Chapter 1
Numbers, Algebra and Geometry

1.2.3 Exercises

- 1(a)

$$2^3 \times 2^{-4} = 2^3 \div 2^4 = 1/2$$

1(b)

$$2^3 \div 2^{-4} = 2^3 \times 2^4 = 2^{3+4} = 2^7$$

1(c)

$$(2^3)^{-4} = 1/(2^3)^4 = 1/2^{12}$$

1(d)

$$3^{1/3} \times 3^{5/3} = 3^{(1/3+5/3)} = 3^2$$

1(e)

$$36^{-1/2} = 1/(36)^{1/2} = 1/6$$

1(f)

$$16^{3/4} = (16^{1/4})^3 = 2^3$$

- **2(a)**

$$(21 + ((4 \times 3) \div 2))$$

2(b)

$$(17 - 6^{(2+3)})$$

2(c)

$$((4 \times 2^3) - ((7 \div 6) \times 2))$$

2(d)

$$((2 \times 3) - (6 \div 4) + 3^{(2^{-5})})$$

- 3(a)

$$\begin{aligned}(7 + 5\sqrt{2})^3 &= (7 + 5\sqrt{2})(7 + 5\sqrt{2})^2 \\ &= (7 + 5\sqrt{2})(99 + 70\sqrt{2}) \\ &= 7 \times 99 + 5 \times 70 \times 2 + (7 \times 70 + 5 \times 99)\sqrt{2} \\ &= 1393 + 985\sqrt{2}\end{aligned}$$

3(b)

$$
\begin{aligned}
(2+\sqrt{2})^4 &= (2+\sqrt{2})^2\,(2+\sqrt{2})^2 \\
&= (6+4\sqrt{2})\,(6+4\sqrt{2}) \\
&= 68+48\sqrt{2}
\end{aligned}
$$

3(c) We are required to find whole numbers x, y so that $x+y\sqrt{2}=\sqrt[3]{7+5\sqrt{2}}$.

$$
\begin{aligned}
x+y\sqrt{2} &= \sqrt[3]{7+5\sqrt{2}} \\
\Rightarrow\quad (x+y\sqrt{2})^3 &= 7+5\sqrt{2} \\
\Rightarrow\quad x^3+3x^2y\sqrt{2}+6xy^2+2y^3\sqrt{2} &= 7+5\sqrt{2} \\
\Rightarrow\quad (x^3+6xy^2)+(3x^2y+2y^3)\sqrt{2} &= 7+5\sqrt{2} \\
\Rightarrow\quad x^3+6xy^2 &= 7 \\
\text{and}\quad 3x^2y+2y^3 &= 5
\end{aligned}
$$

By inspection $x=1$, $y=1$, so $\sqrt[3]{7+5\sqrt{2}}=1+\sqrt{2}$.

3(d) We want to find x, y with $x+y\sqrt{2}=\sqrt{\frac{11}{2}-3\sqrt{2}}$.

$$
\begin{aligned}
x+y\sqrt{2} &= \sqrt{\tfrac{11}{2}-3\sqrt{2}} \\
\Rightarrow\quad (x+y\sqrt{2})^2 &= \tfrac{11}{2}-3\sqrt{2} \\
\Rightarrow\quad x^2+2y^2+2xy\sqrt{2} &= \tfrac{11}{2}-3\sqrt{2} \\
\Rightarrow\quad x^2+2y^2 &= \tfrac{11}{2} \\
\text{and}\quad 2xy &= -3 \\
\text{substituting for } y\quad x^2+2\left(-\tfrac{3}{2x}\right)^2 &= \tfrac{11}{2} \\
\Rightarrow\quad 2x^4-11x^2+9 &= 0 \\
\text{this factorizes:}\quad (2x^2-9)(x^2-1) &= 0
\end{aligned}
$$

Consequently $x=1$ or $x=-1$. ($x=\pm\frac{3}{\sqrt{2}}$ are discarded since they are not rational.) Since $x+y\sqrt{2}$ is to be a positive number we discard $x=1$, $y=-\frac{3}{2}$. So $x=-1$, $y=\frac{3}{2}$.

■ 4

$$
\begin{aligned}
\frac{1}{a+b\sqrt{c}} &= \frac{1}{a+b\sqrt{c}}\times\frac{a-b\sqrt{c}}{a-b\sqrt{c}} \\
&= \frac{a-b\sqrt{c}}{a^2-b^2\sqrt{c}}
\end{aligned}
$$

4(a)

$$
\begin{aligned}
\frac{1}{7+5\sqrt{2}} &= \frac{7-5\sqrt{2}}{7^2-5^2\times 2} \\
&= -7+5\sqrt{2}
\end{aligned}
$$

4(b)

$$\begin{aligned}\frac{2+3\sqrt{2}}{9-7\sqrt{2}} &= (2+3\sqrt{2})\times\frac{9+7\sqrt{2}}{9^2-(-7)^2\times 2}\\ &= \frac{60+41\sqrt{2}}{9^2-(-7)^2\times 2}\\ &= -\frac{60}{17}-\frac{41}{17}\sqrt{2}\end{aligned}$$

4(c)

$$\begin{aligned}\frac{4-2\sqrt{3}}{7-3\sqrt{3}} &= (4-2\sqrt{3})\times\frac{7+3\sqrt{3}}{7^2-3^2\times 3}\\ &= \frac{10-2\sqrt{3}}{22}\\ &= \frac{5}{11}-\frac{\sqrt{3}}{11}\end{aligned}$$

4(d)

$$\begin{aligned}\frac{2+4\sqrt{5}}{4-\sqrt{5}} &= (2+4\sqrt{5})\times\frac{4+\sqrt{5}}{4^2-5}\\ &= \frac{28+18\sqrt{5}}{11}\\ &= \frac{28}{11}+\frac{18}{11}\sqrt{5}\end{aligned}$$

■ 5 $2-1^2=1, 2-\left(\frac{3}{2}\right)^2=-\frac{1}{4}, 2-\left(\frac{7}{5}\right)^2=\frac{1}{25}, 2-\left(\frac{17}{12}\right)^2=-\frac{1}{144}, 2-\left(\frac{41}{29}\right)^2=\frac{1}{841}$, and $2-\left(\frac{99}{70}\right)^2=-\frac{1}{4900}$.

Taking $m=1$, $n=1$ then $\frac{m}{n}=\frac{1}{1}$ and $\frac{m+2n}{m+n}=\frac{3}{2}$

Taking $m=3$, $n=2$ then $\frac{m}{n}=\frac{3}{2}$ and $\frac{m+2n}{m+n}=\frac{7}{5}$

etc.

$$\begin{aligned}2-\left(\frac{m}{n}\right)^2 &= \frac{2n^2-m^2}{n^2}\\ 2-\left(\frac{m+2n}{m+n}\right)^2 &= \frac{m^2-2n^2}{m^2+2mn+n^2}\\ &= -\frac{(2n^2-m^2)}{n^2+(m^2+2mn)}\end{aligned}$$

Observe that m^2+2mn is positive. From this we can see that the error in using $\frac{m+2n}{m+n}$ as an approximation to $\sqrt{2}$ has the opposite sign to the error using $\frac{m}{n}$; also $n^2+(m^2+2mn)>n^2$ so

$$\left|2-\left(\frac{m+2n}{m+n}\right)^2\right|<\left|2-\left(\frac{m}{n}\right)^2\right|$$

Thus the error using $\frac{m+2n}{m+n}$ is smaller than the error in using $\frac{m}{n}$.

The next three terms are obtained as follows:
Taking $m = 99$, $n = 70$ then $\frac{m+2n}{m+n} = \frac{239}{169}$
Taking $m = 239$, $n = 169$ then $\frac{m+2n}{m+n} = \frac{577}{408}$
Taking $m = 577$, $n = 408$ then $\frac{m+2n}{m+n} = \frac{1393}{985}$.

1.2.6 Exercises

■ 6

$$\begin{aligned}(\sqrt{5} + \sqrt{13})^2 &= 5 + 2\sqrt{5}\,\sqrt{13} + 13\\ &= 5 + 2\sqrt{65} + 13\\ &> 5 + 2 \times 8 + 13 = 34\\ &< 5 + 2 \times 9 + 13 = 36\\ (\sqrt{3} + \sqrt{19})^2 &= 3 + 2\sqrt{3}\,\sqrt{19} + 19\\ &= 3 + 2\sqrt{57} + 19\\ &> 3 + 2 \times 7 + 19 = 36\end{aligned}$$

Hence $\sqrt{3} + \sqrt{19} > \sqrt{5} + \sqrt{13}$

■ 7(a) $\{x : |x - 4| \le 6\}$ is the set of points whose distance on the number line from 4 is less than or equal to 6 so it is the set of x satisfying

$$4 - 6 \le x \le 4 + 6$$

i.e.

$$-2 \le x \le 10$$

which is the closed interval $[-2, 10]$.

7(b) $\{x : |x + 3| < 2\}$ is the set of points with distance strictly less than 2 from -3 so it is the set of x satisfying

$$\begin{aligned}-2 - 3 < x < -3 + 2\\ -5 < x < -1\end{aligned}$$

which is the open interval $(-5, -1)$.

7(c) $\{x : |2x - 1| \le 7\}$ is the set of points x with the distance of $2x$ from 1 being less than or equal to 7:

$$\begin{aligned}1 - 7 \le 2x \le 1 + 7\\ \text{equivalently} \quad -6 \le 2x \le 8\\ \text{i.e.} \quad -3 \le x \le 4\end{aligned}$$

which is the interval $[-3, 4]$.

7(d) $\{x : |\frac{1}{4}x + 3| < 3\}$ is the set of x satisfying

$$-3 - 3 < \tfrac{1}{4}x < -3 + 3$$
$$-6 < \tfrac{1}{4}x < 0$$

equivalently, multiplying by 4 $\quad -24 < x < 0$

which is the interval $(-24, 0)$.

■ 8(a) The interval $(1, 7)$ is the set of x satisfying $1 < x < 7$. The midpoint of this interval is 4, so the interval is the set of points with distance strictly less than 3 from 4:

$$4 - 3 < x < 4 + 3$$

This is the set $\{x : |x - 4| < 3\}$.

