2.1

(a) $E = 2T$

$P = 0$

(b) $E = 10T_1 - 2T_2$

$P = 0$

(c) $E = \infty$

$P = \frac{1}{2}$

(d) $E = \infty$

$P = \frac{A^2}{2}$

(e) $E = \infty$

$P = \frac{3}{8}$

(f) $E = T$

$P = 0$

(g) $E = \frac{1}{4a}$

$P = 0$

(h) $E = \frac{A^2 T^3}{4}$

$P = 0$

(i) $E = \infty$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{3}^{3+T} t^{-2/3t} dt = \lim_{T \to \infty} \frac{3\left(3+T\right)^{1/3} - 3\sqrt[3]{3}}{T} = 3 \lim_{T \to \infty} \frac{\left(3+T\right)^{1/3}}{T} = 0$$

2.2

(a) 8

(b) 0

(c) $\dfrac{4}{\pi^2}$

(d) 1

2.3

(a) $$X(s) = \frac{5}{s+1}$$

(b) $$x(t) = 5^{-t} = e^{\ln(5^{-6})} = e^{-t\ln(5)}$$

$$X(s) = \frac{1}{s+\ln(5)}$$

(c) $$\frac{5}{s-1}$$

(d) $$\frac{5}{s+1}$$

2.4

(a) $$x(t) = (t-2)^2 u(t) = \left(t^2 - 4t + 4\right)u(t)$$

$$X(s) = \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s} = \frac{2 - 4s + 4s^2}{s^3}$$

(b) $$x(t) = (t-2)^2 e^{-3t}u(t) = \left(t^2 - 4t + 4\right)e^{-3t}u(t)$$

$$X(s) = \frac{2}{(s+3)^3} - \frac{4}{(s+3)^2} + \frac{4}{s+3} = \frac{26 + 20s + 4s^2}{(s+3)^3}$$

(c) $$x(t) = \cos\left(\omega_0 t - \frac{\pi}{4}\right)u(t)$$

$$= \left[\cos\left(\frac{\pi}{4}\right)\cos\left(\omega_0 t\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\omega_0 t\right)\right]u(t)$$

$$= \left[\frac{1}{\sqrt{2}}\cos\left(\omega_0 t\right) - \frac{1}{\sqrt{2}}\sin\left(\omega_0 t\right)\right]u(t)$$

$$X(s) = \frac{1}{\sqrt{2}}\frac{s}{s^2 + \omega_0^2} + \frac{1}{\sqrt{2}}\frac{\omega_0}{s^2 + \omega_0^2} = \frac{1}{\sqrt{2}}\frac{s + \omega_0}{s^2 + \omega_0^2}$$

(d) $$x(t) = e^{-3t}\cos\left(\omega_0 t + \frac{\pi}{3}\right)u(t)$$

$$= e^{-3t}\left[\cos\left(\frac{\pi}{3}\right)\cos\left(\omega_0 t\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\omega_0 t\right)\right]u(t)$$

$$= e^{-3t}\left[\frac{1}{2}\cos\left(\omega_0 t\right) - \frac{\sqrt{3}}{2}\sin\left(\omega_0 t\right)\right]u(t)$$

$$X(s) = \frac{1}{2}\frac{s+3}{(s+3)^2 + \omega_0^2} - \frac{\sqrt{3}}{2}\frac{\omega_0}{(s+3)^2 + \omega_0^2} = \frac{1}{2}\frac{s + 3 - \sqrt{3}\omega_0}{(s+3)^2 + \omega_0^2}$$

2.5

(a)

$$X(s)=\frac{\frac{1}{2}+j}{s+2-j}+\frac{\frac{1}{2}-j}{s+2+j}$$

$$\begin{aligned}x(t)&=\left[\left(\frac{1}{2}+j\right)e^{-(2-j)t}+\left(\frac{1}{2}-j\right)e^{-(2+j)t}\right]u(t)\\&=e^{-2t}\left[\frac{1}{2}e^{jt}+je^{jt}+\frac{1}{2}e^{-jt}-je^{-jt}\right]u(t)\\&=e^{-2t}\left[\cos(t)-2\sin(t)\right]u(t)\\&=\sqrt{5}e^{-2t}\cos\left(t+63.4^{\circ}\right)u(t)\end{aligned}$$

(b)

$$X(s)=\frac{4}{s+2}-\frac{4}{s+1}+\frac{4}{\left(s+1\right)^2}$$

$$x(t)=\left[4e^{-2t}-4e^{-t}+4te^{-t}\right]u(t)=4\left[e^{-2t}+\left(t-1\right)e^{-t}\right]u(t)$$

(c)

$$X(s)=-\frac{4}{s+2}-\frac{4}{\left(s+2\right)^2}+\frac{4}{s+1}$$

$$x(t)=\left[-4e^{-2t}-4te^{-2t}+4e^{-t}\right]u(t)=4\left[e^{-t}-\left(t+1\right)e^{-2t}\right]u(t)$$

2.6

(a)
$$X(s)=\frac{3}{s+3}+\frac{2}{s+2}+\frac{1}{s+1}$$
$$x(t)=\left[3e^{-3t}+2e^{-2t}+e^{-t}\right]u(t)$$

(b)
$$X(s)=-\frac{229}{s+3}-\frac{135}{(s+3)^2}-\frac{54}{(s+3)^3}+\frac{216}{s+2}-\frac{108}{(s+2)^2}+\frac{13}{s+1}$$
$$x(t)=\left[\left(-229-135t-26t^2\right)e^{-3t}+\left(216-108t\right)e^{-2t}+13e^{-t}\right]u(t)$$

(c)
$$X(s)=\frac{32}{s+3}+\frac{31}{(s+3)^2}+\frac{30}{(s+3)^3}+\frac{21}{s+2}+\frac{20}{(s+2)^2}+\frac{1}{s+1}$$
$$x(t)=\left[\left(32+31t+30t^2\right)e^{-3t}+\left(21+20t\right)e^{-2t}+e^{-t}\right]u(t)$$

2.7

(a) $y''(t)+a_1y'(t)+a_2y(t)=b_1x'(t)+b_0x(t)+$ initial conditions

(b) The poles are at $s=-a_1\pm\sqrt{a_1^2-4a_0}$

(i) For real and distinct poles, we require $a_1^2>4a_0$. In this case, the poles are $s=-a_1\pm\sqrt{a_1^2-4a_0}$.

(ii) For real and repeated poles, we require $a_1^2=4a_0$. In this case, the poles are $s=-a_1, s=-a_1$.

(iii) For complex conjugate poles, we require $a_1^2<4a_0$. In this case, the poles are $s=-a_1\pm j\sqrt{4a_0-a_1^2}$

(c)

(i) $\omega_n=\sqrt{a_0} \quad \zeta=\dfrac{a_1}{2\sqrt{a_0}}$

In general, the poles are at $s=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$

(ii) For real and distinct poles, we require $\zeta>1$. In this case, the poles are $s=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$

(iii) For real and repeated poles, we require $\zeta=1$. In this case, the poles are $s=-\omega_n, s=-\omega_n$.

(iv) For complex conjugate poles, we require $0<\zeta<1$. In this case, the poles are $s=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$

(v) This form is preferred because the nature of the poles is determined by a single parameter.

(d)

(i)

$$H(s)=\frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\overline{\dfrac{b_0}{2\omega_n\sqrt{\zeta^2-1}}}}{s+\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)}-\frac{\overline{\dfrac{b_0}{2\omega_n\sqrt{\zeta^2-1}}}}{s+\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)}$$

$$\begin{aligned}
h(t)&=\frac{b_0}{2\omega_n\sqrt{\zeta^2-1}}\left[e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}-e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t}\right]u(t)\\
&=\frac{b_0e^{-\zeta\omega_n t}}{2\omega_n\sqrt{\zeta^2-1}}\left[e^{\left(\omega_n\sqrt{\zeta^2-1}\right)t}-e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)t}\right]u(t)\\
&=\frac{b_0}{\omega_n\sqrt{\zeta^2-1}}e^{-\omega_n\zeta t}\sinh\left(\left(\omega_n\sqrt{\zeta^2-1}\right)t\right)u(t)
\end{aligned}$$

(ii)

$$H(s)=\frac{b_0}{\left(s+\omega_n\right)^2}$$

$$h(t)=b_0te^{-\omega_n t}u(t)$$

(iii)

$$H(s)=\frac{\overline{\dfrac{b_0}{j2\omega_n\sqrt{1-\zeta^2}}}}{s+\omega_n\left(\zeta-j\sqrt{1-\zeta^2}\right)}-\frac{\overline{\dfrac{b_0}{j2\omega_n\sqrt{1-\zeta^2}}}}{s+\omega_n\left(\zeta+j\sqrt{1-\zeta^2}\right)}$$

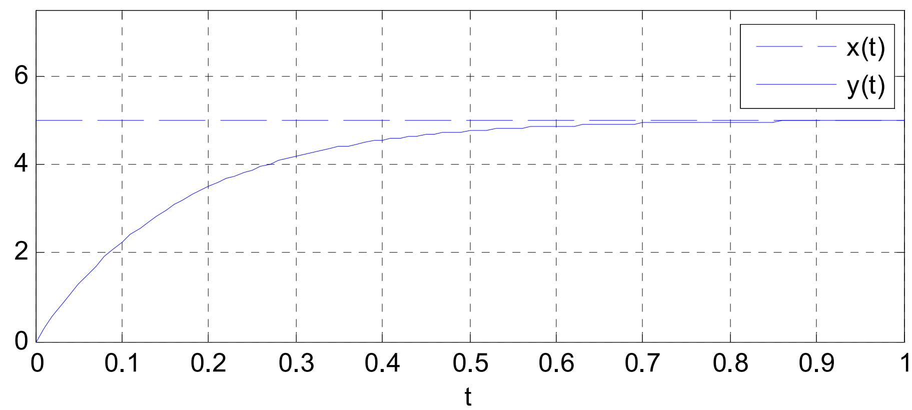
$$\begin{aligned}
h(t)&=\frac{b_0}{j2\omega_n\sqrt{1-\zeta^2}}\left[e^{-\omega_n\zeta t}e^{j\omega_n\sqrt{1-\zeta^2}t}-e^{-\omega_n\zeta t}e^{-j\omega_n\sqrt{1-\zeta^2}t}\right]u(t)\\
&=\frac{b_0e^{-\omega_n\zeta t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t\right)u(t)
\end{aligned}$$

(e) An *overdamped system* does not display any oscillations in its impulse response, whereas the *underdamped system* does display oscillations. Hence, the term *damped* refers to oscillations: if oscillations are present, the oscillations have not been damped; if there are no oscillations, the oscillations have been damped.

2.8

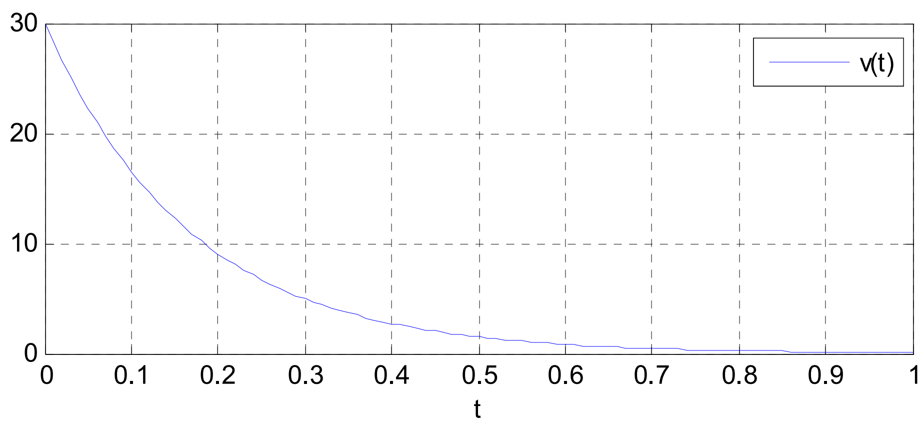
(a)
$$Y(s)=\frac{6}{s+6}X(s)=\frac{6}{s+6}\frac{5}{s}=\frac{5}{s}-\frac{5}{s+6}$$
$$y(t)=5\left[1-e^{-6t}\right]u(t)$$

(b)



(c)
$$V(s)=\frac{6s}{s+6}X(s)=\frac{6s}{s+6}\frac{5}{s}=\frac{30}{s+6}$$
$$v(t)=30e^{-6t}u(t)$$

(d)

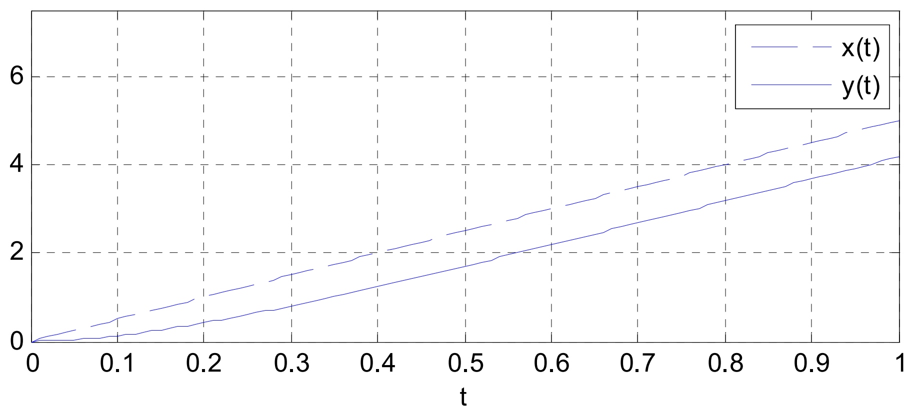


2.9

(a) $$Y(s)=\frac{6}{s+6}X(s)=\frac{6}{s+6}\frac{5}{s^2}=\frac{5}{s^2}-\frac{5/6}{s}+\frac{5/6}{s+6}$$

$$y(t)=\left[5t-\frac{5}{6}\left(1-e^{-6t}\right)\right]u(t)$$

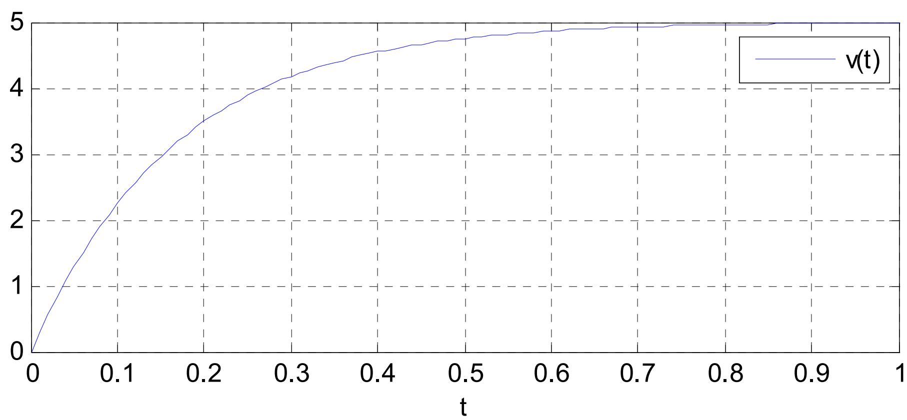
(b)



(c) $$V(s)=\frac{6s}{s+6}X(s)=\frac{6s}{s+6}\frac{5}{s^2}=\frac{30}{s(s+6)}=\frac{5}{s}-\frac{5}{s+6}$$

$$v(t)=5\left[1-e^{-6t}\right]u(t)$$

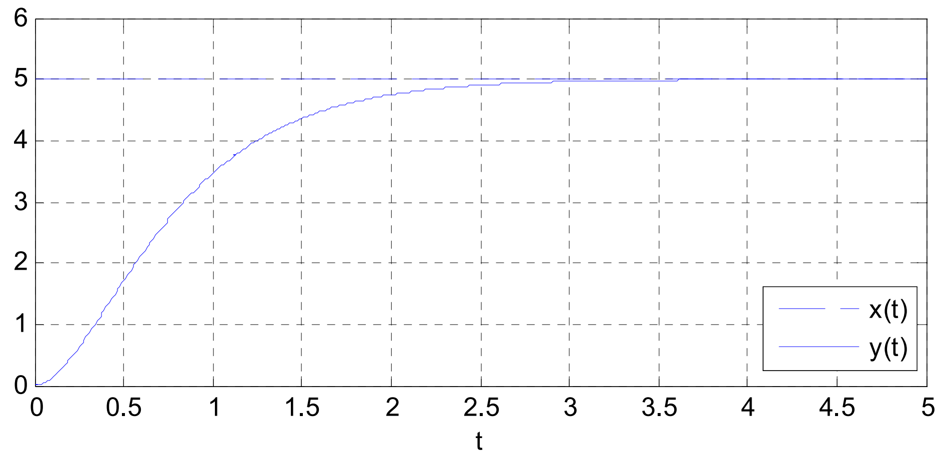
(d)



2.10

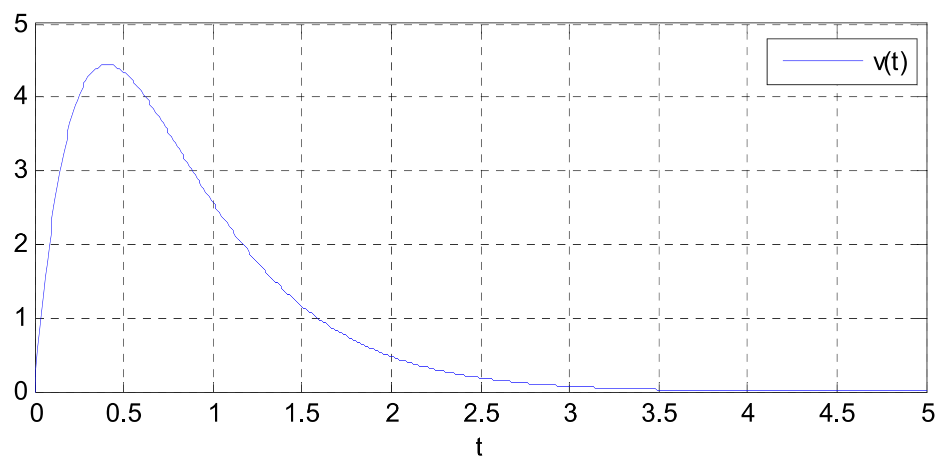
(a)
$$Y(s)=\frac{6}{s^2+5s+6}X(s)=\frac{6}{s^2+5s+6}\frac{5}{s}=\frac{5}{s}-\frac{15}{s+2}+\frac{10}{s+3}$$
$$y(t)=5\left[1-3e^{-2t}+2e^{-3t}\right]u(t)$$

(b)



(c)
$$V(s)=\frac{6s}{s^2+5s+6}X(s)=\frac{6s}{s^2+5s+6}\frac{5}{s}=\frac{30}{s+2}-\frac{30}{s+3}$$
$$v(t)=30\left[e^{-2t}-e^{-3t}\right]u(t)$$

(d)



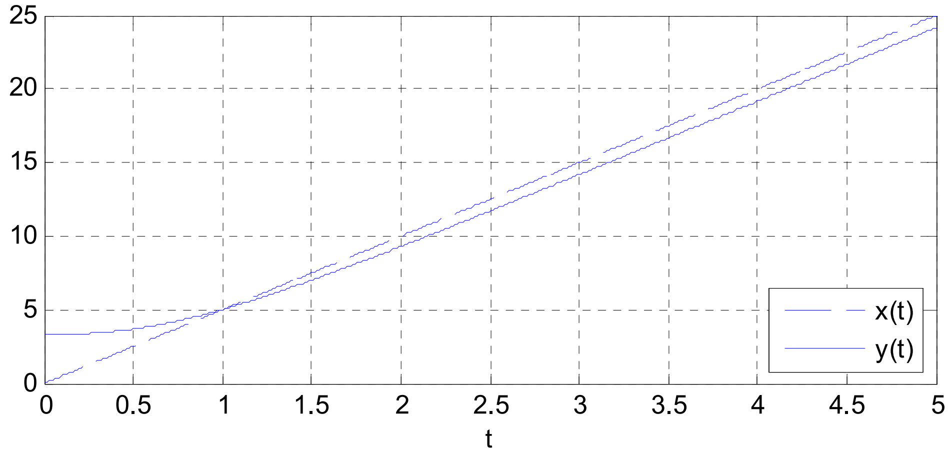
2.11

(a)

$$Y(s)=\frac{6}{s^2+5s+6}X(s)=\frac{6}{s^2+5s+6}\frac{5}{s^2}=\frac{5}{s^2}-\frac{\frac{25}{6}}{s}+\frac{\frac{15}{2}}{s+2}-\frac{\frac{10}{3}}{s+3}$$

$$y(t)=5\left[t-\frac{5}{6}+\frac{3}{2}e^{-2t}-\frac{2}{3}e^{-3t}\right]u(t)$$

(b)

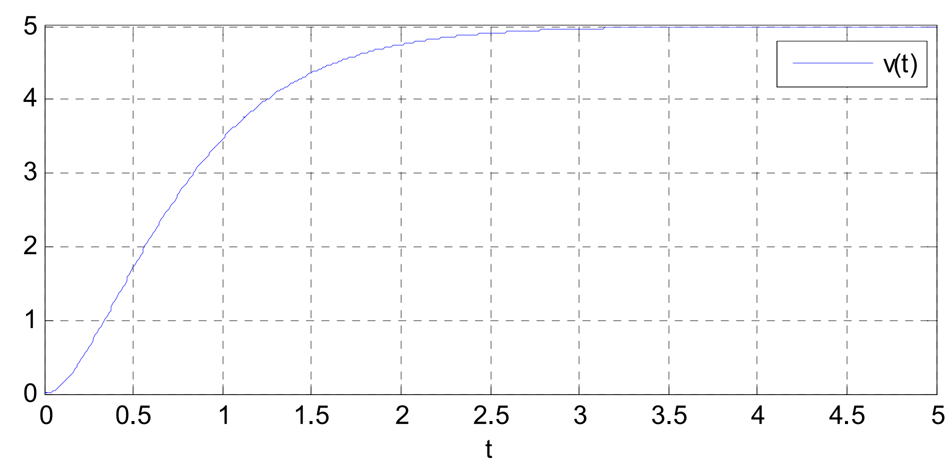


(c)

$$V(s)=\frac{6s}{s^2+5s+6}X(s)=\frac{6s}{s^2+5s+6}\frac{5}{s^2}=\frac{5}{s}-\frac{15}{s+2}+\frac{10}{s+3}$$

$$v(t)=5\left[1-3e^{-2t}+2e^{-3t}\right]u(t)$$

(d)



2.12

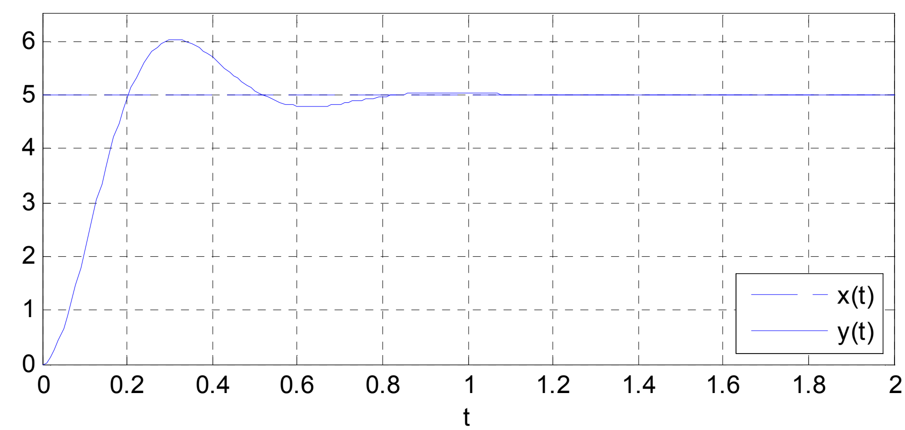
(a)

$$Y(s)=\frac{125}{s^2+10s+125}X(s)=\frac{125}{s^2+10s+125}\frac{5}{s}=\frac{5}{s}-\frac{\frac{-10+j5}{4}}{s+5-j10}+\frac{\frac{-10-j5}{4}}{s+5+j10}$$

$$y(t)=\left[5+e^{-5t}\left(\frac{-10+j5}{4}e^{j10t}-\frac{10+j5}{4}e^{-j10t}\right)\right]u(t)$$

$$=5\left[1-\frac{1}{2}e^{-5t}\left(2\cos\left(10t\right)+\sin\left(10t\right)\right)\right]u(t)=5\left[1-\frac{\sqrt{5}}{2}e^{-5t}\cos\left(10t-26.6^{\circ}\right)\right]u(t)$$

(b)

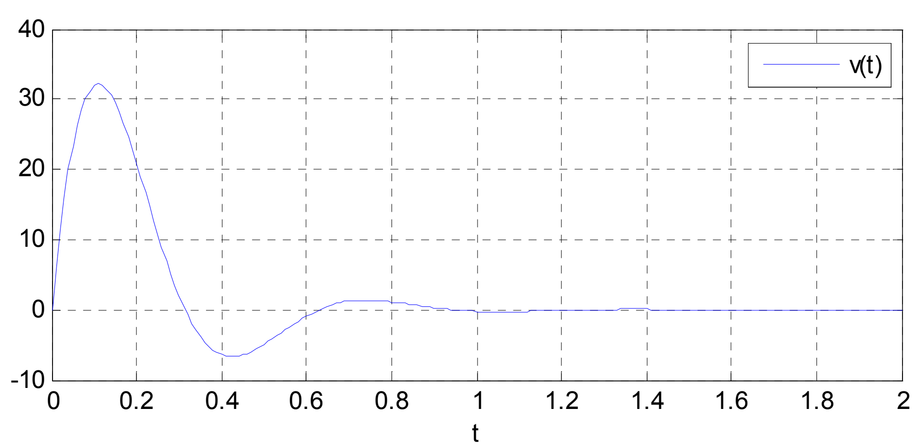


(c)

$$V(s)=\frac{125s}{s^2+10s+125}X(s)=\frac{125s}{s^2+10s+125}\frac{5}{s}=\frac{\frac{125}{j4}}{s+5-j10}-\frac{\frac{125}{j4}}{s+5+j10}$$

$$v(t)=\frac{125}{j4}e^{-5t}\left[e^{j10t}-e^{-j10t}\right]u(t)=\frac{125}{2}e^{-5t}\sin\left(10t\right)u(t)$$

(d)



2.13

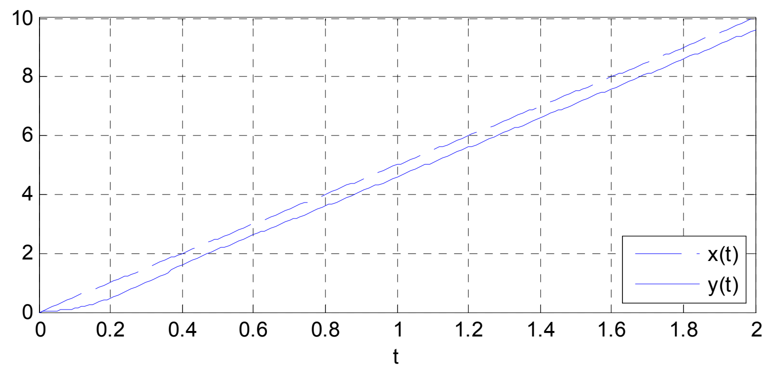
(a)

$$Y(s)=\frac{125}{s^2+10s+125}X(s)=\frac{125}{s^2+10s+125}\frac{5}{s^2}=\frac{-\frac{2}{5}}{s}+\frac{5}{s^2}+\frac{\frac{4+j3}{20}}{s+5-j10}+\frac{\frac{4-j3}{20}}{s+5+j10}$$

$$y(t)=\left[5t-\frac{2}{5}+e^{-5t}\left(\frac{4+j3}{20}e^{j10t}+\frac{4-j3}{20}e^{-j10t}\right)\right]u(t)$$

$$=\left[5t-\frac{2}{5}+e^{-5t}\left(\frac{4}{10}\cos(10t)-\frac{3}{10}\sin(10t)\right)\right]u(t)=\left[5t-\frac{2}{5}+\frac{1}{2}e^{-5t}\cos\left(10t+36.9^\circ\right)\right]u(t)$$

(b)



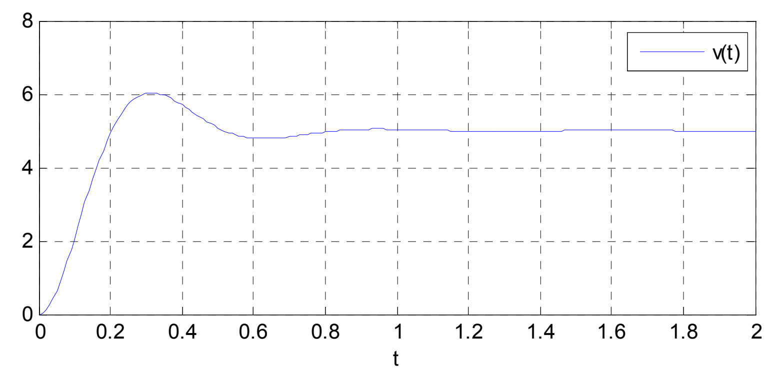
(c)

$$V(s)=\frac{125s}{s^2+10s+125}X(s)=\frac{125s}{s^2+10s+125}\frac{5}{s^2}=\frac{5}{s}-\frac{\frac{-10+j5}{4}}{s+5-j10}+\frac{\frac{-10-j5}{4}}{s+5+j10}$$

$$v(t)=\left[5+e^{-5t}\left(\frac{-10+j5}{4}e^{j10t}-\frac{10+j5}{4}e^{-j10t}\right)\right]u(t)$$

$$=5\left[1-\frac{1}{2}e^{-5t}\left(2\cos(10t)+\sin(10t)\right)\right]u(t)=5\left[1-\frac{\sqrt{5}}{2}e^{-5t}\cos\left(10t-26.6^\circ\right)\right]u(t)$$

(d)



2.14

The transfer function is $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{k}{s+k}$. This system has one pole at $s = -k$. The system is stable if the pole is in the left-half plane. The pole is in the left-half plane when $k > 0$.

