Exercise Set 2.3

1. $f(x)=\dfrac{2x-3}{x-5}$

The expression is in simplified form. We set the denominator equal to zero and solve.

$$x-5=0$$
$$x=5$$

The vertical asymptote is the line $x=5$.

2. $f(x)=\dfrac{x+4}{x-2}$

The expression is in simplified form. The vertical asymptote is the line $x=2$.

3. $f(x)=\dfrac{3x}{x^2-9}$

First, we write the function in simplified form.

$$f(x)=\frac{3x}{(x-3)(x+3)}.$$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$$(x-3)(x+3)=0$$
$$x-3=0 \quad \text{or} \quad x+3=0$$
$$x=3 \quad \text{or} \quad x=-3$$

The vertical asymptotes are the lines $x=-3$ and $x=3$.

4. $f(x)=\dfrac{5x}{x^2-25}=\dfrac{5x}{(x-5)(x+5)}$

The vertical asymptotes are the lines $x=-5$ and $x=5$.

5. $f(x)=\dfrac{x+2}{x^3-6x^2+8x}$

First, we write the function in simplified form.

$$f(x)=\frac{x+2}{x\left(x^2-6x+8\right)} \qquad \text{Factor out } x.$$
$$=\frac{x+2}{x(x-4)(x-2)}$$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$$x(x-2)(x-4)=0$$
$$x=0 \quad \text{or} \quad x-2=0 \quad \text{or} \quad x-4=0$$
$$x=0 \quad \text{or} \quad x=2 \quad \text{or} \quad x=4$$

The vertical asymptotes are the lines $x=0,\ x=2,$ and $x=4$.

6. $f(x)=\dfrac{x+3}{x^3-x}=\dfrac{x+3}{x\left(x^2-1\right)}=\dfrac{x+3}{x(x-1)(x+1)}$

The vertical asymptotes are the lines $x=0,\ x=-1,$ and $x=1$.

7. $f(x)=\dfrac{x+6}{x^2+7x+6}$

First, we write the function in simplified form.

$$f(x)=\frac{x+6}{(x+6)(x+1)}$$
$$=\frac{1}{x+1} \qquad \text{Dividing common terms}$$

Once the expression is in simplified form, we set the denominator equal to zero and solve.

$$x+1=0$$
$$x=-1$$

The vertical asymptote is the line $x=-1$.

8. $f(x)=\dfrac{x+2}{x^2+6x+8}=\dfrac{x+2}{(x+2)(x+4)}=\dfrac{1}{x+4}$

The vertical asymptote is the line $x=-4$.

9. $f(x)=\dfrac{6}{x^2+36}$

The function is in simplified form. The equation $x^2+36=0$ has no real solution; therefore, the function does not have any vertical asymptotes.

10. $f(x)=\dfrac{7}{x^2+49}$

The function is in simplified form. The equation $x^2+49=0$ has no real solution; therefore, the function does not have any vertical asymptotes.

11. $f(x)=\dfrac{6x}{8x+3}$

To find the horizontal asymptote, we consider $\lim\limits_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty,\ \dfrac{b}{ax^n}\to 0$ for any positive integer n.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x}{8x+3}$$

$$= \lim_{x\to\infty} \frac{6x}{8x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \quad \text{Multiplying by a form of 1}$$

$$= \lim_{x\to\infty} \frac{\frac{6x}{x}}{\frac{8x}{x}+\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{6}{8+\frac{3}{x}}$$

$$= \frac{6}{8+0} \quad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$

$$= \frac{6}{8} = \frac{3}{4}.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = \frac{3}{4}.$

The horizontal asymptote is the line $y = \frac{3}{4}$.

12. $f(x) = \frac{3x^2}{6x^2+x}$

Find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{3x^2}{6x^2+x} = \lim_{x\to\infty} \frac{3}{6+\frac{1}{x}} = \frac{3}{6+0} = \frac{1}{2}.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = \frac{1}{2}.$

The horizontal asymptote is the line $y = \frac{1}{2}$.

13. $f(x) = \frac{4x}{x^2-3x}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \frac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x}{x^2-3x}$$

$$= \lim_{x\to\infty} \frac{4x}{x^2-3x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \quad \text{Multiplying by a form of 1}$$

$$= \lim_{x\to\infty} \frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}-\frac{3}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{3}{x^2}}$$

$$= \frac{0}{1+0} \quad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$

$$= 0.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = 0.$

The horizontal asymptote is the line $y = 0$.

14. $f(x) = \frac{2x}{3x^3-x^2}$

Find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{2x}{3x^3-x^2} = \lim_{x\to\infty} \frac{\frac{2}{x^2}}{3-\frac{1}{x}} = \frac{0}{3-0} = 0.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = 0.$

The horizontal asymptote is the line $y = 0$.

15. $f(x) = 5 - \frac{3}{x}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x\to\infty, \frac{b}{ax^n}\to 0$ for any positive integer *n*.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 5 - \frac{3}{x}$$

$$= 5 - 0 \quad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$

$$= 5.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = 5.$

The horizontal asymptote is the line $y = 5$.

16. $f(x) = 4 + \frac{2}{x}$

Find $\lim_{x\to\infty} f(x)$

$\lim_{x\to\infty} 4 + \frac{2}{x} = 4 + 0 = 4$.

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x) = 4.$

The horizontal asymptote is the line $y = 4$.

17. $f(x) = \frac{8x^4 - 5x^2}{2x^3 + x^2}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x \to \infty, \frac{b}{ax^n} \to 0$ for any positive integer .

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{8x^4 - 5x^2}{2x^3 + x^2} \\
&= \lim_{x\to\infty} \frac{8x^4 - 5x^2}{2x^3 + x^2} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \quad \text{Multiplying by a form of 1} \\
&= \lim_{x\to\infty} \frac{8x - \frac{5}{x}}{2 - \frac{1}{x^2}} \\
&= \frac{\lim_{x\to\infty} 8x - \frac{5}{x}}{2 - 0} \\
&= \infty
\end{aligned}$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x) = -\infty.$

The function increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$. Therefore, the function does not have a horizontal asymptote.

18. $f(x) = \frac{6x^3 + 4x}{3x^2 - x}$

Find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{6x^3 + 4x}{3x^2 - x} = \lim_{x\to\infty} \frac{6x + \frac{4}{x}}{3 - \frac{1}{x}} = \frac{\lim_{x\to\infty} 6x + \frac{4}{x}}{3 - 0} = \infty .$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x) = -\infty.$

The function increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$. Therefore, the function does not have a horizontal asymptote.

19. $f(x) = \frac{6x^4 + 4x^2 - 7}{2x^5 - x + 3}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x \to \infty, \frac{b}{ax^n} \to 0$ for any positive integer *n*.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{6x^4 + 4x^2 - 7}{2x^5 - x + 3} \\
&= \lim_{x\to\infty} \frac{6x^4 + 4x^2 - 7}{2x^5 - x + 3} \cdot \frac{\frac{1}{x^5}}{\frac{1}{x^5}} \\
&= \lim_{x\to\infty} \frac{\frac{6}{x} + \frac{4}{x^3} - \frac{7}{x^5}}{2 - \frac{1}{x^4} + \frac{3}{x^5}} \\
&= \frac{0}{2+0} \qquad \left[\text{as } x \to \infty, \frac{b}{ax^n} \to 0\right] \\
&= 0
\end{aligned}$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x) = 0.$

The horizontal asymptote is the line $y = 0$.

20. $f(x) = \frac{4x^3 - 3x + 2}{x^3 + 2x - 4}$

Find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{4x^3 - 3x + 2}{x^3 + 2x - 4} = \lim_{x\to\infty} \frac{4 - \frac{3}{x^2} + \frac{2}{x^3}}{1 + \frac{2}{x^2} - \frac{4}{x^3}} = \frac{4}{1} = 4$$

In a similar manner, it can be shown that $\lim_{x\to-\infty} f(x) = 4.$

The horizontal asymptote is the line $y = 4$.

21. $f(x) = \frac{2x^3 - 4x + 1}{4x^3 + 2x - 3}$

To find the horizontal asymptote, we consider $\lim_{x\to\infty} f(x)$. To find the limit, we will use some algebra and the fact that as $x \to \infty, \frac{b}{ax^n} \to 0$ for any positive integer *n*.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2x^3-4x+1}{4x^3+2x-3}$$

$$= \lim_{x\to\infty}\frac{2x^3-4x+1}{4x^3+2x-3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to\infty}\frac{2-\frac{4}{x^2}+\frac{1}{x^3}}{4+\frac{2}{x^2}-\frac{3}{x^3}}$$

$$= \frac{2-0+0}{4+0-0} \qquad \left[\text{as } x\to\infty, \frac{b}{ax^n}\to 0\right]$$

$$= \frac{2}{4} = \frac{1}{2}$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = \frac{1}{2}$.

The horizontal asymptote is the line $y = \frac{1}{2}$.

22. $f(x) = \dfrac{5x^4-2x^3+x}{x^5-x^3+8}$

Find $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty}\frac{5x^4-2x^3+x}{x^5-x^3+8} = \lim_{x\to\infty}\frac{\frac{5}{x}-\frac{2}{x^2}+\frac{1}{x^5}}{1-\frac{1}{x^2}+\frac{8}{x^5}} = \frac{0}{1} = 0.$$

In a similar manner, it can be shown that

$\lim_{x\to-\infty} f(x) = 0.$

The horizontal asymptote is the line $y = 0$.

23. $f(x) = -\dfrac{5}{x} = -5x^{-1}$

a) *Intercepts.* Since the numerator is the constant -5, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 5x^{-2} = \frac{5}{x^2}$$

$$f''(x) = -10x^{-3} = -\frac{10}{x^3}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = \dfrac{5}{(-1)^2} = 5 > 0$

B: Test 1, $f'(1) = \dfrac{5}{(1)^2} = 5 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

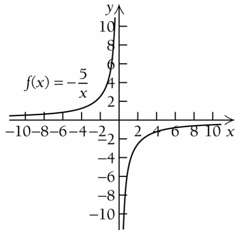
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = -\dfrac{10}{(-1)^3} = 10 > 0$

B: Test 1, $f''(1) = -\dfrac{10}{(1)^3} = -10 < 0$

Therefore, $f(x)$ is concave up on $(-\infty,0)$ and concave down on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



24. $f(x) = \frac{4}{x} = 4x^{-1}$

a) *Intercepts.* Since the numerator is the constant 4, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -4x^{-2} = -\frac{4}{x^2}$$

$$f''(x) = 8x^{-3} = \frac{8}{x^3}$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = -\frac{4}{(-1)^2} = -4 < 0$

B: Test 1, $f'(1) = -\frac{4}{(1)^2} = -4 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

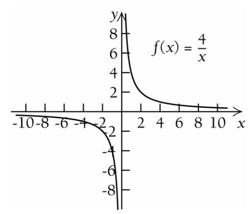
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = \frac{8}{(-1)^3} = -8 < 0$

B: Test 1, $f''(1) = \frac{8}{(1)^3} = 8 > 0$

Therefore, $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

h) *Sketch.*



25. $f(x) = \frac{1}{x-5} = (x-5)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0) = \frac{1}{(0)-5} = -\frac{1}{5}$$

The point $\left(0, -\frac{1}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 5$, so the line $x = 5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -(x-5)^{-2} = \frac{-1}{(x-5)^2}$$

$$f''(x) = 2(x-5)^{-3} = \frac{2}{(x-5)^3}$$

The domain of f is $(-\infty,5)\cup(5,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 5, but 5 is not in the domain of the function, so $x=5$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 5 to divide the real number line into two intervals A: $(-\infty,5)$ and B: $(5,\infty)$, and we test a point in each interval.