8(b) $[-4, -2]$ has midpoint -3 and may be written as

$$-3 - 1 \le x \le -3 + 1$$

i.e. the set $\{x : |x + 3| \le 1\}$.

8(c) $(17, 26)$ has midpoint $21\frac{1}{2}$. It is the set of x such that

$$21\tfrac{1}{2} - 4\tfrac{1}{2} < x < 21\tfrac{1}{2} + 4\tfrac{1}{2}$$

This may be tidied up by multiplying by 2 to give

$$43 - 9 < 2x < 43 + 9$$

This is the set $\{x : |2x - 43| < 9\}$.

8(d) $[-\frac{1}{2}, \frac{3}{4}]$ has midpoint $\frac{1}{8}$, so is the set of x satisfying

$$\tfrac{1}{8} - \tfrac{5}{8} \le x \le \tfrac{1}{2} + \tfrac{5}{8}$$

multiplying by 8 $\quad 1 - 5 \le 8x \le 1 + 5$

This is the set $\{x : |8x - 1| \le 5\}$.

■ 9(a) False: for a counterexample take $a = 1$, $b = 2$, $c = 1$, $d = 3$.

9(b) True: $c < d \Rightarrow -d < -c$ (multiplying by -1 and using 1.2f); also $a < b$ so applying (1.2a) $a - d < b - c$ as required.

9(c) False: take $a = 1$, $b = 4$, $c = -3$, $d = -1$.

9(d) False: take $a = -1$, $b = 1$.
If a, b, c, d are all strictly positive then (a) is false (the counterexample given above will do). However, (c) and (d) are true (1.2f and 1.2g respectively).

■ 10(a) Let T be the total time taken; d_1, d_2 the distances travelled at speeds v_1, v_2 respectively.
Now, $d_1 = v_1\frac{T}{2}$ and $d_2 = v_2\frac{T}{2}$.
The average speed is

$$\begin{aligned} v_a &= \frac{d_1 + d_2}{T} \\ &= \frac{v_1\frac{T}{2} + v_2\frac{T}{2}}{T} \\ &= \frac{v_1 + v_2}{2} \end{aligned}$$

10(b) Let D be the total distance travelled; t_1, t_2 be the times travelled at speeds v_1, v_2 respectively.
$\frac{D}{2} = v_1 t_1 = v_2 t_2$.
The average speed is

$$\begin{aligned} v_b &= \frac{D}{t_1 + t_2} \\ &= \frac{D}{\frac{D}{2v_1} + \frac{D}{2v_2}} \\ &= \frac{2v_1 v_2}{v_1 + v_2} \end{aligned}$$

By the arithmetic–geometric mean inequality of Example 1.9

$$\begin{aligned} \tfrac{1}{4}(v_1 + v_2)^2 &> v_1 v_2 \quad \text{(strict inequality since } v_1 \neq v_2\text{)} \\ \text{so} \quad \frac{v_1 + v_2}{2} &> \frac{2v_1 v_2}{v_1 + v_2} \\ v_a &> v_b \end{aligned}$$

So the average speed is less – and thus the journey longer – if two different speeds are used over equal distances compared with the same two speeds for equal times.

1.3.2 Exercises

■ 11(a)

$$x^3 \times x^{-4} = x^{-1}$$

11(b)

$$\begin{aligned} x^3 \div x^{-4} &= x^3 \times x^4 \\ &= x^7 \end{aligned}$$

11(c)

$$(x^3)^{-4} = x^{-12}$$

11(d)

$$\begin{aligned} x^{1/3} \times x^{5/3} &= x^{(1/3+5/3)} \\ &= x^2 \end{aligned}$$

11(e)

$$(4x^8)^{-1/2} = 1/(2x^4)$$

11(f)

$$\begin{aligned} \left(\frac{3}{2\sqrt{x}}\right)^{-2} &= \left(\frac{2\sqrt{x}}{3}\right)^2 \\ &= \tfrac{4}{9}x \end{aligned}$$

11(g)

$$\sqrt{x}(x^2 - \tfrac{2}{x}) = x^{5/2} - 2/\sqrt{x}$$

11(h)

$$\left(5x^{1/3} - \frac{1}{2x^{1/3}}\right)^2 = 25x^{2/3} - 5 + \frac{1}{4x^{2/3}}$$

11(i)

$$\frac{2x^{1/2} - x^{-1/2}}{x^{1/2}} = 2 - \tfrac{1}{x}$$

11(j)

$$\begin{aligned} \frac{(a^2b)^{1/2}}{(ab^{-2})^2} &= \frac{ab^{1/2}}{a^2b^{-4}} \\ &= a^{-1}b^{9/2} \\ &= b^{9/2}/a \end{aligned}$$

11(k)

$$\begin{aligned} (4ab^2)^{-3/2} &= (2a^{1/2}b)^{-3} \\ &= \frac{1}{8a^{3/2}b^3} \end{aligned}$$

■ 12(a)

$$\begin{aligned} x^2y - xy^2 &= xy(x) - xy(y) \\ &= xy(x - y) \end{aligned}$$

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12(b)

$$x^2yz - xy^2z + 2xyz^2 = xyz(x - y + 2z)$$

12(c)

$$\begin{aligned} ax - 2by - 2ay + bx &= ax + bx - 2ay - 2by \\ &= (a + b)x - 2(a + b)y \\ &= (a + b)(x - 2y) \end{aligned}$$

12(d)

$$\begin{aligned} x^2 + 3x - 10 &= x^2 + (5 - 2)x + (5)(-2) \\ &= (x + 5)(x - 2) \end{aligned}$$

12(e)

$$x^2 - \tfrac{1}{4}y^2 = (x - \tfrac{1}{2}y)(x + \tfrac{1}{2}y)$$

12(f)

$$\begin{aligned} 81x^4 - y^4 &= (9x^2 - y^2)(9x^2 + y^2) \\ &= (3x - y)(3x + y)(9x^2 + y^2) \end{aligned}$$

■ 13(a)

$$\begin{aligned} \frac{x^2 - x - 12}{x^2 - 16} &= \frac{(x+3)(x-4)}{(x+4)(x-4)} \\ &= \frac{x+3}{x+4}, \quad x \neq 4 \end{aligned}$$

13(b)

$$\begin{aligned} \frac{x-1}{x^2 - 2x - 3} - \frac{2}{x+1} &= \frac{x-1}{(x+1)(x-3)} - \frac{2}{x+1} \\ &= \frac{(x-1) - 2(x-3)}{(x+1)(x-3)} \\ &= \frac{5-x}{(x+1)(x-3)} \end{aligned}$$

13(c)

$$\begin{aligned} \frac{1}{x^2 + 3x - 10} + \frac{1}{x^2 + 17x + 60} &= \frac{1}{(x+5)(x-2)} + \frac{1}{(x+5)(x+12)} \\ &= \frac{(x+12) + (x-2)}{(x+5)(x-2)(x+12)} \\ &= \frac{2x+10}{(x+5)(x-2)(x+12)} \\ &= \frac{2}{(x-2)(x+12)} \end{aligned}$$

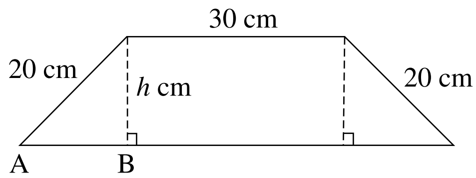
13(d)

$$\begin{aligned}(3x + 2y)(x - 2y) + 4xy &= 3x^2 - 4xy - 4y^2 + 4xy \\ &= 3x^2 - 4y^2 \\ &= \left(\sqrt{3}x - 2y\right)\left(\sqrt{3}x + 2y\right)\end{aligned}$$

■ 14 By Pythagoras' theorem

$AB^2 = 20^2 - h^2$

Hence length of longer side is $30 + 2\sqrt{(20^2 - h^2)}$ and area is $h(30 + \sqrt{(20^2 - h^2)})$ cm^2.



■ 15

$$\begin{aligned}C &= x(300 - 2x)(200 - 2x) \text{ mm}^3 \\ &= x(300 - 2x)(200 - 2x)/1000 \text{ cm}^3 \\ &= 4x(140 - x)(100 - x)/1000 \text{ ml} \\ &= x(150 - x)(100 - x)/250 \text{ ml}\end{aligned}$$

■ 16(a)

$$\begin{aligned}x^2 + x - 12 &= (x + \tfrac{1}{2})^2 - 12 - \tfrac{1}{4} \\ &= (x + \tfrac{1}{2})^2 - \tfrac{49}{4}\end{aligned}$$

16(b)

$$\begin{aligned}3 - 2x + x^2 &= 3 + (1 - x)^2 - 1 \\ &= 2 + (1 - x)^2 \\ &= 2 + (x - 1)^2\end{aligned}$$

16(c)

$$\begin{aligned}(x - 1)^2 - (2x - 3)^2 &= x^2 - 2x + 1 - \{4x^2 - 12x + 9\} \\ &= -3x^2 + 10x - 8 \\ &= -3(x^2 - \tfrac{10}{3}x) - 8 \\ &= -3\left\{\left(x - \tfrac{5}{3}\right)^2 - \tfrac{25}{9}\right\} - 8 \\ &= -3(x - \tfrac{5}{3})^2 + \tfrac{1}{3}\end{aligned}$$

16(d)

$$\begin{aligned}1 + 4x - x^2 &= 1 - (x - 2)^2 + 4 \\ &= 5 - (x - 2)^2\end{aligned}$$

1.3.4 Exercises

■ 17

$$
\begin{aligned}
m &= p\sqrt{\frac{s+t}{s-t}} \\
\Rightarrow \quad m^2 &= p^2\left(\frac{s+t}{s-t}\right) \\
m^2(s-t) &= p^2(s+t) \\
s(m^2-p^2) &= t(m^2+p^2) \\
\Rightarrow \quad s &= (m^2+p^2)t/(m^2-p^2),\ m^2 \neq p^2
\end{aligned}
$$

