 - 2(2)^2 - 4(2) + 3$$
$$= 8 - 8 - 8 + 3$$
$$= -5$$

The critical points $\left(-\frac{2}{3}, \frac{121}{7}\right)$ and $(2,-5)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''\left(-\frac{2}{3}\right)=6\left(-\frac{2}{3}\right)-4=-4-4=-8<0$

So $\left(-\frac{2}{3},\frac{121}{7}\right)$ is a relative maximum.

$f''(2)=6(2)-4=12-4=8>0$

So $(2,-5)$ is a relative minimum.

Then, if we use the points $-\frac{2}{3}$ and 2 to divide the real number line into three intervals, $\left(-\infty,-\frac{2}{3}\right)$, $\left(-\frac{2}{3},2\right)$, and $(2,\infty)$, we know that f is increasing on $\left(-\infty,-\frac{2}{3}\right]$ and on $[2,\infty)$ and f is decreasing on $\left[-\frac{2}{3},2\right]$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$$6x-4=0$$
$$6x=4$$
$$x=\frac{4}{6}=\frac{2}{3}$$

There is a possible inflection point at $x=\frac{2}{3}$.

$$f\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^3-2\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)+3$$
$$=\frac{8}{27}-\frac{8}{9}-\frac{8}{3}+3$$
$$=\frac{8}{27}-\frac{24}{27}-\frac{72}{27}+\frac{81}{27}$$
$$=-\frac{7}{27}$$

Another point on the graph is $\left(\frac{2}{3},-\frac{7}{27}\right)$.

e) To determine concavity we use $\frac{2}{3}$ to divide the real number line into two intervals, A: $\left(-\infty,\frac{2}{3}\right)$ and B: $\left(\frac{2}{3},\infty\right)$, Then test a point in each interval.

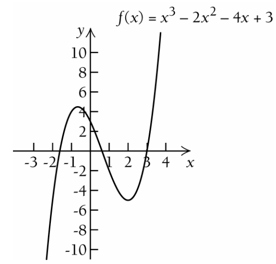
A: Test 0, $f''(0)=6(0)-4=-4<0$

B: Test 1, $f''(1)=6(1)-4=2>0$

We see that f is concave down on $\left(-\infty,\frac{2}{3}\right)$ and concave up on $\left(\frac{2}{3},\infty\right)$, so $\left(\frac{2}{3},-\frac{7}{27}\right)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	-5
-1	4
0	3
1	-2
3	0
4	19



22. $f(x)=x^3-6x^2+9x+1$

a) $f'(x)=3x^2-12x+9$

$f''(x)=6x-12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$3x^2-12x+9=0$$
$$3(x^2-4x+3)=0$$
$$3(x-1)(x-3)=0$$
$$x-1=0 \quad \text{or} \quad x-3=0$$
$$x=1 \quad \text{or} \quad x=3$$

The critical values are 1 and 3.

$f(1)=(1)^3-6(1)^2+9(1)+1=5$

$f(3)=(3)^3-6(3)^2+9(3)+1=1$

So, $(1,5)$ and $(3,1)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(1)=6(1)-12=-6<0$

$f''(3)=6(3)-12=6>0$

Therefore, $(1,5)$ is a relative maximum and $(3,1)$ is a relative minimum. Furthermore, $f(x)$ is increasing on the intervals $(-\infty,1)$ and $(3,\infty)$, and $f(x)$ is decreasing on the interval $(1,3)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers. Solve

$$f''(x)=0$$
$$6x-12=0$$
$$6x=12$$
$$x=2$$
$$f(2)=(2)^3-6(2)^2+9(2)+1=3$$

Thus, the point $(2,3)$ on the graph is a possible inflection point.

e) Use 2 to divide the real number line into two intervals, A: $(-\infty,2)$ and B: $(2,\infty)$. Then test a point in each interval.

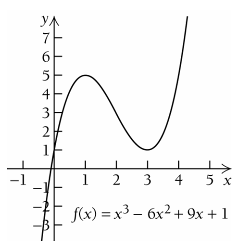
A: Test 0, $f''(0)=6(0)-12=-12<0$

B: Test 3, $f''(3)=6(3)-12=6>0$

Thus, $f(x)$ is concave down on the interval $(-\infty,2)$ and concave up on the interval $(2,\infty)$ and $(2,3)$ is an inflection point.

f) We sketch the graph. Additional points may be found as necessary.