A: Test 4, $f'(4) = \frac{-1}{(4-5)^2} = -\frac{1}{1} = -1 < 0$

B: Test 6, $f'(6) = \frac{-1}{(6-5)^2} = -\frac{1}{1} = -1 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 5, but because 5 is not in the domain of the function, there cannot be an inflection point at 5. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

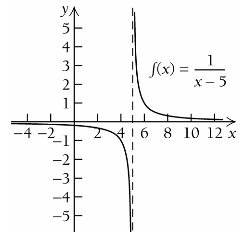
g) *Concavity.* We use 5 to divide the real number line into two intervals A: $(-\infty,5)$ and B: $(5,\infty)$, and we test a point in each interval.

A: Test 4, $f''(4) = \frac{2}{(4-5)^3} = \frac{2}{-1} = -2 < 0$

B: Test 6, $f''(6) = \frac{2}{(6-5)^3} = \frac{2}{1} = 2 > 0$

Therefore, $f(x)$ is concave down on $(-\infty,5)$ and concave up on $(5,\infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



26. $f(x) = \frac{-2}{x-5} = -2(x-5)^{-1}$

a) *Intercepts.* Since the numerator is the constant -2, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0) = \frac{-2}{(0)-5} = \frac{2}{5}$$

The point $\left(0,\frac{2}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=5$, so the line $x=5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2(x-5)^{-2} = \frac{2}{(x-5)^2}$$

$$f''(x) = -4(x-5)^{-3} = \frac{-4}{(x-5)^3}$$

The domain of f is $(-\infty,5)\cup(5,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 5, but 5 is not in the domain of the function, so $x=5$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 5 to divide the real number line into two intervals A: $(-\infty,5)$ and B: $(5,\infty)$, and we test a point in each interval.

A: Test 4, $f'(4) = \frac{2}{(4-5)^2} = \frac{2}{1} = 2 > 0$

B: Test 6, $f'(6) = \frac{2}{(6-5)^2} = \frac{2}{1} = 2 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 5, but because 5 is not in the domain of the function, there cannot be an inflection point at 5. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

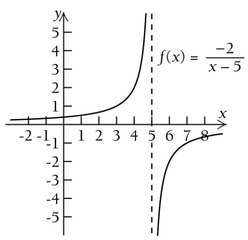
g) *Concavity.* We use 5 to divide the real number line into two intervals A: $(-\infty,5)$ and B: $(5,\infty)$, and we test a point in each interval.

A: Test 4, $f''(4)=\dfrac{-4}{(4-5)^3}=\dfrac{-4}{-1}=4>0$

B: Test 6, $f''(6)=\dfrac{-4}{(6-5)^3}=\dfrac{-4}{1}=-4<0$

Therefore, $f(x)$ is concave up on $(-\infty,5)$ and concave down on $(5,\infty)$.

h) *Sketch*.



27. $f(x)=\dfrac{1}{x+2}=(x+2)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no *x*-intercepts. To find the *y*-intercepts we compute $f(0)$

$$f(0)=\frac{1}{(0)+2}=\frac{1}{2}$$

The point $\left(0,\frac{1}{2}\right)$ is the *y*-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+2)^{-2}=\frac{-1}{(x+2)^2}$$

$$f''(x)=2(x+2)^{-3}=\frac{2}{(x+2)^3}$$

The domain of f is $(-\infty,-2)\cup(2,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of *x* except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f'(-3)=\dfrac{-1}{((-3)+2)^2}=-1<0$

B: Test -1, $f'(-1)=\dfrac{-1}{((-1)+2)^2}=-1<0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

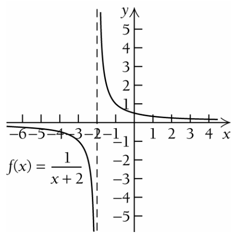
g) *Concavity.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f''(-3)=\dfrac{2}{((-3)+2)^3}=-2<0$

B: Test -1, $f''(-1)=\dfrac{2}{((-1)+2)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,-2)$ and concave up on $(-2,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



28. $f(x)=\dfrac{1}{x-3}=(x-3)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0)=\frac{1}{(0)-3}=-\frac{1}{3}$$

The point $\left(0,-\dfrac{1}{3}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=3$, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x-3)^{-2}=\frac{-1}{(x-3)^2}$$

$$f''(x)=2(x-3)^{-3}=\frac{2}{(x-3)^3}$$

The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f'(2)=\dfrac{-1}{((2)-3)^2}=-1<0$

B: Test 4, $f'(4)=\dfrac{-1}{((4)-3)^2}=-1<0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

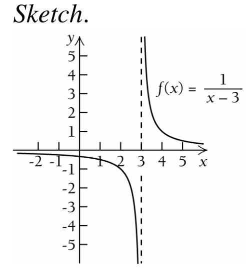
g) *Concavity.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2)=\dfrac{2}{((2)-3)^3}=-2<0$

B: Test 4, $f''(4)=\dfrac{2}{((4)-3)^3}=2>0$

Therefore, $f(x)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$.

h) *Sketch*.



29. $f(x)=\dfrac{-3}{x-3}=-3(x-3)^{-1}$

a) *Intercepts.* Since the numerator is the constant -3, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0)=\frac{-3}{(0)-3}=\frac{3}{3}=1$$

The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 3$, so the line $x = 3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 3(x-3)^{-2} = \frac{3}{(x-3)^2}$$

$$f''(x) = -6(x-3)^{-3} = \frac{-6}{(x-3)^3}$$

The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x = 3$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f'(2) = \dfrac{3}{((2)-3)^2} = 3 > 0$

B: Test 4, $f'(4) = \dfrac{3}{((4)-3)^2} = 3 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

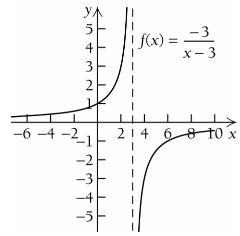
g) *Concavity.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2) = \dfrac{-6}{((2)-3)^3} = 6 > 0$

B: Test 4, $f''(4) = \dfrac{-6}{((4)-3)^3} = -6 < 0$

Therefore, $f(x)$ is concave up on $(-\infty,3)$ and concave down on $(3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



30. $f(x) = \dfrac{-2}{x+5} = -2(x+5)^{-1}$

a) *Intercepts.* Since the numerator is the constant -2, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0) = \frac{-2}{(0)+5} = \frac{-2}{5} = -\frac{2}{5}$$

The point $\left(0,-\frac{2}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = -5$, so the line $x = -5$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2(x+5)^{-2} = \frac{2}{(x+5)^2}$$

$$f''(x) = -4(x+5)^{-3} = \frac{-4}{(x+5)^3}$$

The domain of f is $(-\infty,-5)\cup(-5,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -5, but -5 is not in the domain of the function, so $x = -5$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -5 to divide the real number line into two intervals A: $(-\infty,-5)$ and B: $(-5,\infty)$, and we test a point in each interval.

A: Test -6, $f'(-6) = \dfrac{2}{((-6)+5)^2} = 2 > 0$

B: Test -4, $f'(-4) = \dfrac{2}{((-4)+5)^2} = 2 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at -5, but because -5 is not in the domain of the function, there cannot be an inflection point at -5. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

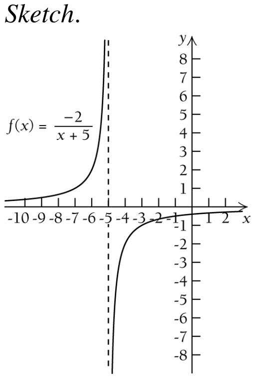
g) *Concavity.* We use -5 to divide the real number line into two intervals A: $(-\infty, -5)$ and B: $(-5, \infty)$, and we test a point in each interval.

A: Test -6, $f''(-6) = \dfrac{-4}{((-6)+5)^3} = 4 > 0$

B: Test -4, $f''(-4) = \dfrac{-4}{((-4)+5)^3} = -4 < 0$

Therefore, $f(x)$ is concave up on $(-\infty, -5)$ and concave down on $(-5, \infty)$.

h) *Sketch*.



31. $f(x) = \dfrac{3x-1}{x}$

a) *Intercepts.* To find the x-intercepts, solve $f(x) = 0$.

$$\frac{3x-1}{x} = 0$$
$$3x - 1 = 0$$
$$3x = 1$$
$$x = \frac{1}{3}$$

Since $x = \frac{1}{3}$ does not make the denominator 0, the x-intercept is $\left(\frac{1}{3}, 0\right)$

The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y = \frac{3}{1}$, or $y = 3$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = \frac{1}{x^2}$$
$$f''(x) = -2x^{-3} = -\frac{2}{x^3}$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = \dfrac{1}{(-1)^2} = 1 > 0$

B: Test 1, $f'(1) = \dfrac{1}{(-1)^2} = 1 > 0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

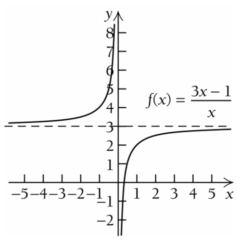
g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = \dfrac{-2}{(-1)^3} = 2 > 0$

B: Test 1, $f''(1) = \dfrac{-2}{(1)^3} = -2 < 0$

Therefore, $f(x)$ is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



32. $f(x) = \dfrac{2x+1}{x}$

a) *Intercepts.* To find the *x*-intercepts, solve $f(x) = 0$.

$$\frac{2x+1}{x} = 0$$
$$2x + 1 = 0$$
$$x = -\frac{1}{2}$$

This value does not make the denominator 0, the *x*-intercept is $\left(-\dfrac{1}{2}, 0\right)$

The number 0 is not in the domain of $f(x)$ so there are no *y*-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y = \dfrac{2}{1}$, or $y = 2$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = -\dfrac{1}{(-1)^2} = -1 < 0$

B: Test 1, $f'(1) = -\dfrac{1}{(-1)^2} = -1 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

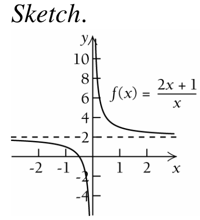
g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = \dfrac{2}{(-1)^3} = -2 < 0$

B: Test 1, $f''(1) = \dfrac{2}{(1)^3} = 2 > 0$

Therefore, $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

h) *Sketch.*



33. $f(x) = x + \frac{2}{x} = \frac{x^2+2}{x}$

a) *Intercepts.* The equation $f(x) = 0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator. As $|x|$ gets very large,

$f(x) = x + \frac{2}{x}$ approaches x. Therefore, $y = x$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x) = 1 - 2x^{-2} = 1 - \frac{2}{x^2}$$

$$f''(x) = 4x^{-3} = \frac{4}{x^3}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The critical points will occur when $f'(x) = 0$.

$$1 - \frac{2}{x^2} = 0$$
$$1 = \frac{2}{x^2}$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

Thus, $-\sqrt{2}$ and $\sqrt{2}$ are critical values. $f(-\sqrt{2}) = -2\sqrt{2}$ and $f(\sqrt{2}) = 2\sqrt{2}$, so the critical points $(-\sqrt{2},-2\sqrt{2})$ and $(\sqrt{2},2\sqrt{2})$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-\sqrt{2}$, 0, and $\sqrt{2}$ to divide the real number line into four intervals
A: $(-\infty,-\sqrt{2})$ B: $(-\sqrt{2},0)$, C: $(0,\sqrt{2})$, and D: $(\sqrt{2},\infty)$.