■ 18

$$
\begin{aligned}
u &= \frac{x^2+t}{x^2-t} \\
\Rightarrow \quad ux^2 - ut &= x^2 + t \\
t + ut &= ux^2 - x^2 \\
t + ut &= ux^2 - x^2 \\
t(1+u) &= x^2(u-1) \\
\Rightarrow \quad t &= x^2(u-1)/(1+u),\ u \neq -1
\end{aligned}
$$

■ 19

$$
\begin{aligned}
\frac{1}{1-t} - \frac{1}{1+t} &= 1 \\
\Rightarrow \quad 1 + t - (1-t) &= (1-t)(1+t) \\
2t &= 1 - t^2 \\
\Rightarrow \quad t^2 + 2t - 1 &= 0 \\
(t+1)^2 &= 2 \\
\Rightarrow \quad t &= -1 \pm \sqrt{2}
\end{aligned}
$$

■ 20

$$
\begin{aligned}
\frac{3c^2+3xc+x^2}{3c^2+3yc+y^2} &= \frac{yV_1}{xV_2} \\
x = 4,\ y = 6,\ V_1 &= 120,\ V_2 = 315 \\
\Rightarrow \quad \frac{3c^2+12c+16}{3c^2+18c+36} &= \frac{720}{1260} \\
\Rightarrow \quad 7(3c^2+12c+16) &= 4(3c^2+18c+36) \\
21c^2 + 84c + 112 &= 12c^2 + 72c + 144 \\
9c^2 + 12c - 32 &= 0 \\
(3c-4)(3c+8) &= 0 \\
c &= 4/3 \quad \text{or} \quad -8/3
\end{aligned}
$$

Since $c > 0$, choose $c = 4/3$.

■ **21**

$$\frac{2p+1}{p+5} + \frac{p-1}{p+1} = 2$$
$$(2p+1)(p+1) + (p-1)(p+5) = 2(p+5)(p+1)$$
$$2p^2 + 3p + 1 + p^2 + 4p - 5 = 2p^2 + 12p + 10$$
$$\Rightarrow \quad p^2 - 5p - 14 = 0$$
$$(p-7)(p+2) = 0$$
$$\Rightarrow \quad p = 7 \quad \text{or} \quad -2$$

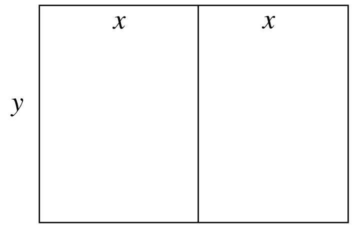
■ **22** Perimeter $= 2(\text{length} + \text{breadth})$
$= 6$ breadth

since length is twice breadth.

Hence breadth is 5 m and length is 10 m.

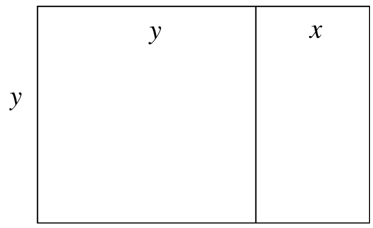
■ **23(a)** Same shape implies ratios of corresponding sides are equal.

$$\Rightarrow \quad \frac{x}{y} = \frac{y}{2x}$$
$$\Rightarrow \quad \frac{y^2}{x^2} = 2$$
$$\Rightarrow \quad \frac{y}{x} = \sqrt{2}$$



23(b) Same shape implies ratios of corresponding sides are equal.

$$\Rightarrow \quad \frac{x}{y} = \frac{y}{x+y}$$
$$\Rightarrow \quad \frac{x}{y}(x+y) = y$$
$$\Rightarrow \quad x^2 + xy = y^2$$
$$\Rightarrow \quad \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right) - 1 = 0$$
$$\Rightarrow \quad \frac{y}{x} = \frac{1+\sqrt{(1+4)}}{2} \quad \left(\text{since } \frac{y}{x} > 0\right)$$
$$\Rightarrow \quad \frac{y}{x} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$



(This result is different from foolscap paper. The description is only an approximation.)

■ **24(a)** In the case $x > 0$, multiplying both sides of the inequality by x gives $5 < 2x$ so the solution is $x > \frac{5}{2}$.

In the case $x < 0$, multiplying by x gives $5 > 2x$ so $x < \frac{5}{2}$, but since $x < 0$ the solution in this case is $x < 0$.

Combining these, the solution is $x < 0$, or $x > \frac{5}{2}$.

24(b) In the case $2 - x > 0$, multiplying by $2 - x$ gives $1 < 2 - x$ so $x < 1$. In the case $2 - x < 0$, multiplying by $2 - x$ gives $1 > 2 - x$ so $x > 1$, but since $2 - x < 0$ the solution is $x > 2$ for this case.
The complete solution set is therefore $\{x : x < 1, \text{ or } x > 2\}$.

24(c) In the case $x - 1 > 0$ multiplying by $x - 1$ gives $3x - 2 > 2(x - 1)$ so $x > 0$.
However, $x - 1 > 0$ (i.e. $x > 1$) so the solution in this case is $x > 1$.
If $x - 1 < 0$, multiplying by $x - 1$ gives $3x - 2 < 2(x - 1)$ so $x < 0$.
The complete solution set is $\{x : x < 0, \text{ or } x > 1\}$.

24(d) If $3x - 2$ and $x + 4$ are either both positive or both negative (i.e. if $(3x - 2)(x + 4) > 0$) then multiplying the inequality by $(3x - 2)(x + 4)$ gives $3(x + 4) > 3x - 2$, i.e. $12 > -2$. This holds for all x. However, $(3x - 2)(x + 4) > 0$ so $x < -4$ or $x > \frac{2}{3}$.
If just one of $3x - 2$, $x + 4$ is negative (i.e. $(3x - 1)(x + 4) < 0$) then multiplying by $(3x - 2)(x + 4)$ gives $3(x + 4) < 3x - 2$. This is equivalent to $12 < -2$. There are no solutions in this case. The solution set is $\{x : x < -4, \text{ or } x > \frac{2}{3}\}$.

■ **25** If $x \geq 0$ then $|x| = x$ so $x^2 < 2 + x$ or equivalently $x^2 - x - 2 < 0$. $x^2 - x - 2$ factorizes as $(x - 2)(x + 1)$ and so is negative for $-1 < x < 2$. Since $x \geq 0$, the solution in this case is $0 \leq x < 2$.
If $x < 0$ then $|x| = -x$ so the inequality becomes $x^2 < 2 - x$ or equivalently $x^2 + x - 2 < 0$. $x^2 + x - 2$ factorizes as $(x + 2)(x - 1)$. $(x + 2)(x - 1) < 0 \Rightarrow -2 < x < 1$. Since $x < 0$ the solution is $-2 < x < 0$.
Combining these two cases, the solution set is $\{x : -2 < x < 2\}$.

■ **26(a)**

$$\begin{aligned} x^2 + 3x - 10 &= (x + \tfrac{3}{2})^2 - 10 - \tfrac{9}{4} \\ &= (x + \tfrac{3}{2})^2 - (\tfrac{7}{2})^2 \\ &\geq -(\tfrac{7}{2})^2 \end{aligned}$$

Equality occurs when $x = -\frac{3}{2}$.

26(b)

$$\begin{aligned} 18 + 4x - x^2 &= 18 - [x^2 - 4x] \\ &= 18 - [(x - 2)^2 - 4] \\ &= 22 - (x - 2)^2 \\ &\leq 22 \end{aligned}$$

Equality occurs when $x = 2$.

26(c)

$$x + \tfrac{4}{x} = \left(\sqrt{x} - \tfrac{2}{\sqrt{x}}\right)^2 + 2 \quad \text{for } x > 0$$
$$\geq 2$$

Equality occurs when $x = 2$.

■ 27(a)

$$\frac{1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$
$$\Rightarrow \quad 1 \equiv A(x-2) + B(x+1)$$

True for all x.
Put $x = -1$, then $1 = A(-3) \Rightarrow A = -\tfrac{1}{3}$.
Put $x = 2$, then $1 = B(3) \Rightarrow B = \tfrac{1}{3}$.

27(b)

$$3x + 2 \equiv A(x-1) + B(x-2)$$

True for all x.
Put $x = 1$, then $5 = B(-1) \Rightarrow B = -5$.
Put $x = 2$, then $8 = A(1) \Rightarrow A = 8$.

27(c)

$$\frac{5x+1}{\sqrt{(x^2+x+1)}} \equiv \frac{A(2x+1)+B}{\sqrt{(x^2+x+1)}}$$
$$\Rightarrow \quad 5x + 1 \equiv A(2x+1) + B$$

True for all x.
Put $x = -\tfrac{1}{2}$, then $-\tfrac{3}{2} = B$.
Put $x = 0$, then $1 = A + B \Rightarrow A = \tfrac{5}{2}$.

■ 28

$$2x^2 - 5x + 12 \equiv A(x-1)^2 + B(x-1) + C$$

True for all x.
Put $x = 1$, then $C = 9$.
Put $x = 0$, then $12 = A - B + C \Rightarrow A - B = 3$.
Put $x = -1$, then $19 = 4A - 2B + C \Rightarrow 2A - B = 5$.
$\Rightarrow A = 2$ and $B = -1$.