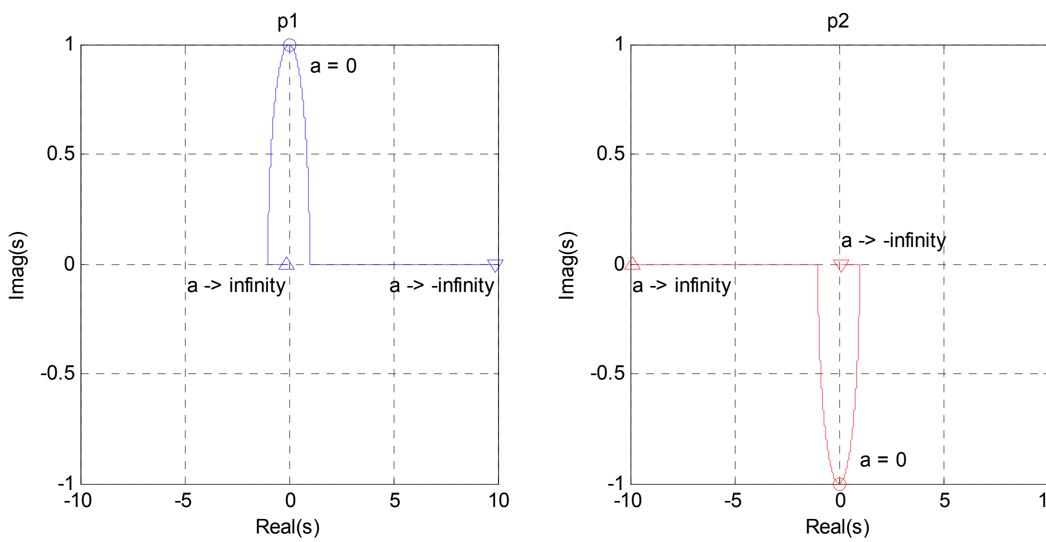
2.15

The system transfer function is $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + as + 1}$. The poles are $s = \frac{-a \pm \sqrt{a^2 - 4}}{2}$.

(a) The system is stable when the poles are in the left-half plane. The poles are in the left-half plane for $a > 0$.

(b) The system impulse response exhibits oscillations when the poles have a non-zero imaginary part. The poles have a non-zero imaginary part when $a^2 - 4 < 0$, which implies $-2 < a < 2$. (Note that the system response is oscillatory *and* stable only for $0 \le a < 2$.)

These results are summarized by the plot below. This plot, known as a "root-locus" plot, plots the location of the poles as a function of *a*.



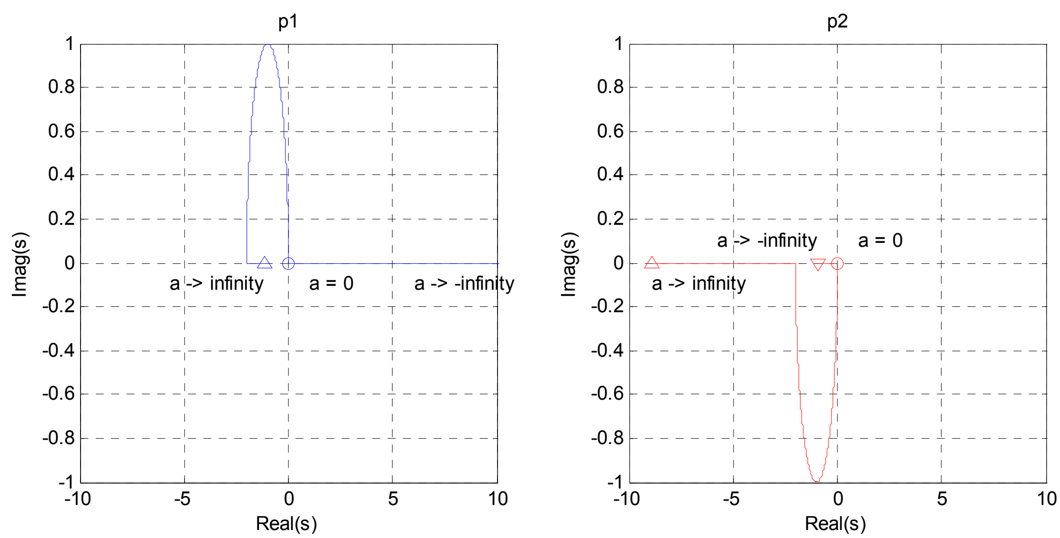
2.16

The system transfer function is $H(s)=\frac{Y(s)}{X(s)}=\frac{a}{s^2+as+a}$. The poles are $s=\frac{-a\pm\sqrt{a(a-4)}}{2}$.

(a) The system is stable when the poles are in the left-half plane. The poles are in the left-half plane for $a>0$.

(b) The system impulse response exhibits oscillations when the poles have a non-zero imaginary part. The poles have a non-zero imaginary part when $a-4<0$ and $a>0$, which implies $0<a<4$. (Note that the system response is oscillatory *and* stable for these values of *a*.)

These results are summarized by the plot below. This plot, known as a "root-locus" plot, plots the location of the poles as a function of *a*.



2.17

$$x(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2}e^{j(\omega_0 t+\theta)} + \frac{A}{2}e^{-j(\omega_0 t+\theta)} = \frac{A}{2}e^{j\theta}e^{j\omega_0 t} + \frac{A}{2}e^{-j\theta}e^{-j\omega_0 t}$$

(a)
$$\begin{aligned} X(j\omega) &= \frac{A}{2}e^{j\theta} \times 2\pi\delta(\omega-\omega_0) + \frac{A}{2}e^{-j\theta} \times 2\pi\delta(\omega+\omega_0) \\ &= A\pi e^{j\theta}\delta(\omega-\omega_0) + A\pi e^{-j\theta}\delta(\omega+\omega_0) \end{aligned}$$

(b)
$$X(f) = \frac{A}{2}e^{j\theta}\delta(f-f_0) + \frac{A}{2}e^{-j\theta}\delta(f+f_0)$$

2.18

$$X(f) = \int_{-\infty}^{0} e^{at} e^{-j2\pi ft} dt + \int_{0}^{\infty} e^{-at} e^{-j2\pi ft} dt$$

$$= \frac{1}{-j2\pi f + a} + \frac{1}{j2\pi f + a}$$

$$= \frac{2a}{a^2 + 4\pi^2 f^2}$$

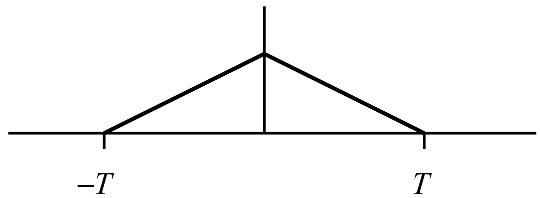
$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{-j\omega + a} + \frac{1}{j\omega + a}$$

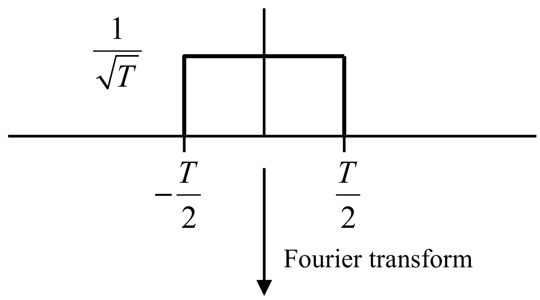
$$= \frac{2a}{a^2 + \omega^2}$$

2.19

The easiest way to compute the Fourier transform is to recognize that $x(t)$, plotted below



can be produced by convolving the function plotted below with itself.



$$\sqrt{T}\,\frac{\sin(\pi fT)}{\pi fT}$$

Thus, the Fourier transform may be computed as follows:

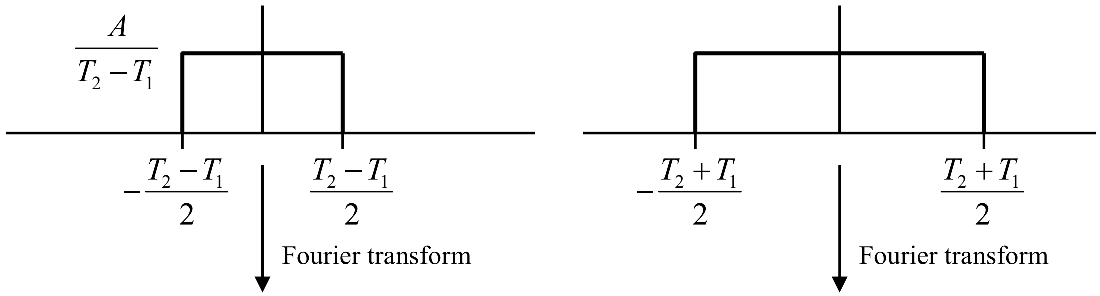
$$X(f) = \sqrt{T}\,\frac{\sin(\pi fT)}{\pi fT}\sqrt{T}\,\frac{\sin(\pi fT)}{\pi fT} = T\,\frac{\sin^2(\pi fT)}{\pi^2 f^2 T^2}$$

Similarly,

$$X(j\omega) = \sqrt{T}\,\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}}\sqrt{T}\,\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}} = T\,\frac{\sin^2\left(\omega\frac{T}{2}\right)}{\left(\omega\frac{T}{2}\right)^2}$$

2.20

The easiest way to compute the Fourier transform is to recognize that $x(t)$ is produced by the convolution of the two functions shown below.



$$(T_2-T_1)\frac{\sin(\pi f(T_2-T_1))}{\pi f(T_2-T_1)} \qquad (T_2+T_1)\frac{\sin(\pi f(T_2+T_1))}{\pi f(T_2+T_1)}$$

Multiplying the two Fourier transforms produces

$$X(f)=A(T_2+T_1)\frac{\sin(\pi f(T_2-T_1))}{\pi f(T_2-T_1)}\frac{\sin(\pi f(T_2+T_1))}{\pi f(T_2+T_1)}$$

Similarly

$$X(j\omega)=A(T_2+T_1)\frac{\sin\left(\omega\left(\frac{T_2-T_1}{2}\right)\right)}{\omega\left(\frac{T_2-T_1}{2}\right)}\frac{\sin\left(\omega\left(\frac{T_2+T_1}{2}\right)\right)}{\omega\left(\frac{T_2+T_1}{2}\right)}$$

2.21

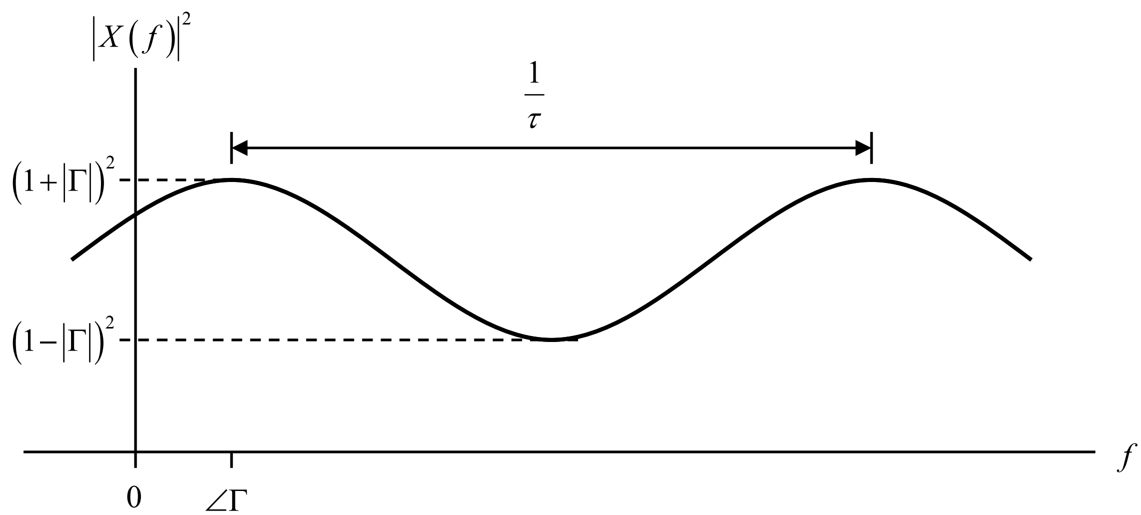
(a) $X(f) = 1 + \Gamma e^{-j2\pi f\tau}$

(b) Let $\Gamma = |\Gamma| e^{j\angle\Gamma}$. Then $X(f) = 1 + |\Gamma| e^{-j(2\pi f\tau - \angle\Gamma)}$

$$\left|X(f)\right|^2 = X(f)X^*(f) = 1 + |\Gamma| e^{j(2\pi f\tau - \angle\Gamma)} + |\Gamma| e^{-j(2\pi f\tau - \angle\Gamma)} + |\Gamma|^2$$

$$= 1 + |\Gamma|^2 + 2|\Gamma|\cos(2\pi f\tau - \angle\Gamma)$$

(c)



2.22

(a) $$Y(j\omega) = 3e^{-j5\omega}\frac{j\omega}{1+j\omega}$$

(b) $$Y(j\omega) = e^{-j3(\omega-5)}\frac{j(\omega-5)}{1+j(\omega-5)}$$

(c) The symmetry properties of $X(j\omega)$ show that $x(t)$ is real. Hence,

$$Y(j\omega) = \frac{-j\omega}{1-j\omega}$$

(d) $$Y(j\omega) = \frac{-\omega^2}{1+j\omega}$$

2.23

(a)

$$x(t) = j5e^{-j2\pi 1000t} + 3e^{j\frac{\pi}{6}}e^{-j2\pi 60t} + 16 + 3e^{-j\frac{\pi}{6}}e^{j2\pi 60t} - j5e^{j2\pi 1000t}$$

$$= 16 + 3\left(e^{-j\left(2\pi 60t - \frac{\pi}{6}\right)} + e^{j\left(2\pi 60t - \frac{\pi}{6}\right)}\right) + j5\left(e^{-j2\pi 1000t} - e^{j2\pi 1000t}\right)$$

$$= 16 + 6\cos\left(2\pi 60t - \frac{\pi}{6}\right) - 10\sin\left(j2\pi 1000t\right)$$

(b) Yes, $x(t)$ is periodic. The period is $\text{LCM}\left\{\frac{1}{60}, \frac{1}{1000}\right\} = \frac{1}{10}$.

2.24

Using the identity $\dfrac{1}{2+j2\pi f} \to e^{-2t}u(t)$ and the property $e^{-j2\pi f t_0} X\left(f\right) \to x\left(t-t_0\right)$ for the last term, we have

$$\begin{aligned} x\left(t\right) &= \frac{1}{2}e^{-j2\pi 2t} + \frac{1}{2}e^{j2\pi 2t} + e^{-2(t+2)}u\left(t+2\right) \\ &= \cos\left(2\pi 2t\right) + e^{-2(t+2)}u\left(t+2\right) \end{aligned}$$

2.25

(a) $X(f) = \dfrac{1}{a + j2\pi f} \quad \rightarrow \quad |X(f)|^2 = \dfrac{1}{a^2 + (2\pi f)^2} \quad \rightarrow \quad \dfrac{1}{a^2 + (2\pi B_3)^2} = \dfrac{1}{2a^2}$

$B_3 = \dfrac{a}{2\pi}$

(b) $X(f) = \dfrac{2a}{a^2 + (2\pi f)^2} \quad \rightarrow \quad \dfrac{2a}{a^2 + (2\pi B_3)^2} = \dfrac{1}{\sqrt{2}} \times \dfrac{2}{a}$

$B_3 = \dfrac{a}{2\pi}\sqrt{\sqrt{2} - 1}$

(c) $X(f) = e^{-\pi f^2} \quad \rightarrow \quad |X(f)|^2 = e^{-2\pi f^2} \quad \rightarrow \quad e^{-2\pi B_3^2} = \dfrac{1}{2}$

$B_3 = \sqrt{\dfrac{\ln(2)}{2\pi}}$

2.26

Using the transform pair

$$2T\frac{\sin(2\pi fT)}{2\pi fT} \quad \leftrightarrow \quad \begin{cases} 1 & -T \le t \le T \\ 0 & \text{otherwise} \end{cases},$$

set $T = \frac{1}{2}$ and apply Parseval's theorem:

$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi f)}{(\pi f)^2}\,df = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1)^2\,dt = 1$$

2.27

(a)
$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{0}^{\infty} e^{-2at}dt = \frac{1}{2a}$$

$$X(f) = \frac{1}{a + j2\pi f} \quad \rightarrow \quad \left|X(f)\right|^2 = \frac{1}{a^2 + (2\pi f)^2}$$

$$\frac{0.9}{2a} = 2\int_{0}^{B_{90}} \frac{df}{a^2 + (2\pi f)^2} = \frac{\tan^{-1}\left(\frac{2\pi B_{90}}{a}\right)}{\pi a}$$

$$B_{90} = \frac{a}{2\pi}\tan\left(\frac{0.9\pi}{a}\right)$$

(b)
$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{0} e^{2at}dt + \int_{0}^{\infty} e^{-2at}dt = \frac{1}{a}$$

$$X(f) = \frac{2a}{a^2 + (2\pi f)^2} \quad \rightarrow \quad \left|X(f)\right|^2 = \frac{4a^2}{\left(a^2 + (2\pi f)^2\right)^2}$$

$$\frac{0.9}{a} = 2\int_{0}^{B_{90}} \frac{4a^2}{\left(a^2 + (2\pi f)^2\right)^2}df = \frac{4B_{90}}{a^2 + (2\pi B_{90})^2} + \frac{2\tan^{-1}\left(\frac{2\pi B_{90}}{a}\right)}{\pi a}$$

(c)
$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} e^{-\pi t^2}dt = 1$$

$$X(f) = e^{-\pi f^2} \quad \rightarrow \quad \left|X(f)\right|^2 = e^{-2\pi f^2}$$

$$0.9 = 2\int_{0}^{B_{90}} e^{-2\pi f^2}df \quad \rightarrow \quad 0.45 = \int_{0}^{B_{90}} e^{-2\pi f^2}df = \int_{0}^{\infty} e^{-2\pi f^2}df - \int_{B_{90}}^{\infty} e^{-2\pi f^2}df$$

Now, using $\int_{0}^{\infty} e^{-\frac{u^2}{2}}du = \frac{\sqrt{2\pi}}{2}$ and $Q(z) = \frac{1}{\sqrt{2\pi}}\int_{z}^{\infty} e^{-\frac{u^2}{2}}du$ the two intergrals are

$$\int_{0}^{\infty} e^{-2\pi f^2}df = \frac{1}{\sqrt{4\pi}}\int_{0}^{\infty} e^{-\frac{u^2}{2}}du = \frac{1}{\sqrt{4\pi}}\frac{\sqrt{2\pi}}{2} = \frac{1}{2\sqrt{2}}$$

$$\int_{B_{90}}^{\infty} e^{-2\pi f^2}df = \frac{1}{\sqrt{4\pi}}\int_{\sqrt{4\pi}B_{90}}^{\infty} e^{-\frac{u^2}{2}}du = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi}}\int_{\sqrt{4\pi}B_{90}}^{\infty} e^{-\frac{u^2}{2}}du = \frac{1}{\sqrt{2}}Q\left(\sqrt{4\pi}B_{90}\right)$$

Putting this together produces, we have that B_{90} is determined from

$$0.45\sqrt{2} - 0.5 = Q\left(\sqrt{4\pi}B_{90}\right)$$

2.28

(a)

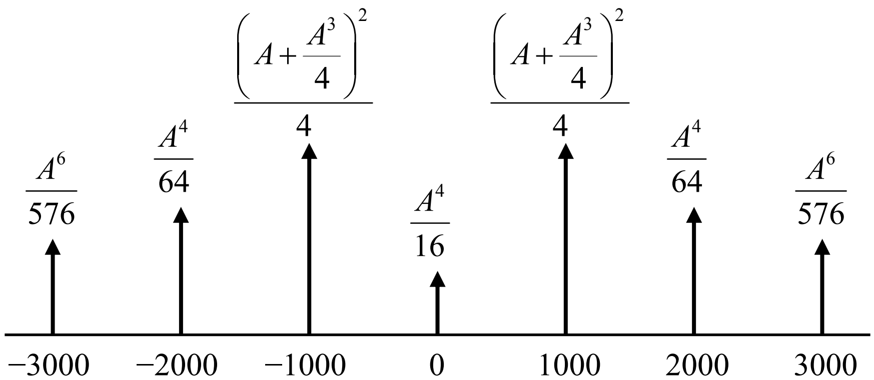
$$y(t) = A\cos(\omega_0 t) + \frac{A^2}{2}\cos^2(\omega_0 t) + \frac{A^3}{3}\cos^3(\omega_0 t)$$

$$= \frac{A^2}{4} + \left(A + \frac{A^3}{4}\right)\cos(\omega_0 t) + \frac{A^2}{4}\cos(2\omega_0 t) + \frac{A^3}{12}\cos(3\omega_0 t)$$

(b)

$$Y(f) = \frac{A^2}{4}\delta(f) + \frac{A + \frac{A^3}{4}}{2}\left[\delta(f+1000) + \delta(f-1000)\right]$$

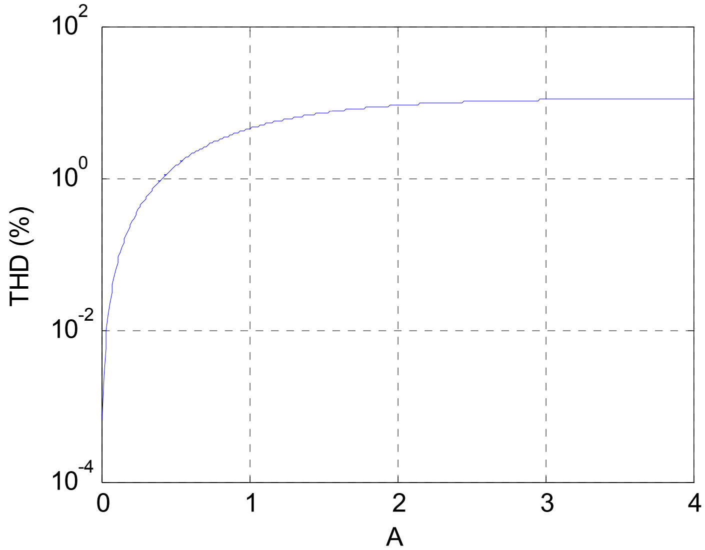
$$+ \frac{A^2}{8}\left[\delta(f+2000) + \delta(f-2000)\right] + \frac{A^3}{24}\left[\delta(f+3000) + \delta(f-3000)\right]$$



(c)

$$\text{THD} = \frac{\frac{A^6}{576} + \frac{A^6}{576} + \frac{A^4}{64} + \frac{A^4}{64}}{\frac{\left(A+\frac{A^3}{4}\right)^2}{4} + \frac{\left(A+\frac{A^3}{4}\right)^2}{4}} \times 100 = \frac{1}{9}\frac{9A^4 + A^6}{16A^2 + 8A^4 + A^6} \times 100$$

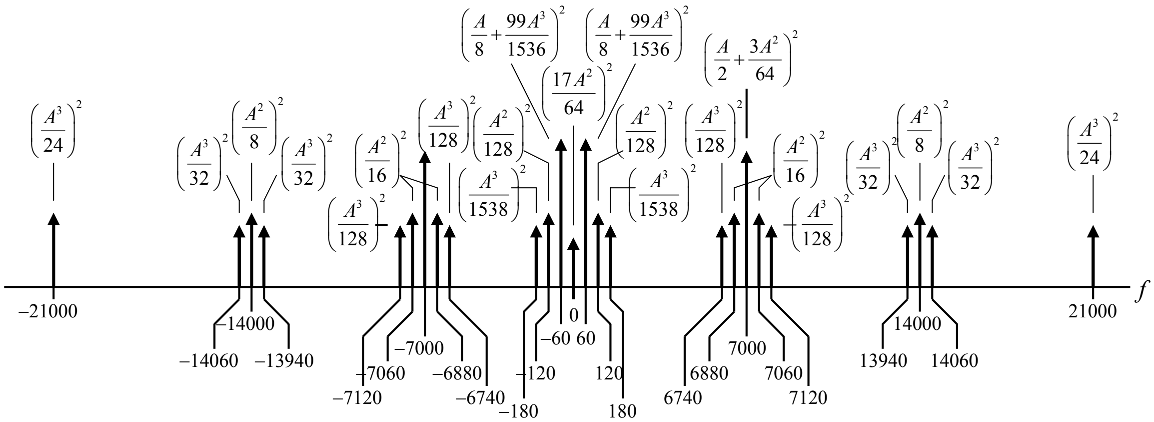
A plot of the THD is shown below. Note that the THD increases as the amplitude of the sinusoid increases.