A: Test -2, $f'(-2) = 1 - \frac{2}{(-2)^2} = \frac{1}{2} > 0$

B: Test -1, $f'(-1) = 1 - \frac{2}{(-1)^2} = -1 < 0$

C: Test 1, $f'(1) = 1 - \frac{2}{(1)^2} = -1 < 0$

D: Test 2, $f'(2) = 1 - \frac{2}{(2)^2} = \frac{1}{2} > 0$

Then $f(x)$ is increasing on $(-\infty,-\sqrt{2}]$ and $[\sqrt{2},\infty)$ and is decreasing on $[-\sqrt{2},0)$ and $(0,\sqrt{2}]$. Therefore, $(-\sqrt{2},-2\sqrt{2})$ is a relative maximum, and $(\sqrt{2},2\sqrt{2})$ is a relative minimum.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

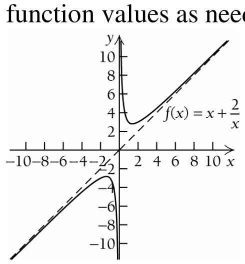
g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = \frac{4}{(-1)^3} = -4 < 0$

B: Test 1, $f''(1) = \frac{4}{(1)^3} = 4 > 0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



34. $f(x) = x + \frac{9}{x}$

a) *Intercepts.* The equation $f(x) = 0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. As $|x|$ gets very large,
$f(x) = x + \frac{9}{x}$ approaches x. Thus, $y = x$ is the slant asymptote.

c) *Derivatives and Domain.*
$$f'(x) = 1 - 9x^{-2} = 1 - \frac{9}{x^2}$$
$$f''(x) = 18x^{-3} = \frac{18}{x^3}$$
The domain of f is $(-\infty, 0) \cup (0, \infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The solution to $f'(x) = 0$ is $x = \pm 3$ Thus, -3 and 3 are critical values.
$f(-3) = -6$ and $f(3) = 6$, so the critical points $(-3, -6)$ and $(3, 6)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use -3, 0, and 3 to divide the real number line into four intervals
A: $(-\infty, -3)$ B: $(-3, 0)$, C: $(0, 3)$, and D: $(3, \infty)$.

A: Test -4, $f'(-4) = \frac{7}{16} > 0$

B: Test -1, $f'(-1) = -8 < 0$

C: Test 1, $f'(1) = -8 < 0$

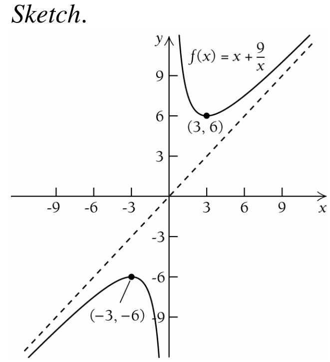
D: Test 4, $f'(4) = \frac{7}{16} > 0$

Then $f(x)$ is increasing on $(-\infty, -3]$ and $[3, \infty)$ and is decreasing on $[-3, 0)$ and $(0, 3]$. Therefore, $(-3, -6)$ is a relative maximum, and $(3, 6)$ is a relative minimum.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use 0 to divide the real number line into two intervals
A: $(-\infty, 0)$ and B: $(0, \infty)$, and we test a point in each interval.
A: Test -1, $f''(-1) = -18 < 0$
B: Test 1, $f''(1) = 18 > 0$
Therefore, $f(x)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

h) *Sketch.*



35. $f(x) = \frac{-1}{x^2} = -x^{-2}$

a) *Intercepts.* Since the numerator is the constant -1, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2x^{-3} = \frac{2}{x^3}$$

$$f''(x) = -6x^{-4} = -\frac{6}{x^4}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = \frac{2}{(-1)^3} = -2 < 0$

B: Test 1, $f'(1) = \frac{2}{(1)^3} = 2 > 0$

Then $f(x)$ is decreasing on $(-\infty,0)$ and is increasing on $(0,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

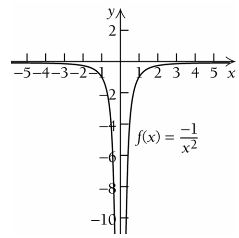
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1) = -\frac{6}{(-1)^4} = -6 < 0$

B: Test 1, $f''(1) = -\frac{6}{(1)^4} = -6 < 0$

Therefore, $f(x)$ is concave down on both intervals.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



36. $f(x) = \frac{2}{x^2} = 2x^{-2}$

a) *Intercepts.* Since the numerator is the constant 2, there are no x-intercepts. The number 0 is not in the domain of the function, so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -4x^{-3} = -\frac{4}{x^3}$$

$$f''(x) = 12x^{-4} = \frac{12}{x^4}$$

The domain of f is $(-\infty,0)\cup(0,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = 4 > 0$

B: Test 1, $f'(1) = -4 < 0$

Then $f(x)$ is increasing on $(-\infty,0)$ and is decreasing on $(0,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

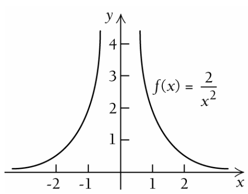
g) *Concavity.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f''(-1)=\dfrac{12}{(-1)^4}=12>0$

B: Test 1, $f''(1)=\dfrac{12}{(1)^4}=12>0$

Therefore, $f(x)$ is concave up on both intervals.

h) *Sketch.*



37. $f(x)=\dfrac{x}{x+2}$

a) *Intercepts.* To find the x-intercepts, solve $f(x)=0$.

$$\frac{x}{x+2}=0$$
$$x=0$$

Since $x=0$ does not make the denominator 0, the x-intercept is $(0,0)$. $f(0)=0$, so the y-intercept is $(0,0)$ also.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{1}$, or $y=1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=\frac{2}{(x+2)^2}$$

$$f''(x)=-4(x+2)^{-3}=-\frac{4}{(x+2)^3}$$

The domain of f is $(-\infty,-2)\cup(2,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f'(-3)=\dfrac{2}{((-3)+2)^2}=2>0$

B: Test -1, $f'(-1)=\dfrac{2}{((-1)+2)^2}=2>0$

Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

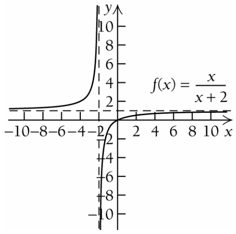
g) *Concavity.* We use -2 to divide the real number line into two intervals A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.

A: Test -3, $f''(-3)=-\dfrac{4}{((-3)+2)^3}=4>0$

B: Test -1, $f''(-1)=-\dfrac{4}{((-1)+2)^3}=-4<0$

Therefore, $f(x)$ is concave up on $(-\infty,-2)$ and concave down on $(-2,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



38. $f(x) = \dfrac{x}{x-3}$

a) *Intercepts.* The numerator is 0 for $x = 0$ and since this value of *x* does not make the denominator 0, the *x*-intercept is $(0,0)$.

$f(0) = 0$, so the *y*-intercept is $(0,0)$ also.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 3$, so the line $x = 3$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so

$y = \frac{1}{1}$, or $y = 1$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = \frac{-3}{(x-3)^2}$$

$$f''(x) = 6(x-3)^{-3} = \frac{6}{(x-3)^3}$$

The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of *x* except 3, but 3 is not in the domain of the function, so $x = 3$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f'(2) = -3 < 0$

B: Test 4, $f'(4) = -3 < 0$

Then $f(x)$ is decreasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points*. $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

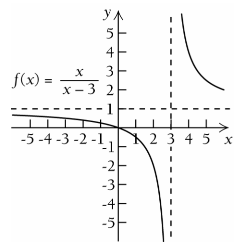
g) *Concavity.* We use 3 to divide the real number line into two intervals
A: $(-\infty,3)$ and B: $(3,\infty)$, and we test a point in each interval.

A: Test 2, $f''(2) = -6 < 0$

B: Test 4, $f''(4) = 6 > 0$

Therefore, $f(x)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



39. $f(x) = \dfrac{-1}{x^2+2} = -(x^2+2)^{-1}$

a) *Intercepts.* Since the numerator is the constant -1, there are no *x*-intercepts.

$f(0) = \dfrac{-1}{(0)^2+2} = -\dfrac{1}{2}$, so the *y*-intercept is $\left(0,-\dfrac{1}{2}\right)$.

b) *Asymptotes.*
Vertical. $x^2 + 2 = 0$ has no real solution, so there are no vertical asymptotes.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = 2x(x^2+2)^{-2} = \frac{2x}{(x^2+2)^2}$$

$$f''(x) = \frac{-6x^2+4}{(x^2+2)^3}$$

The domain of f is $\mathbb{R}$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all real numbers. Solve $f'(x)=0$

$$\frac{2x}{\left(x^2+2\right)^2}=0$$
$$2x=0$$
$$x=0$$

The critical value is 0. From step (a) we found $\left(0,-\frac{1}{2}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1)=-\frac{2}{9}<0$

B: Test 1, $f'(1)=\frac{2}{9}>0$

Then $f(x)$ is decreasing on $(-\infty,0]$ and is increasing on $[0,\infty)$. Thus $\left(0,-\frac{1}{2}\right)$ is a relative minimum.

f) *Inflection points.* $f''(x)$ exists for all real numbers. Solve $f''(x)=0$.

$$\frac{-6x^2+4}{\left(x^2+2\right)^3}=0$$
$$-6x^2+4=0$$
$$-6x^2=-4$$
$$x^2=\frac{2}{3}$$
$$x=\pm\sqrt{\frac{2}{3}}$$

$f\left(-\sqrt{\frac{2}{3}}\right)=-\frac{3}{8}$ and $f\left(\sqrt{\frac{2}{3}}\right)=-\frac{3}{8}$

So, $\left(-\sqrt{\frac{2}{3}},-\frac{3}{8}\right)$ and $\left(\sqrt{\frac{2}{3}},-\frac{3}{8}\right)$ are possible points of inflection.

g) *Concavity.* We use $-\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{2}{3}}$ to divide the real number line into three intervals

A: $\left(-\infty,-\sqrt{\frac{2}{3}}\right)$ B: $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$,

and C: $\left(\sqrt{\frac{2}{3}},\infty\right)$

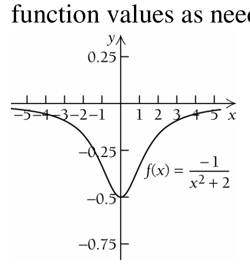
A: Test -1, $f''(-1)=-\frac{2}{27}<0$

B: Test 0, $f''(0)=\frac{1}{2}>0$

C: Test 1, $f''(1)=-\frac{2}{27}<0$

Therefore, $f(x)$ is concave down on $\left(-\infty,-\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{\frac{2}{3}},\infty\right)$ and concave up on $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$. Therefore the points $\left(-\sqrt{\frac{2}{3}},-\frac{3}{8}\right)$ and $\left(\sqrt{\frac{2}{3}},-\frac{3}{8}\right)$ are points of inflection.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



40. $f(x)=\frac{1}{x^2+3}=\left(x^2+3\right)^{-1}$

a) *Intercepts.* Since the numerator is the constant 1, there are no x-intercepts.