1.3.7 Exercises

■ 29(a)

$$\sum_{k=0}^{4} a_k = a_0 + a_1 + a_2 + a_3 + a_4$$
$$= 2 - 1 - 4 + 5 + 3$$
$$= 5$$

29(b)

$$\sum_{i=1}^{3} a_i = a_1 + a_2 + a_3$$
$$= -1 + 5 - 4$$
$$= 0$$

29(c)

$$\sum_{k=1}^{2} a_k b_k = a_1 b_1 + a_2 b_2$$
$$= (-1)(1) + (-4)(2)$$
$$= -9$$

29(d)

$$\sum_{j=0}^{4} b_j^2 = (1)^2 + (1)^2 + (2)^2 + (-1)^2 + (2)^2$$
$$= 11$$

29(e)

$$\prod_{k=1}^{3} a_k = a_1 a_2 a_3$$
$$= (-1)(-4)(5)$$
$$= 20$$

29(f)

$$\prod_{k=1}^{4} b_k = b_1 b_2 b_3 b_4$$
$$= (1)(2)(-1)(2)$$
$$= -4$$

■ 30(a)

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$= 120$$

30(b)

$$3!/4! = 3 \cdot 2 \cdot 1/(4 \cdot 3 \cdot 2 \cdot 1)$$
$$= \tfrac{1}{4}$$

30(c)

$$7!/(3! \times 4!) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1/(3 \cdot 2 \cdot 1 \times 4 \cdot 3 \cdot 2 \cdot 1)$$
$$= 7 \cdot 5 = 35$$

30(d)

$$\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10$$

30(e)

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

30(f)

$$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70$$

■ 31(a)

$$(x-3)^4 = x^4 + 4x^3(-3) + 6x^2(-3)^2 + 4x(-3)^3 + (-3)^4$$
$$= x^4 - 12x^3 + 54x^2 - 108x + 81$$

31(b)

$$(x + \tfrac{1}{2})^3 = x^3 + 3x^2(\tfrac{1}{2}) + 3x(\tfrac{1}{2})^2 + (\tfrac{1}{2})^3$$
$$= x^3 + \tfrac{3}{2}x^2 + \tfrac{3}{4}x + \tfrac{1}{8}$$

31(c)

$$(2x+3)^5 = (2x)^5 + 5(2x)^4(3) + 10(2x)^3(3)^2 + 10(2x)^2(3)^3 + 5(2x)^1(3)^4 + (3)^5$$
$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$$

31(d)

$$(3x+2y)^4 = (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4$$
$$= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$$

1.4.4 Exercises

■ **32(a)** The line with gradient m through (x_0, y_0) has equation $(y - y_0) = m(x - x_0)$. In this case the equation is

$$y - 1 = \tfrac{3}{2}(x - 2)$$
$$\text{equivalently} \quad y = \tfrac{3}{2}x - 2$$

32(b) The line has equation $y - 3 = -2(x + 2)$, equivalently $y = -2x - 1$.

32(c) The gradient is $m = \frac{7-2}{3-1} = \frac{5}{2}$ so the line has equation

$$y - 2 = \tfrac{5}{2}(x - 1)$$
$$\text{or} \quad y = \tfrac{5}{2}x - \tfrac{1}{2}$$

32(d) The gradient is $m = \frac{3-0}{0-5} = -\frac{3}{5}$ so the equation of the line is

$$y = -\tfrac{3}{5}(x - 5)$$
$$y = -\tfrac{3}{5}x + 3$$

32(e) The required line has the same gradient as the line $3y - x = 5$ (equivalently $y = \frac{x}{3} + \frac{5}{3}$); this is $\frac{1}{3}$. Thus our line has equation

$$y - 1 = \tfrac{1}{3}(x - 1)$$
$$y = \tfrac{1}{3}x + \tfrac{2}{3}$$

32(f) Lines perpendicular to $3y - x = 5$ have gradient $-\frac{1}{1/3} = -3$. The required line has equation

$$y - 1 = -3(x - 1)$$
$$y = -3x + 4$$

■ **33** The equation of the circle centre (x_0, y_0), radius r, is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

so the equation of this circle is

$$(x - 1)^2 + (y - 2)^2 = 5^2$$

■ **34** Completing the square in x and then in y

$$x^2 + y^2 + 4x - 6y = 3$$

may be written as

$$(x + 2)^2 + (y - 3)^2 = 16$$

the equation of a circle centre $(-2, 3)$, radius 4.

■ **35** The radius, r, of the circle is the distance between (–2, 3) and (1, –1) so

$$\begin{aligned} r^2 &= (-1-3)^2 + (1--2)^2 \\ &= 4^2 + 3^2 \\ &= 5^2 \end{aligned}$$

So the equation of the circle is

$$(x+2)^2 + (y-3)^2 = 25$$

which is the same as $x^2 + y^2 + 4x - 6y = 12$.

■ **36** The equation of the circle, C, is $(x - x_0)^2 + (y - y_0)^2 = r^2$. We are required to find x_0, y_0 and r.

(1,0) lies on C so $(1 - x_0)^2 + y_0^2 = r^2$ (1)

(3,4) lies on C so $(3 - x_0)^2 + (4 - y_0)^2 = r^2$ (2)

(5,0) lies on X so $(5 - x_0)^2 + y_0^2 = r^2$ (3)

Subtracting (3) from (1) gives $(1 - x_0)^2 - (5 - x_0)^2 = 0$ so $(6 - 2x_0) \times -4 = 0$ and so $x_0 = 3$.

Now (1) and (2) become $y_0^2 = r^2 - 4$ and $(4 - y_0)^2 = r^2$ respectively. Therefore

$$\begin{aligned} (4 - y_0)^2 &= y_0^2 + 4 \\ \Rightarrow \quad y_0 &= \tfrac{3}{2} \end{aligned}$$

Now (2) becomes $(4 - \frac{3}{2})^2 = r^2$ so $r = \frac{5}{2}$. Having obtained x_0, y_0, r we can write down the equation of the circle. It is

$$(x-3)^2 + (y - \tfrac{3}{2})^2 = \tfrac{25}{4}$$

Multiplying out this is

$$x^2 + y^2 - 6x - 3y + 5 = 0$$

■ **37** Let tangent have equation $y = m(x - 1) + 2$

so that it passes through (1,2). This line cuts the circle $x^2 + y^2 - 4x - 1 = 0$ where

$$x^2 + [m(x-1) + 2]^2 - 4x - 1 = 0$$

Simplifying this gives

$$(m^2 + 1)x^2 + (2m - 2m^2 - 4)x + m^2 - 2m + 3 = 0$$

This has two distinct roots except where the line is a tangent where there is a repeated root at $x = 1$:

$x^2 + [mx + (2 - m)]^2 - 4x - 1 = 0$

$x^2 + m^2x^2 + 2mx\,(2 - m) + (2 - m)^2 - 4x - 1 = 0$

$x^2(m^2 + 1) + 2x\,(2m - m^2 - 2) + 3 - 4m + m^2 = 0$

$m^2 + 1$	$4m - 2m^2 - 4$	$3 - 4m + m^2$
0	$m^2 + 1$	$4m - m^2 - 3$
$m^2 + 1$	$4m - m^2 - 3$	0
0	$m^2 + 1$	
$m^2 + 1$	$4m - 2$	$m = \frac{1}{2}$
$y = \frac{1}{2}(x - 1) + 2 = \frac{1}{2}(x + 3)$		

■ 38 Let x be the distance (in cm) along one of the wires from the point of intersection of the wires; y be the distance from the intersection along the other wire. If the midpoint of the rod has coordinates (x, y) then, since it is the midpoint, the end on the x-axis must be at $2x$ and the end on the y-axis at $2y$. Since the rod has length 50 cm

$$(2x)^2 + (2y)^2 = (50)^2$$
$$\text{so} \quad x^2 + y^2 = (25)^2$$

1.4.6 Exercises

■ 39

$$y^2 = 4ax \text{ has focus at } x = a \text{ and directrix at } x = -a$$
$$3y^2 = 8x \quad \Rightarrow \quad y^2 = \tfrac{8}{3}x \quad \text{and} \quad a = \tfrac{2}{3}$$

Focus is at $(\frac{2}{3}, 0)$, directrix is $x = -\frac{2}{3}$.

$$x = \tfrac{2}{3} \quad \Rightarrow \quad y^2 = \tfrac{16}{19} \quad \Rightarrow \quad y = \pm\tfrac{4}{3}$$

Hence latus rectum has length $\frac{8}{3}$.

■ 40

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ has } b < a$$

so in this case the role of x and y is interchanged since

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With $b < a$ the foci are at $(\pm ae, 0)$, $e^2 = 1 - \dfrac{b^2}{a^2}$, directrices $x = \pm a/e$.
Here $a = 5$, $b = 4$, $e = 3/5$ and the foci are at $(0, \pm 3)$; the directrices are $y = \pm 25/3$, the major axis is 10 and the minor axis 8.

■ 41 For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the foci are $(\pm ae, 0)$, the eccentricity $e = 1 + \frac{b^2}{a^2}$, the directrices are $x = \pm a/e$. Here $\frac{x^2}{16} - \frac{y^2}{9} = 1$, so that the foci are at $(\pm 5, 0)$ with eccentricity $e = \frac{5}{4}$.
The vertices occur where $y = 0$, giving $x = \pm 4$ and the asymptotes are given by

$$\frac{x^2}{16} - \frac{y^2}{9} = 0, \text{ i.e. } y = \pm\frac{3}{4}x$$

1.5.4 Exercises

■ 42

$$\begin{aligned} 110110.101_2 &= 2^5 + 2^4 + 2^2 + 2^1 + 2^{-1} + 2^{-3} \\ &= 54.625_{10} \end{aligned}$$

■ 43

$$\begin{aligned} 16\ 321 &= 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^0 \\ &= 11111111000001_2 \end{aligned}$$

$$\begin{aligned} 16\ 321 &= 3 \times 8^4 + 7 \times 8^3 + 7 \times 8^2 + 8^0 \\ &= 37701_8 \end{aligned}$$

To convert from binary to octal: take the first three entries immediately to the right and include the 2^0 term in the binary expansion; this may be considered as a three-digit binary number; convert this number into octal; the resulting octal number is the 8^0 term of the octal expansion. Now do the same with the next three digits of the binary expansion to get the 8^1 term of the octal expansion, and so on.

$$101_2 = 5_8,\ 100_2 = 4_8,\ 011_2 = 3_8, \text{ and } 1_2 = 1_8 \text{ so}$$
$$[1011100101101_2 = 13455_8]$$

■ 44

$$\begin{aligned} 30.6 &= 2^4 + 2^3 + 2^2 + 2^1 + 2^{-1} + 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + 2^{-12} + 2^{-13} \\ &= 11110.1001100110011\ldots_2 \end{aligned}$$

$$\begin{aligned} 30.6 &= 3 \times 8 + 6 \times 8^0 + 4 \times 8^{-1} + 6 \times 8^{-2} + 3 \times 8^{-3} + 8^{-4} + 4 \times 8^{-5} \\ &\quad + 6 \times 8^{-6} + 3 \times 8^{-7} + 8^{-8} + \ldots \\ &= 36.46314631\ldots_8 \end{aligned}$$

The rule works in this case as well: $100_2 = 4_8$, $110_2 = 6_8$, $011_2 = 3_8$ and $001_2 = 1_8$.