x	$f(x)$
–2	–49
–1	–15
0	1
4	5
5	21



23. $f(x)=3x^4+4x^3$

a) $f'(x)=12x^3+12x^2$

$f''(x)=36x^2+24x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$$12x^3+12x^2=0$$
$$12x^2(x+1)=0$$
$$12x^2=0 \quad \text{or} \quad x+1=0$$
$$x=0 \quad \text{or} \quad x=-1$$

The critical values are -1 and 0.

$$f(-1)=3(-1)^4+4(-1)^3=-1$$
$$f(0)=3(0)^4+4(0)^3=0$$

The critical points $(-1,-1)$ and $(0,0)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$$f''(-1)=36(-1)^2+24(-1)=36-24$$
$$=12>0$$

So $(-1,-1)$ is a relative minimum.

$$f''(0)=36(0)^2+24(0)=0$$

The test fails. We will use the First Derivative Test. We use 0 to divide the interval $(-1,\infty)$ into two intervals, A: $(-1,0)$ and B: $(0,\infty)$, and test a point in each interval.

A: Test $-\frac{1}{2}$,

$$f'\left(-\frac{1}{2}\right)=12\left(-\frac{1}{2}\right)^3+12\left(-\frac{1}{2}\right)^2$$
$$=\frac{3}{2}>0$$

B: Test 2,

$$f'(2)=12(2)^3+12(2)^2=144>0$$

f is increasing on both intervals $[-1,0]$ and $[0,\infty)$. Therefore, $(0,0)$ is not a relative extremum. Since $(-1,-1)$ is a relative minimum, we know that f is decreasing on $(-\infty,-1]$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$$f''(x)=0$$
$$36x^2+24x=0$$
$$12x(3x+2)=0$$
$$12x=0 \quad \text{or} \quad 3x+2=0$$
$$x=0 \quad \text{or} \quad x=-\frac{2}{3}$$

There are a possible inflection points at $x=-\frac{2}{3}$ and $x=0$.

$$f\left(-\frac{2}{3}\right)=3\left(-\frac{2}{3}\right)^4+4\left(-\frac{2}{3}\right)^3$$
$$=3\left(\frac{16}{81}\right)+4\left(-\frac{8}{27}\right)$$
$$=\frac{16}{27}-\frac{32}{27}$$
$$=-\frac{16}{27}$$

$f(0)=3(0)^4+4(0)^3=0$

This gives one additional point $\left(-\frac{2}{3},-\frac{16}{27}\right)$ on the graph.

e) To determine concavity we use $-\frac{2}{3}$ and 0 to divide the real number line into three intervals, A: $\left(-\infty,-\frac{2}{3}\right)$, B: $\left(-\frac{2}{3},0\right)$, and C: $(0,\infty)$. Then test a point in each interval.

A: Test -1,

$f''(-1)=36(-1)^2+24(-1)=12>0$

B: Test $-\frac{1}{2}$,

$$f''\left(-\frac{1}{2}\right)=36\left(-\frac{1}{2}\right)^2+24\left(-\frac{1}{2}\right)$$
$$=-3<0$$

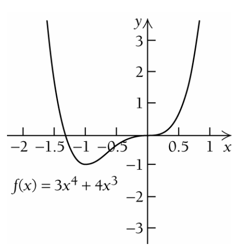
C: Test 1,

$f''(1)=36(1)^2+24(1)=60>0$

We see that *f* is concave up on the intervals $\left(-\infty,-\frac{2}{3}\right)$ and $(0,\infty)$, and concave down on the interval $\left(-\frac{2}{3},0\right)$, so both $\left(-\frac{2}{3},-\frac{16}{27}\right)$ and $(0,0)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
–2	16
1	7
2	80



24. $f(x)=x^4-2x^3$

a) $f'(x)=4x^3-6x^2$

$f''(x)=12x^2-12x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve

$$f'(x)=0$$
$$4x^3-6x^2=0$$
$$2x^2(2x-3)=0$$
$$2x^2=0 \quad \text{or} \quad 2x-3=0$$
$$x=0 \quad \text{or} \quad x=\frac{3}{2}$$

The critical values are 0 and $\frac{3}{2}$.

$f(0)=(0)^4-2(0)^3=0$

$f\left(\frac{3}{2}\right)=\left(\frac{3}{2}\right)^4-2\left(\frac{3}{2}\right)^3=-\frac{27}{16}$

So, $(0,0)$ and $\left(\frac{3}{2},-\frac{27}{16}\right)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0)=12(0)^2-12(0)=0$

The Second Derivative Test fails, we will have to use the First Derivative Test for $x=0$. Divide $\left(-\infty,\frac{3}{2}\right)$ into two intervals, A: $(-\infty,0)$ and B: $\left(0,\frac{3}{2}\right)$, and test a point in each interval.