$f(0)=\frac{1}{3}$, so the y-intercept is $\left(0,\frac{1}{3}\right)$.

b) *Asymptotes.*

Vertical. $x^2+3=0$ has no real solution, so there are no vertical asymptotes.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -\frac{2x}{\left(x^2+3\right)^2}$$

$$f''(x) = \frac{6x^2-6}{\left(x^2+3\right)^3}$$

The domain of f is $\mathbb{R}$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all real numbers. $f'(x)=0$ for $x=0$, so 0 is a critical value. From step (a) we already know $\left(0,\frac{1}{3}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test -1, $f'(-1) = \frac{1}{8} > 0$

B: Test 1, $f'(1) = -\frac{1}{8} < 0$

Then $f(x)$ is increasing on $(-\infty,0]$ and is decreasing on $[0,\infty)$. Thus $\left(0,\frac{1}{3}\right)$ is a relative maximum.

f) *Inflection points.* $f''(x)$ exist for all real numbers. $f''(x)=0$ for $x=\pm 1$, so -1 and 1 are possible inflection points.

$f(-1) = \frac{1}{4}$ and $f(1) = \frac{1}{4}$

So, $\left(-1,\frac{1}{4}\right)$ and $\left(1,\frac{1}{4}\right)$ are possible points of inflection.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals A: $(-\infty,-1)$ B: $(-1,1)$, and C: $(1,\infty)$

A: Test -2, $f''(-2) = \frac{18}{343} > 0$

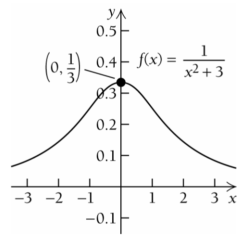
B: Test 0, $f''(0) = -\frac{2}{9} < 0$

C: Test 2, $f''(2) = \frac{18}{343} > 0$

Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and $(1,\infty)$, and concave down on $(-1,1)$.

Thus the points $\left(-1,\frac{1}{4}\right)$ and $\left(1,\frac{1}{4}\right)$ are points of inflection.

h) *Sketch.*



41. $f(x) = \frac{x+3}{x^2-9} = \frac{x+3}{(x+3)(x-3)} = \frac{1}{x-3}$, $x \neq \pm 3$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x = \pm 3$.

a) *Intercepts.* $f(x)=0$ has no solution. $x=-3$ is not in the domain of the function. Therefore, there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0) = \frac{1}{(0)-3} = -\frac{1}{3}$$

The point $\left(0,-\frac{1}{3}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-3$ or $x=3$, however, $x=-3$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.

$$\lim_{x\to -3}\frac{x+3}{x^2-9} = \lim_{x\to -3}\frac{1}{x-3} = \frac{1}{-3-3} = -\frac{1}{6}.$$

Because the limit exists, the line $x=-3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(-3,-\frac{1}{6}\right)$.

An open circle is drawn at $\left(-3,-\frac{1}{6}\right)$ to show that it is not part of the graph.
The denominator is 0 for $x=3$ and the numerator is not 0 at this value, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -(x-3)^{-2} = \frac{-1}{(x-3)^2}$$

$$f''(x) = 2(x-3)^{-3} = \frac{2}{(x-3)^3}$$

The domain of f is $(-\infty,-3)\cup(-3,\infty)$ as determined in step (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x = 3$ is not a critical value. The equation $f'(x) = 0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -3 and 3 to divide the real number line into three intervals

A: $(-\infty,-3)$ B: $(-3,3)$ and C: $(3,\infty)$.

We notice that $f'(x) < 0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-3)$, $(-3,3)$, and $(3,\infty)$.

Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -3 and 3 to divide the real number line into three intervals

A: $(-\infty,-3)$ B: $(-3,3)$ and C: $(3,\infty)$ and we test a point in each interval.

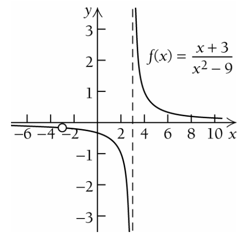
A: Test -4, $f''(-4) = \frac{2}{((-4)-3)^3} = -\frac{2}{343} < 0$

B: Test 2, $f''(2) = \frac{2}{((2)-3)^3} = -2 < 0$

C: Test 4, $f''(4) = \frac{2}{((4)-3)^3} = 2 > 0$

Therefore, $f(x)$ is concave down on $(-\infty,-3)$ and $(-3,3)$ and concave up on $(3,\infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



42. $f(x) = \frac{x-1}{x^2-1} = \frac{1}{x+1}, \quad x \neq \pm 1$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x = \pm 1$.

a) *Intercepts.* $f(x) = 0$ has no solution, so there are no x-intercepts. To find the y-intercepts we compute $f(0)$

$$f(0) = \frac{1}{(0)+1} = 1$$

The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*

Vertical. In the original function, the denominator is 0 for $x = -1$ or $x = 1$, however, $x = 1$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.

$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$. Because the limit exists, the line $x = 1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(1, \frac{1}{2}\right)$.

An open circle is drawn at this point to show that it is not part of the graph.

The denominator is 0 for $x = -1$ and the numerator is not 0 at this value, so the line $x = -1$ is a vertical asymptote.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y = 0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = -(x+1)^{-2} = \frac{-1}{(x+1)^2}$$

$$f''(x) = 2(x+1)^{-3} = \frac{2}{(x+1)^3}$$

The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined above.

d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 and 1 to divide the real number line into three intervals

A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.

We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals

A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.

and we test a point in each interval.

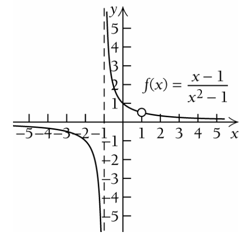
A: Test -2, $f''(-2) = -2 < 0$

B: Test 0, $f''(0) = 2 > 0$

C: Test 2, $f''(2) = \frac{2}{27} > 0$

Therefore, $f(x)$ is concave down on $(-\infty,-1)$ and concave up on $(-1,1)$ and $(1,\infty)$.

h) *Sketch.*



43. $f(x) = \dfrac{x-1}{x+2}$

a) *Intercepts.* To find the x-intercepts, solve $f(x)=0$.

$$\frac{x-1}{x+2} = 0$$

$$x = 1$$

Since $x=1$ does not make the denominator 0, the x-intercept is $(1,0)$.

$f(0) = \dfrac{0-1}{0+2} = -\dfrac{1}{2}$, so the y-intercept is $\left(0,-\dfrac{1}{2}\right)$.

b) *Asymptotes.*

Vertical. The denominator is 0 for $x=-2$, so the line $x=-2$ is a vertical asymptote.

Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{1}$, or $y=1$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x) = \frac{3}{(x+2)^2}$$

$$f''(x) = -\frac{6}{(x+2)^3}$$

The domain of f is $(-\infty,-2)\cup(-2,\infty)$ as determined in part (b).

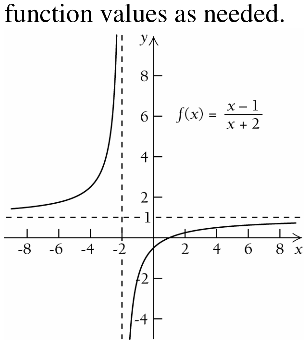
d) *Critical Points.* $f'(x)$ exists for all values of x except -2, but -2 is not in the domain of the function, so $x=-2$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -2 to divide the real number line into two intervals
A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.
A: Test -3, $f'(-3)=3>0$
B: Test -1, $f'(-1)=3>0$
Then $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -2, but because -2 is not in the domain of the function, there cannot be an inflection point at -2. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -2 to divide the real number line into two intervals
A: $(-\infty,-2)$ and B: $(-2,\infty)$, and we test a point in each interval.
A: Test -3, $f''(-3)=6>0$
B: Test -1, $f''(-1)=-6<0$
Therefore, $f(x)$ is concave up on $(-\infty,-2)$ and concave down on $(-2,\infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



44. $f(x)=\dfrac{x-2}{x+1}$

a) *Intercepts.* $f(x)=0$ for $x=2$ and this value does not make the denominator 0, the *x*-intercept is $(2,0)$. $f(0)=-2$, so the *y*-intercept is $(0,-2)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x=-1$, so the line $x=-1$ is a vertical asymptote.
Horizontal. The numerator and the denominator have the same degree, so $y=\dfrac{1}{1}$, or $y=1$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$$f'(x)=\frac{3}{(x+1)^2}$$
$$f''(x)=-\frac{6}{(x+1)^3}$$
The domain of f is $(-\infty,-1)\cup(-1,\infty)$ as determined in part (b).

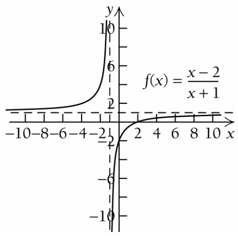
d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 to divide the real number line into two intervals
A: $(-\infty,-1)$ and B: $(-1,\infty)$. We notice that $f'(x)>0$ for all real numbers; therefore, $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 to divide the real number line into two intervals
A: $(-\infty,-1)$ and B: $(-1,\infty)$, and we test a point in each interval.
A: Test -2, $f''(-2)=6>0$
B: Test 0, $f''(0)=-6<0$
Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and concave down on $(-1,\infty)$.

h) *Sketch.*



45. $f(x) = \dfrac{x^2 - 4}{x+3}$

a) *Intercepts.* To find the *x*-intercepts, solve $f(x) = 0$.

$$\frac{x^2 - 4}{x+3} = 0$$
$$x^2 - 4 = 0$$
$$x = \pm 2$$

Neither of these values make the denominator 0, so the *x*-intercepts are $(-2, 0)$ and $(2, 0)$. $f(0) = \dfrac{0^2 - 4}{0+3} = -\dfrac{4}{3}$, so the *y*-intercept is $\left(0, -\dfrac{4}{3}\right)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = -3$, so the line $x = -3$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. Divide the numerator by the denominator.

$$\begin{array}{r|l} & x \qquad -3 \\ \hline x+3 & x^2 \qquad -4 \\ & \underline{x^2 + 3x} \\ & \quad -3x - 4 \\ & \quad \underline{-3x - 9} \\ & \qquad\quad 5 \end{array}$$

$$f(x) = x - 3 + \frac{5}{x+3}$$

As $|x|$ gets very large, $f(x)$ approaches $x - 3$, so $y = x - 3$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x) = \frac{x^2 + 6x + 4}{(x+3)^2}$$
$$f''(x) = \frac{10}{(x+3)^3}$$

The domain of f is $(-\infty, -3) \cup (-3, \infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -3, but -3 is not in the domain of the function, so $x = -3$ is not a critical value. Solve $f'(x) = 0$.

$$\frac{x^2 + 6x + 4}{(x+3)^2} = 0$$
$$x^2 + 6x + 4 = 0$$
$$x = -3 \pm \sqrt{5} \quad \text{Using the Quadratic Formula}$$

$x \approx -5.236$ or $x \approx -0.764$
$f(-5.236) \approx -10.472$ and $f(-0.764) \approx -1.528$, so $(-5.236, -10.472)$ and $(-0.764, -1.528)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use -5.236, -3, and -0.764 to divide the real number line into four intervals
A: $(-\infty, -5.236)$, B: $(-5.236, -3)$,
C: $(-3, -0.764)$, and D: $(-0.764, \infty)$
We test a point in each interval.