■ 45(a)

$$\begin{array}{r} 100011.011_2 \\ +\ 1011.001_2 \\ \hline 101110.100_2 \end{array}$$

45(b)

$$\begin{array}{r} 111.10011_2 \\ \times\ 10.111_2 \\ \hline 0.11110011 \\ 1.1110011 \\ 11.110011 \\ +\ 1111.0011 \\ \hline 10101.11010101_2 \end{array}$$

■ 46(a) 3 dp, 6 sf.

46(b) 30 dp, 3 sf.

46(c) 0 dp, 5 sf.

46(d) 0 dp, 3 sf.

46(e) 0 dp, 4 sf.

46(f) 10 dp, 3 sf.

■ 47 The absolute error bound for the lengths $h = 1$, $b = 2$ is 0.005. The error table for the calculation of $\sqrt{h^2 + b^2}$ is as follows:

Label	*Value*	*Absolute error bound*	*Relative error bound*
h	1	0.005	0.005
b	1	0.005	0.0025
h^2	1	0.01	0.01
b^2	4	0.02	0.005
$h^2 + b^2$	5	0.03	0.006
$\sqrt{h^2 + b^2}$	2.236 068	0.007	0.003

The absolute error bound for $\sqrt{h^2 + b^2}$ is 0.007 so the value 2.236 07 is not sensible. The hypotenuse has length 2.236 ± 0.007. The angle of the triangle is stated as 90°; this is only known up to the accuracy of the angle-measuring tool used. This may be the cause of further error.

■ 48(a) The absolute error bound is $\varepsilon_a = \frac{5}{60}$ min.
The relative error bound is $r_a = \frac{\varepsilon_a}{a} = \frac{1}{420}$.

48(b) Absolute error $\varepsilon_a = 35 \times \frac{4}{100}$ min $= \frac{7}{5}$ min.
Relative error $r_a = \frac{7}{5} \times \frac{1}{35} = \frac{1}{25}$

48(c) Absolute error $\varepsilon_a = 0.005$
Relative error $r_a = \frac{0.005}{0.58} = \frac{1}{116}$

■ 49 Absolute error = 12.9576 × 0.0003 = 0.0039.
Let x be the exact value. We know that

$$12.9576 - 0.0039 \leq x \leq 12.9576 + 0.0039$$
$$12.9537 \leq x \leq 12.9615$$

So we can only state x to 3 significant figures: $x = 12.9$.

■ 50(a)

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	1.316	5×10^{-4}	3.8×10^{-4}
b	5.713	5×10^{-4}	8.8×10^{-5}
c	8.010	5×10^{-4}	6.2×10^{-5}
$a - b + c$	3.613	0.0015	4.2×10^{-4}

Correctly rounded $a - b + c = 3.61$

50(b)

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	2.51	0.005	2×10^{-3}
b	1.01	0.005	5×10^{-3}
ab	2.5351	0.018	7×10^{-3}

Correctly rounded $ab = 2.5$

50(c)

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	19.61	0.005	2.55×10^{-4}
b	21.53	0.005	2.32×10^{-4}
c	18.67	0.005	2.68×10^{-4}
$a + b - c$	22.47	0.015	6.67×10^{-4}

Correctly rounded $a + b - c = 22.5$

■ 51

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	12.42	0.005	4.03×10^{-4}
b	5.675	0.0005	8.81×10^{-5}
c	15.63	0.005	3.20×10^{-4}
$\frac{ab}{c}$	4.5095	0.0037	8.11×10^{-4}

$\frac{ab}{c} = 4.5095 \pm 0.0037$

Correctly rounded this is 4.51.

■ 52

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	4.99	0.005	0.001
b	5.01	0.005	0.001
$a + b$	10	0.01	0.001
$a - b$	−0.02	0.01	0.5
ab	24.9999	0.05	0.002
$\frac{a}{b}$	0.996	0.002	0.002

■ 53

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	3.251	0.0005	1.5×10^{-4}
b	3.115	0.0005	1.6×10^{-4}
$a - b$	0.136	0.001	7.4×10^{-3}
c	0.112	0.0005	4.5×10^{-3}
$\frac{a-b}{c}$	1.214 29	0.015	1.2×10^{-2}
d	9.21	0.005	5.4×10^{-4}
$d + \frac{a-b}{c}$	10.424 29	0.02	1.9×10^{-3}

So $9.21 + \frac{3.251-3.115}{0.112} = 10.4243 \pm 0.02$
Correctly rounded this is 10.4.

■ 54

Label	*Value*	*Absolute error bound*	*Relative error bound*
u	1.135	0.0005	4.4×10^{-4}
v	2.332	0.0005	2.1×10^{-4}
uv	2.646 82	1.7×10^{-3}	6.5×10^{-4}
$u + v$	3.467	0.001	2.9×10^{-4}
$\frac{uv}{u+v}$	0.763 43	7.2×10^{-4}	9.4×10^{-4}

$\frac{uv}{u+v} = 0.763\ 43 \pm 0.000\ 72$
Correctly rounded this is 0.76.

■ 55(a)

Term	*Value* (4 dp)	*Error bound*	
1	1.0000	0.0000	
−1.65	−1.6500	0.0050	
$+\frac{1}{2}(1.65)^2$	+1.3612	0.0082	(i)
$-\frac{1}{6}(1.65)^3$	−0.7487	0.0068	(ii)
$+\frac{1}{24}(1.65)^4$	+0.3088	0.0037	(iii)
E	0.2713	0.0237	

so $0.2476 \le E \le 0.2950$.

(i) Error $\le \frac{1}{2}[1.65 \times 0.0050 + 1.65 \times 0.0050]$ $(\varepsilon_{ab} = a\varepsilon_b + b\varepsilon_a)$

(ii) Error $\le \frac{1}{3}[\frac{1}{2}(1.65)^2 \times 0.0050 + 1.65 \times 0.0050 + 1.65 \times 0.0082]$
$(\frac{1}{6}(1.65)^3 = \frac{1}{3}\times\frac{1}{2}(1.65)^2 \times 1.65)$

(iii) Error $\le \frac{1}{4}[\frac{1}{6}(1.65)^3 \times 0.0050 + 1.65 \times 0.0050 + 1.65 \times 0.0068]$
$(\frac{1}{24}(1.65)^4 = \frac{1}{4}\times\frac{1}{6}(1.65)^3 \times 1.65)$

We can also obtain these by arguing that if ε is the error in 1.65, then $(1.65 + \varepsilon)^n = 1.65^n + n(1.65)^n\varepsilon$ + terms in ε^2, ε^3, etc. Ignoring these latter terms, error in $(1.65)^n$ is bounded by $n(1.65)^n 0.005$.

55(b)

Term	*Value* (4 dp)	*Error bound*	
$-\frac{1}{6}+\frac{1}{24}1.65 = t_1$	−0.0979	0.0002	(i)
$1.65 \times t_1 = t_2$	−1.1615	0.0008	(ii)
$\frac{1}{2} + t_2 = t_3$	+0.3385	0.0008	(iii)
$1.65 \times t_3 = t_4$	+0.5585	0.0030	(iv)
$-1 + t_4 = t_5$	+0.4415	0.0030	(v)
$1.65 \times t_5 = t_6$	−0.7285	0.0072	(vi)
$1 + t_6 = E$	0.2715	0.0072	(vii)

so $0.2643 \le E \le 0.2787$.

(i) Error ≤ error in 0.1667 + $\frac{1}{24}$ (error in 1.65) ≃ 0.000 24, round to 0.0002

(ii) Error $\le 1.65 \times 0.0002 + t_1 \times 0.0050$ (iv), (vi) are similar

(iii) Error = error in t_2 (v), (vii) are similar

It is obvious that (b) gives us a better guarantee of accuracy.

1.5.6 Exercises

■ 56

$$\begin{aligned}
&(((10^1(0.1000) + 10^1(0.1000)) + 10^0(0.5000)) + 10^0(0.1667)) + 10^{-1}(0.4167) \\
&\quad = ((10^1(0.2000) + 10^0(0.5000)) + 10^0(0.1667)) + 10^{-1}(0.4167) \\
&\quad = (10^1(0.2500) + 10^0(0.1667)) + 10^{-1}(0.4167) \\
&\quad = (10^1(0.2500) + 10^1(0.016\,67)) + 10^{-1}(0.4167) \\
&\quad = 10^1(0.2667) + 10^{-1}(0.4167) \\
&\quad = 10^1(0.2667) + 10^1(0.004\,167) \\
&\quad = 10^1(0.2709)
\end{aligned}$$

$$\begin{aligned}
&(((10^{-1}(0.4167) + 10^0(0.1667)) + 10^0(0.5000)) + 10^1(0.1000)) + 10^1(0.1000) \\
&\quad = ((10^0(0.2084) + 10^0(0.5000)) + 10^1(0.1000)) + 10^1(0.1000) \\
&\quad = (10^0(0.7084) + 10^1(0.1000)) + 10^1(0.1000) \\
&\quad = 10^1(0.1708) + 10^1(0.1000) \\
&\quad = 10^1(0.2708)
\end{aligned}$$

The first method is less accurate: adding the large numbers first means that the small numbers will be swamped.