A: Test -1,

$f'(-1)=4(-1)^3-6(-1)^2=-10<0$

B: Test 1,

$f'(1)=4(1)^3-6(1)^2=-2<0$

Since, f is decreasing on both intervals, $(0,0)$ is not a relative extremum.

We use the Second Derivative Test for $x=\frac{3}{2}$

$$f''\left(\frac{3}{2}\right)=12\left(\frac{3}{2}\right)^2-12\left(\frac{3}{2}\right)=9>0$$

Therefore, $\left(\frac{3}{2},-\frac{27}{16}\right)$ is a relative minimum.

Furthermore, $f(x)$ is increasing on the interval $\left(\frac{3}{2},\infty\right)$, and $f(x)$ is decreasing on the intervals $(-\infty,0)$ and $\left(0,\frac{3}{2}\right)$.

d) Find the points of inflection. $f''(x)$ exists for all real numbers. Solve

$$f''(x)=0$$
$$12x^2-12x=0$$
$$12x(x-1)=0$$
$$12x=0 \quad \text{or} \quad x-1=0$$
$$x=0 \quad \text{or} \quad x=1$$
$$f(0)=0 \quad \text{found earlier}$$
$$f(1)=(1)^4-2(1)^3=-1$$

Thus, the points $(0,0)$ and $(1,-1)$ on the graph are possible inflection points.

e) Use 0 and 1 to divide the real number line into two intervals, A: $(-\infty,0)$, B: $(0,1)$, and C: $(1,\infty)$. Then test a point in each interval.

A: Test -1,
$$f''(1)=12(-1)^2-12(-1)=24>0$$

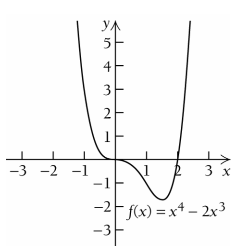
B: Test $\frac{1}{2}$,
$$f''\left(\frac{1}{2}\right)=12\left(\frac{1}{2}\right)^2-12\left(\frac{1}{2}\right)=-3<0$$

C: Test 2,
$$f''(2)=12(2)^2-12(2)=24>0$$

Thus, $f(x)$ is concave up on the intervals $(-\infty,0)$ and $(1,\infty)$ and concave down on the interval $(0,1)$. Also, the points $(0,0)$ and $(1,-1)$ are inflection points.

f) Using the preceding information, we sketch the graph. Additional points may be found as necessary.

x	$f(x)$
-2	32
-1	3
2	0
3	27



25. $f(x)=x^3-6x^2-135x$

a) $f'(x)=3x^2-12x-135$

$f''(x)=6x-12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$$3x^2-12x-135=0$$
$$x^2-4x-45=0$$
$$(x-9)(x+5)=0$$
$$x-9=0 \quad \text{or} \quad x+5=0$$
$$x=9 \quad \text{or} \quad x=-5$$

The critical values are -5 and 9.

$$f(-5)=(-5)^3-6(-5)^2-135(-5)$$
$$=-125-150+675$$
$$=400$$
$$f(9)=(9)^3-6(9)^2-135(9)$$
$$=729-486-1215$$
$$=-972$$

The critical points $(-5,400)$ and $(9,-972)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$$f''(-5)=6(-5)-12=-30-12$$
$$=-42<0$$

The critical point $(-5,400)$ is a relative maximum.

$$f''(9)=6(9)-12=54-12$$
$$=42>0$$

The critical point $(9,-972)$ is a relative minimum.

If we use the points -5 and 9 to divide the real number line into three intervals $(-\infty,-5)$, $(-5,9)$, and $(9,\infty)$ we see that $f(x)$ is increasing on the intervals $(-\infty,-5)$ and $(9,\infty)$ and $f(x)$ is decreasing on the interval $(-5,9)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x)=0$.

$$6x-12=0$$
$$6x=12$$
$$x=2$$

The only possible inflection point is 2.

$$f(2)=(2)^3-(2)^2-135(2)$$
$$=8-24-270$$
$$=-286$$

The point $(2,-286)$ is a possible inflection point on the graph.

e) To determine concavity we use 2 to divide the real number line into two intervals, $A:(-\infty,2)$ and B: $(2,\infty)$, Then test a point in each interval.