A: Test -6, $f'(-6) = \dfrac{4}{9} > 0$

B: Test -4, $f'(-4) = -4 < 0$

C: Test -2, $f'(-2) = -4 < 0$

D: Test 0, $f'(0) = \dfrac{4}{9} > 0$

Then $f(x)$ is increasing on the intervals $(-\infty, -5.236]$ and $[-0.764, \infty)$, and is decreasing on the intervals $[-5.226, -3)$ and $(-3, -0.764]$. Therefore, $(-5.236, -10.472)$ is a relative maximum and $(-0.764, -1.528)$ is a relative minimum.

f) *Inflection points.* $f''(x)$ does not exist at -3, but because -3 is not in the domain of the function, there cannot be an inflection point at -3. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

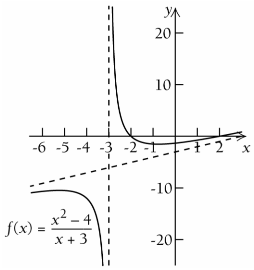
g) *Concavity.* We use -3 to divide the real number line into two intervals
A: $(-\infty, -3)$ and B: $(-3, \infty)$, and we test a point in each interval.

A: Test -4, $f''(-4) = -10 < 0$

B: Test -2, $f''(-2) = 10 > 0$

Therefore, $f(x)$ is concave down on $(-\infty, -3)$ and concave up on $(-3, \infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



46. $f(x) = \dfrac{x^2 - 9}{x+1}$

a) *Intercepts.* The numerator is 0 for $x = -3$ or $x = 3$ and neither of these values make the denominator 0, so the *x*-intercepts are $(-3,0)$ and $(3,0)$.

$f(0) = -9$, so the *y*-intercept is $(0,-9)$.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = -1$, so the line $x = -1$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. By dividing the numerator by the denominator, we get

$$f(x) = x - 1 - \frac{8}{x+1}$$

As $|x|$ gets very large, $f(x)$ approaches $x-1$, so $y = x-1$ is the slant asymptote.

c) *Derivatives and Domain.*

$$f'(x) = \frac{x^2 + 2x + 9}{(x+1)^2}$$

$$f''(x) = -\frac{16}{(x+1)^3}$$

The domain of f is $(-\infty,-1) \cup (-1,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x = -1$ is not a critical value. $f'(x) = 0$ has no real solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 to divide the real number line into two intervals

A: $(-\infty,-1)$ and B: $(-1,\infty)$

We test a point in each interval.

A: Test -2, $f'(-2) = 9 > 0$

B: Test 0, $f'(0) = 9 > 0$

Then $f(x)$ is increasing on both intervals.

Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x) = 0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 to divide the real number line into two intervals

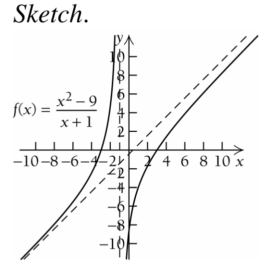
A: $(-\infty,-1)$ and B: $(-1,\infty)$, and we test a point in each interval.

A: Test -2, $f''(-2) = 16 > 0$

B: Test 0, $f''(0) = -16 < 0$

Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and concave down on $(-1,\infty)$.

h) *Sketch.*



47. $f(x) = \dfrac{x+1}{x^2 - 2x - 3} = \dfrac{x+1}{(x-3)(x+1)} = \dfrac{1}{x-3}$, $x \neq -1$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x = -1$ and $x = 3$

a) *Intercepts.* $f(x) = 0$ has no solution.

$x = -1$ is not in the domain of the function. Therefore, there are no *x*-intercepts. To find the *y*-intercepts we compute $f(0)$

$$f(0) = \frac{0+1}{(0)^2 - 2(0) - 3} = -\frac{1}{3}$$

The point $\left(0, -\frac{1}{3}\right)$ is the *y*-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-1$ or $x=3$, however, $x=-1$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.
$$\lim_{x\to -1}\frac{x+1}{x^2-2x-3}=\lim_{x\to -1}\frac{1}{x-3}=\frac{1}{-1-3}=-\frac{1}{4}$$
Because the limit exists, the line $x=-1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(-1,-\frac{1}{4}\right)$. An open circle is drawn at this point to show that it is not part of the graph.
The denominator is 0 for $x=3$ and the numerator is not 0 at this value, so the line $x=3$ is a vertical asymptote.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*
$$f'(x)=-(x-3)^{-2}=\frac{-1}{(x-3)^2}$$
$$f''(x)=2(x-3)^{-3}=\frac{2}{(x-3)^3}$$
The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in part (b).

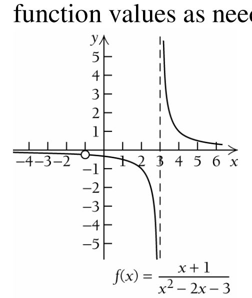
d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.*
We use -1 and 3 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,3)$ and C: $(3,\infty)$.
We notice that $f'(x)<0$ for all real numbers, so $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,3)$, and $(3,\infty)$.
Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at 3, but because 3 is not in the domain of the function, there cannot be an inflection point at 3. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 3 to divide the real number line into three intervals
A: $(-\infty,-1)$ B: $(-1,3)$ and C: $(3,\infty)$, and we test a point in each interval.
A: Test -2, $f''(-2)=-\frac{2}{125}<0$
B: Test 2, $f''(2)=-2<0$
C: Test 4, $f''(4)=2>0$
Therefore, $f(x)$ is concave down on $(-\infty,-1)$ and $(-1,3)$ and concave up on $(3,\infty)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



48. $f(x)=\frac{x-3}{x^2+2x-15}=\frac{1}{x+5}$
We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=-5$ and $x=3$

a) *Intercepts.* $f(x)=0$ has no solution.
$x=3$ is not in the domain of the function. Therefore, there are no x-intercepts. To find the y-intercepts we compute $f(0)$
$$f(0)=\frac{0-3}{(0)^2+2(0)-15}=\frac{1}{5}$$
The point $\left(0,\frac{1}{5}\right)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-5$ or $x=3$, however, $x=3$ also made the numerator equal to 0.

We look at the limits to determine if there are vertical asymptotes at these points.

$$\lim_{x\to 3}\frac{x-3}{x^2+2x-15}=\lim_{x\to 3}\frac{1}{x+5}=\frac{1}{8}$$

Because the limit exists, the line $x=3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $\left(3,\frac{1}{8}\right)$. An open circle is drawn at this point to show that it is not part of the graph.

The denominator is 0 for $x=-5$ and the numerator is not 0 at this value, so the line $x=-5$ is a vertical asymptote.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+5)^{-2}=\frac{-1}{(x+5)^2}$$

$$f''(x)=2(x+5)^{-3}=\frac{2}{(x+5)^3}$$

The domain of f is $(-\infty,-5)\cup(-5,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -5, but -5 is not in the domain of the function, so $x=-5$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -5 and 3 to divide the real number line into three intervals A: $(-\infty,-5)$, B: $(-5,3)$, and C: $(3,\infty)$. We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-5)$, $(-5,3)$, and $(3,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -5, but because -5 is not in the domain of the function, there cannot be an inflection point at -5. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -5 and 3 to divide the real number line into three intervals A: $(-\infty,-5)$ B: $(-5,3)$ and C: $(3,\infty)$, and we test a point in each interval.

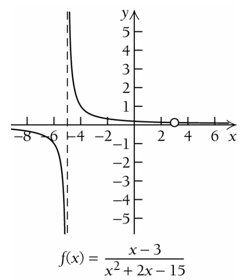
A: Test -6, $f''(-6)=-2<0$

B: Test -4, $f''(-4)=2>0$

C: Test 4, $f''(4)=\frac{2}{729}>0$

Therefore, $f(x)$ is concave down on $(-\infty,-5)$ and concave up on $(-5,3)$ and $(3,\infty)$.

h) *Sketch.*



49. $f(x)=\frac{2x^2}{x^2-16}$

a) *Intercepts.* The numerator is 0 for $x=0$ and this value does not make the denominator 0, the x-intercept is $(0,0)$

$f(0)=0$, so the y-intercept is $(0,0)$ also.

b) *Asymptotes.*

Vertical. The denominator is 0 when

$$x^2-16=0$$
$$x^2=16$$
$$x=\pm 4$$

So the lines $x=-4$ and $x=4$ are vertical asymptotes.

Horizontal. The numerator and the denominator have the same degree, so $y=\frac{2}{1}$, or $y=2$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-\frac{64x}{(x^2-16)^2}$$

$$f''(x)=\frac{192x^2+1024}{(x^2-16)^3}$$

The domain of f is $(-\infty,-4)\cup(-4,4)\cup(4,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except $x=-4$ and $x=4$, but -4 and 4 are not in the domain of the function, so $x=-4$ and $x=4$ are not critical values. $f'(x)=0$ for $x=0$ so, $(0,0)$ is the only critical point.

e) *Increasing, decreasing, relative extrema.* We use -4, 0, and 4 to divide the real number line into four intervals
A: $(-\infty,-4)$ B: $(-4,0)$, C: $(0,4)$, and D: $(4,\infty)$
We test a point in each interval.

A: Test -5, $f'(-5)=\dfrac{320}{81}>0$

B: Test -1, $f'(-1)=\dfrac{64}{225}>0$

C: Test 1, $f'(1)=-\dfrac{64}{225}<0$

D: Test 5, $f'(5)=-\dfrac{320}{81}<0$

Then $f(x)$ is increasing on the intervals $(-\infty,-4)$ and $(-4,0]$, and is decreasing on the intervals $[0,4)$ and $(4,\infty)$. Thus, there is a relative maximum at $(0,0)$.

f) *Inflection points.* $f''(x)$ does not exist at -4 and 4, but because -4 and 4 are not in the domain of the function, there cannot be an inflection point at -4 or 4. The equation $f''(x)=0$ has no real solution; therefore, there are no inflection points.

g) *Concavity.* We use -4 and 4 to divide the real number line into three intervals
A: $(-\infty,-4)$ B: $(-4,4)$ and C: $(4,\infty)$, and we test a point in each interval.