10^2
10^0
10^-2
10^-4
THD (%)
0
1
2
3
4
A

2.29

(a)
$$x(t)=\frac{A}{4}\cos(\omega_0 t)+A\cos(\omega_1 t)$$
$$x^2(t)=\frac{17A^2}{32}+\frac{A^2}{32}\cos(2\omega_0 t)+\frac{A^2}{32}\cos(2\omega_1 t)+\frac{A^2}{4}\cos((\omega_1-\omega_0)t)+\frac{A^2}{4}\cos((\omega_1+\omega_0)t)$$
$$\begin{aligned}x^3(t)=&\frac{99A^3}{256}\cos(\omega_0 t)+\frac{27A^3}{32}\cos(\omega_1 t)+\frac{A^3}{256}\cos(3\omega_0 t)+\frac{A^3}{4}\cos(3\omega_1 t)\\&+\frac{3A^3}{64}\cos((\omega_1-2\omega_0)t)+\frac{3A^3}{64}\cos((\omega_1+2\omega_0)t)\\&+\frac{3A^3}{16}\cos((2\omega_1-\omega_0)t)+\frac{3A^3}{16}\cos((2\omega_1+\omega_0)t)\end{aligned}$$
$$\begin{aligned}y(t)=&\frac{17A^2}{64}+\left(\frac{A}{4}+\frac{99A^3}{768}\right)\cos(\omega_0 t)+\left(A+\frac{3A^3}{32}\right)\cos(\omega_1 t)+\frac{A^2}{64}\cos(2\omega_0 t)+\frac{A^2}{4}\cos(2\omega_1 t)\\&+\frac{A^3}{768}\cos(3\omega_0 t)+\frac{A^3}{12}\cos(3\omega_1 t)+\frac{A^2}{8}\cos((\omega_1-\omega_0)t)+\frac{A^2}{8}\cos((\omega_1+\omega_0)t)\\&+\frac{A^3}{64}\cos((\omega_1-2\omega_0)t)+\frac{A^3}{64}\cos((\omega_1+2\omega_0)t)\\&+\frac{A^3}{16}\cos((2\omega_1-\omega_0)t)+\frac{A^3}{16}\cos((2\omega_1+\omega_0)t)\end{aligned}$$

(b)
$$\begin{aligned}Y(f)=&\frac{17A^2}{64}\delta(f)+\left(\frac{A}{8}+\frac{99A^3}{1536}\right)\left[\delta(f+60)+\delta(f-60)\right]\\&+\left(\frac{A}{2}+\frac{3A^3}{64}\right)\left[\delta(f+7000)+\delta(f-7000)\right]\\&+\frac{A^2}{128}\left[\delta(f+120)+\delta(f-120)\right]+\frac{A^2}{8}\left[\delta(f+14000)+\delta(f-14000)\right]\\&+\frac{A^3}{1538}\left[\delta(f+180)+\delta(f-180)\right]+\frac{A^3}{24}\left[\delta(f+21000)+\delta(f-21000)\right]\\&+\frac{A^2}{16}\left[\delta(f+6740)+\delta(f-6740)\right]+\frac{A^2}{16}\left[\delta(f+7060)+\delta(f-7060)\right]\\&+\frac{A^3}{128}\left[\delta(f+6880)+\delta(f-6880)\right]+\frac{A^3}{128}\left[\delta(f+7120)+\delta(f-7120)\right]\\&+\frac{A^3}{32}\left[\delta(f+13940)+\delta(f-13940)\right]+\frac{A^3}{32}\left[\delta(f+14060)+\delta(f-14060)\right]\end{aligned}$$

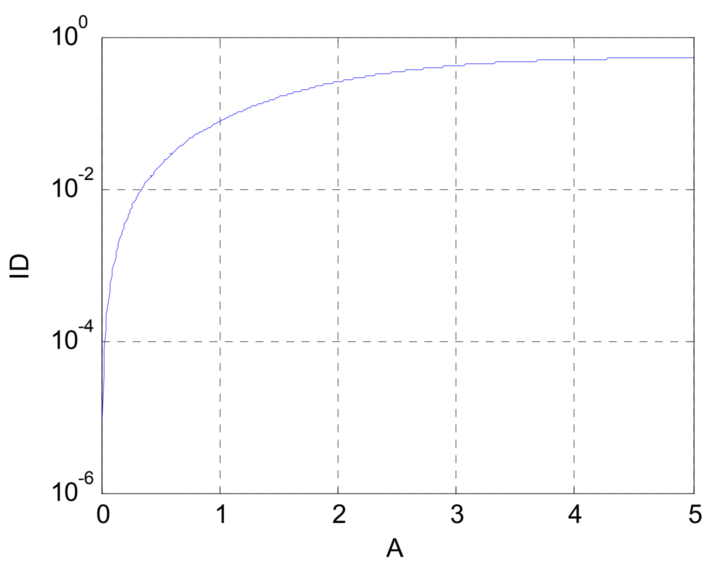


(c)

$$\text{ID} = \frac{2\left(\frac{A^2}{128}\right)^2 + 2\left(\frac{A^3}{1538}\right)^2 + 2\left(\frac{A^3}{128}\right)^2 + 2\left(\frac{A^2}{16}\right)^2 + 2\left(\frac{A^2}{16}\right)^2 + 2\left(\frac{A^3}{128}\right)^2 + 2\left(\frac{A^3}{32}\right)^2 + 2\left(\frac{A^2}{8}\right)^2 + 2\left(\frac{A^3}{32}\right)^2}{2\left(\frac{A}{8} + \frac{99A^2}{1536}\right)^2 + 2\left(\frac{A}{2} + \frac{3A^2}{64}\right)^2}$$

$$= \frac{\left(\frac{A^2}{128}\right)^2 + \left(\frac{A^3}{1538}\right)^2 + 2\left(\frac{A^3}{128}\right)^2 + 2\left(\frac{A^2}{16}\right)^2 + 2\left(\frac{A^3}{32}\right)^2 + \left(\frac{A^2}{8}\right)^2}{\left(\frac{A}{8} + \frac{99A^2}{1536}\right)^2 + \left(\frac{A}{2} + \frac{3A^2}{64}\right)^2}$$

A plot of the ID is shown below. Note that the ID is a function of *A*.



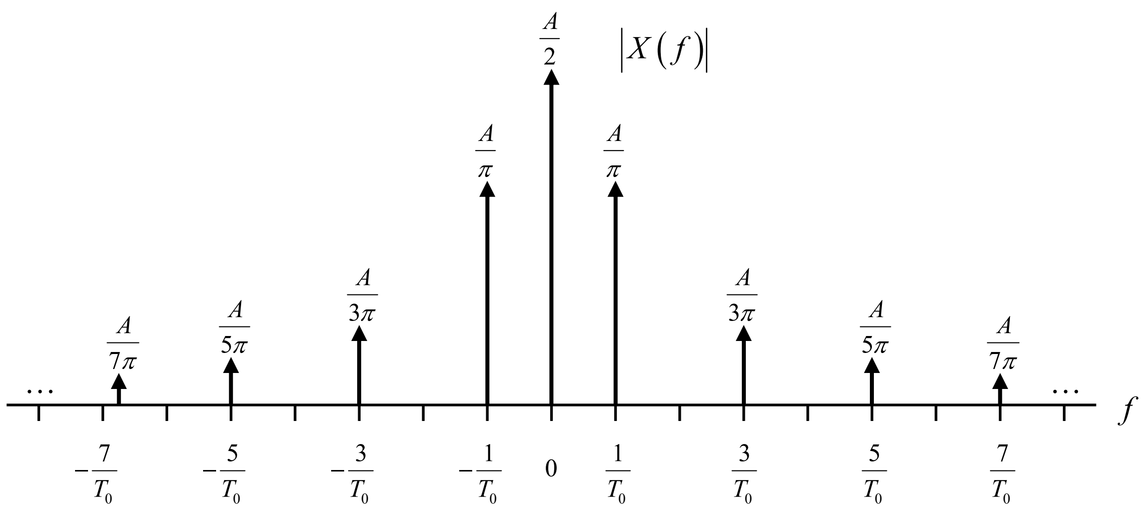
2.30

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k=\frac{1}{T_0}\int_0^{\frac{T_0}{2}} A e^{-j\frac{2\pi k}{T_0}t}\,dt=\frac{A}{2}\frac{\sin\left(\frac{\pi k}{2}\right)}{\frac{\pi k}{2}}e^{-j\frac{\pi k}{2}}=\begin{cases}\frac{A}{2} & k=0\\ 0 & k,\ \text{even}\\ \frac{A}{\pi k}(-j)^k & k,\ \text{odd}\end{cases}$$

$$X(f)=\sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T_0}\right)$$

$$\left|X(f)\right|=\sum_{k=-\infty}^{\infty} \left|c_k\right|\delta\left(f-\frac{k}{T_0}\right)$$



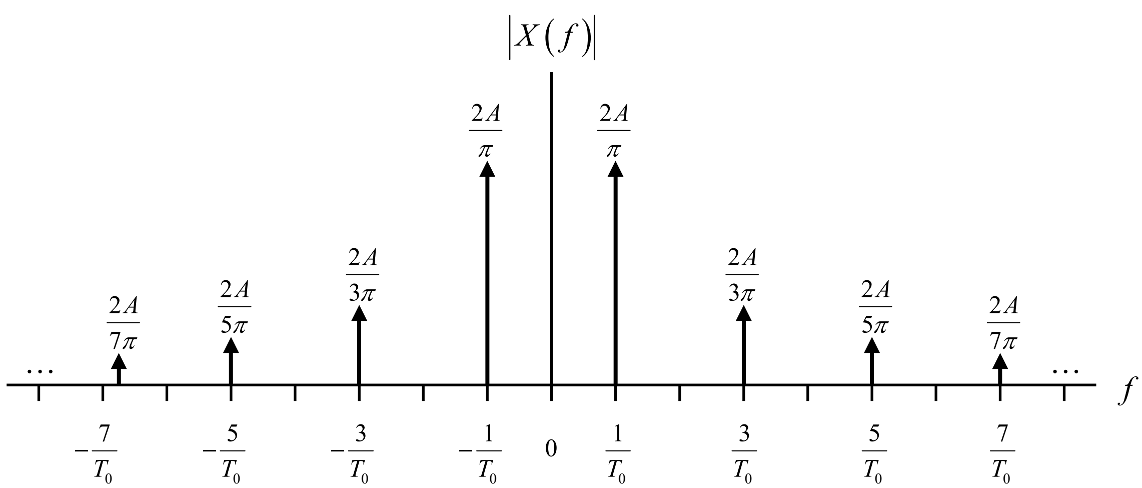
2.31

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k=\frac{1}{T_0}\int_0^{\frac{T_0}{2}} A e^{-j\frac{2\pi k}{T_0}t}dt-\frac{1}{T_0}\int_{\frac{T_0}{2}}^{T_0} A e^{-j\frac{2\pi k}{T_0}t}dt=-A\frac{\sin\left(\frac{\pi k}{2}\right)}{\frac{\pi k}{2}}(-1)^k e^{-j\frac{\pi k}{2}}=\begin{cases}0 & k=0\\ 0 & k,\ \text{even}\\ -\frac{2A}{\pi k}(-j)^k & k,\ \text{odd}\end{cases}$$

$$X(f)=\sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T_0}\right)$$

$$\left|X(f)\right|=\sum_{k=-\infty}^{\infty} \left|c_k\right|\delta\left(f-\frac{k}{T_0}\right)$$



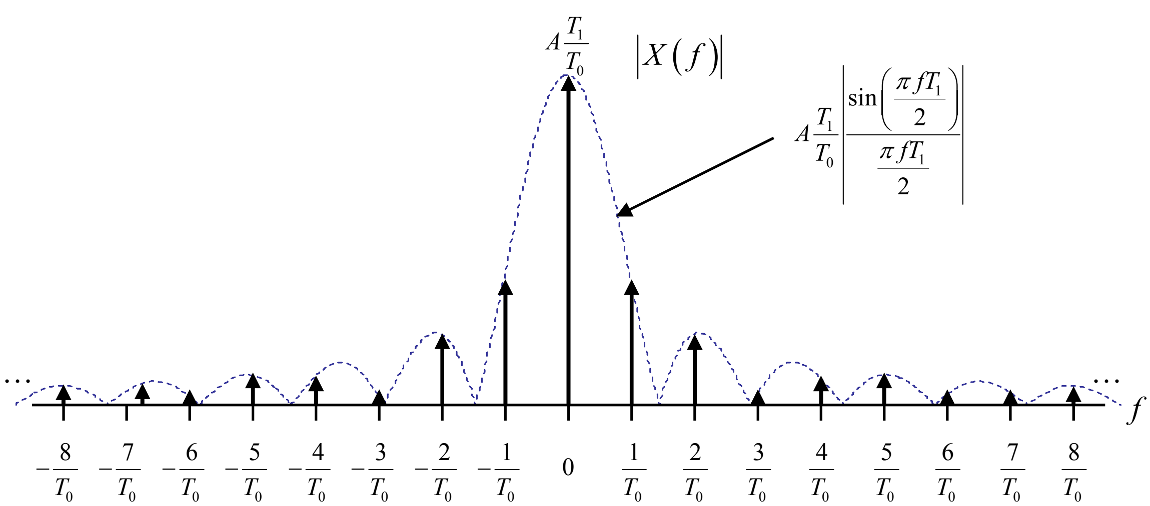
2.32

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k = \frac{1}{T_0}\int_0^{T_1} A e^{-j\frac{2\pi k}{T_0}t}\,dt = A\frac{T_1}{T_0}\frac{\sin\left(\frac{\pi k}{2}\frac{T_1}{T_0}\right)}{\frac{\pi k}{2}\frac{T_1}{T_0}}e^{-j\pi k\frac{T_1}{T_0}}$$

$$X\left(f\right) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T_0}\right)$$

$$\left|X\left(f\right)\right| = \sum_{k=-\infty}^{\infty} \left|c_k\right| \delta\left(f - \frac{k}{T_0}\right)$$



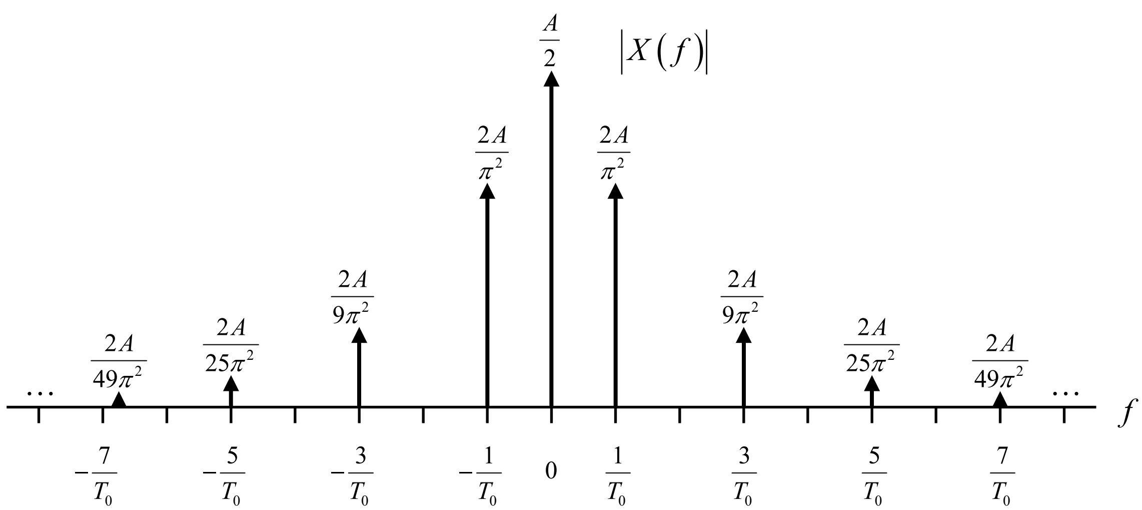
2.33

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k=\frac{1}{T_0}\int_0^{\frac{T_0}{2}}\frac{2A}{T_0}te^{-j\frac{2\pi k}{T_0}t}dt+\frac{1}{T_0}\int_{\frac{T_0}{2}}^{T_0}\left(2A-\frac{2A}{T_0}t\right)e^{-j\frac{2\pi k}{T_0}t}dt=\frac{A}{\pi^2k^2}\left[(-1)^k-1\right]=\begin{cases}\frac{A}{2} & k=0\\ 0 & k,\ \text{even}\\ -\frac{2A}{\pi^2k^2} & k,\ \text{odd}\end{cases}$$

$$X(f)=\sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T_0}\right)$$

$$\left|X(f)\right|=\sum_{k=-\infty}^{\infty}\left|c_k\right|\delta\left(f-\frac{k}{T_0}\right)$$



2.34

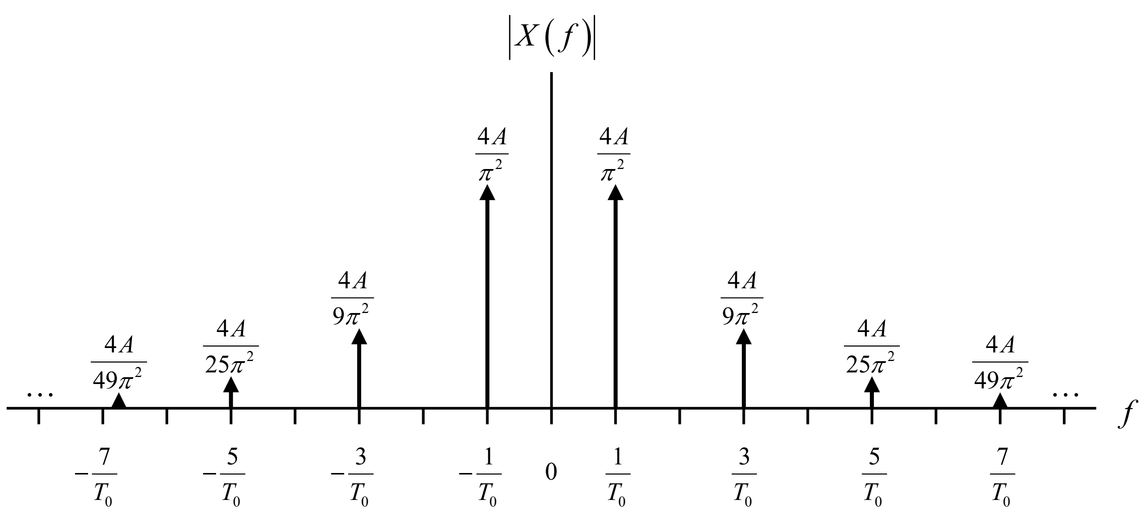
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} \frac{4A}{T_0} t e^{-j\frac{2\pi k}{T_0}t} dt + \frac{1}{T_0}\int_{\frac{T_0}{4}}^{\frac{3T_0}{4}} \left(2A - \frac{4A}{T_0}t\right) e^{-j\frac{2\pi k}{T_0}t} dt = -j\frac{4A}{\pi^2 k^2}\sin\left(\frac{\pi k}{2}\right)$$

$$= \begin{cases} 0 & k = 0 \\ 0 & k, \text{ even} \\ -j\dfrac{4A}{\pi^2 k^2}\sin\left(\dfrac{\pi k}{2}\right) & k, \text{ odd} \end{cases}$$

$$X(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T_0}\right)$$

$$|X(f)| = \sum_{k=-\infty}^{\infty} |c_k| \delta\left(f - \frac{k}{T_0}\right)$$



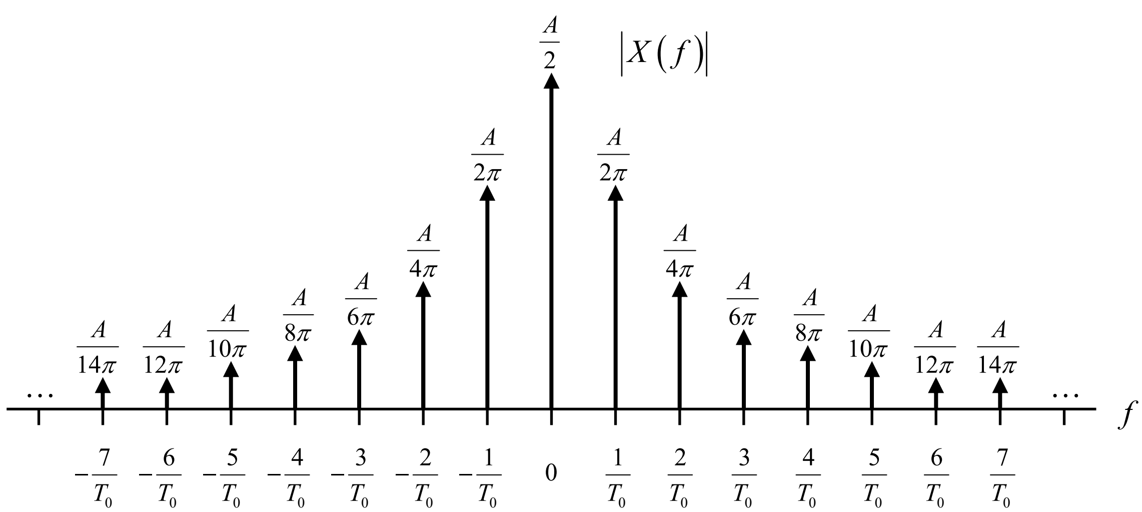
2.35

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} \frac{A}{T_0} t e^{-j\frac{2\pi k}{T_0}t}\, dt = \begin{cases} \dfrac{A}{2} & k = 0 \\ j\dfrac{A}{2\pi k} & k \neq 0 \end{cases}$$

$$X\left(f\right) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T_0}\right)$$

$$\left|X\left(f\right)\right| = \sum_{k=-\infty}^{\infty} \left|c_k\right| \delta\left(f - \frac{k}{T_0}\right)$$



2.36

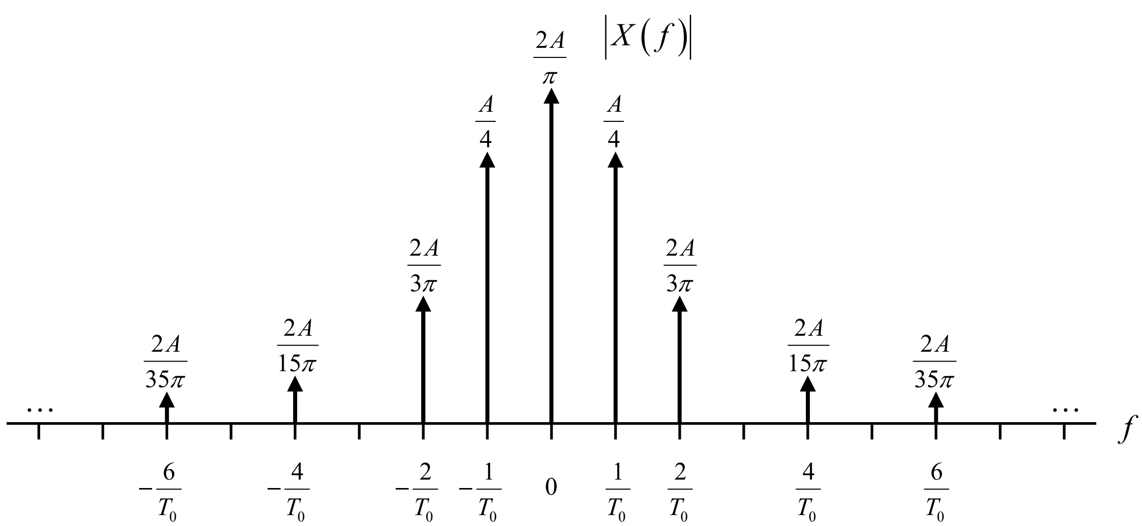
$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T_0}t}$$

$$c_k=\frac{1}{T_0}\int_0^{T_0} A\sin\left(\frac{2\pi}{T_0}t\right)e^{-j\frac{2\pi k}{T_0}t}\,dt=\begin{cases}-\dfrac{A}{\pi\left(k^2-1\right)}\left[\left(-1\right)^k+1\right] & k\neq 1\\ -j\dfrac{A}{4} & k=1\end{cases}$$

$$=\begin{cases}\dfrac{A}{\pi} & k=0\\ -j\dfrac{A}{4} & k=1\\ 0 & k,\ \text{odd}, k\neq 1\\ -\dfrac{2A}{\pi\left(k^2-1\right)} & k,\ \text{even}\end{cases}$$

$$X\left(f\right)=\sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{k}{T_0}\right)$$

$$\left|X\left(f\right)\right|=\sum_{k=-\infty}^{\infty} \left|c_k\right|\delta\left(f-\frac{k}{T_0}\right)$$