■ 57

$$\begin{aligned}
10^{-2}(0.3251) \times 10^{-5}(0.2011) &= 10^{-7}(0.065\,378) \\
&= 10^{-8}(0.653\,78) \\
&= 10^{-8}(0.6538)
\end{aligned}$$

$$\begin{aligned}
10^{-1}(0.2168) \div 10^{2}(0.3211) &= 10^{-3}(0.675\,179) \\
&= 10^{-3}(0.6752)
\end{aligned}$$

■ 58

$10^4(0.1000) + 10^2(0.1234) = 10^4(0.101\,234) \to 10^4(0.1012)$

$10^4(0.1012) - 10^4(0.1013) = 10^4(-0.0001) \to \underline{10^1(-0.1000)}$

The exact answer is

$$1000 + 12.34 - 1013 = -0.66$$

and so the relative error is, in absolute value,

$$(1 - 0.66)/0.66 = 0.34/0.66 \approx \tfrac{1}{2} \text{ or } 50\%$$

This large value is due to *cancellation* – the error introduced by this addition is amplified by the subtraction of two nearly equal numbers.

1.7 Review Exercises

■ 1(a)

$$
\begin{aligned}
& Q = \frac{\sqrt{H}}{K}\sqrt{\frac{A^2D^2}{A^2+D^2}} \\
\Rightarrow\quad & Q^2 = \frac{H}{K^2}\cdot\frac{A^2D^2}{A^2+D^2} \\
\Rightarrow\quad & Q^2K^2(A^2+D^2) = HA^2D^2 \\
\Rightarrow\quad & A^2(Q^2K^2 - HD^2) = -Q^2K^2D^2 \\
\Rightarrow\quad & A = \frac{\pm QKD}{\sqrt{(HD^2 - Q^2K^2)}}
\end{aligned}
$$

1(b)

$$
\begin{aligned}
& \frac{1}{x+2} - \frac{2}{x} = \frac{3}{x-1} \\
\Rightarrow\quad & x(x-1) - 2(x+2)(x-1) = 3(x+2)x \\
\Rightarrow\quad & x^2 - x - 2x^2 - 2x + 4 = 3x^2 + 6x \\
\Rightarrow\quad & 4x^2 + 9x - 4 = 0 \\
\Rightarrow\quad & x = [-9 \pm \sqrt{(81+64)}]/8 \\
& \ \ = (-9 \pm \sqrt{145})/8
\end{aligned}
$$

■ 2(a)

$$
\begin{aligned}
ax - 2x - a + 2 &= (a-2)x - (a-2) \\
&= (a-2)(x-1)
\end{aligned}
$$

2(b)

$$
\begin{aligned}
a^2 - b^2 + 2bc - c^2 &= a^2 - (b-c)^2 \\
&= (a-b+c)(a+b-c)
\end{aligned}
$$

2(c)

$$
\begin{aligned}
4k^2 + 4kl + l^2 - 9m^2 &= (2k+l)^2 - 9m^2 \\
&= (2k+l-3m)(2k+l+3m)
\end{aligned}
$$

2(d)

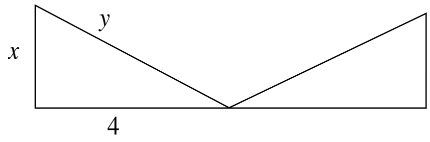
$$
p^2 - 3pq + 2q^2 = (p-2q)(p-q)
$$

2(e)

$$
\begin{aligned}
l^2 + lm + ln + mm &= l(l+m) + n(l+m) \\
&= (l+n)(l+m)
\end{aligned}
$$

■ **3(a)** Let x cm be distance of mass below peg, then by Pythagoras' theorem

$$y^2 = x^2 + 4^2$$



Since total length is 16 cm we have

$$\begin{aligned} 2x + 2y &= 16 \\ \Rightarrow \quad x + \sqrt{(x^2 + 16)} &= 8 \\ \Rightarrow \quad \sqrt{(x^2 + 16)} &= 8 - x \\ \Rightarrow \quad x^2 + 16 &= (8 - x)^2 \\ &= 64 - 16x + x^2 \\ \Rightarrow \quad 16x &= 48 \Rightarrow x = 3 \end{aligned}$$

Hence mass rises by 1 cm.

3(b) Shaded area is $5x(2x - 5) - 40$ square units. Hence

$$\begin{aligned} 10x^2 - 25x - 40 &= 10 \\ \Rightarrow \quad 2x^2 - 5x - 10 &= 0 \\ \Rightarrow \quad x &= \frac{5 + \sqrt{(25 + 80)}}{4} \quad \text{since } x \text{ is positive} \\ &= 3.812 \end{aligned}$$

■ **4(a)**

$$\begin{aligned} Z^2 &= R^2 + \left(2\pi nL - \frac{1}{2\pi nC}\right)^2 \\ \Rightarrow \quad 2\pi nL - \frac{1}{2\pi nC} &= \pm\sqrt{(Z^2 - R^2)} \\ \Rightarrow \quad 2\pi nL &= \frac{1}{2\pi nC} \pm \sqrt{(Z^2 - R^2)} \\ \Rightarrow \quad L &= \frac{1}{4\pi^2 n^2 C} \pm \frac{\sqrt{(Z^2 - R^2)}}{2\pi n} \end{aligned}$$

4(b) $n = 50,\ R = 15,\ C = 10^{-4}$ gives

$$L = \frac{10^4}{10^4 \pi^2} \pm \frac{\sqrt{(Z^2 - 225)}}{100\pi}$$

$$\begin{aligned} Z = 20 \Rightarrow L &= \frac{1}{\pi^2} \pm \frac{\sqrt{175}}{100\pi} \\ &= \frac{1}{\pi^2} \pm \frac{\sqrt{7}}{20\pi} \\ &= \frac{20 \pm \pi\sqrt{7}}{20\pi^2} \\ &= 0.1434 \text{ or } 0.0592 \end{aligned}$$

$$Z = 100 \Rightarrow L = \frac{1}{\pi^2} \pm \frac{\sqrt{9775}}{100\pi}$$
$$= \frac{1}{\pi^2} \pm \frac{\sqrt{391}}{20\pi}$$

Since $L > 0$, we have

$$L = \frac{20 + \pi\sqrt{391}}{20\pi^2}$$
$$= 0.4160$$

■ 5(a)

$$(3\sqrt{2} - 2\sqrt{3})^2 = 9 \times 2 - 2 \times 6 \times \sqrt{2}\sqrt{3} + 4 \times 3$$
$$= 30 - 12\sqrt{6}$$

5(b)

$$(\sqrt{5} + 7\sqrt{3})(2\sqrt{5} - 3\sqrt{3}) = 2 \times 5 - 3\sqrt{3}\sqrt{5} + 14\sqrt{3}\sqrt{5} - 21 \times 3$$
$$= -53 + 11\sqrt{15}$$

5(c)

$$\frac{4 + 3\sqrt{2}}{5 + \sqrt{2}} = \frac{4 + 3\sqrt{2}}{5 + \sqrt{2}} \times \frac{5 - \sqrt{2}}{5 - \sqrt{2}}$$
$$= \frac{14 + 11\sqrt{2}}{23}$$

5(d)

$$\frac{\sqrt{3} + \sqrt{2}}{2 - \sqrt{3}} = \frac{\sqrt{3} + \sqrt{2}}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$
$$= \frac{2\sqrt{3} + 2\sqrt{2} + 3 + \sqrt{6}}{1}$$
$$= 3 + 2\sqrt{2} + 2\sqrt{3} + \sqrt{6}$$

5(e)

$$\frac{1}{1 + \sqrt{2} - \sqrt{3}} = \frac{1}{1 + \sqrt{2} - \sqrt{3}} \times \frac{1 - \sqrt{2} + \sqrt{3}}{1 - \sqrt{2} + \sqrt{3}}$$
$$= \frac{1 - \sqrt{2} + \sqrt{3}}{-4 + 2\sqrt{6}}$$
$$= \frac{(1 - \sqrt{2} + \sqrt{3})(-4 - 2\sqrt{6})}{-8}$$
$$= \frac{-4 - 2\sqrt{6} + 4\sqrt{2} + 4\sqrt{3} - 4\sqrt{3} - 6\sqrt{2}}{-8}$$
$$= \frac{1}{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}$$

■ 6

$$\begin{aligned} \sqrt{11+2\sqrt{30}} &= \sqrt{m}+\sqrt{n} \\ \Rightarrow \quad 11+2\sqrt{30} &= m+n+2\sqrt{mn} \\ \Rightarrow \quad m+n &= 11 \\ mn &= 30 \end{aligned}$$

So $m = 5$, $n = 6$.

■ 7

$$\begin{aligned} \sqrt{n+1}-\sqrt{n} &= (\sqrt{n+1}-\sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{n+1+\sqrt{(n+1)n}-\sqrt{(n+1)n}-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \qquad \cdots\cdots (1) \end{aligned}$$

Now, $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n-1}}$

so $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n-1}}$

and using the identity (1)

$$\sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n}-\sqrt{n-1}$$

as required.

Using the above

$$2(\sqrt{n+1}-\sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n}-\sqrt{n-1})$$

$$\text{so} \quad 2\sum_{n=1}^{10000}(\sqrt{n+1}-\sqrt{n}) < \sum_{n=1}^{10000}\frac{1}{\sqrt{n}} < 2\sum_{n=1}^{10000}(\sqrt{n}-\sqrt{n-1})$$

However,

$$\sum_{n=1}^{10000}(\sqrt{n+1}-\sqrt{n}) = \sqrt{10001}-1 \qquad \text{since adjacent terms cancel}$$

$$\text{and} \quad \sum_{n=1}^{10000}(\sqrt{n}-\sqrt{n-1}) = \sqrt{10000}$$

$$\text{so} \quad 198.01 < \sum_{1}^{1000}\frac{1}{\sqrt{n}} < 200$$

■ 8(a)

$$\{x : 4x^2-3 < 4x\} = \{x : 4x^2-4x-3 < 0\}$$

$$\begin{aligned} 4x^2-4x-3 &< 0 \\ \Rightarrow \quad (2x-3)(2x+1) &< 0 \\ \text{so} \quad -\tfrac{1}{2} < x &< \tfrac{3}{2} \end{aligned}$$

This is the interval $(-\frac{1}{2}, \frac{3}{2})$.