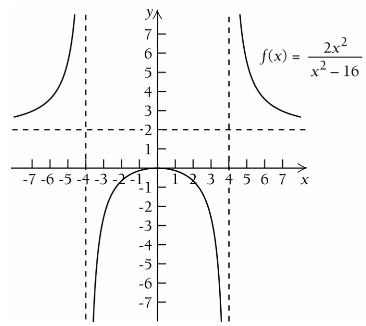
A: Test -5, $f''(-5)=\dfrac{5824}{729}>0$

B: Test 0, $f''(0)=-\dfrac{1}{4}<0$

C: Test 5, $f''(5)=\dfrac{5824}{729}>0$

Therefore, $f(x)$ is concave up on the intervals $(-\infty,-4)$ and $(4,\infty)$ and concave down on the interval $(-4,4)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



50. $f(x)=\dfrac{x^2+x-2}{2x^2-2}=\dfrac{(x+2)(x-1)}{2(x+1)(x-1)}=\dfrac{x+2}{2(x+1)}$

We write the expression in simplified form noting that the domain is restricted to all real numbers except for $x=\pm1$.

a) *Intercepts.* $f(x)=0$ for $x=-2$; therefore, the x-intercept is $(-2,0)$.

$$f(0)=\frac{(0)^2+(0)-2}{2(0)^2-2}=1$$

The point $(0,1)$ is the y-intercept.

b) *Asymptotes.*
Vertical. In the original function, the denominator is 0 for $x=-1$ or $x=1$, however, $x=1$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.
$\lim\limits_{x\to1}\dfrac{x^2+x-2}{2x^2-2}=\lim\limits_{x\to1}\dfrac{x+2}{2(x+1)}=\dfrac{3}{4}$. Because the limit exists, the line $x=1$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a “hole” at the point $\left(1,\dfrac{3}{4}\right)$. An open circle is drawn at this point to show that it is not part of the graph. The denominator is 0 for $x=-1$ and the numerator is not 0 at this value, so the line $x=-1$ is a vertical asymptote.

Horizontal. The numerator and the denominator have the same degree, so $y=\frac{1}{2}$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-(x+1)^{-2}=\frac{-1}{2(x+1)^2}$$

$$f''(x)=2(x+1)^{-3}=\frac{1}{(x+1)^3}$$

The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -1, but -1 is not in the domain of the function, so $x=-1$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -1 and 1 to divide the real number line into three intervals

A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.

We notice that $f'(x)<0$ for all real numbers, $f(x)$ is decreasing on all three intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ does not exist at -1, but because -1 is not in the domain of the function, there cannot be an inflection point at -1. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals

A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$.

and we test a point in each interval.

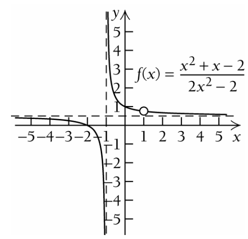
A: Test -2, $f''(-2)=-1<0$

B: Test 0, $f''(0)=1>0$

C: Test 2, $f''(2)=\frac{1}{27}>0$

Therefore, $f(x)$ is concave down on the interval $(-\infty,-1)$ and concave up on the intervals $(-1,1)$ and $(1,\infty)$.

h) *Sketch.*



51. $f(x)=\frac{1}{x^2-1}$

a) *Intercepts.* Since the numerator is a constant 1, there are no x-intercepts.

$$f(0)=\frac{1}{(0)^2-1}=-1$$

The point $(0,-1)$ is the y-intercept.

b) *Asymptotes.*

Vertical. The denominator $x^2-1=(x-1)(x+1)$ is 0 for $x=-1$ or $x=1$, so the lines $x=-1$ and $x=1$ are vertical asymptotes.

Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.

Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=\frac{-2x}{(x^2-1)^2}$$

$$f''(x)=\frac{2(3x^2+1)}{(x^2-1)^3}$$

The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except -1 and 1, but these values are not in the domain of the function, so $x=-1$ and $x=1$ are not critical values. $f'(x)=0$ for $x=0$. From step (a) we know $f(0)=-1$, so the critical point is $(0,-1)$.

e) *Increasing, decreasing, relative extrema.* We use $-1,\ 0,$ and 1 to divide the real number line into four intervals A: $(-\infty,-1)$, B: $(-1,0)$, C: $(0,1)$, and D: $(1,\infty)$, and we test a point in each interval.

A: Test -2, $f'(-2)=\frac{4}{9}>0$

B: Test $-\frac{1}{2}, f'\left(-\frac{1}{2}\right)=\frac{16}{9}>0$

C: Test $\frac{1}{2},\ f'\left(\frac{1}{2}\right)=-\frac{16}{9}<0$

D: Test $2, f'(2)=-\frac{4}{9}<0$

We see that $f(x)$ is increasing on the intervals $(-\infty,-1)$ and $(-1,0]$, and is decreasing on the intervals $[0,1)$ and $(1,\infty)$. Therefore, $(0,-1)$ is a relative maximum.

f) *Inflection points.* $f''(x)$ does not exist at -1 and 1, but because these values are not in the domain of the function, there cannot be an inflection point at -1 or 1. The equation $f''(x)=0$ has no real solution; therefore, there are no inflection points.

g) *Concavity.* We use -1 and 1 to divide the real number line into three intervals A: $(-\infty,-1)$ B: $(-1,1)$ and C: $(1,\infty)$. and we test a point in each interval.

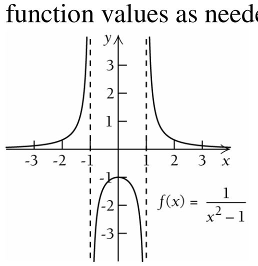
A: Test $-2, f''(-2)=\frac{26}{27}>0$

B: Test $0, f''(0)=-2<0$

C: Test $2, f''(2)=\frac{26}{27}>0$

Therefore, $f(x)$ is concave up on $(-\infty,-1)$ and $(1,\infty)$, and concave down on $(-1,1)$.

h) *Sketch.* Use the preceding information to sketch the graph. Compute additional function values as needed.



52. $f(x)=\frac{10}{x^2+4}$

a) *Intercepts.* Since the numerator is the constant 10, there are no x-intercepts. $f(0)=\frac{5}{2}$, so the y-intercept is $\left(0,\frac{5}{2}\right)$.

b) *Asymptotes.*
Vertical. $x^2+4=0$ has no real solution, so there are no vertical asymptotes.
Horizontal. The degree of the numerator is less than the degree of the denominator, so $y=0$ is the horizontal asymptote.
Slant. There is no slant asymptote since the degree of the numerator is not one more than the degree of the denominator.

c) *Derivatives and Domain.*

$$f'(x)=-\frac{20x}{\left(x^2+4\right)^2}$$

$$f''(x)=\frac{20\left(3x^2-4\right)}{\left(x^2+4\right)^3}$$

The domain of f is $\mathbb{R}$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all real numbers. $f'(x)=0$ for $x=0$, so 0 is a critical value. From step (a) we already know $\left(0,\frac{5}{2}\right)$ is on the graph.

e) *Increasing, decreasing, relative extrema.* We use 0 to divide the real number line into two intervals A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.

A: Test $-1, f'(-1)=\frac{4}{5}>0$

B: Test $1,\ f'(1)=-\frac{4}{5}<0$

Then $f(x)$ is increasing on $(-\infty,0]$ and is decreasing on $[0,\infty)$. Thus $\left(0,\frac{5}{2}\right)$ is a relative maximum.

f) *Inflection points.* $f''(x)$ exist for all real numbers. $f''(x)=0$ for $x=\pm\frac{2}{\sqrt{3}}$

$$f\left(-\frac{2}{\sqrt{3}}\right)=\frac{15}{8} \text{ and } f\left(\frac{2}{\sqrt{3}}\right)=\frac{15}{8}$$

So, $\left(-\frac{2}{\sqrt{3}}, \frac{15}{8}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{15}{8}\right)$ are possible inflection points.

g) *Concavity.* We use $-\frac{2}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ to divide the real number line into three intervals
A: $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ B: $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, and
C: $\left(\frac{2}{\sqrt{3}}, \infty\right)$.

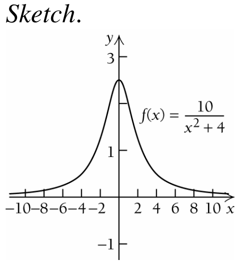
A: Test -2, $f''(-2) = \frac{5}{16} > 0$

B: Test 0, $f''(0) = -\frac{5}{4} < 0$

C: Test 2, $f''(2) = \frac{5}{16} > 0$

Therefore, $f(x)$ is concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and concave down on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Thus, the points $\left(-\frac{2}{\sqrt{3}}, \frac{15}{8}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{15}{8}\right)$ are points of inflection.

h) *Sketch.*



53. $f(x) = \frac{x^2+1}{x}$

a) *Intercepts.* The equation $f(x) = 0$ has no real solutions, so there are no x-intercepts. The number 0 is not in the domain of $f(x)$ so there are no y-intercepts.

b) *Asymptotes.*
Vertical. The denominator is 0 for $x = 0$, so the line $x = 0$ is a vertical asymptote.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator. When we divide the numerator by the denominator we have
$f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$. As $|x|$ gets very large, $f(x) = x + \frac{1}{x}$ approaches x. Therefore, $y = x$ is the slant asymptote.

c) *Derivatives and Domain.*
$$f'(x) = \frac{x^2-1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
The domain of f is $(-\infty, -0) \cup (0, \infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 0, but 0 is not in the domain of the function, so $x = 0$ is not a critical value. The critical points will occur when $f'(x) = 0$.
$$\frac{x^2-1}{x^2} = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$
Thus, -1 and 1 are critical values.
$f(-1) = -2$ and $f(1) = 2$, so the critical points $(-1, -2)$ and $(1, 2)$ are on the graph.

e) *Increasing, decreasing, relative extrema.*
We use $-1, 0,$ and 1 to divide the real number line into four intervals
A: $(-\infty, -1)$ B: $(-1, 0)$, C: $(0, 1)$, and D: $(1, \infty)$. We test a point in each interval.

A: Test -2, $f'(-2) = \frac{3}{4} > 0$

B: Test $-\frac{1}{2}$, $f'\left(-\frac{1}{2}\right) = -3 < 0$

C: Test $\frac{1}{2}$, $f'\left(\frac{1}{2}\right) = -3 < 0$

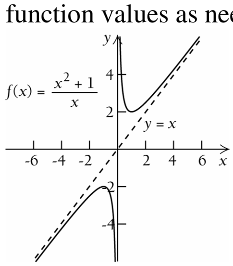
D: Test 2, $f'(2) = \frac{3}{4} > 0$

Then $f(x)$ is increasing on $(-\infty,-1]$ and $[1,\infty)$ and is decreasing on $[-1,0)$ and $(0,1]$. Therefore, $(-1,-2)$ is a relative maximum, and $(1,2)$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at 0, but because 0 is not in the domain of the function, there cannot be an inflection point at 0. The equation $f''(x)=0$ has no solution; therefore, there are no inflection points.

g) *Concavity*. We use 0 to divide the real number line into two intervals
A: $(-\infty,0)$ and B: $(0,\infty)$, and we test a point in each interval.
A: Test -1, $f''(-1)=-2<0$
B: Test 1, $f''(1)=2>0$
Therefore, $f(x)$ is concave down on $(-\infty,0)$ and concave up on $(0,\infty)$.

h) *Sketch*. Use the preceding information to sketch the graph. Compute additional function values as needed.