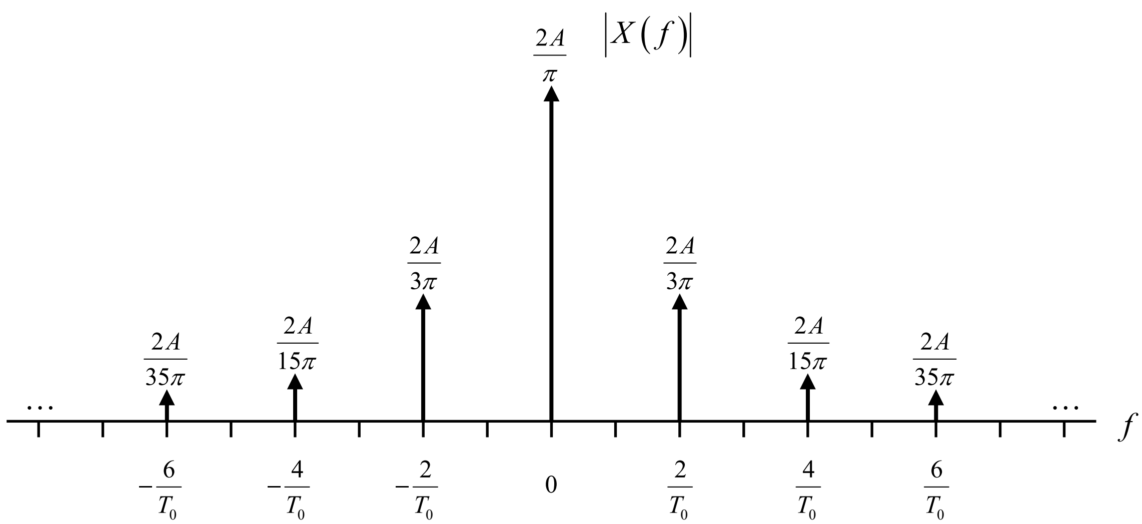
2.37

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{j\frac{4\pi k}{T_0}t}$$

$$c_k=\frac{2}{T_0}\int_0^{T_0} A\sin\left(\frac{2\pi}{T_0}t\right)e^{-j\frac{4\pi k}{T_0}t}\,dt=-\frac{2A}{\pi\left(4k^2-1\right)}$$

$$X(f)=\sum_{k=-\infty}^{\infty} c_k\delta\left(f-\frac{2k}{T_0}\right)$$

$$\left|X(f)\right|=\sum_{k=-\infty}^{\infty} \left|c_k\right|\delta\left(f-\frac{2k}{T_0}\right)$$



2.38

$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is periodic with period T. Hence it can be represented by a Fourier series

$p(t)=\sum_{k=-\infty}^{\infty}c_k e^{j\frac{2\pi k}{T}t}$ where the Fourier series coefficients are

$$c_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\sum_{n=-\infty}^{\infty}\delta(t-nT)e^{-j\frac{2\pi k}{T}t}dt=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{-j\frac{2\pi k}{T}t}dt=\frac{1}{T}$$

Thus, $p(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\frac{2\pi k}{T}t}$

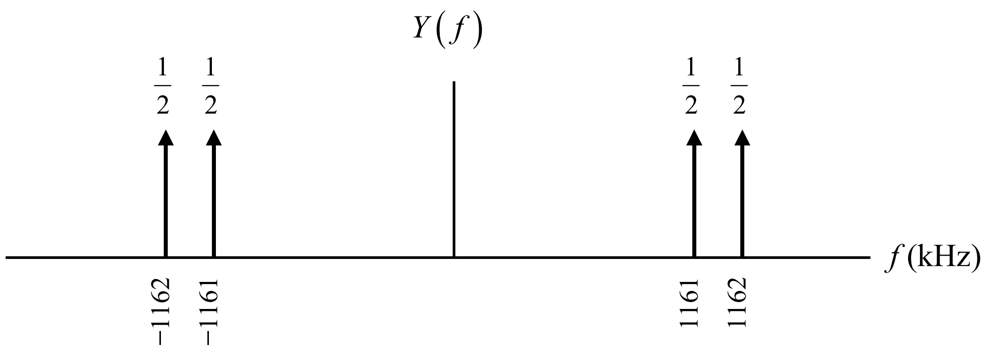
Now using the transform pair $e^{j\frac{2\pi k}{T}t}\ \rightarrow\ 2\pi\delta\left(\omega-\frac{2\pi k}{T}\right)$ we have

$$P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right).$$

2.39

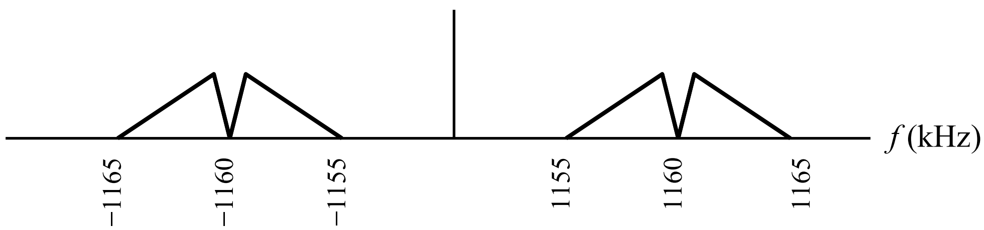
The key to solving this problem is to understand what $H(f)$ does to $\cos(2\pi f_1 t)$. The Fourier transform of $\cos(2\pi f_1 t)$ is $\frac{1}{2}\delta(f+f_1)+\frac{1}{2}\delta(f-f_1)$. When this is the input, the Fourier transform of the output of $H(f)$ is $-\frac{j}{2}\delta(f+f_1)+\frac{j}{2}\delta(f-f_1)$ whose inverse Fourier transform is $-\frac{j}{2}e^{-j2\pi f_1 t}+\frac{j}{2}e^{j2\pi f_1 t}=-\sin(2\pi f_1 t)$. Armed with this relationship, we may now compute $y(t)$:

$$
\begin{aligned}
y(t) &= \cos(2\pi f_0 t)\left[\cos(2\pi f_1 t)+\cos(2\pi f_2 t)\right]+\sin(2\pi f_0 t)\left[-\sin(2\pi f_1 t)-\sin(2\pi f_2 t)\right] \\
&= \frac{1}{2}\cos\left(2\pi(f_0-f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0-f_2)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_2)t\right) \\
&\quad -\frac{1}{2}\cos\left(2\pi(f_0-f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_1)t\right)-\frac{1}{2}\cos\left(2\pi(f_0-f_2)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_2)t\right) \\
&= \cos\left(2\pi(f_0+f_1)t\right)+\cos\left(2\pi(f_0+f_2)t\right)
\end{aligned}
$$

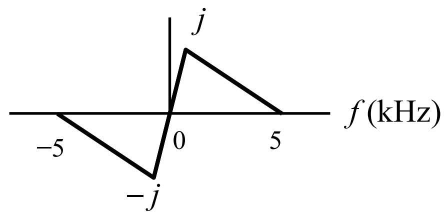


2.40

The Fourier transform of the upper input to the adder is



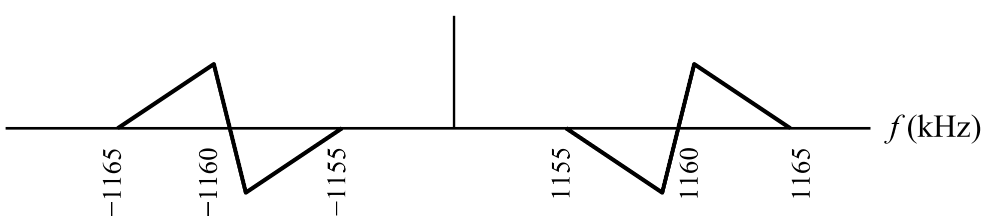
The Fourier transform of the output of $H(f)$ is



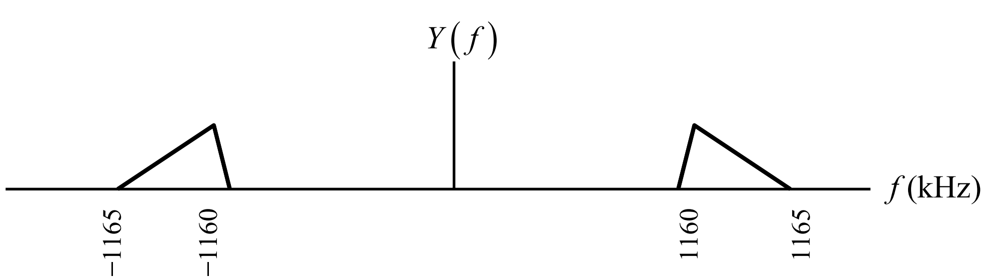
Now, if $z(t) \leftrightarrow Z(f)$ are a Fourier transform pair, then

$$z(t)\sin(2\pi f_0 t) \leftrightarrow \frac{1}{j2} Z(f - f_0) - \frac{1}{j2} Z(f + f_0)$$

Applying this relationship to the output of $H(f)$ produces the Fourier transform of the lower input to the adder:



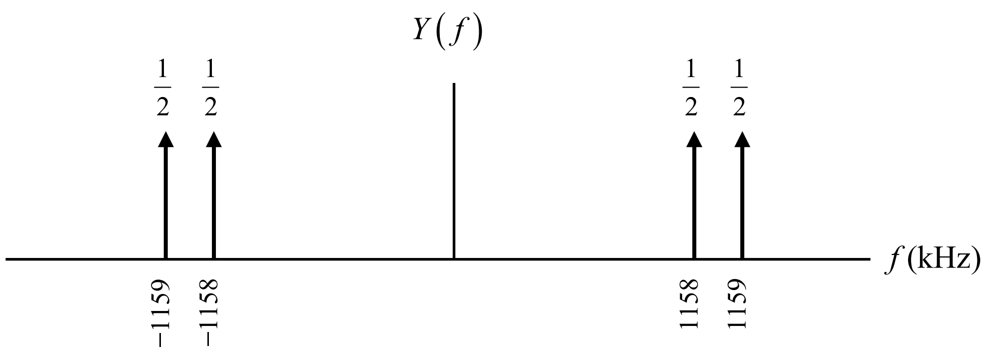
The result is



2.41

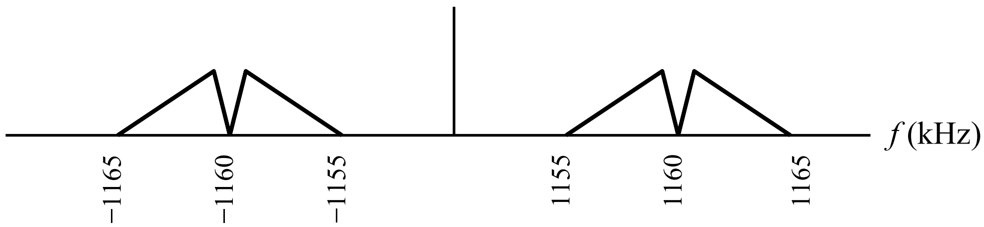
The key to solving this problem is to understand what $H(f)$ does to $\cos(2\pi f_1 t)$. The Fourier transform of $\cos(2\pi f_1 t)$ is $\frac{1}{2}\delta(f+f_1)+\frac{1}{2}\delta(f-f_1)$. When this is the input, the Fourier transform of the output of $H(f)$ is $-\frac{j}{2}\delta(f+f_1)+\frac{j}{2}\delta(f-f_1)$ whose inverse Fourier transform is $-\frac{j}{2}e^{-j2\pi f_1 t}+\frac{j}{2}e^{j2\pi f_1 t}=-\sin(2\pi f_1 t)$. Armed with this relationship, we may now compute $y(t)$:

$$\begin{aligned}
y(t) &= \cos(2\pi f_0 t)\left[\cos(2\pi f_1 t)+\cos(2\pi f_2 t)\right]-\sin(2\pi f_0 t)\left[-\sin(2\pi f_1 t)-\sin(2\pi f_2 t)\right] \\
&= \frac{1}{2}\cos\left(2\pi(f_0-f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0-f_2)t\right)+\frac{1}{2}\cos\left(2\pi(f_0+f_2)t\right) \\
&\quad +\frac{1}{2}\cos\left(2\pi(f_0-f_1)t\right)-\frac{1}{2}\cos\left(2\pi(f_0+f_1)t\right)+\frac{1}{2}\cos\left(2\pi(f_0-f_2)t\right)-\frac{1}{2}\cos\left(2\pi(f_0+f_2)t\right) \\
&= \cos\left(2\pi(f_0-f_1)t\right)+\cos\left(2\pi(f_0-f_2)t\right)
\end{aligned}$$

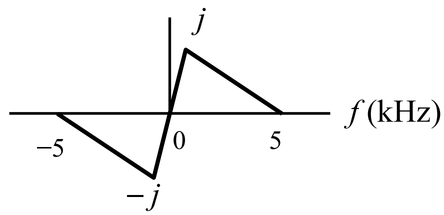


2.42

The Fourier transform of the upper input to the adder is



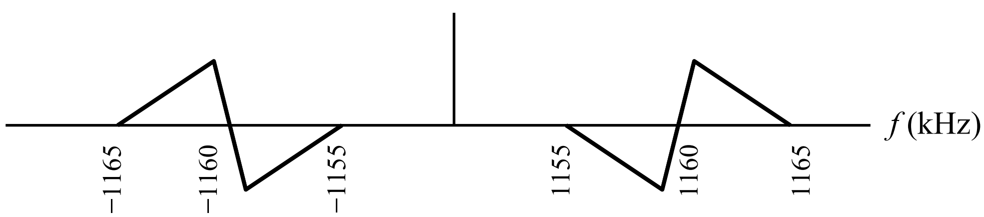
The Fourier transform of the output of $H(f)$ is



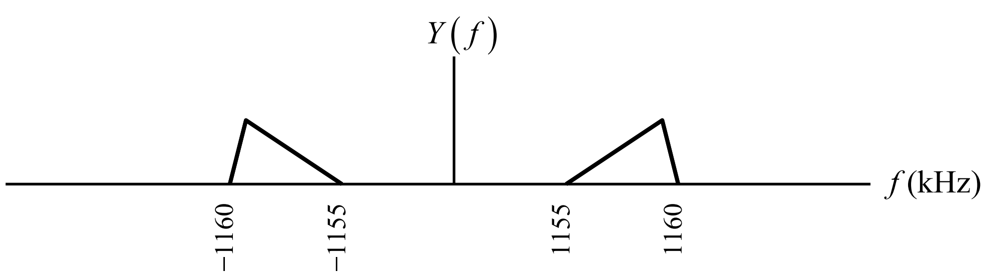
Now, if $z(t) \leftrightarrow Z(f)$ are a Fourier transform pair, then

$$z(t)\sin(2\pi f_0 t) \leftrightarrow \frac{1}{j2}Z(f-f_0) - \frac{1}{j2}Z(f+f_0)$$

Applying this relationship to the output of $H(f)$ produces the Fourier transform of the lower input to the adder:

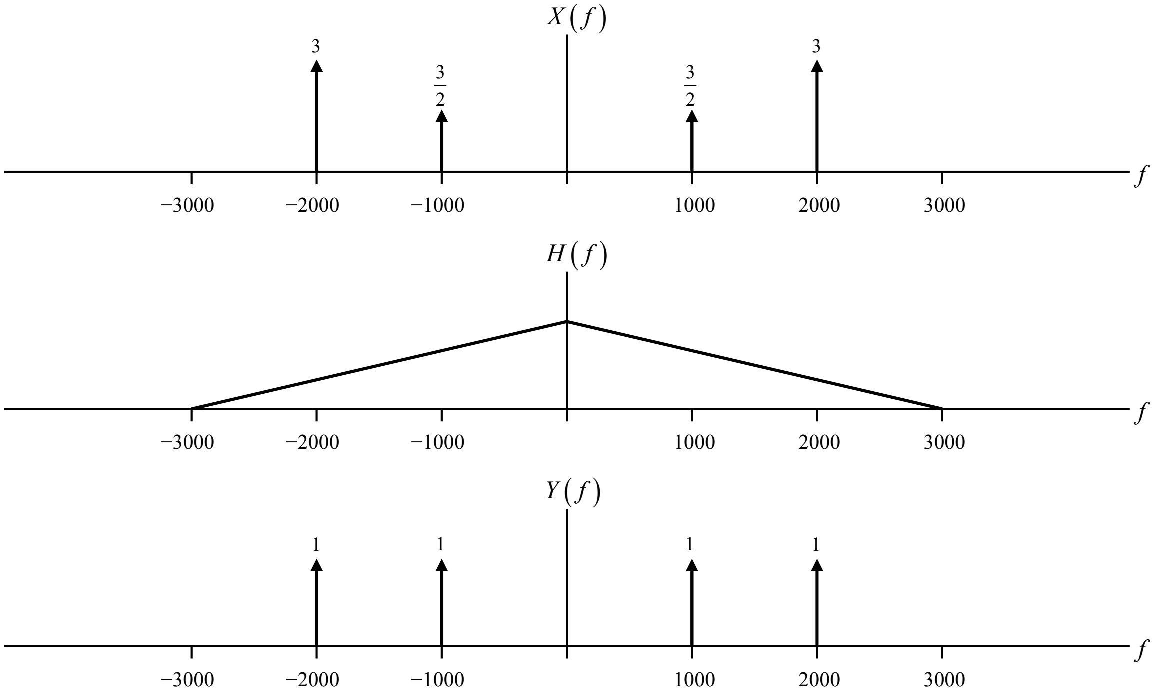


The result is



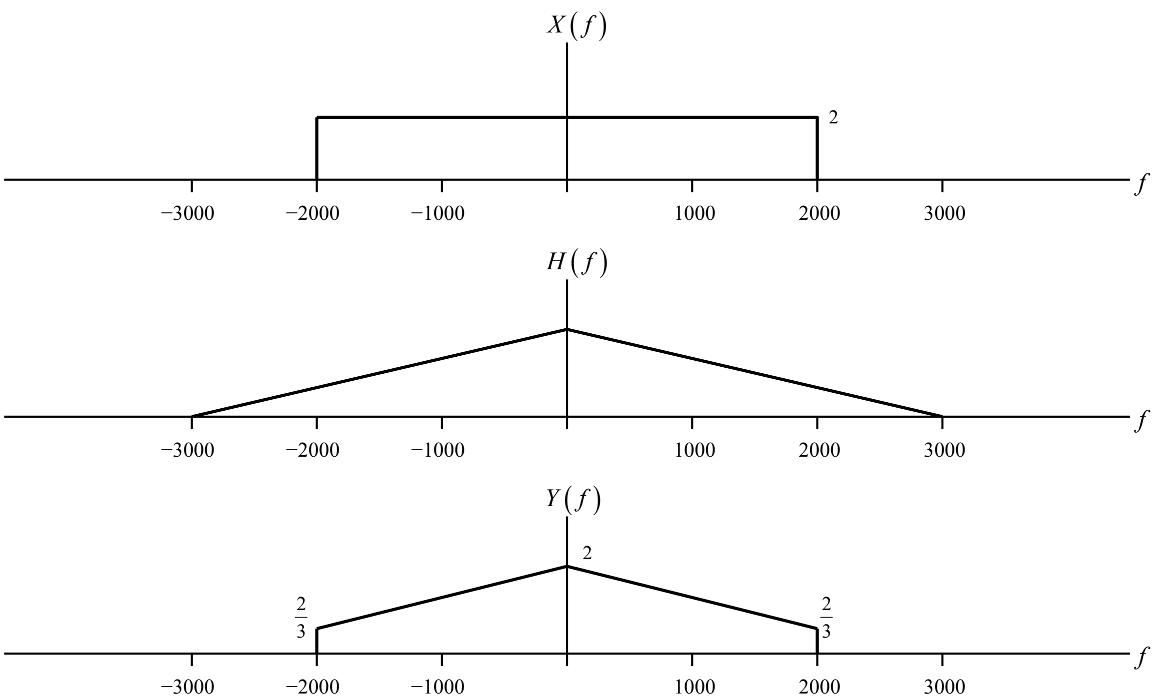
2.43

(a)



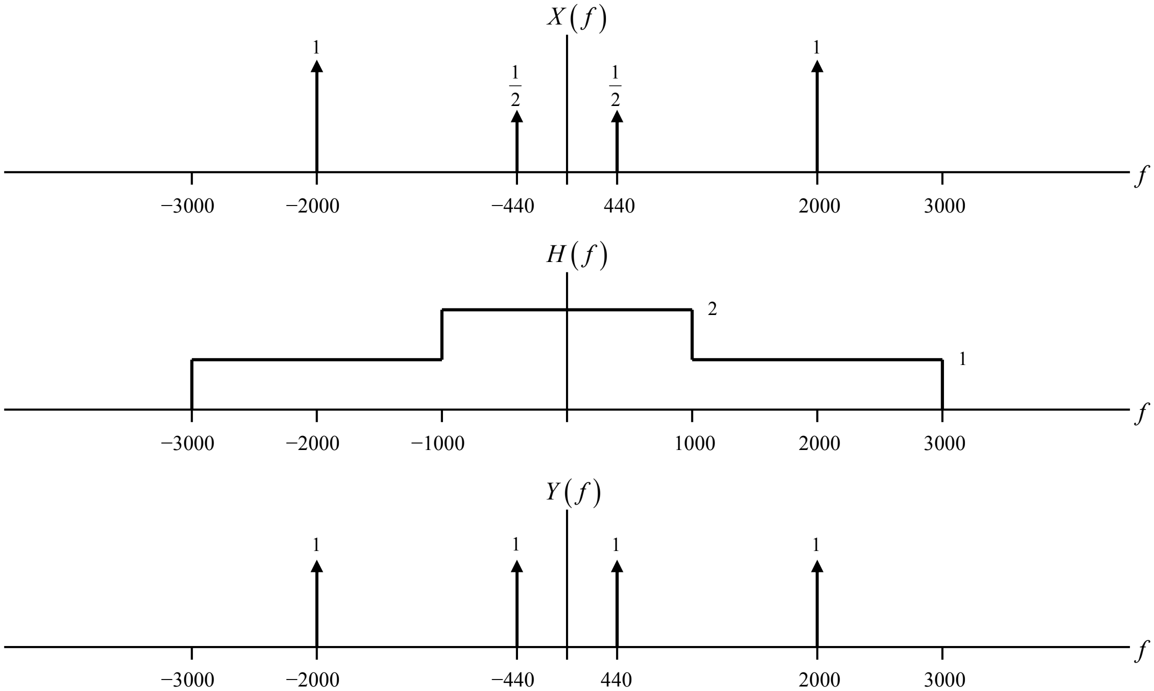
$$Y\left(f\right)=2\cos\left(2\pi 1000t\right)+2\cos\left(2\pi 2000t\right)$$

(b)



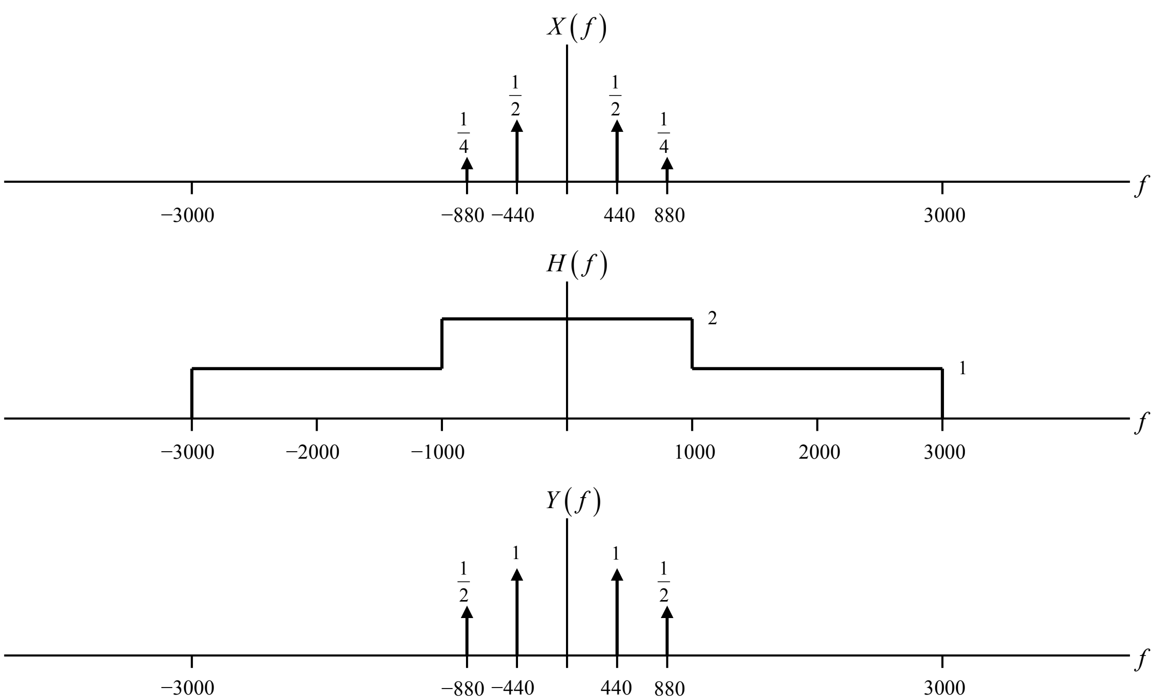
2.44

(a)



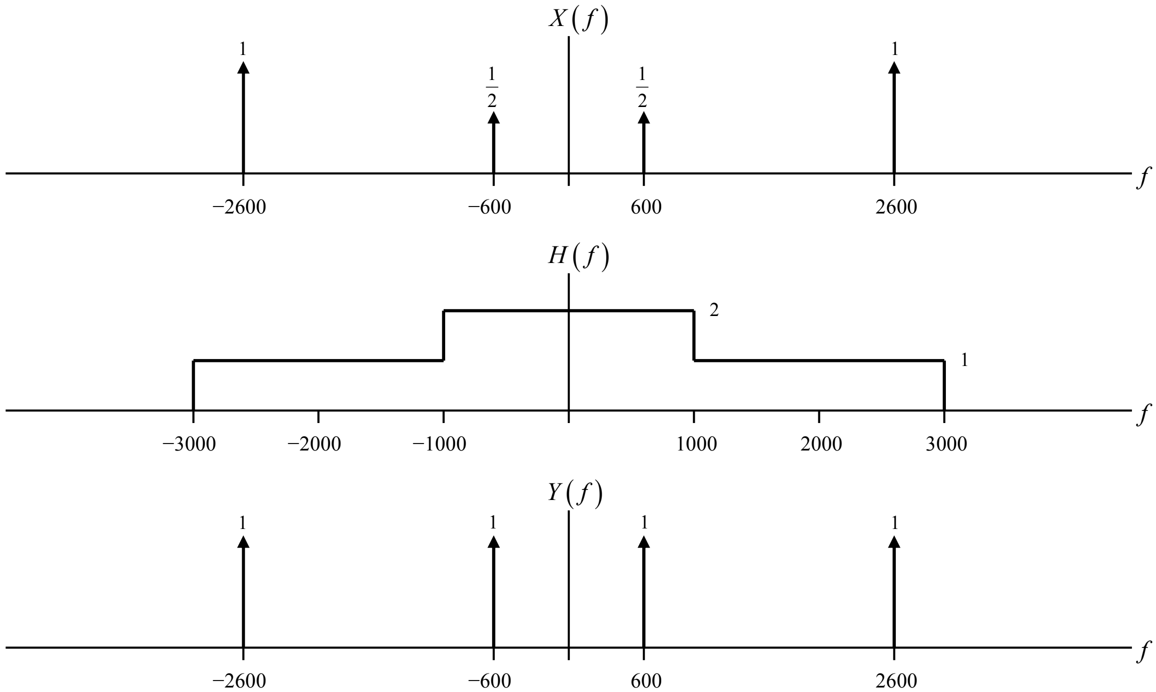
$$y(t) = 2\cos(2\pi 440t) + 2\cos(2\pi 2000t)$$