8(b) Case 1: $(x+2)(x-1) > 0$ (i.e. $x < -2$ or $x > 1$) multiplying by $(x+2)(x-1)$ gives

$$(x-1) > 2(x+2)$$
$$\text{so} \quad x < -5$$

Case 2: $(x+2)(x-1) < 0$ (i.e. $-2 < x < 1$)

$$\text{we get} \quad (x-1) < 2(x+2)$$
$$\text{so} \quad x > -5$$

Since $-2 < x < 1$ the solution in this case is $-2 < x < 1$.
Therefore the complete solution is

$$(-\infty, -5) \cup (-2, 1)$$

8(c)

$$|x+1| < 2$$
$$\Rightarrow \quad -1-2 < x < -1+2$$
$$\text{so} \quad -3 < x < 1$$

The solution set is $(-3, 1)$.

8(d) Case 1: $x \geq -1$

$$x + 1 < 1 + \tfrac{1}{2}x$$
$$\Rightarrow \quad x < 0$$

so in this case the solution is $-1 \leq x < 0$.
Case 2: $x < -1$

$$-x - 1 < 1 + \tfrac{1}{2}x$$
$$\Rightarrow \quad x > -\tfrac{4}{3}$$
$$\text{so} \quad -\tfrac{4}{3} < x < -1$$

Hence the complete solution set is $(-\frac{4}{3}, 0)$.

■ 9 $L \geq p$ where p is the perimeter of the circle of area A.
For the circle

$$2\pi r = p$$
$$\pi r^2 = A$$
$$\text{So} \quad p^2 = 4\pi A$$

Since $L \geq p$, $L^2 \geq p^2$ and so $L^2 \geq 4\pi A$ as required.
A square of area A has perimeter $L = 4\sqrt{A}$

$$16A > 4\pi A \text{ since } \quad \pi < 4$$

and the inequality holds.

A semicircle of area A has perimeter L where

$$L^2 = \tfrac{2}{\pi}(2+\pi)^2 A$$

$$\text{but} \qquad \tfrac{2}{\pi}(2+\pi)^2 \simeq 16.8 > 4\pi$$

so again the inequality holds.

■ 10 $\left(\dfrac{x+y}{2}\right)^2 \geq xy$

Substituting $x = \frac{1}{2}(a+b)$ and $y = \frac{1}{2}(c+d)$ gives

$$\left(\frac{a+b+c+d}{4}\right)^2 \geq \frac{1}{4}(a+b)(c+d)$$

Squaring both sides (remembering that a, b, c and $d >$) gives

$$\left(\frac{a+b}{2}\right)^2\left(\frac{c+d}{2}\right)^2 \leq \left(\frac{a+b+c+d}{4}\right)^4$$

Now $\left(\dfrac{a+b}{2}\right)^2 \geq ab$ and $\left(\dfrac{c+d}{2}\right)^2 \geq cd$

$$\text{so that} \qquad abcd \leq \left(\frac{a+b}{2}\right)^2\left(\frac{c+d}{2}\right)^2 \leq \left(\frac{a+b+c+d}{4}\right)^4$$

Hence $\dfrac{a+b+c+d}{4} \geq \sqrt[4]{(abcd)}$

■ 11 $a+c < b+c$ since $a < b$

so $\frac{a+c}{b+c} < 1$ as $b+c > 0$

also $a < b$

$\Rightarrow$ $ac < bc$ since $c > 0$

$\Rightarrow$ $ab + ac < ab + bc$

therefore

$$a(b+c) < b(a+c)$$

$$\frac{a}{b} < \frac{a+b}{b+c} \qquad \text{since } b > 0,\ b+c > 0$$

Combining these two gives

$$\frac{a}{b} < \frac{a+b}{b+c} < 1$$

If $a > b$ and $b > 0$

then $a+c > b+c$ so $\frac{a+c}{b+c} > 1$ as $b+c > 0$.

Also $ba + bc < ab + ac$

so $\frac{a+c}{b+c} < \frac{a}{b}$

and therefore $\frac{a}{b} > \frac{a+c}{b+c} > 1$

■ 12(a)

$$\binom{n}{n_1}\binom{n_2+n_3}{n_2} = \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n_2+n_3)!}{n_2!(n_2+n_3-n_2)!}$$
$$= \frac{n!}{n_1!(n_2+n_3)!}\cdot\frac{(n_2+n_3)!}{n_2!n_3!}$$
$$= \frac{n!}{n_1!n_2!n_3!}$$

12(b)

(i)

$$\left(1-\frac{x}{2}\right)^5 = \left(1-\frac{5}{1}\cdot\frac{x}{2}+\frac{5\cdot4}{1\cdot2}\cdot\frac{x^2}{4}-\frac{5\cdot4\cdot3}{1\cdot2\cdot3}\cdot\frac{x^3}{8}+\frac{5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4}\cdot\frac{x^4}{16}-\frac{5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5}\cdot\frac{x^5}{32}\right)$$
$$= 1-\frac{5}{2}x+\frac{5}{2}x^2-\frac{5}{4}x^3+\frac{5}{16}x^4-\frac{1}{32}x^5$$

(ii)

$$(3-2x)^6 = 3^6 - 6\cdot3^5\cdot2x + 15\cdot3^4\cdot2^2x^2 - 20\cdot3^3\cdot2^3x^3$$
$$+ 15\cdot3^2\cdot2^4\cdot x^4 - 6\cdot3\cdot2^5x^5 + 2^6x^6$$
$$= 729 - 2916x + 4860x^2 - 4320x^3$$
$$+ 2160x^4 - 576x^5 + 64x^6$$

■ 13(a)

$$\sum_{n=-2}^{3}[n^{n+1} + 3(-1)^n] = [(-2)^{-1} + 3(-1)^2] + [(-1)^0 + 3(-1)^{-1}] + [(0)^1 + 3(-1)^0]$$
$$+ [(1)^2 + 3(-1)^1] + [(2)^3 + 3(-1)^2] + [(3)^4 + 3(-1)^3]$$
$$= [-\tfrac{1}{2} + 3] + [1-3] + [0+3] + [1-3] + [8+3] + [81-3]$$
$$= -\tfrac{1}{2} + 1 + 1 + 8 + 81 = 90.5$$

13(b) Number of dots inside third L shape is 9. A further seven dots are needed to extend to a 4×4 square.

$$P_r = 2(r-1)+1$$
$$= 2r-1$$

$$\sum_{r=1}^{n}P_r = \sum_{r=1}^{n}(2r-1) = 2\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$$
$$= 2\cdot\tfrac{n}{2}(n+1)-n$$
$$= n^2+n-n$$
$$= n^2$$

■ 14(a)

$$\frac{y+11}{5+11} = \frac{x+6}{2+6}$$
$$\Rightarrow \quad y + 11 = 2(x + 6)$$
$$\Rightarrow \quad y = 2x + 1$$

14(b)

$$y = \tfrac{1}{3}(x - 4) - 1$$
$$\Rightarrow \quad y = \tfrac{1}{3}x - \tfrac{7}{3}$$

14(c)

$$y = \tfrac{1}{3}x - \tfrac{7}{3}$$
$$x = 0 \Rightarrow y = -\tfrac{7}{3} \quad \text{(required intercept)}$$
$$y = 2x + 1 \text{ has slope } 2 \quad \text{(required slope)}$$

Hence line is $y = 2x - \frac{7}{3}$.

■ 15 Since circle touches y-axis at (0, 3), its equation has the form

$$(x - a)^2 + (y - 3)^2 = a^2$$

where a is a number.
Since circle passes through (1, 0), we have

$$(1 - a)^2 + 9 = a^2$$

and hence $a = 5$.
Since circle passes through (1, 0), we have

$$(1 - a)^2 + 9 = a^2$$

and hence $a = 5$.
So equation of circle is

$$(x - 5)^2 + (y - 3)^2 = 25$$

■ 16(a)

$$x^2 + y^2 + 2x - 4y + 1 = 0$$
$$\Rightarrow \quad (x + 1)^2 + (y - 2)^2 + 1 = 1 + 4$$
$$\Rightarrow \quad (x + 1)^2 + (y - 2)^2 = 4$$

Centre (−1, 2), radius 2

16(b)

$$4x^2 - 4x + 4y^2 + 12y + 9 = 0$$
$$\Rightarrow \quad x^2 - x + y^2 + 3y + \tfrac{9}{4} = 0$$
$$\Rightarrow \quad (x - \tfrac{1}{2})^2 + (y + \tfrac{3}{2})^2 + \tfrac{9}{4} = \tfrac{1}{4} + \tfrac{9}{4}$$
$$\Rightarrow \quad (x - \tfrac{1}{2})^2 + (y + \tfrac{3}{2})^2 = \tfrac{1}{4}$$
Centre $(\tfrac{1}{2}, -\tfrac{3}{2})$, radius $\tfrac{1}{2}$

16(c)

$$9x^2 + 6x + 9y^2 - 6y = 25$$
$$\Rightarrow \quad x^2 + \tfrac{2}{3}x + y^2 - \tfrac{2}{3}y = \tfrac{25}{9}$$
$$\Rightarrow \quad (x + \tfrac{1}{3})^2 + (y - \tfrac{1}{3})^2 = \tfrac{25}{9} + \tfrac{1}{9} + \tfrac{1}{9}$$
$$\Rightarrow \quad (x + \tfrac{1}{3})^2 + (y - \tfrac{1}{3})^2 = \tfrac{27}{9}$$
Centre $(-\tfrac{1}{3}, \tfrac{1}{3})$, radius $\sqrt{3}$

■ 17(i)

$$y^2 = 8x + 4y - 12$$
$$\Rightarrow \quad y^2 - 4y = 8x - 12$$
$$(y - 2)^2 = 8x - 12 + 4$$
$$(y - 2)^2 = 8(x - 1)$$

(a) vertex (1, 2)
(b) focus (1 + 2, 2) = (3, 2) (cf. $y^2 = 4ax$)
(c) directrix $x = (1 - 2) = -1$
(d) axis of symmetry $y = 2$

17(ii)