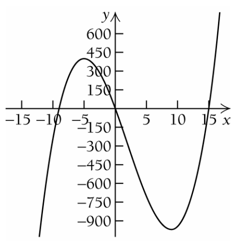
A: Test 0, $f''(0)=6(0)-12=-12<0$

B: Test 3, $f''(3)=6(3)-12=6>0$

We see that *f* is concave down on the interval $(-\infty,2)$ and concave up on the interval $(2,\infty)$, Therefore $(2,-286)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−11	−572
−10	−250
−3	324
0	0
2	−286
5	−700
15	0
16	400



$f(x)=x^3-6x^2-135x$

26. $f'(x)=x^3-3x^2-144x-140$

a) $f'(x)=3x^2-6x-144$

$f''(x)=6x-6$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$3x^2-6x-144=0$$
$$x^2-2x-48=0$$
$$(x+6)(x-8)=0$$
$$x+6=0 \quad \text{or } x-8=0$$
$$x=-6 \quad \text{or} \quad x=8 \quad \text{Critical Values}$$
$$f(-6)=400$$
$$f(8)=-972$$

So the critical points $(-6,400)$ and $(8,-972)$ are on the graph.

c) Using the Second Derivative Test, we have:

$f''(-6)=-42<0$

$f''(8)=42>0$

So, $(-6,400)$ is a relative maximum and $(8,-972)$ is a relative minimum. Therefore, $f(x)$ is increasing on the intervals $(-\infty,-6)$ and $(8,\infty)$ and $f(x)$ is decreasing on the interval $(-6,8)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$$f''(x)=0$$
$$6x-6=0$$
$$x=1$$
$$f(1)=-286$$

The possible inflection point $(1,-286)$ is on the graph.

e) To determine concavity, use 1 to divide the real number line into two intervals, $A:(-\infty,1)$ and B: $(1,\infty)$ and test a point in each interval.

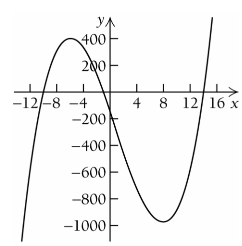
A: Test 0, $f''(0)=6(0)-6=-6<0$

B: Test 2, $f''(2)=6(2)-6=6>0$

Therefore, $f(x)$ is concave down on the interval $(-\infty,1)$ and concave up on the interval $(1,\infty)$. The point $(1,-286)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−12	−572
−10	0
−3	238
0	−140
4	−700
14	0
15	400



$f(x)=x^3-3x^2-144x-140$

27. $f(x)=x^4-4x^3+10$

a) $f'(x)=4x^3-12x^2$

$f''(x)=12x^2-24x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of *x*, so the only critical points of *f* are where $f'(x)=0$.

$4x^3-12x^2=0$

$4x^2(x-3)=0$

$4x^2=0$ or $x-3=0$

$x=0$ or $x=3$

The critical values are 0 and 3.

$f(0)=(0)^4-4(0)^3+10=10$

$f(3)=(3)^4-4(3)^3+10=-17$

The critical points $(0,10)$ and $(3,-17)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0)=12(0)^2-24(0)=0$

The test fails, we will use the First Derivative Test.

Divide $(-\infty,3)$ into two intervals,

A: $(-\infty,0)$ and B: $(0,3)$, and test a point in each interval.

A: Test -1,

$f'(-1)=4(-1)^3-12(-1)^2=-16<0$

B: Test 1, $f'(1)=4(1)^3-12(1)^2=-8<0$

Since, f is decreasing on both intervals, $(0,10)$ is not a relative extremum.

We use the Second Derivative Test for $x=3$.

$f''(3)=12(3)^2-24(3)=36>0$

The critical point $(3,-17)$ is a relative minimum.

When we applied the First Derivative Test, we saw that $f(x)$ was decreasing on the intervals $(-\infty,0]$ and $[0,3]$. Since $(3,-17)$ is a relative minimum, we know that $f(x)$ is increasing on $[3,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of *x*, so the only possible inflection points occur when $f''(x)=0$.

$12x^2-24x=0$

$12x(x-2)=0$

$12x=0$ or $x-2=0$

$x=0$ or $x=2$

$f(0)=(0)^4-4(0)^3+10=10$

$f(2)=(2)^4-4(2)^3+10=-6$

The points $(0,10)$ and $(2,-6)$ are possible inflection points on the graph.

e) To determine concavity we use 0 and 2 to divide the real number line into three intervals,

A : $(-\infty,0)$, B: $(0,2)$, and C: $(2,\infty)$, Then test a point in each interval.