(b)



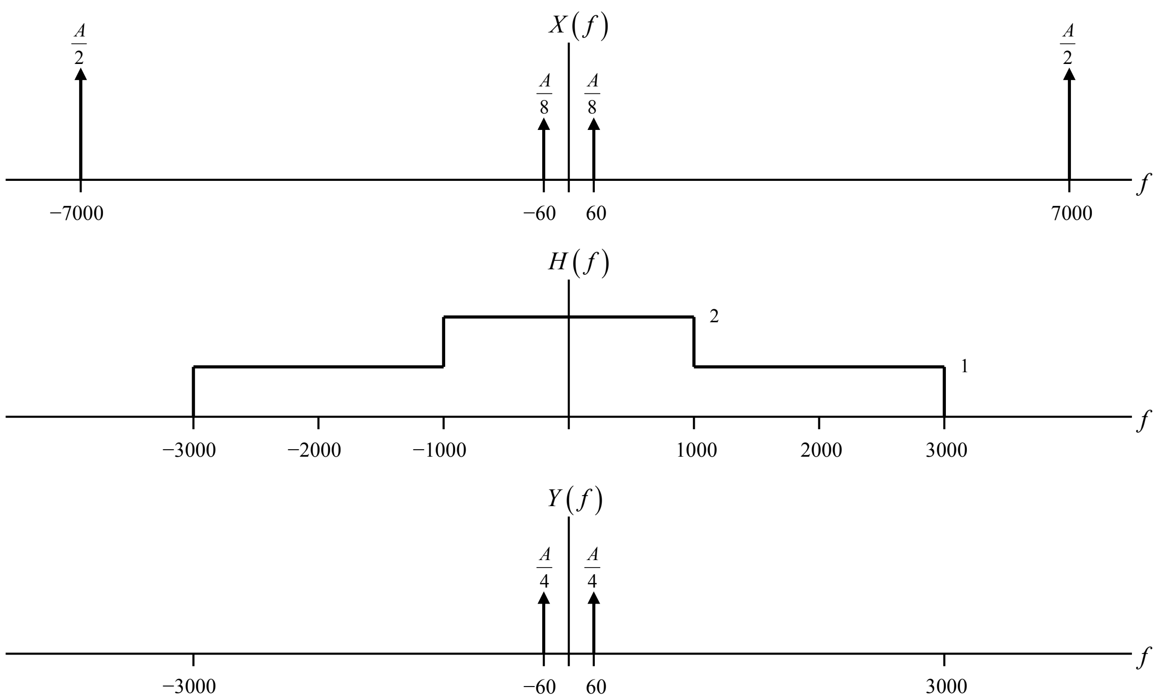
$$y(t) = 2\cos(2\pi 440t) + \cos(2\pi 880t)$$

(c)



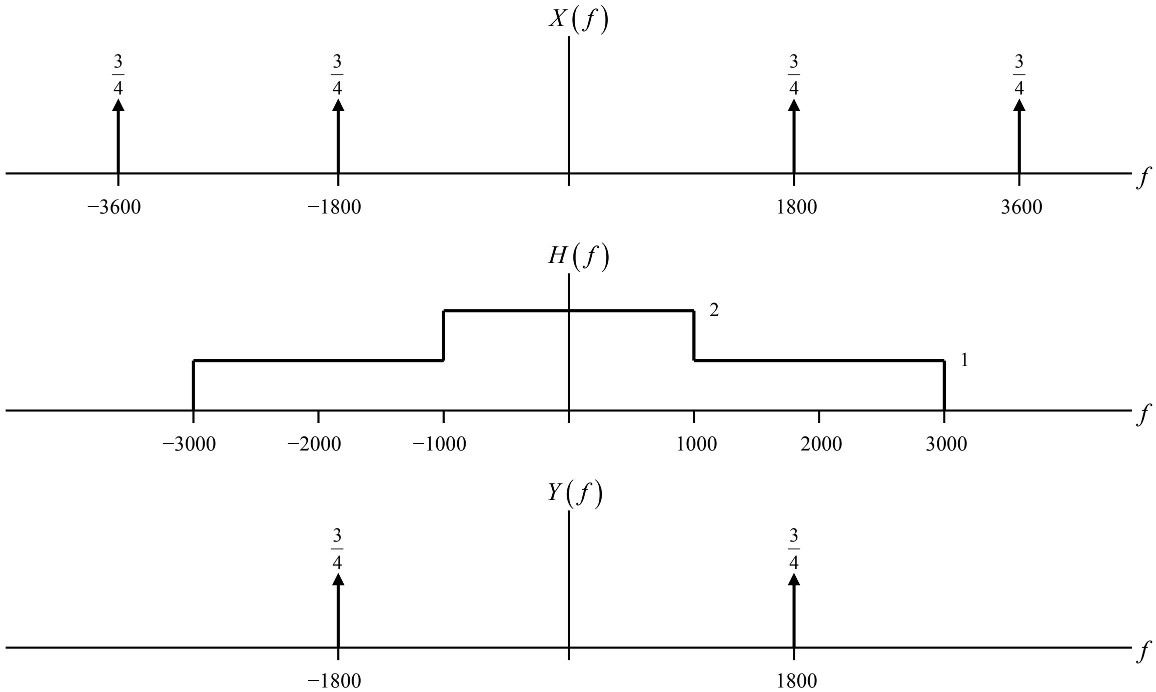
$$y(t) = 2\cos(2\pi 600t) + 2\cos(2\pi 2600t)$$

(d)



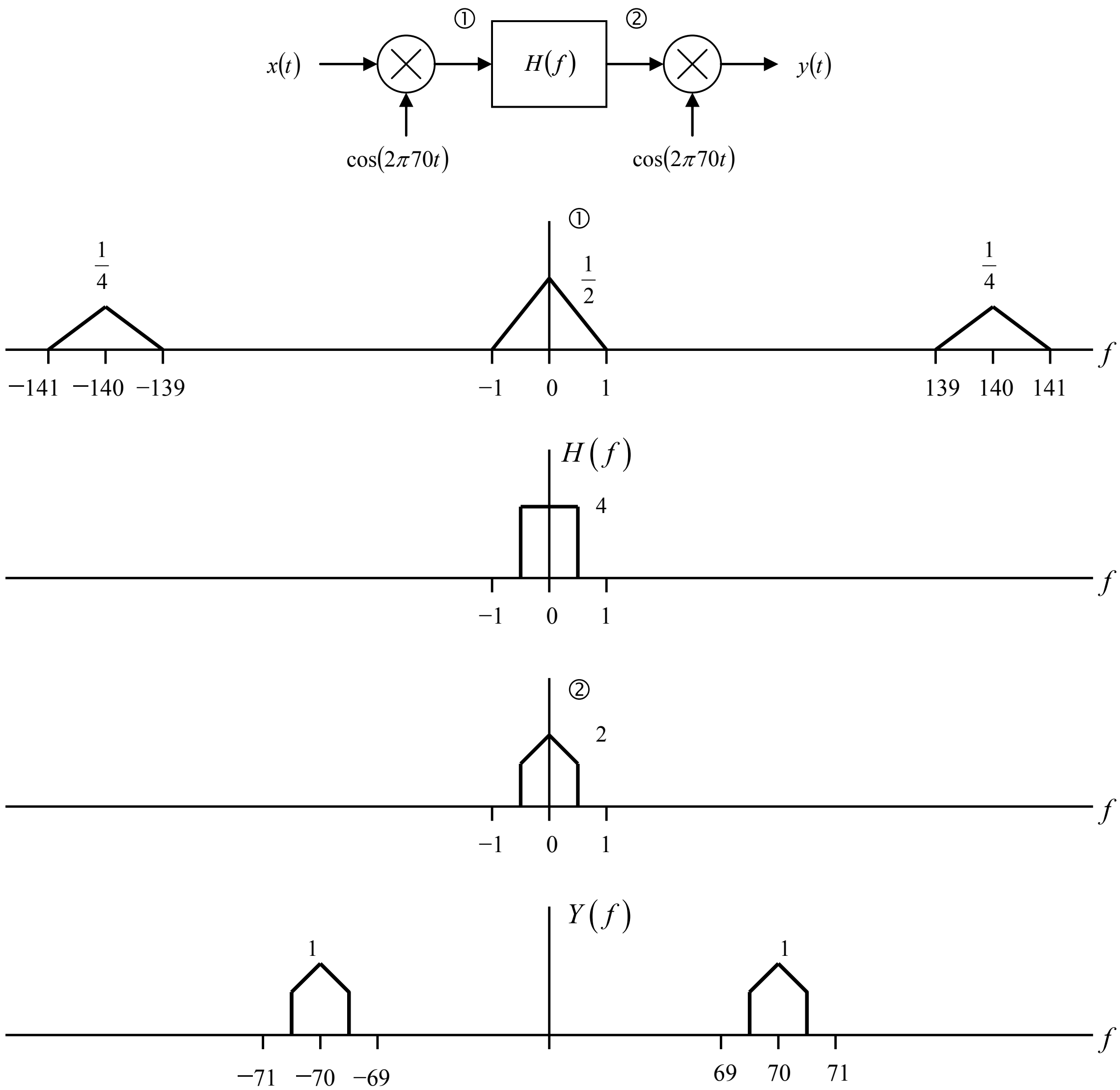
$$y(t) = \frac{A}{2}\cos(2\pi 60t)$$

(e)



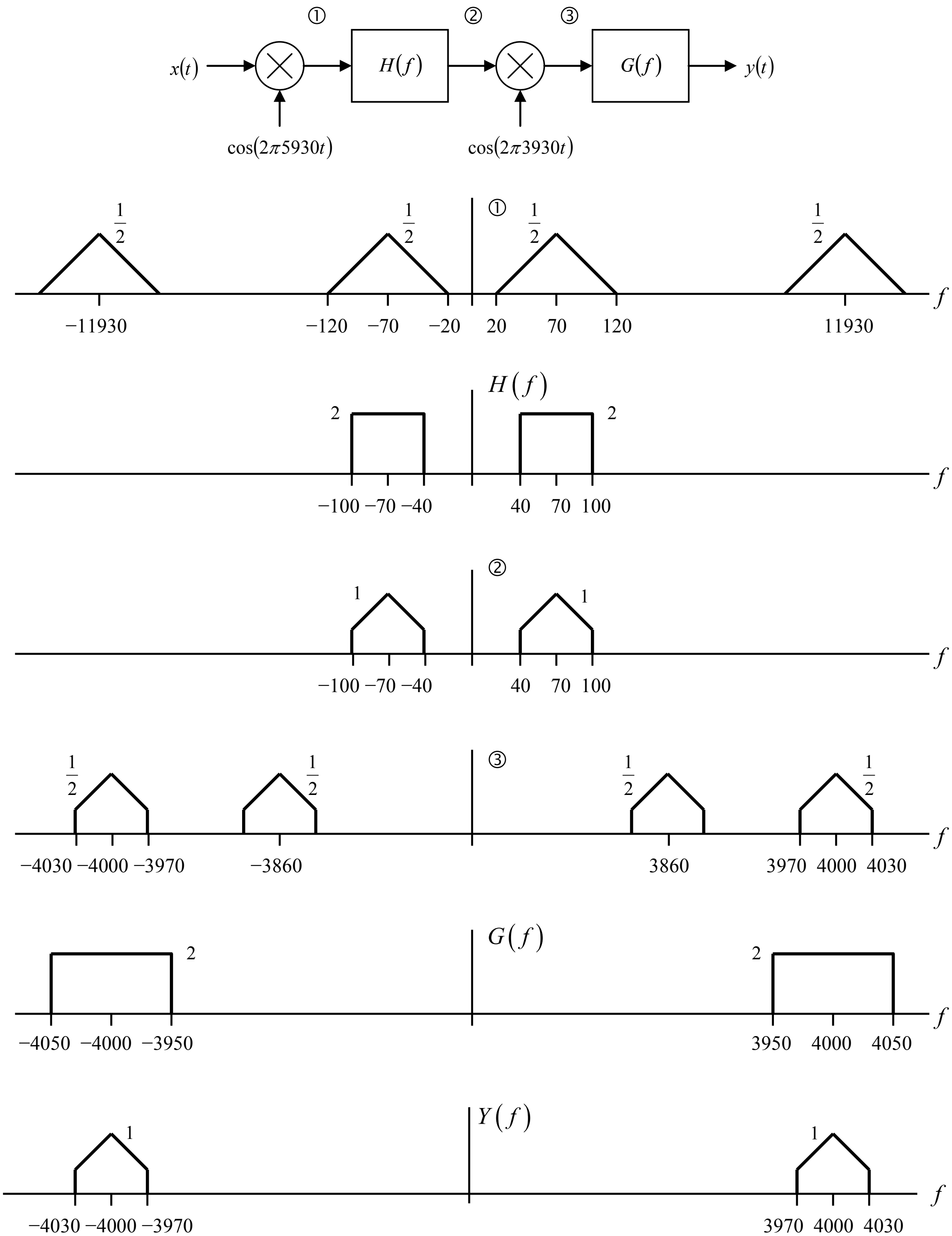
$$y(t) = \frac{3}{2}\cos(2\pi 1800 t)$$

2.45



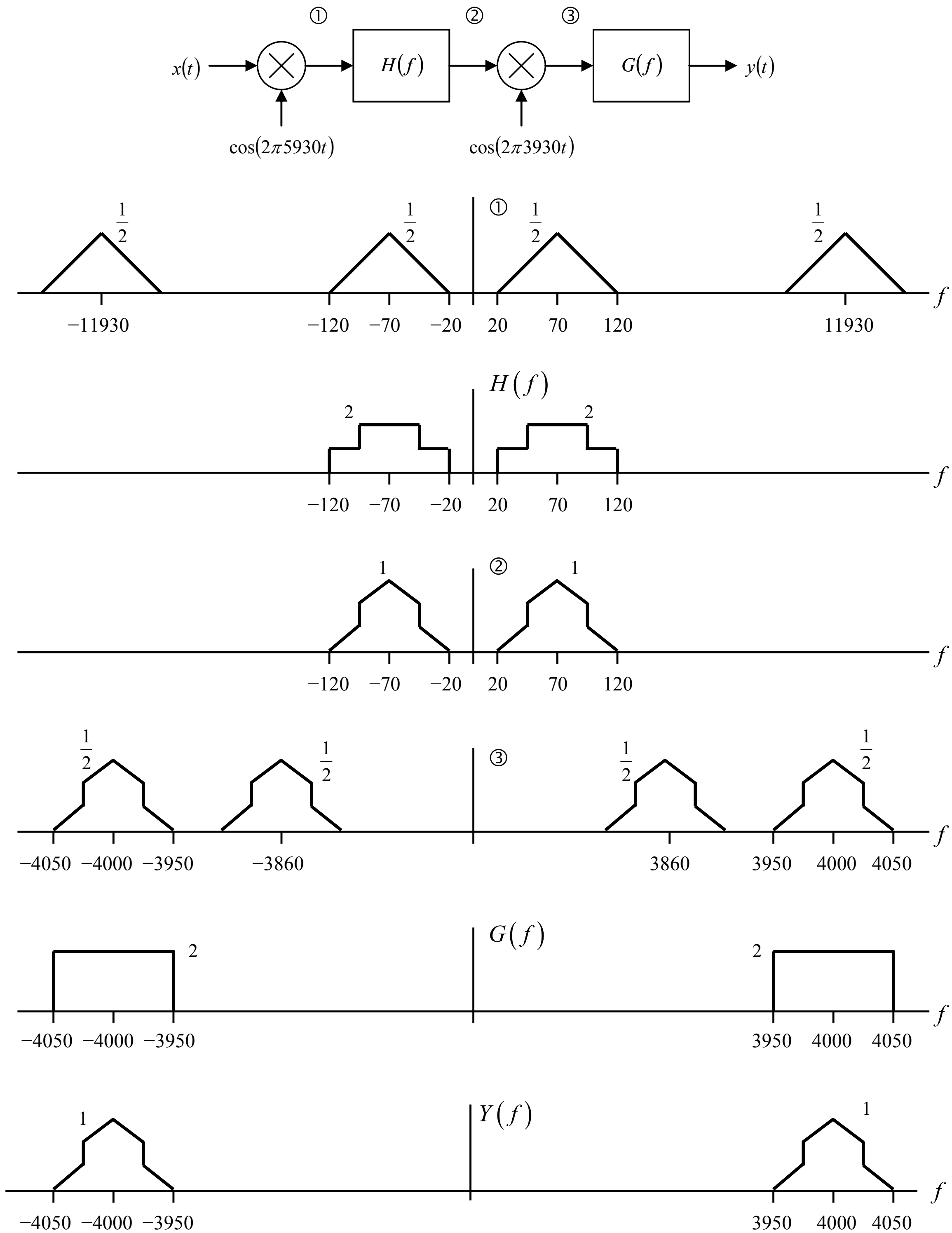
①
②
$x(t)$
$H(f)$
$y(t)$
$\cos(2\pi 70t)$
$\cos(2\pi 70t)$
①
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
f
−141 −140 −139
−1 0 1
139 140 141
$H(f)$
4
f
−1 0 1
②
2
f
−1 0 1
$Y(f)$
1
1
f
−71 −70 −69
69 70 71

2.46



①
②
③
x(t)
H(f)
G(f)
y(t)
cos(2π5930t)
cos(2π3930t)
1/2
−11930
−120 −70 −20
20 70 120
11930
f
H(f)
2
−100 −70 −40
40 70 100
1
−4030 −4000 −3970
−3860
3860
3970 4000 4030
G(f)
−4050 −4000 −3950
3950 4000 4050
Y(f)

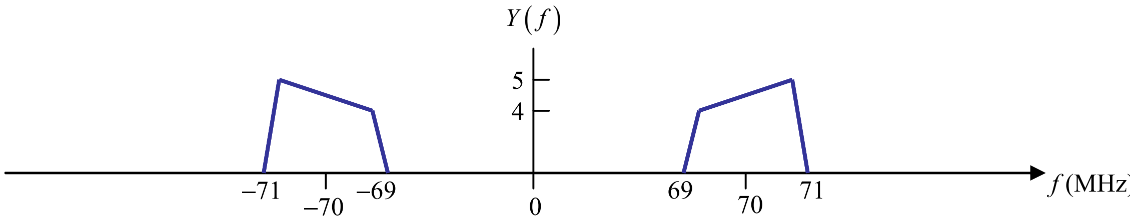
2.47



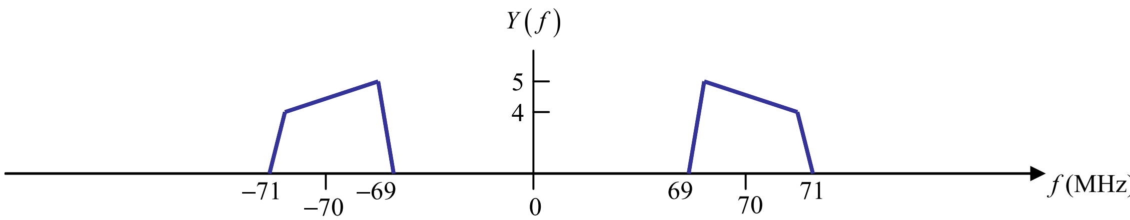
①
②
③
$x(t)$
$H(f)$
$G(f)$
$y(t)$
$\cos(2\pi 5930t)$
$\cos(2\pi 3930t)$
$\frac{1}{2}$
-11930
-120 -70 -20 20 70 120
11930
f
$H(f)$
2
1
-4050 -4000 -3950
-3860
3860
3950 4000 4050
$G(f)$
$Y(f)$

2.48

(a)



(b)



(c)

$$
\begin{aligned}
r(t)\cos(2\pi f_0 t) &= \left[I(t)\cos(2\pi f_1 t) - Q(t)\sin(2\pi f_1 t)\right]\cos(2\pi f_0 t) \\
&= \frac{I(t)}{2}\cos\left(2\pi(f_1 - f_0)t\right) + \frac{I(t)}{2}\cos\left(2\pi(f_1 + f_0)t\right) \\
&\quad - \frac{Q(t)}{2}\sin\left(2\pi(f_1 - f_0)t\right) - \frac{Q(t)}{2}\sin\left(2\pi(f_1 + f_0)t\right)
\end{aligned}
$$

$$
y(t) = \frac{I(t)}{2}\cos\left(2\pi(f_1 - f_0)t\right) - \frac{Q(t)}{2}\sin\left(2\pi(f_1 - f_0)t\right)
$$

In part (a), $f_0 = 930$ so that

$$
y(t) = \frac{I(t)}{2}\cos(2\pi 70t) - \frac{Q(t)}{2}\sin(2\pi 70t)
$$

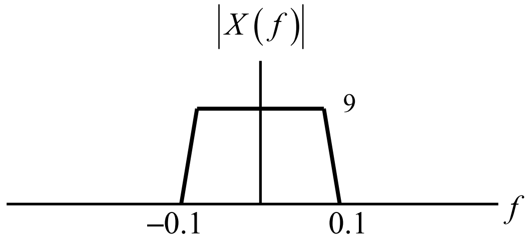
In part (b), $f_0 = 1070$ so that

$$
y(t) = \frac{I(t)}{2}\cos(2\pi 70t) + \frac{Q(t)}{2}\sin(2\pi 70t)
$$

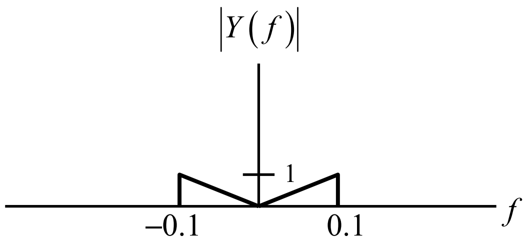
The difference is the sign of the "quadrature term" (the term involving $\sin(2\pi ft)$).

2.49

(a)

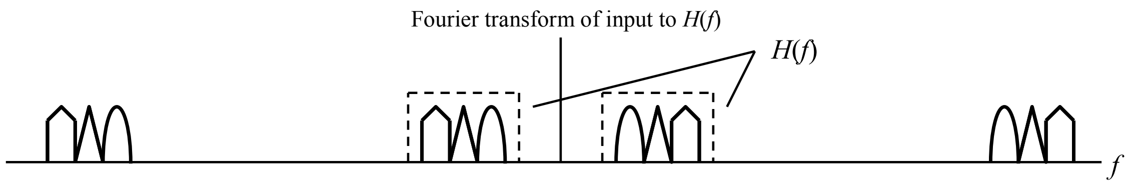


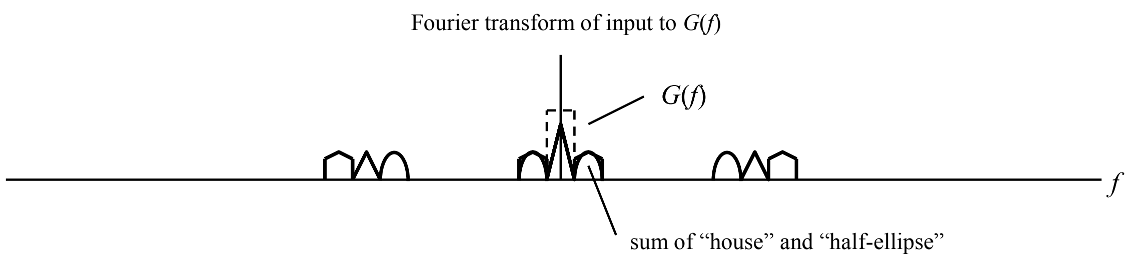
(b)

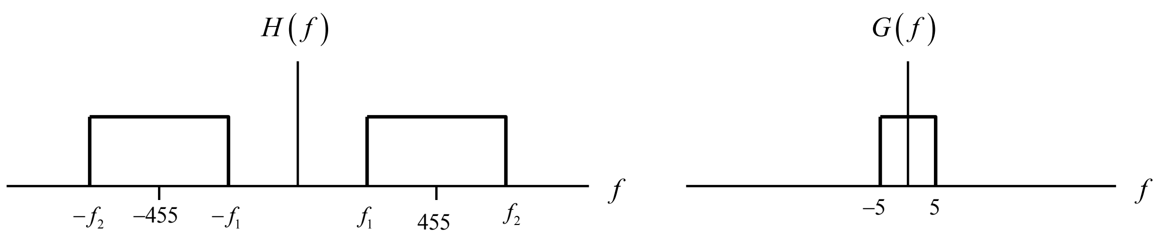


2.50

(a)

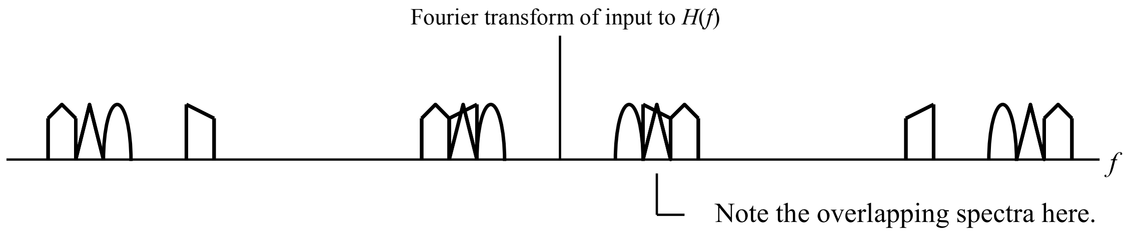






$$0 \le f_1 \le 450$$
$$460 \le f_2 \le 1850$$

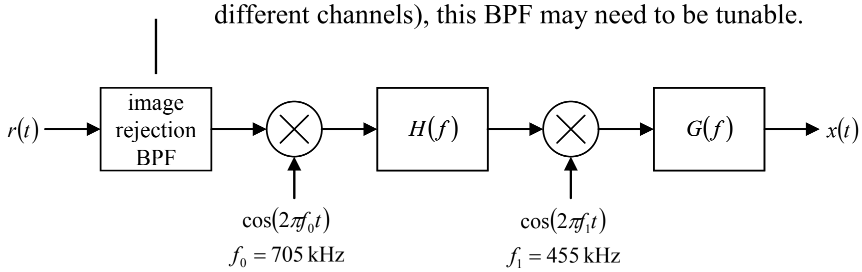
(b)



Note the overlapping spectra here. This is bad because there is not way to use a filter to isolate the desired spectrum from the combination.