54. $f(x)=\dfrac{x^3}{x^2-1}$

a) *Intercepts.* The numerator is 0 for $x=0$ and this value does not make the denominator 0, the *x*-intercept is $(0,0)$
$f(0)=0$, so the *y*-intercept is $(0,0)$ also.

b) *Vertical.* The denominator $x^2-1=(x-1)(x+1)$ is 0 for $x=-1$ or $x=1$, so the lines $x=-1$ and $x=1$ are vertical asymptotes.
Horizontal. The degree of the numerator is greater than the degree of the denominator, so there are no horizontal asymptotes.
Slant. The degree of the numerator is exactly one greater than the degree of the denominator. When we divide the numerator by the denominator we have
$f(x)=\dfrac{x^3}{x^2-1}=x+\dfrac{x}{x^2-1}$. As $|x|$ gets very large, $f(x)=x+\dfrac{x}{x^2-1}$ approaches *x*.
Therefore, $y=x$ is the slant asymptote.

c) *Derivatives and Domain.*
$$f'(x)=\frac{x^4-3x^2}{(x^2-1)^2}$$
$$f''(x)=\frac{2x^3+6x}{(x^2-1)^3}$$
The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of *x* except -1 and 1, but these values are not in the domain of the function, so $x=-1$ and $x=1$ are not a critical values. $f'(x)=0$ for $x=-\sqrt{3}$, $x=0$, and $x=\sqrt{3}$. Therefore, $f(-\sqrt{3})=-\dfrac{3\sqrt{3}}{2}$, $f(0)=0$, and $f(\sqrt{3})=\dfrac{3\sqrt{3}}{2}$. The critical points $\left(-\sqrt{3},-\dfrac{3\sqrt{3}}{2}\right)$, $(0,0)$ and $\left(\sqrt{3},\dfrac{3\sqrt{3}}{2}\right)$ are on the graph.

e) *Increasing, decreasing, relative extrema.* We use $-\sqrt{3}$, -1, 0, 1, and $\sqrt{3}$ to divide the real number line into six intervals
A: $(-\infty,-\sqrt{3})$ B: $(-\sqrt{3},-1)$, C: $(-1,0)$, D: $(0,1)$, E: $(1,\sqrt{3})$, and F:$(\sqrt{3},\infty)$. We test a point in each interval.

A: Test -2, $f'(-2)=\dfrac{4}{9}>0$

B: Test $-\dfrac{3}{2}$, $f'\left(-\dfrac{3}{2}\right)=-\dfrac{27}{25}<0$

C: Test $-\dfrac{1}{2}$, $f'\left(-\dfrac{1}{2}\right)=-\dfrac{11}{9}<0$

D: Test $\dfrac{1}{2}$, $f'\left(\dfrac{1}{2}\right)=-\dfrac{11}{9}<0$

E: Test $\dfrac{3}{2}$, $f'\left(\dfrac{3}{2}\right)=-\dfrac{27}{25}<0$

F: Test 2, $f'(2)=\dfrac{4}{9}>0$

Then $f(x)$ is increasing on the intervals $\left(-\infty,-\sqrt{3}\right]$ and $\left[\sqrt{3},\infty\right)$ and is decreasing on the intervals $\left[-\sqrt{3},-1\right)$, $(-1,0]$, $[0,1)$, and $\left(1,\sqrt{3}\right]$. Therefore, $(0,0)$ is not a relative extremum, $\left(-\sqrt{3},\dfrac{-3\sqrt{3}}{2}\right)$ is a relative maximum, and $\left(\sqrt{3},\dfrac{3\sqrt{3}}{2}\right)$ is a relative minimum.

f) *Inflection points*. $f''(x)$ does not exist at -1 and 1, but because these values are not in the domain of the function, there cannot be an inflection point at -1 or 1.

$f''(x)=0$ when $x=0$. We know that $f(0)=0$, so there is a possible inflection point at $(0,0)$.

g) *Concavity*. We use $-1,\ 0,$ and 1 to divide the real number line into four intervals A: $(-\infty,-1)$ B: $(-1,0)$, C: $(0,1)$, and D: $(1,\infty)$, and we test a point in each interval.

A: Test -2, $f''(-2)=-\dfrac{28}{27}<0$

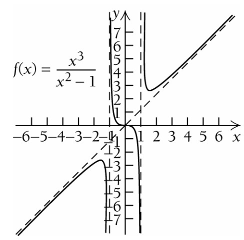
B: Test $-\dfrac{1}{2}$, $f''\left(-\dfrac{1}{2}\right)=\dfrac{208}{27}>0$

C: Test $\dfrac{1}{2}$, $f''\left(\dfrac{1}{2}\right)=-\dfrac{208}{27}<0$

D: Test 2, $f''(2)=\dfrac{28}{27}>0$

Therefore, $f(x)$ is concave down on $(-\infty,0)$ and $(0,1)$ and concave up on $(-1,0)$ and $(1,\infty)$. Thus, the point $(0,0)$ is a point of inflection.

h) *Sketch*.



55. $f(x)=\dfrac{x^2-9}{x-3}$

We write the expression in simplified form:

$f(x)=\dfrac{(x-3)(x+3)}{x-3}=x+3,\quad x\neq 3$

Note that the domain is restricted to all real numbers except for $x=3$.

a) *Intercepts.* The numerator is 0 when $x=-3$ or $x=3$ however, $x=3$ is not in the domain of $f(x)$, so the *x*-intercept is $(-3,0)$.

$f(0)=\dfrac{0^2-9}{(0)-3}=\dfrac{-9}{-3}=3$

The point $(0,3)$ is the *y*-intercept.

b) *Asymptotes.*

In simplified form $f(x)=x+3$, a linear function everywhere except $x=3$. So there are no asymptotes of any kind.

In the original function, the denominator is 0 for $x=3$, however, $x=3$ also made the numerator equal to 0. We look at the limits to determine if there is a vertical asymptote at this point.

$\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=\lim\limits_{x\to 3}x+3=6$. Because the limit exists, the line $x=3$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a "hole" at the point $(3,6)$.

An open circle is drawn at this point to show that it is not part of the graph.

c) *Derivatives and Domain.*

$f'(x)=1,\quad x\neq 3$

$f''(x)=0$

The domain of f is $(-\infty,3)\cup(3,\infty)$ as determined in part (b).

d) *Critical Points.* $f'(x)$ exists for all values of x except 3, but 3 is not in the domain of the function, so $x=3$ is not a critical value. The equation $f'(x)=0$ has no solution, so there are no critical points.

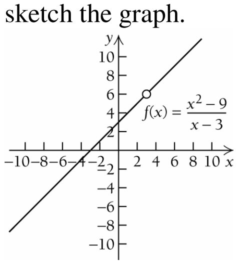
e) *Increasing, decreasing, relative extrema.* We use 3 to divide the real number line into two intervals A: $(-\infty,3)$ and B: $(3,\infty)$.

We notice that $f'(x)>0$ for all real numbers, $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ is constant; therefore, there are no points of inflection.

g) *Concavity.* $f''(x)$ is 0; therefore, there is no concavity.

h) *Sketch.* Use the preceding information to sketch the graph.



Note: In the preceding problem, we could have noticed that the graph of $f(x)=\dfrac{x^2-9}{x-3}$ is the graph of $f(x)=x+3$ with the exception of the point $(3,6)$ which is a removable discontinuity. We simply need to graph $f(x)=x+3$ with a hole at the point $(3,6)$ and determine all other aspects of the graph of $f(x)$ from the linear graph.

56. $f(x)=\dfrac{x^2-16}{x+4}=x-4, \quad x\neq -4$

Notice that $f(x)=x-4$ for all values of x except $x=-4$, where it is undefined. The graph of $f(x)$ will be the graph of $y=x-4$ except at the point $x=-4$.

a) *Intercepts.* $f(x)=0$ when $x=4$, so the x-intercept is $(4,0)$.

$f(0)=-4$

The point $(0,-4)$ is the y-intercept.

b) *Asymptotes.*

In simplified form $f(x)=x-4$, a linear function everywhere except $x=-4$. So there are no asymptotes of any kind.

In the original function, the denominator is 0 for $x=-4$, however, $x=-4$ also made the numerator equal to 0. We look at the limits to determine if there are vertical asymptotes at these points.

$\lim\limits_{x\to -4}\dfrac{x^2-16}{x+4}=\lim\limits_{x\to -4}x-4=-8$. Because the limit exists, the line $x=-4$ is not a vertical asymptote. Instead, we have a removable discontinuity, or a “hole” at the point $(-4,-8)$.

An open circle is drawn at $(-4,-8)$ to show that it is not part of the graph.

c) *Derivatives and Domain.*

$f'(x)=1, \quad x\neq -4$

$f''(x)=0$

d) *Critical Points.* There are no critical points.

e) *Increasing, decreasing, relative extrema.* We use -4 to divide the real number line into two intervals

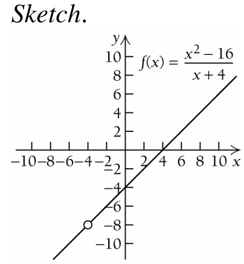
A: $(-\infty,-4)$ and B: $(-4,\infty)$.

We notice that $f'(x)>0$ for all real numbers, $f(x)$ is increasing on both intervals. Since there are no critical points, there are no relative extrema.

f) *Inflection points.* $f''(x)$ is constant; therefore, there are no points of inflection.

g) *Concavity.* $f''(x)$ is 0; therefore, there is no concavity.

h) *Sketch.*



57. $V(t) = 50 - \dfrac{25t^2}{(t+2)^2}$

a) $V(0) = 50 - \dfrac{25(0)^2}{((0)+2)^2} = 50 - 0 = 50$

The inventory's value after 0 months is \$50.

$V(5) = 50 - \dfrac{25(5)^2}{((5)+2)^2} = 50 - \dfrac{625}{49} \approx 37.24$

The inventory's value after 5 months is \$37.24.

$V(10) = 50 - \dfrac{25(10)^2}{((10)+2)^2} = 50 - \dfrac{2500}{144} \approx 32.64$

The inventory's value after 10 months is \$32.64.

$V(70) = 50 - \dfrac{25(70)^2}{((70)+2)^2} = 50 - \dfrac{122,500}{5184} \approx 26.37$

The inventory's value after 70 months is \$26.37.

b) Find $V'(t)$ and $V''(t)$.

$$V'(t) = -\frac{(t+2)^2(50t) - 25t^2(2(t+2)(1))}{((t+2)^2)^2}$$
$$= -\frac{(t+2)\left[(t+2)(50t) - 25t^2(2)\right]}{(t+2)^4}$$
$$= -\frac{50t^2 - 100t - 50t^2}{(t+2)^3}$$
$$= -\frac{100t}{(t+2)^3}$$

$$V''(t) = -\frac{(t+2)^3(100) - 100t\left[3(t+2)^2(1)\right]}{((t+2)^3)^2}$$
$$= -\frac{(t+2)^2\left[100(t+2) - 100t(3)\right]}{(t+2)^6}$$
$$= -\frac{100t + 200 - 300t}{(t+2)^4}$$
$$= -\frac{-200t + 200}{(t+2)^4}$$
$$= \frac{200t - 200}{(t+2)^4}$$

$V'(t)$ exists for all values of t in $[0, \infty)$.