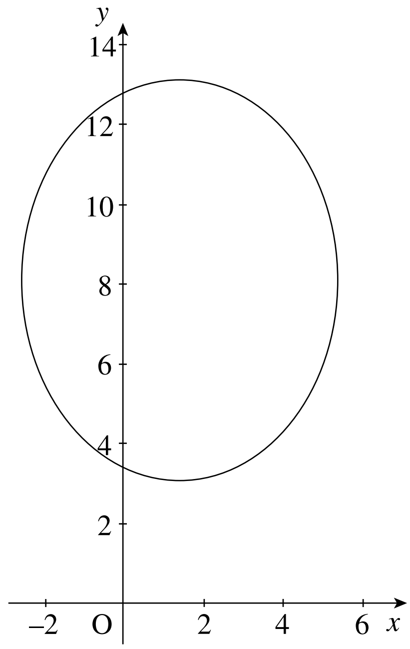
$$x^2 + 12y + 4x = 8$$
$$\Rightarrow \quad 12y = 8 - x^2 - 4x$$
$$\Rightarrow \quad 12y = 8 - (x + 2)^2 + 4$$
$$\Rightarrow \quad 12(y - 1) = -(x + 2)^2$$

(a) vertex (–2, 1)
(b) focus (–2, 1 – 3) = (–2, –2)
(c) directrix $y = (1 + 3) = 4$
(d) axis of symmetry $x = -2$

■ 18

$$25x^2 + 16y^2 - 100x - 256y + 724 = 0$$
$$\Rightarrow \quad 25x(x^2 - 4x) + 16(y^2 - 16) + 724 = 0$$
$$\Rightarrow \quad 25(x - 2)^2 + 16(y - 8)^2 = 100 + 1024 - 724$$
$$\Rightarrow \quad 25x(x - 2)^2 + 16(y - 8)^2 = 400$$
$$\Rightarrow \quad \frac{(x - 2)^2}{16} + \frac{(y - 8)^2}{25} = 1$$

Centre of ellipse is (2, 8).
Major axis of length 10, minor axis of length 8.
Eccentricity is $\sqrt{\left(1-\left(\frac{4}{5}\right)^2\right)} = \frac{3}{5}$, vertices are $(2, 8 \pm 5) = (2, 3)$ or $(2, 13)$.
Foci are $(2, 8 \pm \frac{3}{5} \times 5) = (2, 5)$ or $(2, 11)$.
Directrices are $y = 8 \pm \frac{5}{3} \times 5 = \frac{-1}{3}$ or $\frac{49}{3}$.



■ **19**

$$\begin{aligned} 10.386\,23 &= 10 + 12^0 + 4 \times 12^{-1} + 7 \times 12^{-2} + 7 \times 12^{-3} + 4 \times 12^{-4} \\ &\quad + 10 \times 12^{-5} + 4 \times 12^{-6} + 7 \times 12^{-7} + 7 \times 12^{-8} \\ &\quad + 4 \times 12^{-9} + 10 \times 12^{-10} + \cdots \\ &= \Delta.4774\Delta4774\Delta \ldots \\ &\qquad \text{where } \Delta_{12} = 10_{10} \end{aligned}$$

■ **20** Suppose x has absolute error bound ε_x and relative error bound r_x. An approximation for x is at worst

$$\tilde{x} = x + \varepsilon_x$$

$$\text{now} \quad \sqrt{\tilde{x}} = (x + \varepsilon_x)^{1/2} = x^{1/2}\left(1 + \frac{\varepsilon_x}{x}\right)^{1/2}$$

$$= x^{1/2}(1 + r_x)^{1/2}$$

$$\simeq x^{1/2}(1 + \tfrac{1}{2} r_x)$$

So the absolute error for $\sqrt{x}$ is approximately $\frac{1}{2}\sqrt{x}r_x$ and the relative error is

$$\frac{\frac{1}{2}\sqrt{x}r_x}{\sqrt{x}} = \frac{r_x}{2}$$

Label	*Value*	*Absolute error bound*		*Relative error bound*
a	7.01	0.005	$\rightarrow$	0.0007
$\sqrt{a}$	2.6476	0.0009	$\leftarrow$	0.000 35
b	52.13	0.005	$\rightarrow$	0.0001
$\sqrt{b}$	7.2201	0.0004	$\leftarrow$	0.000 05
c	0.010 11	0.000 005	$\rightarrow$	0.0005
$\sqrt{c}$	0.100 55	0.000 03	$\leftarrow$	0.000 25
d	5.631×10^{11}	5×10^{7}	$\rightarrow$	0.000 09
$\sqrt{d}$	750 399.89	34	$\leftarrow$	0.000 045

Correctly rounded values:

$\sqrt{a}$	$\sqrt{b}$	$\sqrt{c}$	$\sqrt{d}$
2.65	7.22	0.101	7.504×10^{5}

■ **21** The positive root is

$$x = \frac{-5.7 + \sqrt{(5.7)^2 + 4\times1.4\times2.3}}{2\times1.4}$$

The error table for this calculation is

Label	*Value*	*Absolute error bound*		*Relative error bound*
a	1.4	0.05	$\rightarrow$	0.036
b	5.7	0.05	$\rightarrow$	0.0088
c	–2.3	0.05	$\rightarrow$	0.022
b^2	32.49	0.58	$\leftarrow$	0.018
ac	–3.22	0.19	$\leftarrow$	0.058
$b^2 - 4ac$	45.37	1.34	$\rightarrow$	0.03
$\sqrt{b^2 - 4ac}$	6.736	0.10	$\leftarrow$	0.015
$-b + \sqrt{b^2 - 4ac}$	1.036	0.15	$\rightarrow$	0.14
$\frac{-b+\sqrt{b^2-4ac}}{2a}$	0.37	0.065	$\leftarrow$	0.176

$$x = 0.37 \pm 0.07$$

■ 22

Label	*Value*	*Absolute error bound*		*Relative error bound*
u	3.00	0.005	→	0.0017
v	4.00	0.005	→	0.0013
$\frac{1}{u}$	0.333 33	0.000 57	←	0.0017
$\frac{1}{v}$	0.25	0.000 33	←	0.0013
$\frac{1}{u}+\frac{1}{v}$	0.583 33	0.0009	→	0.0015
$f=\frac{1}{(\frac{1}{u}+\frac{1}{v})}$	1.7143	0.0026	←	0.0015
uv	12.00	0.036	←	0.003
$u+v$	7.00	0.01	→	0.0014
$f=\frac{uv}{u+v}$	1.7143	0.0075	←	0.0044

Method (a) gives the more accurate value

$$f = 1.714 \pm 0.0026$$

(b) gives

$$f = 1.714 \pm 0.0074$$

■ 23 The representation to base b has more digits than the decimal representation so $b < 10$. The digit 5 appears in the base b representation so $b > 5$. The possibilities are 6, 7, 8, 9. Since the last digit of the base b representation is 2 the remainder on division by b is 2. Only 6 has this property. So $b = 6$.

■ 24 Area of card $A(h, w) = (\frac{1}{2}w + 70 + w + 70 + \frac{1}{2}w + 5)(5 + 35 + h + 35 + 5)$

$= (2w + 145)(h + 80)$

Capacity is $h \times w \times 70$ and equals 1 136 000 mm^3.

$\Rightarrow \quad hw = 113\,600/7$

$A(h, w) = 2wh + 160w + 145h + 11\,600$

$= 145h + 160w + 2(113\,600)/7 + 11\,600$

$= C(h, w) + 308\,400/7$

$$\frac{145h + 160w}{2} \geq \sqrt{\{145h \times 160w\}}$$

with equality where $145h = 160w$, that is $w = \dfrac{145h}{160}$.

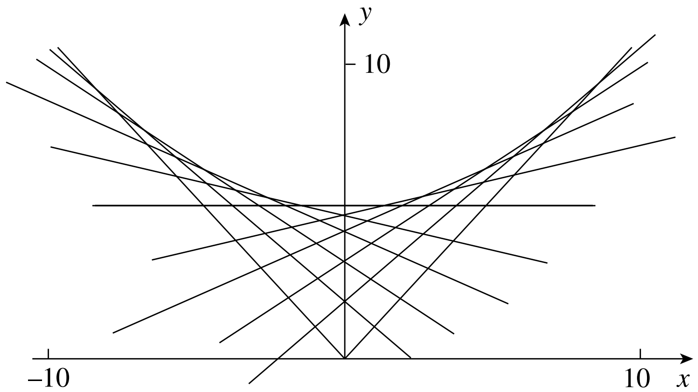
Hence $C(h, w) \geq \sqrt{\{145h \times 145h\}}$

where h is given by $h\left(\dfrac{145h}{160}\right) = 113\,600/7$.

Hence $h = 133.8$ and $w = \dfrac{145}{160} \times 133.8 = 121.3$

Practical values are $h = 134$ and $w = 121$.

■ 25(a)



25(b) $\dfrac{y-p}{10-2p} = \dfrac{x+p}{10-p+p}$; $10y - 10p = (10 - 2p)x + 10p - 2p^2$

$\Rightarrow \quad 5y = (5 - p)x + 10p - p^2$

25(c) The value of p such that a member of the family of lines passes through (x_0, y_0) is given by $p^2 + (x_0 - 10)p + 5(y_0 - x_0) = 0$.
This quadratic has two real roots if $(x_0 - 10)^2 > 20(y_0 - x_0)$
i.e. if $x_0^2 - 20x_0 + 100 > 20y_0 - 20x_0$
i.e. if $x_0^2 > 20(y_0 - 5)$
The equation has no real roots if $x_0^2 < 20(y_0 - 5)$ i.e. no line of the family passes through (x_0, y_0).
When $x_0^2 = 20(y_0 - 5)$, the equation has one double root and one line of the family passes through (x_0, y_0).
The equation of the envelope is $y = x^2/20 + 5$.

25(d) $FP^2 = x^2 + (y - 10)^2$; $PQ^2 = y^2$; FP = PQ gives

$x^2 + (y - 10)^2 = y^2 \quad \Rightarrow \quad x^2 + y^2 - 20y + 100 = y^2$

$\Rightarrow \quad y = x^2/20 + 5$