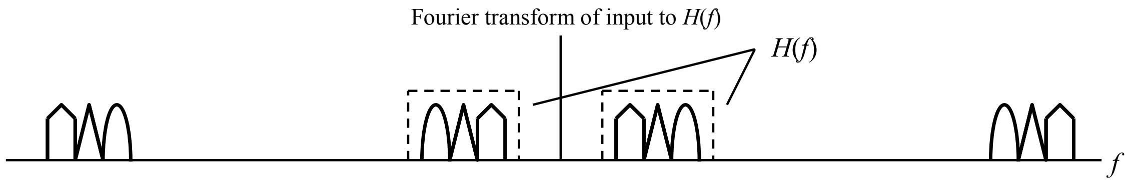
It is not possible to design an $H(f)$ and $G(f)$ to produce the desired output. The signal centered at 250 Hz must be removed before the mixer preceding the filter $H(f)$. The typical solution is to add an additional filter as follows:

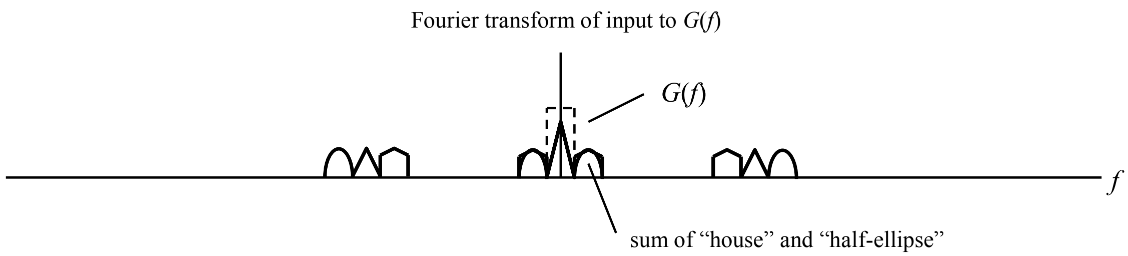
This filter needs to remove the spectral content that interferes with the desired signal. In tunable systems (i.e., systems that need to isolate different channels), this BPF may need to be tunable.

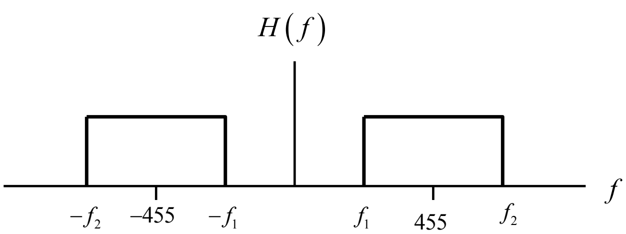


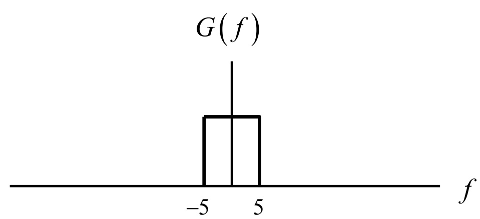
2.51

(a)



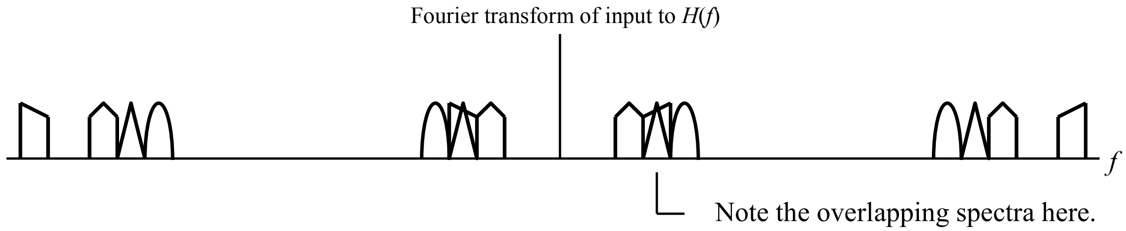






$$0 \le f_1 \le 450$$
$$460 \le f_2 \le 1850$$

(b)



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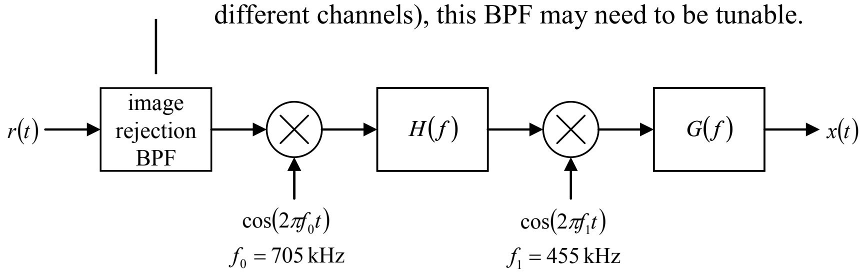
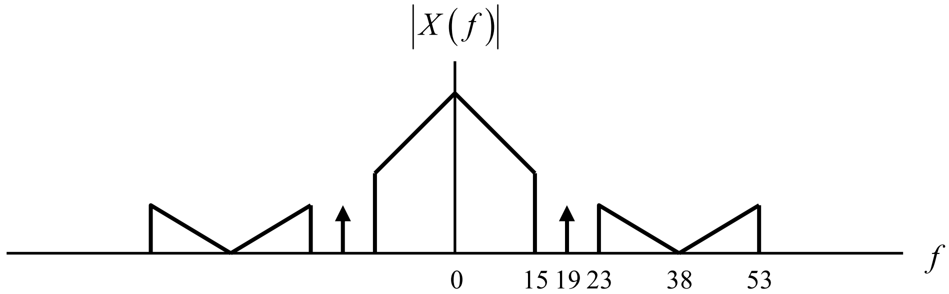
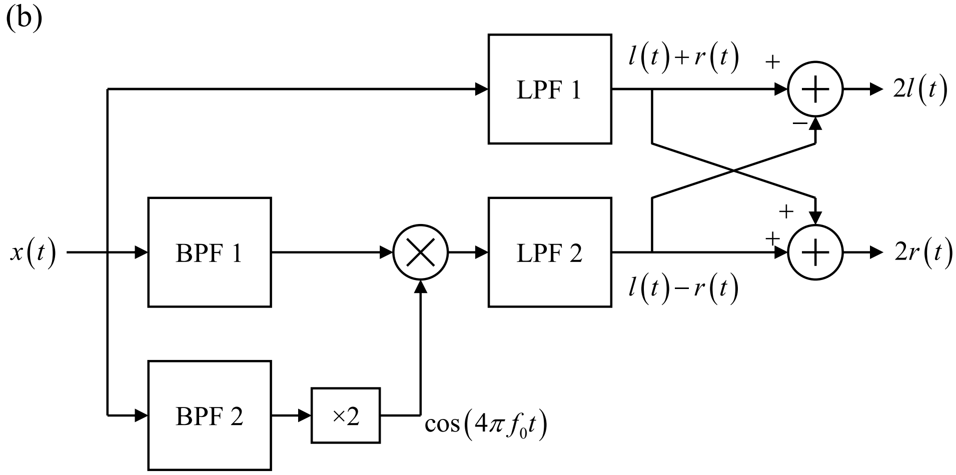


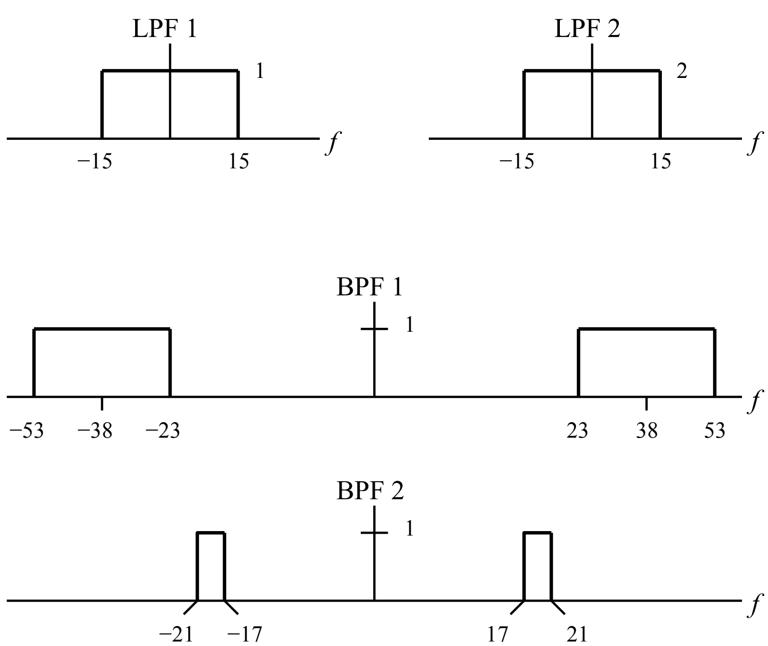
2.52

(a)



(b)





(c) $l(t) + r(t)$ is available at the output of LPF 1.

2.53

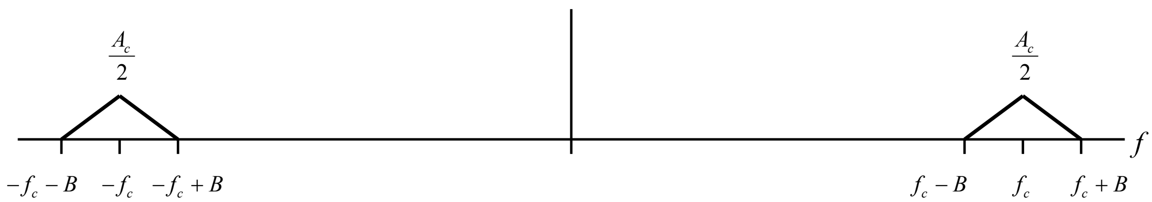
(a) Using $x(t)e^{j2\pi f_0 t} \to X(f - f_0)$, we have

$$x(t)\cos(2\pi f_0 t) = x(t)\left[\frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2}x(t)e^{j2\pi f_0 t} + \frac{1}{2}x(t)e^{-j2\pi f_0 t}$$

$$\to \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

(b)

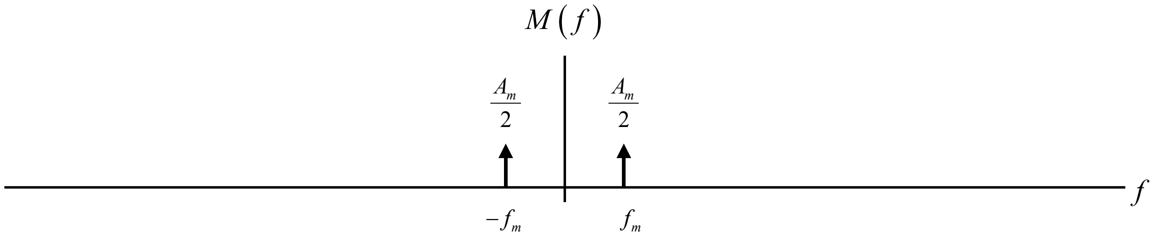


(c) $f_c > B$

(d) The minimum theoretical approach would be to space the carriers 2*B* Hz apart. But, this would require ideal low-pass filters. Real filters require a transition band thus necessitating a larger spacing. In commercial broadcast AM, for example, the channel assignments are every 2*B* = 10 kHz, but broadcast stations are not assigned to adjacent channels in any given geographical area. Instead, the closest spacing is every other channel slot.

2.54

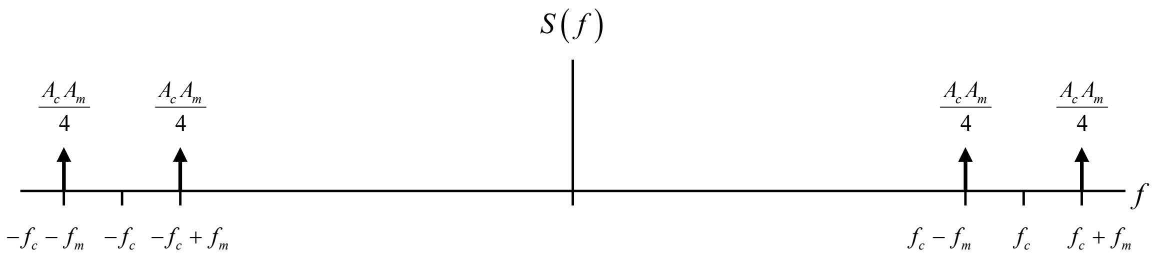
(a)



(b)

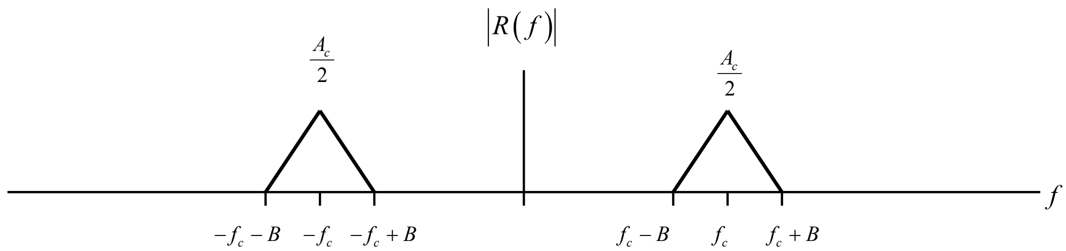
$$s(t) = A_m \cos(2\pi f_m t) A_c \cos(2\pi f_c t)$$
$$= \frac{A_m A_c}{2}\cos(2\pi(f_c - f_m)t) + \frac{A_m A_c}{2}\cos(2\pi(f_c + f_m)t)$$
$$S(f) = \frac{A_m A_c}{4}\left[\delta(f - f_c - f_m) + \delta(f - f_c + f_m) + \delta(f + f_c + f_m) + \delta(f + f_c - f_m)\right]$$

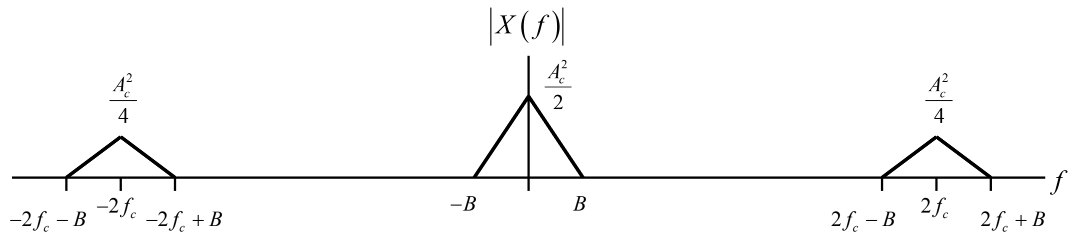
(c)



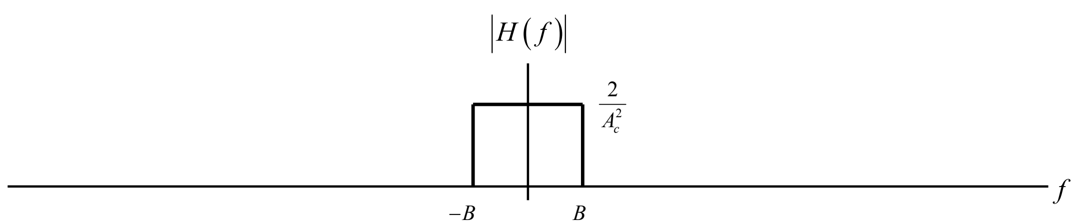
2.55

(a) i.





ii. The filter must satisfy the conditions illustrated below.



(b) i.

$$x(t) = \frac{A^2}{2} m(t) + \frac{A^2}{2} \cos(4\pi f_c t)$$

ii. The filter removes the double frequency term and scales the baseband term by $\frac{2}{A^2}$. Thus the filter output is

$$y(t) = m(t)$$

(c) i.

$$x(t) = \frac{A_c^2}{2} m(t) \cos(\theta) + \frac{A_c^2}{2} m(t) \cos(4\pi f_c t + \theta)$$
$$y(t) = m(t) \cos(\theta)$$

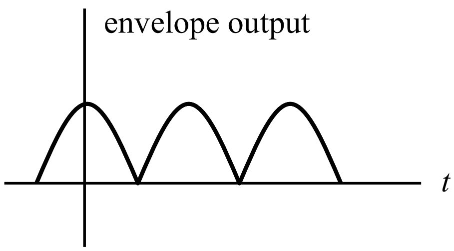
ii. The phase offset scales the output by the constant $\cos(\theta)$. This does not cause any distortion, especially when θ is small. As $\theta \to \frac{\pi}{2}$, the result is disastrous. There are only two solutions: either θ must be known (or estimated from the received signal) or the modulation format should be altered to allow non-coherent detection.

2.56

(a) $A_c m(t)\cos(2\pi f_c t+\theta) = A_c A_m \cos(2\pi f_m t)\cos(2\pi f_c t+\theta)$

$$\text{envelope} = \left|A_c A_m \cos(2\pi f_m t)\right|$$

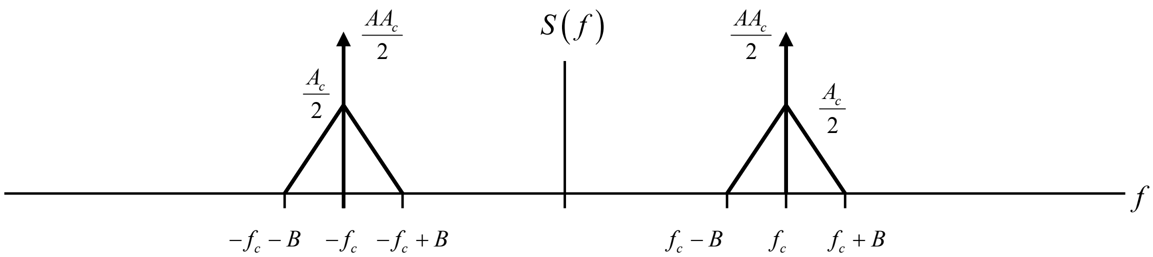
(b)



(c) The envelope detector does not preserve sign information. The solution is to reformat the signal so that the envelope is never negative.

2.57

(a)



(b) Write the modulated signal as

$$s(t) = \left[A + m(t)\right] A_c \cos(2\pi f_c t) = AA_c \cos(2\pi f_c t) + A_c m(t) \cos(2\pi f_c t)$$

The first term is a replica of the unmodulated carrier. Because this replica is part of the transmitted signal, the term "transmitted carrier" is used.

(c) The envelope detector output is $A_c \left|A + m(t)\right|$. Assuming $A + m(t) > 0$ for all t, the output may be written as $A_c A + A_c m(t)$. The term $A_c A$ needs to be subtracted from the envelope detector output to produce an appropriately scaled version of the desired signal. (In other words, the envelope detector needs to be *AC-coupled*.)

The condition that guarantees correct output is $A + \min\{m(t)\} > 0$.

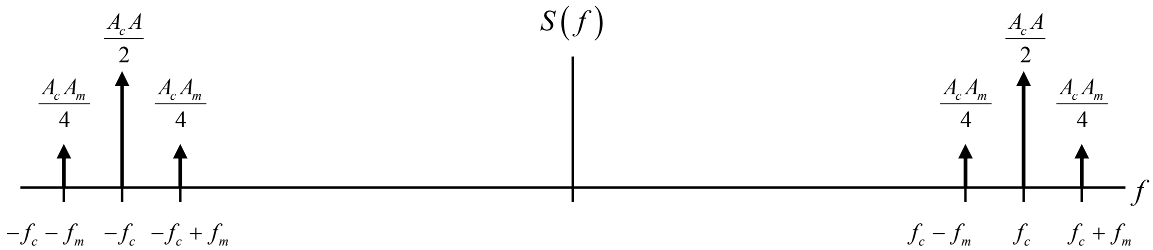
2.58

(a) $A > A_m$

(b)
$$A(f) = A_c A\delta(f) + \frac{A_c A_m}{2}\delta(f + f_m) + \frac{A_c A_m}{2}\delta(f - f_m)$$

(c)
$$S(f) = \frac{1}{2}A(f + f_c) + \frac{1}{2}A(f - f_c)$$
$$= \frac{A_c A}{2}\delta(f + f_c) + \frac{A_c A_m}{2}\delta(f + f_c + f_m) + \frac{A_c A_m}{2}\delta(f + f_c - f_m)$$
$$\frac{A_c A}{2}\delta(f - f_c) + \frac{A_c A_m}{2}\delta(f - f_c + f_m) + \frac{A_c A_m}{2}\delta(f - f_c - f_m)$$

(d)



2.59

(a)

$$s(t) = A_c A_m \cos(2\pi f_m t)\cos(2\pi f_c t)$$
$$= \frac{A_c A_m}{2}\cos(2\pi(f_c - f_m)t) + \frac{A_c A_m}{2}\cos(2\pi(f_c + f_m)t)$$
$$s^2(t) = \frac{A_c^2 A_m^2}{8} + \frac{A_c^2 A_m^2}{8}\cos(4\pi(f_c - f_m)t)$$
$$+ \frac{A_c^2 A_m^2}{4}\cos(4\pi f_c t) + \frac{A_c^2 A_m^2}{4}\cos(4\pi f_m t)$$
$$+ \frac{A_c^2 A_m^2}{8} + \frac{A_c^2 A_m^2}{8}\cos(4\pi(f_c + f_m)t)$$
$$P_m = \frac{1}{T_m}\int_0^{T_m} s^2(t)\,dt = \frac{A_c^2 A_m^2}{4}$$
$$P_{\text{tot}} = P_m$$
$$\text{power ratio} = \frac{P_m}{P_{\text{tot}}} = 1$$

(b)

$$s(t) = A_c A\cos(2\pi f_c t) + A_c A_m \cos(2\pi f_m t)\cos(2\pi f_c t)$$
$$= A_c A\cos(2\pi f_c t) + \frac{A_c A_m}{2}\cos(2\pi(f_c - f_m)t) + \frac{A_c A_m}{2}\cos(2\pi(f_c + f_m)t)$$
$$s^2(t) = \frac{A_c^2 A^2}{2} + \frac{A_c^2 A^2}{2}\cos(4\pi f_c t)$$
$$+ \frac{A_c^2 A_m^2}{8} + \frac{A_c^2 A_m^2}{8}\cos(4\pi(f_c - f_m)t) + \frac{A_c^2 A_m^2}{8} + \frac{A_c^2 A_m^2}{8}\cos(4\pi(f_c + f_m)t)$$
$$+ \frac{A_c^2 A A_m}{2}\cos(2\pi(2f_c - f_m)t) + \frac{A_c^2 A A_m}{2}\cos(4\pi f_m t)$$
$$+ \frac{A_c^2 A A_m}{2}\cos(2\pi(2f_c + f_m)t) + \frac{A_c^2 A A_m}{2}\cos(4\pi f_m t)$$
$$+ \frac{A_c^2 A_m^2}{4}\cos(4\pi f_c t) + \frac{A_c^2 A_m^2}{4}\cos(4\pi f_m t)$$

$$P_m = \frac{A_c A_m}{4} \quad \text{(from part (a))}$$

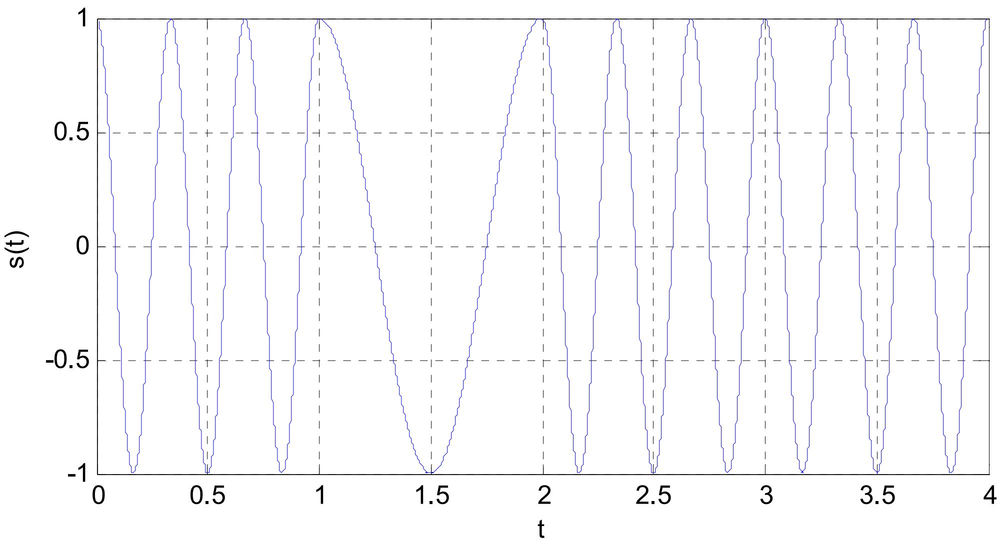
$$P_{\text{tot}} = \frac{1}{T_m}\int_0^{T_m} s^2\left(t\right)dt = \frac{A_c^2 A^2}{2} + \frac{A_c^2 A_m^2}{4}$$

$$\text{power ratio} = \frac{P_m}{P_{\text{tot}}} = \frac{\frac{A_c^2 A_m^2}{4}}{\frac{A_c^2 A^2}{2} + \frac{A_c^2 A_m^2}{4}} = \frac{1}{2\left(\frac{A}{A_m}\right)^2 + 1}$$

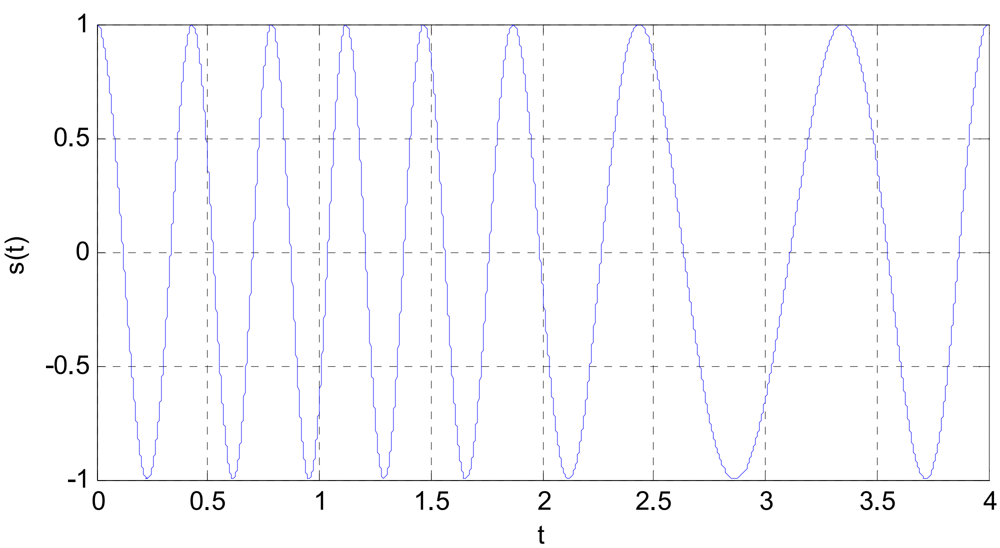
$$A \geq A_m \rightarrow \text{power ratio} \leq \frac{1}{3}$$

2.60

(a)



(b)



2.61

(a) $$\theta(t) = 2\pi\Delta f\int_0^t A_m \cos\left(2\pi f_m\tau\right)d\tau = \frac{A_m\Delta f}{f_m}\sin\left(2\pi f_m t\right)$$

Let $\beta = \dfrac{A_m\Delta f}{f_m}$, then $\theta(t) = \beta\sin\left(2\pi f_m t\right)$ and $s(t) = A_c\cos\left(2\pi f_c t + \beta\sin\left(2\pi f_m t\right)\right)$.