Solve $V'(t) = 0$.

$$-\frac{100t}{(t+2)^3} = 0$$
$$-100t = 0$$
$$t = 0$$

Since $t = 0$ is an endpoint of the domain, there cannot be a relative extrema at $t = 0$. We notice that $V'(t) < 0$ for all x in the domain, therefore, $V(t)$ is decreasing over the interval $[0, \infty)$. Since $V(t)$ is decreasing, the *absolute* maximum value of the inventory will be \$50 when $t = 0$.

c) Using the techniques of this section, we find the following additional information:

Intercepts. There are no t-intercepts in $[0, \infty)$. The V-intercept is the point $(0, 50)$

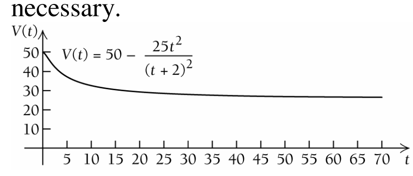
Asymptotes. There are no vertical asymptotes in $[0, \infty)$.

The line $V = 25$ is a horizontal asymptote. There are no slant asymptotes.

Increasing, decreasing, relative extrema. We have already seen that $V(t)$ is decreasing over the interval $[0, \infty)$. There are no relative extrema.

Inflection points, concavity. $V''(t)$ exist for all values of t in $[0, \infty)$. $V''(t) = 0$, when $t = 1$. We use this to split the domain into two intervals. A: $(0, 1)$ and B: $(1, \infty)$. Testing points in each interval, we see $V(t)$ is concave down on $(0, 1)$ and concave up on $(1, \infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



d) $\boxed{tw}$ Yes; The value below which V will never fall is $\lim_{t \to \infty} V(t) = 25$. We observed this from the horizontal asymptote on the graph.

58. $C(x)=3x^2+80$

a) $A(x)=\dfrac{C(x)}{x}=\dfrac{3x^2+80}{x}=3x+\dfrac{80}{x}$

b) Using the techniques of this section we find the following information. We will only consider the values of x is $(0,\infty)$.

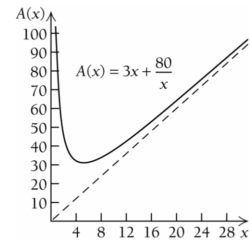
Intercepts. None.
Asymptotes. $x=0$ is the vertical asymptote. There is no horizontal asymptote. The line $y=3x$ is the slant asymptote.

Increasing, decreasing, relative extrema.
$A'(x)=3-\dfrac{80}{x^2}$. $A'(x)$ is not defined for $x=0$, however that value is outside the domain of the function. $A'(x)=0$ when $x=\sqrt{\dfrac{80}{3}}$, and $A\left(\sqrt{\dfrac{80}{3}}\right)=2\sqrt{240}$.

Using $x=\sqrt{\dfrac{80}{3}}$ to divide the interval $(0,\infty)$ into two intervals, $\left(0,\sqrt{\dfrac{80}{3}}\right)$ and $\left(\sqrt{\dfrac{80}{3}},\infty\right)$, and testing a point in each interval, we find that $A(x)$ is decreasing on $\left(0,\sqrt{\dfrac{80}{3}}\right]$ and increasing on $\left[\sqrt{\dfrac{80}{3}},\infty\right)$. Therefore, the point $\left(\sqrt{\dfrac{80}{3}},2\sqrt{240}\right)$ is a relative minimum.

Inflection points, concavity.
$A''(x)=\dfrac{160}{x^3}$ exist for all values of t in $(0,\infty)$. The equation $A''(x)=0$ has no real solution, so there are no possible points of inflection. Furthermore, $A''(x)>0$ for all x in the domain, so $A(x)$ is concave up on $(0,\infty)$.

We use this information to sketch the graph. Additional values may be computed as necessary.



c) The degree of the numerator is exactly one greater than the degree of the denominator. When we divide the numerator by the denominator we have

$A(x)=\dfrac{3x^2+80}{x}=3x+\dfrac{80}{x}$. As $|x|$ gets very large, $A(x)=3x+\dfrac{80}{x}$ approaches $3x$.

Therefore, $y=3x$ is the slant asymptote.

This means that when a large number of units are produced, the average cost can be estimated by multiplying the number of units produced by 3.

59. $C(p)=\dfrac{48{,}000}{100-p}$

a) $C(0)=\dfrac{48{,}000}{100-0}=\dfrac{48{,}000}{100}=480$

The cost of removing 0% of the pollutants from a chemical spill is \$480.

$C(20)=\dfrac{48{,}000}{100-(20)}=\dfrac{48{,}000}{80}=600$

The cost of removing 20% of the pollutants from a chemical spill is \$600.

$C(80)=\dfrac{48{,}000}{100-(80)}=\dfrac{48{,}000}{20}=2400$

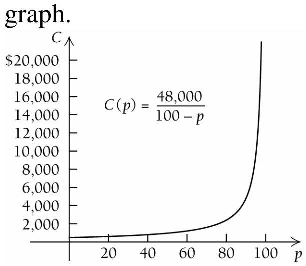
The cost of removing 80% of the pollutants from a chemical spill is \$2400.

$C(90)=\dfrac{48{,}000}{100-(90)}=\dfrac{48{,}000}{10}=4800$

The cost of removing 90% of the pollutants from a chemical spill is \$4800.

b) The domain of C is $0\le p<100$ since it is not possible to remove less than 0% or more than 100% of the pollutants, and $C(p)$ is not defined for $p=100$.

c) Using the techniques of this section we find the following additional information.
Intercepts. No p-intercepts. The point $(0,480)$ is the C-intercept.
Asymptotes. *Vertical.* $p=100$
Horizontal. $C=0$
Slant. None
Increasing, decreasing, relative extrema. $C(p)$ is increasing over the interval $[0,100)$. There are no relative extrema.
Inflection points, concavity. $C(p)$ is concave up on the interval $(0,100)$. there are no inflection points.
We use this information and compute other function values as necessary to sketch the graph.



d) $\boxed{tw}$ From the result in part (b) we see that there is a vertical asymptote at $p=100$. This means that the cost of cleaning up the spill increases without bound as the amount of pollutants removed approaches 100%. The company will not be able to afford to clean up 100% of the pollutants.

60. $C(x)=5000+600x$

$R(x)=-\frac{1}{2}x^2+1000x$

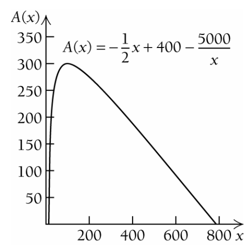
a) $P(x)=R(x)-C(x)$

$$=-\frac{1}{2}x^2+1000x-(5000+600x)$$
$$=-\frac{1}{2}x^2+400x-5000$$

b) $A(x)=\frac{P(x)}{x}$

$$=\frac{-\frac{1}{2}x^2+400x-5000}{x}$$
$$=-\frac{1}{2}x+400-\frac{5000}{x}$$

c) Using the techniques of this section we find the following additional information.
Intercepts. The x-intercepts are $(12.70,0)$ and $(787.30,0)$. There is no P-intercept.
Asymptotes. *Vertical.* $x=0$
Horizontal. None
Slant. $y=-\frac{1}{2}x+400$
Increasing, decreasing, relative extrema. $A(x)$ is increasing over the interval $(0,100]$ and decreasing over the interval $[100,\infty)$. The point $(100,300)$ is a relative maximum.
Inflection points, concavity. $A(x)$ is concave down on the interval $(0,\infty)$. There are no inflection points.
We use this information and compute other function values as necessary to sketch the graph.



d) As $|x|$ gets very large, $A(x)$ approaches $-\frac{1}{2}x+400$. Therefore, $y=-\frac{1}{2}x+400$ is the slant asymptote. This represents the average profit for x items, when x is a large number of items.

61. $P(x)=\frac{2.632}{1+0.116x}$

a) $P(10)=\frac{2.632}{1+0.116(10)}=1.21851852$

In 1980, the purchasing power of a dollar was \$1.22.

$P(20)=\frac{2.632}{1+0.116(20)}=0.79277108$

In 1990, the purchasing power of a dollar was \$0.79.

$P(40)=\frac{2.632}{1+0.116(40)}=0.4666667$

In 2010, the purchasing power of a dollar was \$0.47.

b) Solve $P(x) = 0.50$

$$\frac{2.632}{1+0.116x} = 0.50$$
$$2.632 = 0.50(1+0.116x)$$
$$2.632 = 0.50 + 0.058x$$
$$2.132 = 0.058x$$
$$36.7586 = x$$

36.8 years after 1970, or in 2006, the purchasing power of a dollar will be \$0.50.

c) Find $\lim_{x\to\infty} P(x)$.

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{2.632}{1+0.116x}$$
$$= \lim_{x\to\infty} \frac{2.632}{1+0.116x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$= \lim_{x\to\infty} \frac{\frac{2.632}{x}}{\frac{1}{x}+0.116}$$
$$= \frac{0}{0+0.116} = 0$$

$\lim_{x\to\infty} P(x) = 0$.

62. $A(t) = \dfrac{A_0}{t^2+1}$

If we assume 100 cc is the initial amount injected, then $A(t) = \dfrac{100}{t^2+1}$.

a) $A(0) = \dfrac{100}{(0)^2+1} = 100$

After 0 hours (at the time of injection) there are 100 cc's of medication in the bloodstream.

$A(1) = \dfrac{100}{(1)^2+1} = 50$

After 1 hour, there are 50 cc's of the medication in the bloodstream.

$A(2) = \dfrac{100}{(2)^2+1} = 20$

After 2 hours, there are 20 cc's of the medication in the bloodstream.

$A(7) = \dfrac{100}{(7)^2+1} = 2$

After 7 hours, there are 2 cc's of the medication in the bloodstream.

$A(10) = \dfrac{100}{(10)^2+1} \approx 0.99009901$

After 10 hours, there is approximately 0.9901 cc's of the medication in the bloodstream.

b) $A'(t) = \dfrac{-200t}{(t^2+1)^2}$. Notice that $A'(t) < 0$ for all t in the interval $(0,\infty)$, so there are no relative extrema, and $A(t)$ is decreasing on the interval $[0,\infty)$. Therefore, the maximum value of $A(t)$ is the initial value of 100cc at the time of injection.

c) Using the techniques of this section we find the following additional information.

Intercepts. There are no t-intercepts. The A-intercept is $(0,100)$.

Asymptotes. *Vertical.* None
Horizontal. $y = 0$
Slant. None

Increasing, decreasing, relative extrema.
$A(t)$ is decreasing over the interval $[0,\infty)$.There are no relative extrema

Inflection points, concavity.
$A''(t) = \dfrac{-200(3t^2-1)}{(t^2+1)^3}$ exists for all t in the interval $[0,\infty)$. $A''(t) = 0$ when $3t^2 - 1 = 0$. The only solution to the equation on $[0,\infty)$ is $t = \dfrac{1}{\sqrt{3}}$. We divide the interval $(0,\infty)$ into two intervals $\left(0,\dfrac{1}{\sqrt{3}}\right)$ and $\left(\dfrac{1}{\sqrt{3}},\infty\right)$ and test a point in each interval. $A(t)$ is concave down on the interval $\left(0,\dfrac{1}{\sqrt{3}}\right)$ and is concave up on the interval $\left(\dfrac{1}{\sqrt{3}},\infty\right)$. The point $\left(\dfrac{1}{\sqrt{3}},75\right)$ is an inflection point.

We use this information and compute other function values as necessary to sketch the graph.