A: Test -1,

$f''(-1)=12(-1)^2-24(-1)=36>0$

B: Test 1,

$f''(1)=12(1)^2-24(1)=-12<0$

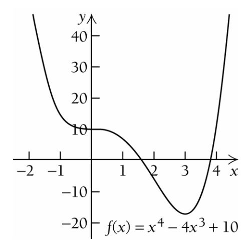
C: Test 3,

$f''(3)=12(3)^2-24(3)=36>0$

We see that *f* is concave up on the intervals $(-\infty,0)$ and $(2,\infty)$ and concave down on the interval $(0,2)$. Therefore both $(0,10)$ and $(2,-6)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	58
-1	15
1	7
4	10
5	135



28. $f(x)=\frac{4}{3}x^3-2x^2+x$

a) $f'(x)=4x^2-4x+1$

$f''(x)=8x-4$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$f'(x)=0$

$4x^2-4x+1=0$

$(2x-1)^2=0$

$2x-1=0$

$x=\frac{1}{2}$ Critical Value

$f\left(\frac{1}{2}\right)=\frac{1}{6}$

So the critical point $\left(\frac{1}{2},\frac{1}{6}\right)$ is on the graph.

c) Using the Second Derivative Test, we have: $f''\left(\frac{1}{2}\right)=0$ and the test fails. We will use the First Derivative Test.
Divide the real number line into two intervals, A: $\left(-\infty,\frac{1}{2}\right)$ and B: $\left(\frac{1}{2},\infty\right)$, and test a point in each interval.

A: Test 0, $f'(0)=4(0)^2-4(0)+1=1>0$

B: Test 1, $f'(1)=4(1)^2-4(1)+1=1>0$

Since, f is increasing on both intervals, $\left(\frac{1}{2},\frac{1}{6}\right)$ is not a relative extremum. However, we now know that $f(x)$ is increasing over the intervals $\left(-\infty,\frac{1}{2}\right)$ and $\left(\frac{1}{2},\infty\right)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$$f''(x)=0$$
$$8x-4=0$$
$$x=\frac{1}{2}$$
$$f\left(\frac{1}{2}\right)=\frac{1}{6}$$

The possible inflection point is $\left(\frac{1}{2},\frac{1}{6}\right)$.

e) To determine concavity, use $\frac{1}{2}$ to divide the real number line into two intervals, A: $\left(-\infty,\frac{1}{2}\right)$ and B: $\left(\frac{1}{2},\infty\right)$ and test a point in each interval.

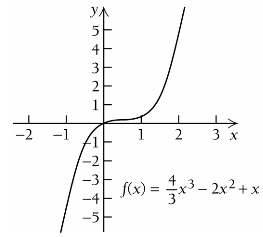
A: Test 0, $f''(0)=8(0)-4=-4<0$

B: Test 1, $f''(1)=8(1)-4=4>0$

Therefore, $f(x)$ is concave down on the interval $\left(-\infty,\frac{1}{2}\right)$ and concave up on the interval $\left(\frac{1}{2},\infty\right)$. The point $\left(\frac{1}{2},\frac{1}{6}\right)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-6	-366
-3	-57
-1	$-\frac{13}{3}$
0	0
1	$\frac{1}{3}$
3	21
6	222



29. $f(x)=x^3-6x^2+12x-6$

a) $f'(x)=3x^2-12x+12$

$f''(x)=6x-12$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$$3x^2-12x+12=0$$
$$x^2-4x+4=0 \quad \text{Dividing by 3}$$
$$(x-2)^2=0$$
$$x-2=0$$
$$x=2$$

The critical value is 2.

$f(2)=(2)^3-6(2)^2+12(2)-6=2$

The critical point $(2,2)$ is on the graph.

c) Applying the Second Derivative Test, we have:

$f''(2)=6(2)-12=0$

The test fails, we will use the First Derivative Test.
Divide the real line into two intervals, A: $(-\infty,2)$ and B: $(2,\infty)$, and test a point in each interval.

A: Test 0,

$f'(0)=3(0)^2-12(0)+12=12>0$

B: Test 3,

$f'(3)=3(3)^2-12(3)+12=3>0$

Since, f is increasing on both intervals, $(2,2)$ is not a relative extremum.

When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-\infty, 2)$ and $(2, \infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of *x*, so the only possible inflection points occur when $f''(x) = 0$.

$$6x - 12 = 0$$
$$6x = 12$$
$$x = 2$$

We have already seen that $f(2) = 2$,so the point $(2,2)$ is a possible inflection point on the graph.

e) To determine concavity we use 2 to divide the real number line into two intervals, A : $(-\infty, 2)$ and B: $(2, \infty)$, Then test a point in each interval.