(b) Using the identity $e^{jX} = \cos(X) + j\sin(X)$, we see that $\cos(X) = \mathrm{Re}\left\{e^{jX}\right\}$. Thus, $A_c\cos\left(2\pi f_c t + \theta(t)\right) = A_c\mathrm{Re}\left\{e^{j(2\pi f_c t + \theta(t))}\right\} = A_c\mathrm{Re}\left\{e^{j\theta(t)}e^{j2\pi f_c t}\right\}$. Substitute $\theta(t) = \beta\sin\left(2\pi f_m t\right)$ to obtain the desired result.

(c) $$c_k = \frac{1}{T_0}\int_0^{T_0}\tilde{s}(t)e^{-j2\pi k f_m t}dt = \frac{1}{T_0}\int_0^{T_0}e^{j\beta\sin(2\pi f_m t)}e^{-j2\pi k f_m t}dt$$

Using the substitution $x = 2\pi f_m t$, the integral becomes

$$c_k = \frac{1}{2\pi}\int_0^{2\pi}e^{j\left[\beta\sin(x) - kx\right]}dx = J_k\left(\beta\right)$$

Thus $s(t) = A_c\sum_{k=-\infty}^{\infty}J_k\left(\beta\right)\cos\left(2\pi\left(f_c + kf_m\right)t\right)$

(d) $$S\left(f\right) = \frac{A_c}{2}\sum_{k=-\infty}^{\infty}J_k\left(\beta\right)\left[\delta\left(f + f_c + kf_m\right) + \delta\left(f - f_c - kf_m\right)\right]$$

The bandwidth is infinite.

(e) The power at the carrier frequency is determined by the $k = 0$ in the answer to part (d). The $k = 0$ term is

$$\frac{A_c}{2}J_0\left(\beta\right)\left[\delta\left(f + f_c\right) + \delta\left(f - f_c\right)\right]$$

The condition of zero power at the carrier frequency occurs when $J_0\left(\beta\right) = 0$. Thus the curious and interesting situation occurs at the values of β corresponding to the zeros of $J_0\left(\cdot\right)$. The first five zeros are $\beta = 2.4048, 5.5201, 8.6537, 11.7915, 14.9309$.

2.62

(a)

$$S(f)=\frac{A_c}{2}\sum_{k=-\infty}^{\infty}J_k(\beta)\left[\delta(f+f_c+kf_m)+\delta(f-f_c-kf_m)\right]$$

$$\left|S(f)\right|^2=\frac{A_c^2}{4}\sum_{k=-\infty}^{\infty}J_k^2(\beta)\left[\delta(f+f_c+kf_m)+\delta(f-f_c-kf_m)\right]$$

$$P=\int_{-\infty}^{\infty}\left|S(f)\right|^2 df=\frac{A_c^2}{4}\sum_{k=-\infty}^{\infty}J_k^2(\beta)\int_{-\infty}^{\infty}\left[\delta(f+f_c+kf_m)+\delta(f-f_c-kf_m)\right]df$$

$$=\frac{A_c^2}{4}\sum_{k=-\infty}^{\infty}J_k^2(\beta)\times 2=\frac{A_c^2}{2}\sum_{k=-\infty}^{\infty}J_k^2(\beta)$$

(b) The table of the Bessel functions is

	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8
B = 1	0.58553	0.19364	0.01320	0.00038	0.00001	0.00000	0.00000	0.00000	0.00000
B = 2	0.05013	0.33261	0.12449	0.01663	0.00116	0.00005	0.00000	0.00000	0.00000
B = 5	0.03154	0.10731	0.00217	0.13310	0.15306	0.06819	0.01717	0.00285	0.00034

The table of the power ratio R for different values of K is

$$R=J_0^2(\beta)+2\sum_{k=1}^{K}J_k^2(\beta)$$

	K=0	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8
B = 1	0.58553	0.97282	**0.99922**	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000
B = 2	0.05013	0.71535	0.96433	**0.99759**	0.99990	1.00000	1.00000	1.00000	1.00000
B = 5	0.03154	0.24616	0.25049	0.51670	0.82282	0.95921	**0.99356**	0.99926	0.99993

The bold numbers indicate the smallest value of K which captures 98% of the power. The values are

$$K_{98}=2 \quad \text{for} \quad \beta=1$$
$$K_{98}=3 \quad \text{for} \quad \beta=2$$
$$K_{98}=6 \quad \text{for} \quad \beta=5$$

from which we deduce that $K_{98}=\beta+1$. Thus we have

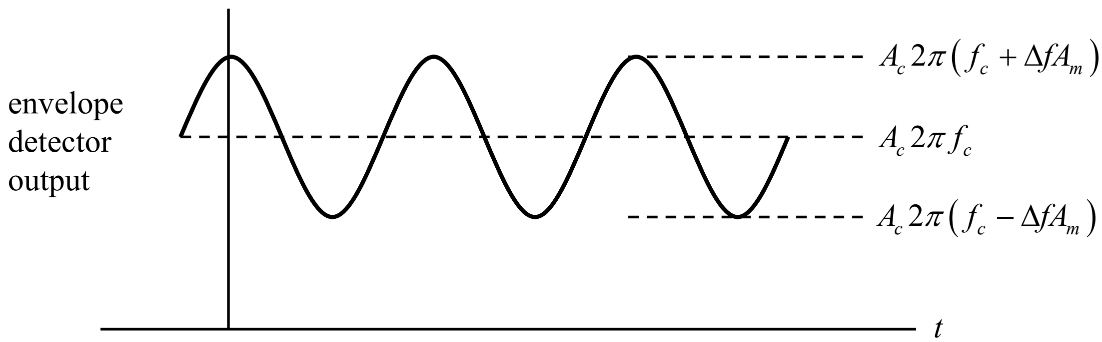
$$\text{BW}=2K_{98}f_m=2(\beta+1)f_m$$

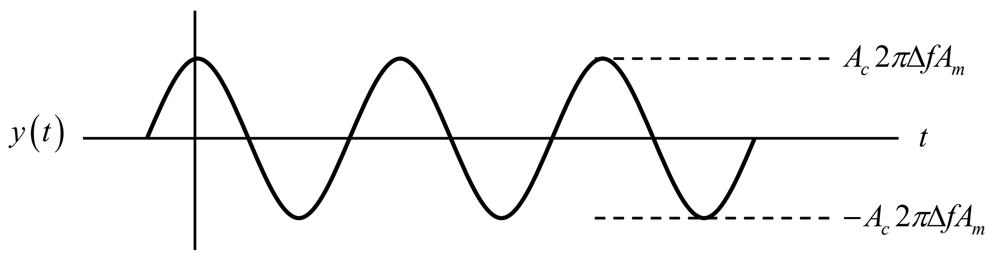
2.63

(a) $$\frac{d}{dt} A_c \cos\left(2\pi f_c t + \theta(t)\right) = A_c\left[2\pi f_c + \theta'(t)\right]\sin\left(2\pi f_c t + \theta(t)\right)$$
$$= A_c\left[2\pi f_c + 2\pi\Delta f m(t)\right]\sin\left(2\pi f_c t + \theta(t)\right)$$

(b) The envelope detector output is $A_c\left|2\pi f_c + 2\pi\Delta f m(t)\right|$. To produce the desired output, we need $f_c \geq \Delta f m(t)$ for all t.

(c)





2.64

(a) $$H_{\text{FM}}(s)=\frac{sk_pF(s)}{s+k_0k_pF(s)}$$

(b) $V(s)=H_{\text{FM}}(s)\Theta(s)=s\Theta(s) \quad \rightarrow \quad v(t)=\dfrac{d}{dt}\theta(t)$ which is the desired result.

(c) $$\frac{sk_pF(s)}{s+k_0k_pF(s)}=s \quad \rightarrow \quad F(s)=\frac{s}{k_p\left(1-k_0\right)}$$

The filter is a differentiator.

$k_0=1$ should be avoided.

2.65

(a) $$h(n)=\left(-\frac{1}{2}\right)^{n}u(n)-\frac{1}{2}\left(-\frac{1}{2}\right)^{n-1}u(n-1)=\delta(n)+\left(-1\right)^{n}\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

(b) $$h(n)=\left(-\frac{1}{2}\right)^{n}u(n)+\frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}u(n-1)$$

(c) $$h(n)=\left(-\frac{1}{2}\right)^{n-2}u(n-2)$$

(d) $h(n)=-\delta(n-1)+2\delta(n-2)-3\delta(n-3)+4\delta(n-4)$

2.66

(a) $$h(n) = 2\left(\frac{1}{2}\right)^{n-1} u(n-1) = \left(\frac{1}{2}\right)^{n} u(n-1)$$

(b) $$H(z) = \frac{(3-j4)z^{-1}}{1-\left(\frac{1}{3}+j\frac{1}{4}\right)z^{-1}} + \frac{(3+j4)z^{-1}}{1-\left(\frac{1}{3}-j\frac{1}{4}\right)z^{-1}}$$

$$h(n) = (3-j4)\left(\frac{1}{3}+j\frac{1}{4}\right)^{n-1} u(n-1) + (3+j4)\left(\frac{1}{3}-j\frac{1}{4}\right)^{n-1} u(n-1)$$

$$\left(\frac{1}{3}+j\frac{1}{4}\right) = \frac{5}{12}e^{j\theta} \qquad 3-j4 = 5e^{-j\left(\frac{\pi}{2}-\theta\right)} = -j5e^{j\theta}$$

$$\left(\frac{1}{3}-j\frac{1}{4}\right) = \frac{5}{12}e^{-j\theta} \qquad 3+j4 = 5e^{j\left(\frac{\pi}{2}-\theta\right)} = j5e^{-j\theta}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$h(n) = 10\left(\frac{5}{12}\right)^{n-1} \sin(n\theta) u(n-1)$$

(c) $$H(z) = \frac{\left(1-j\frac{3}{4}\right)z^{-1}}{1-\left(\frac{1}{4}+j\frac{1}{3}\right)z^{-1}} + \frac{\left(1+j\frac{3}{4}\right)z^{-1}}{1-\left(\frac{1}{4}-j\frac{1}{3}\right)z^{-1}}$$

$$h(n) = \left(1-j\frac{3}{4}\right)\left(\frac{1}{4}+j\frac{1}{3}\right)^{n-1} u(n-1) + \left(1+j\frac{3}{4}\right)\left(\frac{1}{4}-j\frac{1}{3}\right)^{n-1} u(n-1)$$

$$\left(\frac{1}{4}+j\frac{1}{3}\right) = \frac{5}{12}e^{j\theta} \qquad 1-j\frac{3}{4} = \frac{5}{4}e^{-j\left(\frac{\pi}{2}-\theta\right)} = -j\frac{5}{4}e^{j\theta}$$

$$\left(\frac{1}{4}-j\frac{1}{3}\right) = \frac{5}{12}e^{-j\theta} \qquad 1+j\frac{3}{4} = \frac{5}{4}e^{j\left(\frac{\pi}{2}-\theta\right)} = j\frac{5}{4}e^{j\theta}$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$h(n) = \frac{5}{2}\left(\frac{5}{12}\right)^{n-1} \sin(n\theta) u(n-1)$$

(d)

$$H(z) = \frac{2z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} + \frac{2z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

$$h(n) = \left[4(n-1)\left(\frac{1}{2}\right)^{n-1} + 2\left(\frac{1}{4}\right)^{n-1}\right]u(n-1)$$

2.67

(a) $Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}$

(b) $$Y(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

(c) $$Y(z) = \frac{z}{z^2 - z - 1} = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}$$

$$= \frac{\frac{5+\sqrt{5}}{10}}{z - \frac{1+\sqrt{5}}{2}} + \frac{\frac{5-\sqrt{5}}{10}}{z - \frac{1-\sqrt{5}}{2}}$$

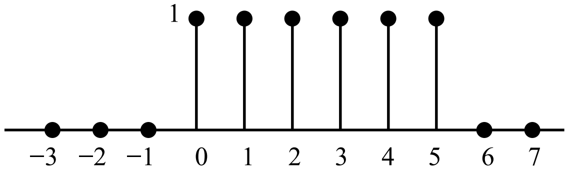
$$= \frac{\frac{5+\sqrt{5}}{10} z^{-1}}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} + \frac{\frac{5-\sqrt{5}}{10} z^{-1}}{1 - \frac{1-\sqrt{5}}{2} z^{-1}}$$

$$y(n) = \left[\frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \right] u(n-1)$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \right] u(n-1)$$

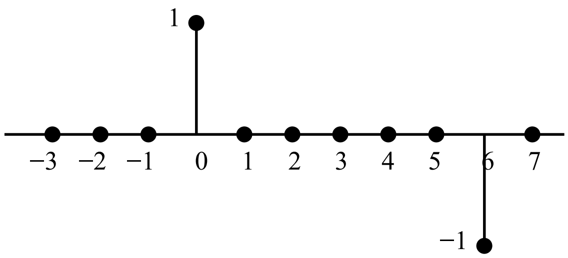
2.68

(a)



(b) $X(z) = \sum_{n=0}^{5} z^{-n} = \dfrac{1-z^{-6}}{1-z^{-1}}$

(c)



(d) $G(z) = 1 - z^{-6}$

(e) $G(z) = X(z) - z^{-1}X(z) = (1-z^{-1})X(z) = (1-z^{-1})\dfrac{1-z^{-6}}{1-z^{-1}} = 1 - z^{-6}$

This answer is the same as that from part (d).

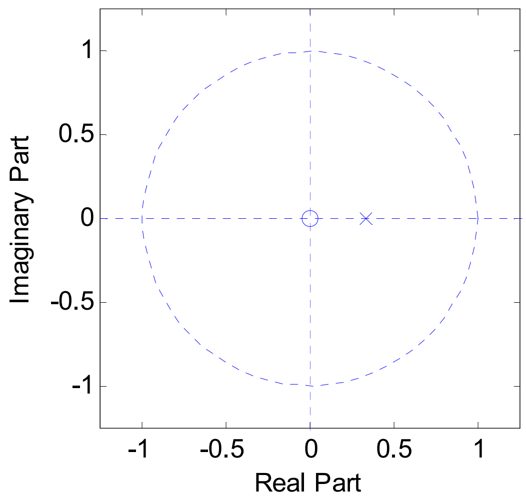
(f) $X(z) = \dfrac{1}{1-z^{-1}}G(z) = \dfrac{1-z^{-6}}{1-z^{-1}}$

This answer is the same as that from part (b).

2.69

(a) $$H(z)=\frac{1}{1-\frac{1}{3}z^{-1}}$$

(b)



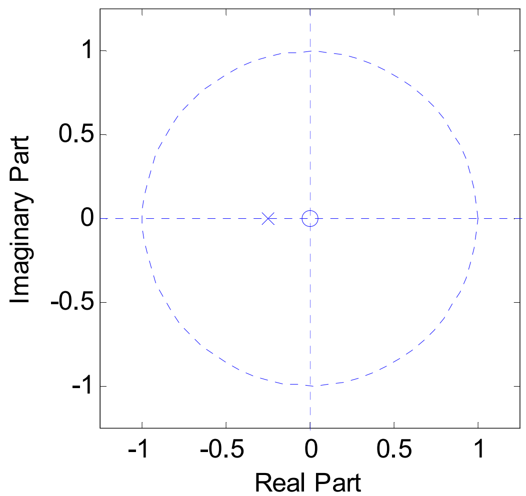
(c)
$$h(n)=\left(\frac{1}{3}\right)^{n}u(n)$$

$$\begin{aligned}
y(n)&=h(n)*x(n)\\
&=h(n)*\left[\delta(n)+\delta(n-1)+\delta(n-2)\right]\\
&=h(n)+h(n-1)+h(n-2)\\
&=\begin{cases}0 & n<0\\ 1 & n=0\\ \frac{4}{3} & n=1\\ \frac{13}{9}\left(\frac{1}{3}\right)^{n-2} & n\geq 2\end{cases}
\end{aligned}$$

2.70

(a) $$H(z)=\frac{1}{1+\frac{1}{4}z^{-1}}$$

(b)



(c)
$$h(n)=\left(-\frac{1}{4}\right)^{n}u(n)$$
$$\begin{aligned}y(n)&=h(n)*x(n)\\&=h(n)*\left[\delta(n)+\delta(n-1)+\delta(n-2)\right]\\&=h(n)+h(n-1)+h(n-2)\\&=\begin{cases}0 & n<0\\ 1 & n=0\\ \frac{3}{4} & n=1\\ \frac{13}{16}\left(-\frac{1}{4}\right)^{n-2} & n\geq 2\end{cases}\end{aligned}$$

2.71

(a) IIR

(b) FIR

(c)

$$H(z) = H_1(z)H_2(z) = \frac{6z^{-1} - 8z^{-2} + \frac{7}{3}z^{-3}}{1 - \frac{11}{6}z^{-1} + z^{-2} - \frac{1}{6}z^{-3}}$$

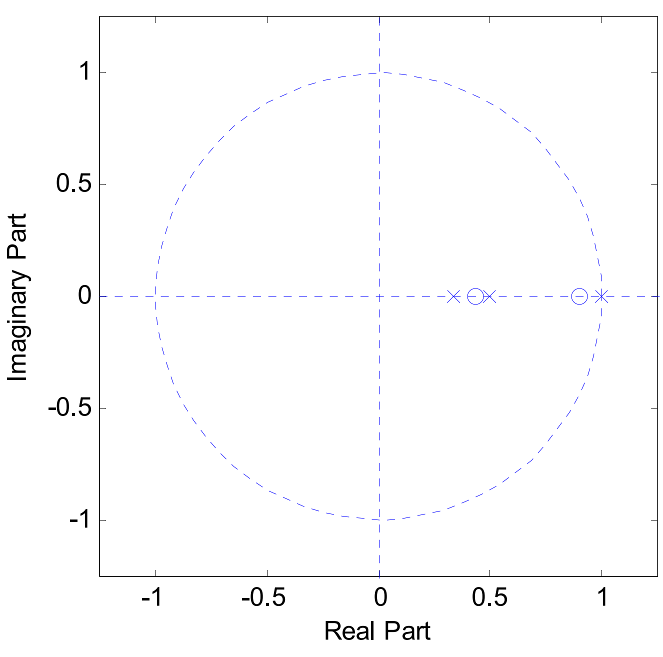
(d) IIR

(e)

$$y(n) = \frac{11}{6}y(n-1) - y(n-2) + \frac{1}{6}y(n-3) + 6x(n-1) - 8x(n-2) + \frac{7}{3}x(n-3)$$

(f)

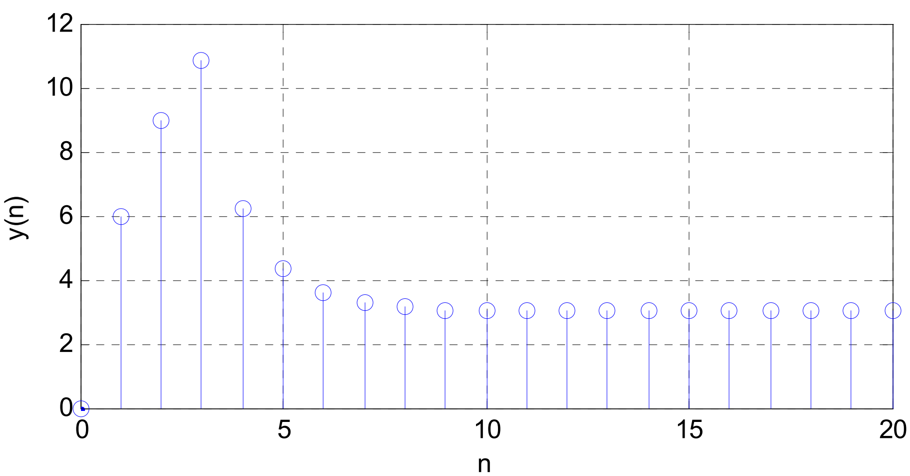
$$H(z) = \frac{z^{-1}\left(1 - \left(\frac{2}{3} + \frac{\sqrt{2}}{6}\right)z^{-1}\right)\left(1 - \left(\frac{2}{3} - \frac{\sqrt{2}}{6}\right)z^{-1}\right)}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$



(g) $$H(z)=\frac{z^{-1}}{1-z^{-1}}+\frac{2z^{-1}}{1-\frac{1}{2}z^{-1}}+\frac{3z^{-1}}{1-\frac{1}{3}z^{-1}}$$

$$h(n)=\left[1+2\left(\frac{1}{2}\right)^{n-1}+3\left(\frac{1}{3}\right)^{n-1}\right]u(n-1)$$

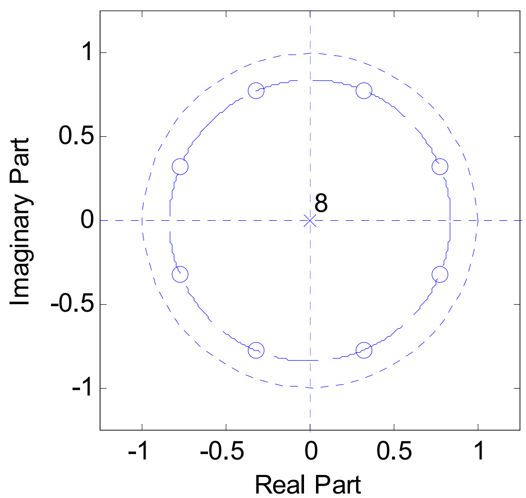
(h) $$\begin{aligned} y(n) &= h(n)*x(n) \\ &= h(n)*\left[\delta(n)+\delta(n-1)+\delta(n-2)\right] \\ &= h(n)+h(n-1)+h(n-2) \\ &= \begin{cases} 0 & n=0 \\ 6 & n=1 \\ 9 & n=2 \\ 3+\frac{7}{4}\left(\frac{1}{2}\right)^{n-4}+\frac{13}{9}\left(\frac{1}{3}\right)^{n-4} & n\ge 3 \end{cases} \end{aligned}$$



2.72

(a) $H(z) = 1 - \alpha z^{-8}$

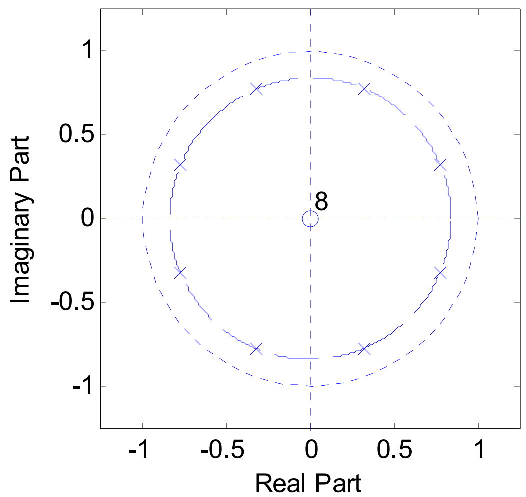
$H(z)$ has 8 zeros equally spaced on a circle of radius $\alpha^{\frac{1}{8}}$ and 8 poles at the origin as shown below.



The ROC is the entire z-plane except for $z = 0$.

(b) $G(z) = \dfrac{1}{H(z)} = \dfrac{1}{1-\alpha z^{-8}}$

$G(z)$ has 8 poles equally spaced on a circle of radius $\alpha^{\frac{1}{8}}$ and 8 zeros at the origin as shown below. The ROC for a causal stable system is $|z| > \alpha^{\frac{1}{8}}$.