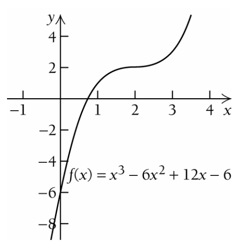
A: Test 0, $f''(0) = 6(0) - 12 = -12 < 0$

B: Test 3, $f''(3) = 6(3) - 12 = 6 > 0$

We see that $f(x)$ is concave down on the interval $(-\infty, 2)$ and concave up on the interval $(2, \infty)$. Therefore, the point $(2,2)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−1	−25
0	−6
1	1
3	3
4	10



30. $f(x) = x^3 + 3x + 1$

a) $f'(x) = 3x^2 + 3$

$f''(x) = 6x$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers, and the equation $f'(x) = 0$ has no real solution. $\left[3x^2 + 3 > 0 \text{ for all } x\right]$. Thus, $f(x)$ has no critical points.

c) Since $f'(x) > 0$ for all real numbers, $f(x)$ is increasing over the entire domain, $(-\infty, \infty)$.

d) Find the points of inflection. Since $f''(x)$ exists for all real numbers, solve:

$$f''(x) = 0$$
$$6x = 0$$
$$x = 0$$
$$f(0) = 1$$

So the point $(0,1)$ is a possible inflection point on the graph.

e) To determine concavity, we use 0 to divide the real number line into two intervals. A : $(-\infty, 0)$ and B: $(0, \infty)$, Then test a point in each interval.

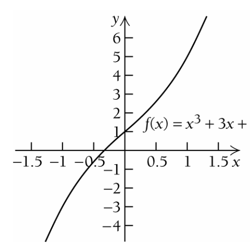
A: Test −1, $f''(-1) = 6(-1) = -6 < 0$

B: Test 1, $f''(1) = 6(1) = 6 > 0$

We see that $f(x)$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$. Therefore, the point $(0,1)$ is an inflection point.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
−2	−13
−1	−3
1	5
2	15



31. $f(x) = 5x^3 - 3x^5$

a) $f'(x) = 15x^2 - 15x^4$

$f''(x) = 30x - 60x^3$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of *x*, so the only critical points of *f* are where $f'(x) = 0$.

$$15x^2 - 15x^4 = 0$$
$$15x^2(1 - x^2) = 0$$
$$15x^2 = 0 \quad \text{or} \quad 1 - x^2 = 0$$
$$x = 0 \quad \text{or} \quad x = \pm 1$$

The critical values are $-1,\ 0,$ and 1.

$f(-1) = 5(-1)^3 - 3(-1)^5 = -2$

$f(0) = 5(0)^3 - 3(0)^5 = 0$

$f(1) = 5(1)^3 - 3(1)^5 = 2$

The critical points $(-1,-2)$, $(0,0)$ and $(1,2)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(-1) = 30(-1) - 60(-1)^3 = 30 > 0$

So, the critical point $(-1,-2)$ is a relative minimum.

$f''(0) = 30(0) - 60(0)^3 = 0$

The test fails, we will use the First Derivative Test.

Divide $(-1,1)$ into two intervals, A: $(-1,0)$ and B: $(0,1)$, and test a point in each interval.

A: Test $-\frac{1}{2}$,

$$f'\left(-\frac{1}{2}\right) = 15\left(-\frac{1}{2}\right)^2 - 15\left(-\frac{1}{2}\right)^4 = \frac{45}{16} > 0$$

B: Test $\frac{1}{2}$,

$$f'\left(\frac{1}{2}\right) = 15\left(\frac{1}{2}\right)^2 - 15\left(\frac{1}{2}\right)^4 = \frac{45}{16} > 0$$

Since, f is increasing on both intervals, $(0,0)$ is not a relative extremum.

We use the Second Derivative Test for $x = 1$.

$f''(1) = 30(1) - 60(1)^3 = -30 < 0$

The critical point $(1,2)$ is a relative maximum.

When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-1,0)$ and $(0,1)$. Since $(-1,-2)$ is a relative minimum, we know that $f(x)$ is decreasing on $(-\infty,-1)$. Since $(1,2)$ is a relative maximum, we know that $f(x)$ is decreasing on $(1,\infty)$.

d) Find the points of inflection. $f''(x)$ exists for all values of x, so the only possible inflection points occur when $f''(x) = 0$.

$30x - 60x^3 = 0$

$30x(1 - 2x^2) = 0$

$30x = 0 \quad \text{or} \quad 1 - 2x^2 = 0$

$x = 0 \quad \text{or} \quad x^2 = \frac{1}{2}$

$x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

$$f\left(-\frac{1}{\sqrt{2}}\right) = 5\left(-\frac{1}{\sqrt{2}}\right)^3 - 3\left(-\frac{1}{\sqrt{2}}\right)^5 = -1.237$$

$f(0) = 5(0)^3 - 3(0)^5 = 0$

$$f\left(\frac{1}{\sqrt{2}}\right) = 5\left(\frac{1}{\sqrt{2}}\right)^3 - 3\left(\frac{1}{\sqrt{2}}\right)^5 = 1.237$$

The points $\left(-\frac{1}{\sqrt{2}},-1.237\right)$, $(0,0)$ and $\left(\frac{1}{\sqrt{2}},1.237\right)$ are possible inflection points on the graph.

e) To determine concavity we use $-\frac{1}{\sqrt{2}}$, 0, and $\frac{1}{\sqrt{2}}$ to divide the real number line into four intervals, A: $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$, B: $\left(-\frac{1}{\sqrt{2}},0\right)$, C: $\left(0,\frac{1}{\sqrt{2}}\right)$, and D: $\left(\frac{1}{\sqrt{2}},\infty\right)$.

Then test a point in each interval.

A: Test -1, $\quad f''(-1) = 30(-1) - 60(-1)^3 = 30 > 0$

B: Test $-\frac{1}{2}$,

$$f''\left(-\frac{1}{2}\right) = 30\left(-\frac{1}{2}\right) - 60\left(-\frac{1}{2}\right)^3 = -\frac{15}{2} < 0$$

C: Test $\frac{1}{2}$,

$$f''\left(\frac{1}{2}\right)=30\left(\frac{1}{2}\right)-60\left(\frac{1}{2}\right)^3$$
$$=\frac{15}{2}>0$$

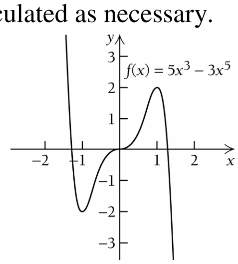
D: Test 1 ,

$$f''(1)=30(1)-60(1)^3$$
$$=-30<0$$

We see that f is concave up on the intervals $\left(-\infty,-\frac{1}{\sqrt{2}}\right)$ and $\left(0,\frac{1}{\sqrt{2}}\right)$ and concave down on the intervals $\left(-\frac{1}{\sqrt{2}},0\right)$ and $\left(\frac{1}{\sqrt{2}},\infty\right)$. Therefore, the points $\left(-\frac{1}{\sqrt{2}},-1.237\right)$, $(0,0)$ and $\left(\frac{1}{\sqrt{2}},1.237\right)$ are inflection points.

e) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-2	56
$-\frac{1}{2}$	$-\frac{17}{32}$
$\frac{1}{2}$	$\frac{17}{32}$
2	-56



32. $f(x)=20x^3-3x^5$

a) $f'(x)=60x^2-15x^4$

$f''(x)=120x-60x^3$

The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all real numbers. Solve:

$$f'(x)=0$$
$$60x^2-15x^4=0$$
$$15x^2(4-x^2)=0$$
$$15x^2=0 \quad \text{or} \quad 4-x^2=0$$
$$x=0 \quad \text{or} \quad x=\pm 2$$
$$f(-2)=-64$$
$$f(0)=0$$
$$f(2)=64$$

So the critical points $(-2,-64)$, $(0,0)$, and $(2,64)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(-2)=120(-2)-60(-2)^3=240>0$

So, the critical point $(-2,-64)$ is a relative minimum.

$f''(0)=120(0)-60(0)^3=0$

The test fails, we will use the First Derivative Test.

Divide $(-2,2)$ into two intervals,

A: $(-2,0)$ and B: $(0,2)$, and test a point in each interval.

A: Test -1,

$f'(-1)=60(-1)^2-15(-1)^4=45>0$

B: Test 1,

$f'(1)=60(1)^2-15(1)^4=45>0$

Since, f is increasing on both intervals, $(0,0)$ is not a relative extremum.

We use the Second Derivative Test for $x=2$.

$f''(2)=120(2)-60(2)^3=-240<0$

The critical point $(2,64)$ is a relative maximum.

c) When we applied the First Derivative Test, we saw that $f(x)$ was increasing on the intervals $(-2,0)$ and $(0,2)$. Since $(-2,-64)$ is a relative minimum, we know that $f(x)$ is decreasing on $(-\infty,-2)$. Since $(2,64)$ is a relative maximum, we know that $f(x)$ is decreasing on $(2,\infty)$.

d) Find the inflection points. $f''(x)$ exists for all real numbers. Solve:

$$120x-60x^3=0$$
$$60x(2-x^2)=0$$
$$60x=0 \quad \text{or} \quad 2-x^2=0$$
$$x=0 \quad \text{or} \quad x^2=2$$
$$x=0 \quad \text{or} \quad x=\pm\sqrt{2}$$
$$f(-\sqrt{2})=-28\sqrt{2}$$
$$f(0)=0$$
$$f(\sqrt{2})=28\sqrt{2}$$

The points $\left(-\sqrt{2},-28\sqrt{2}\right)$, $(0,0)$ and $\left(\sqrt{2},28\sqrt{2}\right)$ are possible inflection points on the graph.

e) To determine concavity we use $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the real number line into four intervals,
A : $\left(-\infty,-\sqrt{2}\right)$, B: $\left(--\sqrt{2},0\right)$, C: $\left(0,\sqrt{2}\right)$, and D: $\left(\sqrt{2},\infty\right)$

Then test a point in each interval.

A: Test -2, $f''(-2)=120(-2)-60(-2)^3$
$= 240 > 0$

B: Test -1, $f''(-1)=120(-1)-60(-1)^3$
$= -60 < 0$

C: Test 1, $f''(1)=120(1)-60(1)^3$
$= 60 > 0$

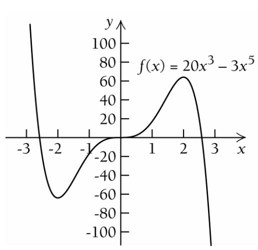
D: Test 2 , $f''(2)=120(2)-60(2)^3$
$= -240 < 0$

We see that f is concave up on the intervals $\left(-\infty,-\sqrt{2}\right)$ and $\left(0,\sqrt{2}\right)$ and concave down on the intervals $\left(-\sqrt{2},0\right)$ and $\left(\sqrt{2},\infty\right)$.

Therefore, the points $\left(-\sqrt{2},-28\sqrt{2}\right)$, $(0,0)$ and $\left(\sqrt{2},28,\sqrt{2}\right)$ are inflection points.

f) We sketch the graph using the preceding information. Additional function values may also be calculated as necessary.

x	$f(x)$
-3	189
-1	-17
1	17
3	-189



33. $f(x)=x^2(3-x)^2$
$= x^2\left(9-6x+x^2\right)$
$= 9x^2-6x^3+x^4$

a) $f'(x)=18x-18x^2+4x^3$
$f''(x)=18-36x+12x^2$
The domain of f is $\mathbb{R}$.

b) $f'(x)$ exists for all values of x, so the only critical points of f are where $f'(x)=0$.

$18x-18x^2+4x^3=0$
$2x\left(9-9x+2x^2\right)=0$
$2x(3-2x)(3-x)=0$
$2x=0$ or $3-2x=0$ or $3-x=0$
$x=0$ or $x=\dfrac{3}{2}$ or $x=3$

The critical values are $0,\ \dfrac{3}{2}$, and 3 .

$f(0)=(0)^2(3-(0))^2=0$

$f\left(\dfrac{3}{2}\right)=\left(\dfrac{3}{2}\right)^2\left(3-\left(\dfrac{3}{2}\right)\right)^2=\dfrac{81}{16}$

$f(3)=(3)^2(3-(3))^2=0$

The critical points $(0,0)$, $\left(\dfrac{3}{2},\dfrac{81}{16}\right)$,and $(3,0)$ are on the graph.

c) Applying the Second Derivative Test, we have:

$f''(0)=18-36(0)+12(0)^2=18>0$

So, the critical point $(0,0)$ is a relative minimum.

$f''\left(\dfrac{3}{2}\right)=18-36\left(\dfrac{3}{2}\right)+12\left(\dfrac{3}{2}\right)^2=-9<0$

So, the critical point $\left(\dfrac{3}{2},\dfrac{81}{16}\right)$ is a relative maximum.

$f''(3)=18-36(3)+12(3)^2=18>0$

So, the critical point $(3,0)$ is a relative minimum.

We use the points $0,\ \dfrac{3}{2}$, and 3 to divide the real number line into four intervals, $(-\infty,0)$, $\left(0,\dfrac{3}{2}\right)$, $\left(\dfrac{3}{2},3\right)$, and $(3,\infty)$, we know that $f(x)$ is decreasing on the intervals $(-\infty,0]$ and $\left[\dfrac{3}{2},3\right]$, and $f(x)$ is increasing on the intervals $\left[0,\dfrac{3}{2}\right]$ and $[3,\infty)$.