(c) $$g(n) = \begin{cases} \alpha^n & n = \text{a multiple of } 8 \\ 0 & \text{otherwise} \end{cases}$$

2.73

(a) $y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$

(b) The poles are $z = \frac{a_1}{2}\left(1 \pm \sqrt{1+4\frac{a_2}{a_1^2}}\right)$

i. For real, distinct poles we require $a_2 > -\frac{a_2^2}{4}$. The poles are $z = \frac{a_1}{2}\left(1 \pm \sqrt{1+4\frac{a_2}{a_1^2}}\right)$.

$$H(z) = b_0 + \frac{\dfrac{\frac{a_1 b_0}{2} + 2\frac{a_2}{a_1}b_0 + \left(\frac{a_1 b_0}{2} + \frac{a_2}{a_1}b_0\right)\sqrt{1+4\frac{a_2}{a_1^2}}}{1+4\frac{a_2}{a_1^2}} z^{-1}}{1 - \frac{a_1}{2}\left(1+\sqrt{1+4\frac{a_2}{a_1^2}}\right)z^{-1}}$$

$$+ \frac{\dfrac{\frac{a_1 b_0}{2} + 2\frac{a_2}{a_1}b_0 - \left(\frac{a_1 b_0}{2} + \frac{a_2}{a_1}b_0\right)\sqrt{1+4\frac{a_2}{a_1^2}}}{1+4\frac{a_2}{a_1^2}} z^{-1}}{1 - \frac{a_1}{2}\left(1+\sqrt{1+4\frac{a_2}{a_1^2}}\right)z^{-1}}$$

$$h(n) = b_0 \delta(n)$$

$$+ \left[\frac{\frac{a_1 b_0}{2} + 2\frac{a_2}{a_1}b_0 + \left(\frac{a_1 b_0}{2} + \frac{a_2}{a_1}b_0\right)\sqrt{1+4\frac{a_2}{a_1^2}}}{1+4\frac{a_2}{a_1^2}}\right]\left(\frac{a_1}{2}\left(1+\sqrt{1+4\frac{a_2}{a_1^2}}\right)\right)^{n-1} u(n-1)$$

$$+ \left[\frac{\frac{a_1 b_0}{2} + 2\frac{a_2}{a_1}b_0 - \left(\frac{a_1 b_0}{2} + \frac{a_2}{a_1}b_0\right)\sqrt{1+4\frac{a_2}{a_1^2}}}{1+4\frac{a_2}{a_1^2}}\right]\left(\frac{a_1}{2}\left(1-\sqrt{1+4\frac{a_2}{a_1^2}}\right)\right)^{n-1} u(n-1)$$

ii. For real, repeated poles, we require $-a_1^2 = 4a_2$. The poles are $z = \frac{a_1}{2}, \frac{a_1}{2}$

$$H(z) = b_0 + \frac{b_0 a_1 z^{-1} - \frac{1}{4} b_0 a_1^2 z^{-2}}{\left(1 - \frac{a_1}{2} z^{-1}\right)^2} = b_0 + 2b_0 \frac{\frac{a_1}{2} z^{-1}}{\left(1 - \frac{a_1}{2} z^{-1}\right)^2} - b_0 \frac{a_1}{2} z^{-1} \frac{\frac{a_1}{2} z^{-1}}{\left(1 - \frac{a_1}{2} z^{-1}\right)^2}$$

$$h(n) = b_0 \delta(n) + 2b_0 n \left(\frac{a_1}{2}\right)^n u(n) - b_0 \frac{a_1}{2}(n-1)\left(\frac{a_1}{2}\right)^{n-1} u(n-1)$$

iii. For complex conjugate poles, we require $a_1^2 + 4a_2 < 0$. Let $\theta^2 = a_1^2 + 4a_2$, then the poles are $z = \frac{a_1}{2} \pm j\frac{\theta}{2}$.

$$H(z) = b_0 + \frac{\left[\frac{a_1 b_0}{2} - j\left(\frac{a_1^2 b_0}{2\theta} + \frac{a_2 b_0}{\theta}\right)\right] z^{-1}}{1 - \left(\frac{a_1}{2} + j\frac{\theta}{2}\right) z^{-1}} + \frac{\left[\frac{a_1 b_0}{2} + j\left(\frac{a_1^2 b_0}{2\theta} + \frac{a_2 b_0}{\theta}\right)\right] z^{-1}}{1 - \left(\frac{a_1}{2} - j\frac{\theta}{2}\right) z^{-1}}$$

$$h(n) = b_0 \delta(n) + \left[\frac{a_1 b_0}{2} - j\left(\frac{a_1^2 b_0}{2\theta} + \frac{a_2 b_0}{\theta}\right)\right]\left(\frac{a_1}{2} + j\frac{\theta}{2}\right)^{n-1} u(n-1)$$
$$+ \left[\frac{a_1 b_0}{2} + j\left(\frac{a_1^2 b_0}{2\theta} + \frac{a_2 b_0}{\theta}\right)\right]\left(\frac{a_1}{2} - j\frac{\theta}{2}\right)^{n-1} u(n-1)$$

(c)

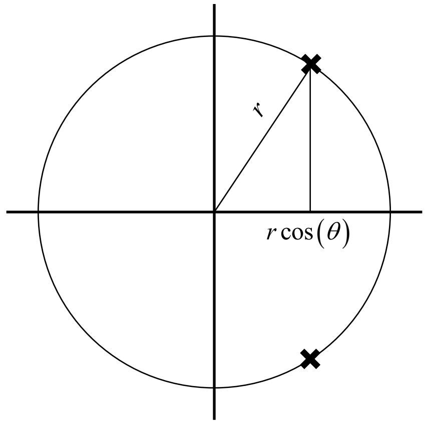
i. The poles are $z = \frac{a_1}{2} \pm j\frac{1}{2}\sqrt{-a_1^2 - 4a_2} = re^{j\theta}$ from which we obtain

$$r = \sqrt{-a_2}$$

$$\theta = \tan^{-1}\left(\sqrt{-1 - 4\frac{a_2}{a_1^2}}\right)$$

ii. $B(z) = 1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}$

iii.



The location of the pole intersects the real axis at $-\frac{1}{2}$ times the coefficient of z^{-1}.

The distance from the origin is the square root of the coefficient of z^{-2}.

iv.

$$H(z) = b_0 + b_0 \frac{\frac{-jre^{j2\theta}}{2\sin(\theta)} z^{-1}}{1 - re^{j\theta} z^{-1}} + b_0 \frac{\frac{jre^{-j2\theta}}{2\sin(\theta)} z^{-1}}{1 - re^{-j\theta} z^{-1}}$$

$$h(n) = b_0 \delta(n) + b_0 \frac{r^n}{\sin(\theta)} \sin((n+1)\theta) u(n-1)$$

(d)

The case of real, repeated poles corresponds to $\theta = 0$. In this case, $r = \sqrt{-a_2} = \sqrt{\frac{a_1^2}{4}} = \frac{a_1}{2}$.

Also note that $\lim_{\theta \to 0} \frac{\sin((n+1)\theta)}{\sin(\theta)} = n+1$. Now, the answer to part (c)-iv becomes

$$\begin{aligned}
h(n) &= b_0 \delta(n) + b_0 \frac{r^n}{\sin(\theta)} \sin((n+1)\theta) u(n-1) \\
&= b_0 \delta(n) + b_0 (n+1) r^n u(n-1) \\
&= b_0 \delta(n) + b_0 (n-1+2) r^n u(n-1) \\
&= b_0 \delta(n) + 2b_0 r^n u(n) + b_0 r (n-1) r^{n-1} u(n-1)
\end{aligned}$$

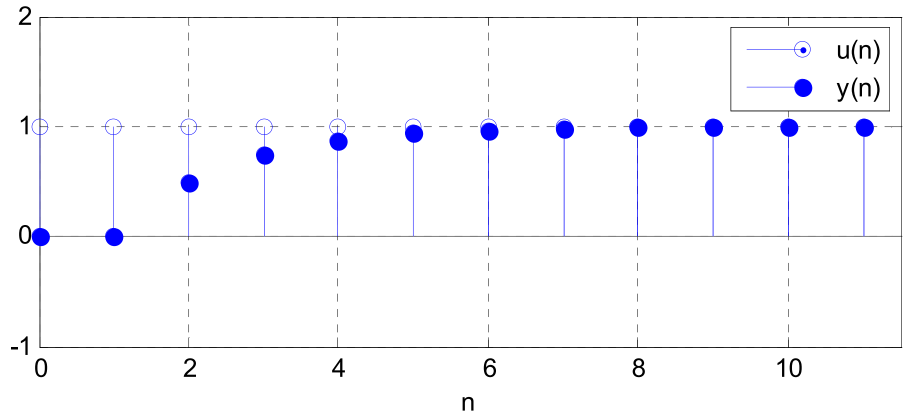
Using $r = \frac{a_1}{2}$ gives the answer from part (c)-iv.

2.74

(a) $F(z)=0.5$

$$Y(z)=\frac{0.5z^{-1}}{1-0.5z^{-1}}X(z)=\frac{0.5z^{-1}}{1-0.5z^{-1}}\frac{1}{1-z^{-1}}=\frac{z^{-1}}{1-z^{-1}}-\frac{z^{-1}}{1-0.5z^{-1}}$$

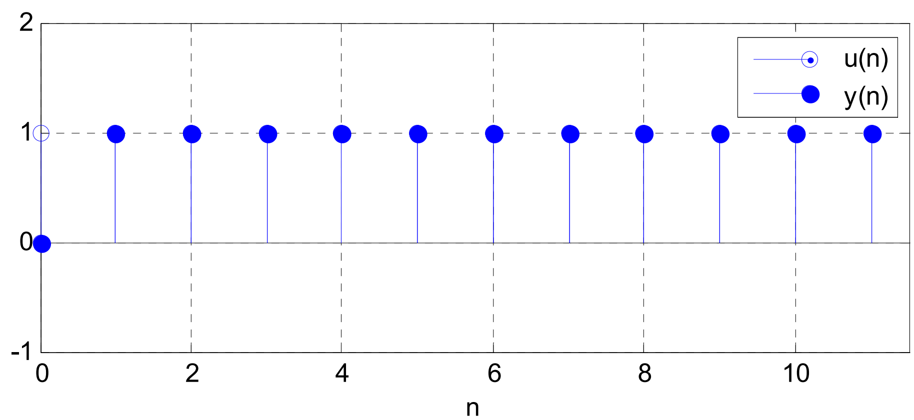
$$y(n)=\left[1-(0.5)^{n-1}\right]u(n-1)$$



(b) $F(z)=1$

$$Y(z)=z^{-1}X(z)=\frac{z^{-1}}{1-z^{-1}}$$

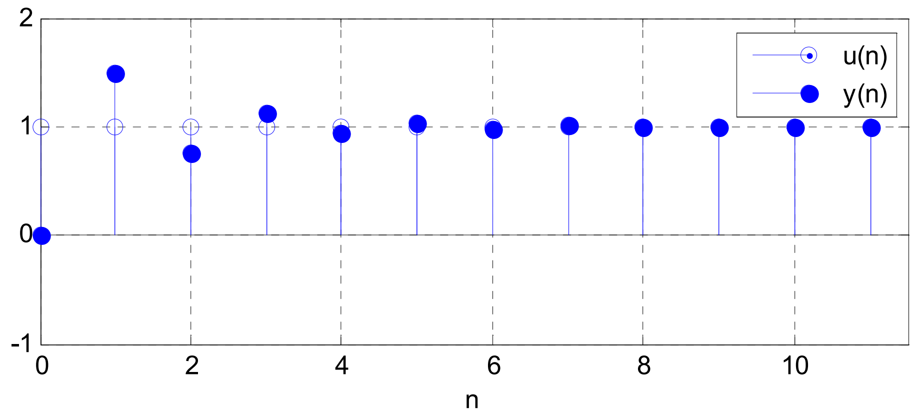
$$y(n)=u(n-1)$$



(c) $F(z) = 1.5$

$$Y(z) = \frac{1.5z^{-1}}{1+0.5z^{-1}} X(z) = \frac{1.5z^{-1}}{1+0.5z^{-1}} \frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}} + \frac{0.5z^{-1}}{1+0.5z^{-1}}$$

$$y(n) = \left[1 + 0.5(-0.5)^{n-1}\right] u(n-1)$$



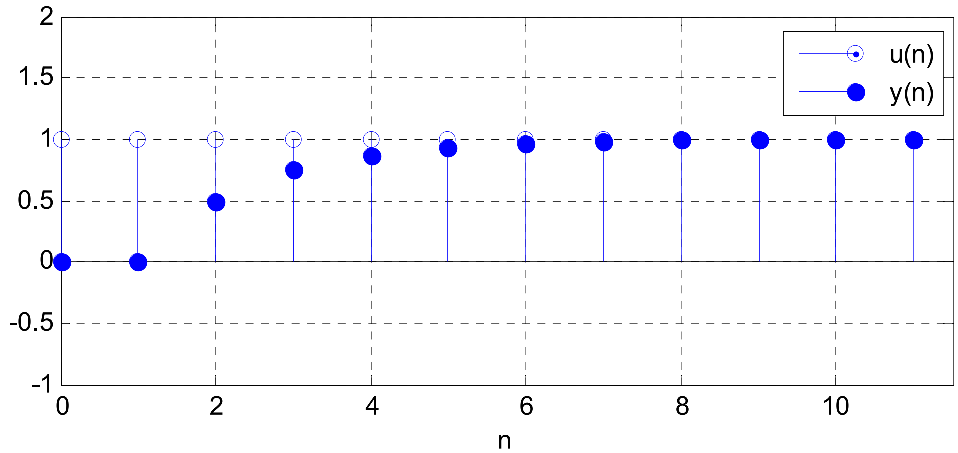
(d) Unlike the case with continuous-time systems, a first-order system can show an oscillatory transient reponse.

2.75

(a)

$$V(z)=\frac{\frac{1}{2}\left(1-z^{-1}\right)}{1-\left(1-\frac{1}{2}\right)z^{-1}}\times\frac{1}{1-z^{-1}}=\frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$$

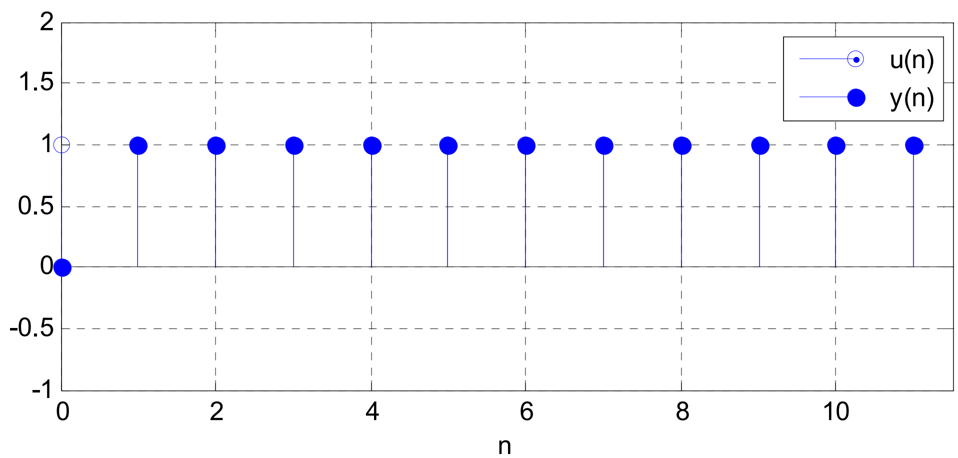
$$v(n)=\frac{1}{2}\left(\frac{1}{2}\right)^{n}u(n)$$



(b)

$$V(z)=\frac{1\times\left(1-z^{-1}\right)}{1-(1-1)z^{-1}}\times\frac{1}{1-z^{-1}}=1$$

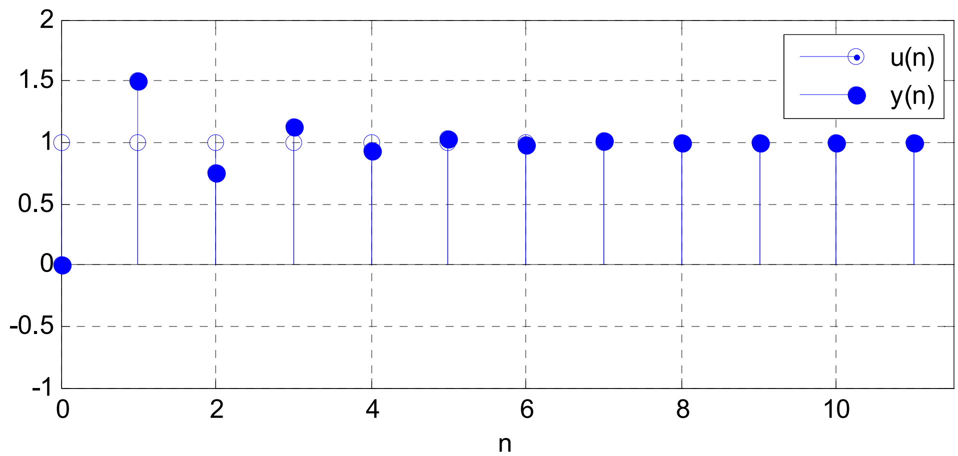
$$v(n)=\delta(n)$$



(c)

$$V(z) = \frac{\frac{3}{2}\left(1 - z^{-1}\right)}{1 - \left(1 - \frac{3}{2}\right) z^{-1}} \times \frac{1}{1 - z^{-1}} = \frac{\frac{3}{2}}{1 + \frac{1}{2} z^{-1}}$$

$$v(n) = \frac{3}{2}\left(-\frac{1}{2}\right)^{n} u(n)$$



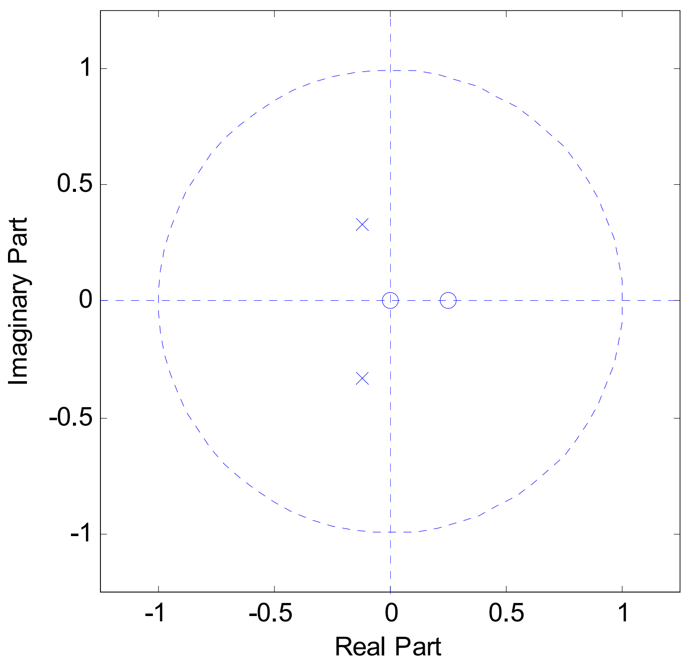
(d) Even though this is a first-order system, it can still exhibit oscillations.

2.76

(a)

$$H(z)=\frac{1-\frac{1}{4}z^{-1}}{1+\frac{1}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

(b)



(c)

$$Y(z)=1+\frac{\left(-\frac{7}{308}+j\frac{9\sqrt{7}}{308}\right)z^{-1}}{1+\left(\frac{1}{8}-j\frac{\sqrt{7}}{8}\right)z^{-1}}+\frac{\left(-\frac{7}{308}-j\frac{9\sqrt{7}}{308}\right)z^{-1}}{1+\left(\frac{1}{8}+j\frac{\sqrt{7}}{8}\right)z^{-1}}+\frac{\frac{6}{11}z^{-1}}{1-z^{-1}}$$

$$y(n)=\delta(n)+\left[\begin{array}{l}\left(-\frac{7}{308}+j\frac{9\sqrt{7}}{308}\right)\left(-\frac{1}{8}+j\frac{\sqrt{7}}{8}\right)^{n-1}\\ \quad+\left(-\frac{7}{308}-j\frac{9\sqrt{7}}{308}\right)\left(-\frac{1}{8}-j\frac{\sqrt{7}}{8}\right)^{n-1}\\ \qquad+\frac{6}{11}\end{array}\right]u(n-1)$$

2.77

The closed loop transfer function is

$$\frac{Y(z)}{X(z)} = \frac{1-0.25z^{-1}}{1+\left(0.75-K\right)z^{-1}+0.25Kz^{-2}}$$
$$= \frac{z\left(z-0.25\right)}{z^2+\left(0.75-K\right)z+0.125K}$$

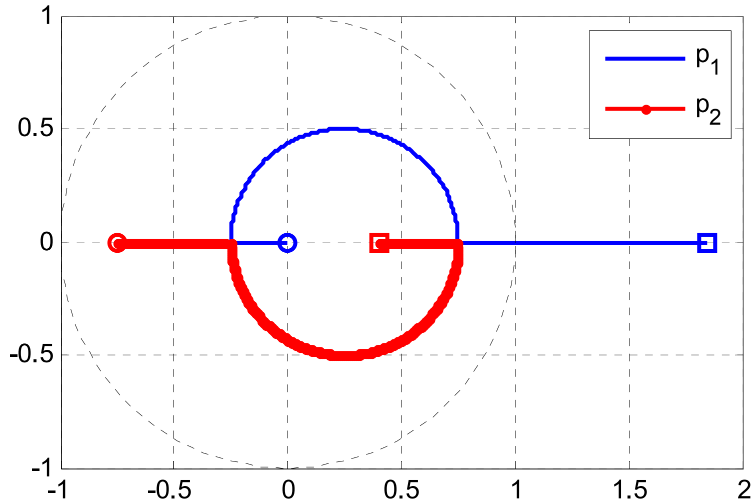
This system has two poles at

$$p_1, p_2 = \frac{-\left(0.75-K\right) \pm \sqrt{\left(0.75-K\right)^2 - K}}{2}$$

The goal is to find the values of K for which both poles are inside the unit circle. That is, find the values of K for which $|p_1|<1$ and $|p_2|<1$. The following Matlab script plots the poles for $0 \le K \le 3$:

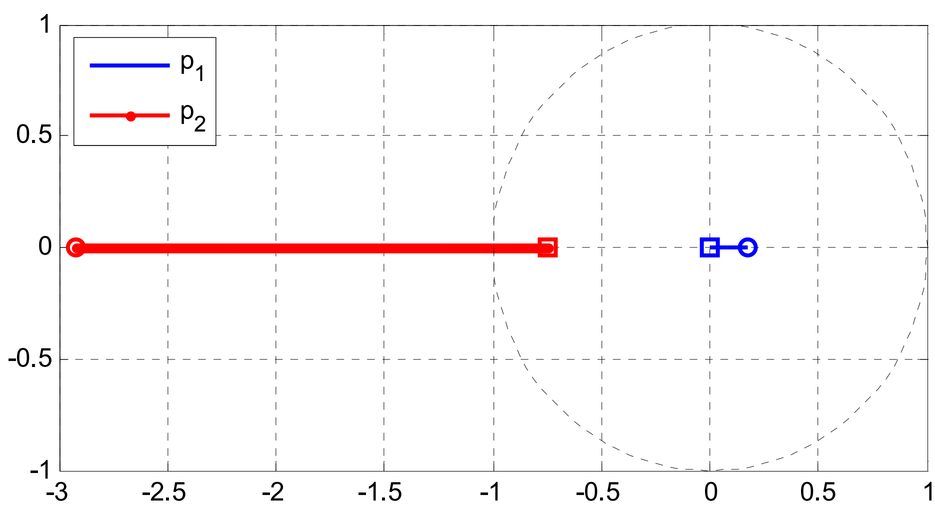
```
%% ex2_77

K = 0:0.001:3;
p1 = 0.5*(K-0.75 + sqrt((0.75-K).^2-K));
p2 = 0.5*(K-0.75 - sqrt((0.75-K).^2-K));

plot(real(p1),imag(p1),'b-',real(p2),imag(p2),'r.-',...
    real(p1(1)),imag(p1(1)),'bo',...
    real(p1(end)),imag(p1(end)),'bs',...
    real(p2(1)),imag(p2(1)),'ro',...
    real(p2(end)),imag(p2(end)),'rs','LineWidth',2);
legend('p_1','p_2');
grid on;
%% plot unit circle for reference
hold on; plot(exp(j*2*pi*[0:0.01:1]),'k:'); hold off;
```

This script produces the following plot



From this plot, we see that p_2 is always inside the unit circle and that p_1 is outside the unit circle when K is too large. Using the data vector p1 produced by the script, we see that p_1 is outside the unit circle for $K \geq 2.334$.

Repeating the same script for $K < 0$ produces the following plot



Form this plot, we see that p_1 is always inside the unit circle and that p_2 is outside the unit circle when K gets too negative. Using the data vector p2 produced by the script we see that p_2 is outside the unit circle for $K \leq -0.2$

Putting this all together, the system is stable for $-0.2 < K < 2.334$

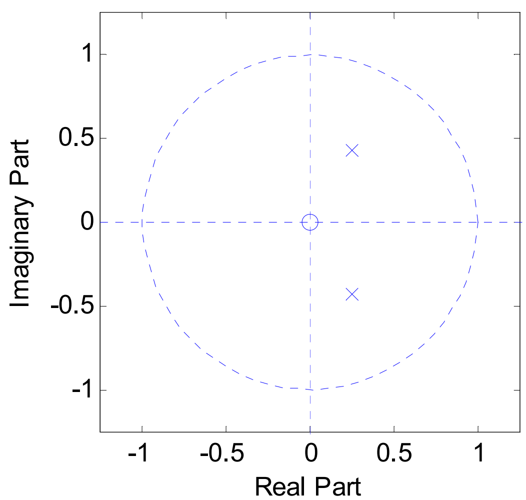
2.78

(a)

$$H(z) = \frac{\frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

(b)

There is one zero at $z = 0$ and two poles at $z = \frac{1}{4} \pm j\frac{\sqrt{3}}{4}$



(c)

$$Y(z) = \frac{\frac{-1-j\sqrt{3}}{12}z^{-1}}{1-\left(\frac{1}{4}+j\frac{\sqrt{3}}{4}\right)z^{-1}} + \frac{\frac{-1+j\sqrt{3}}{12}z^{-1}}{1-\left(\frac{1}{4}-j\frac{\sqrt{3}}{4}\right)z^{-1}} + \frac{\frac{2}{3}z^{-1}}{1-z^{-1}}$$

$$y(n) = \left[\left(\frac{-1-j\sqrt{3}}{12}\right)\left(\frac{1}{4}+j\frac{\sqrt{3}}{4}\right)^{n-1} + \left(\frac{-1+j\sqrt{3}}{12}\right)\left(\frac{1}{4}-j\frac{\sqrt{3}}{4}\right)^{n-1} + \frac{2}{3}\right]u(n-1)$$

$$= \left[\frac{1}{3}\left(\frac{1}{2}\right)^{n-1}\cos\left(\theta(n-1)+\phi\right) + \frac{2}{3}\right]u(n-1)$$

$$\theta = \tan^{-1}\left(\sqrt{3}\right)$$

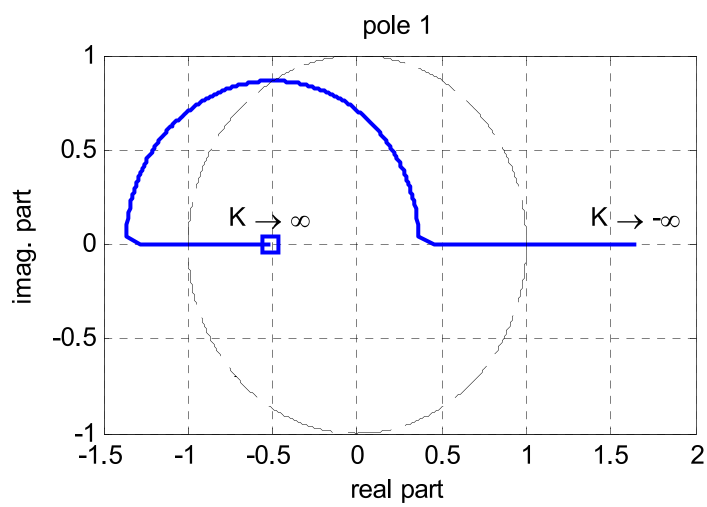
$$\phi = -\tan^{-1}\left(\sqrt{3}\right)$$

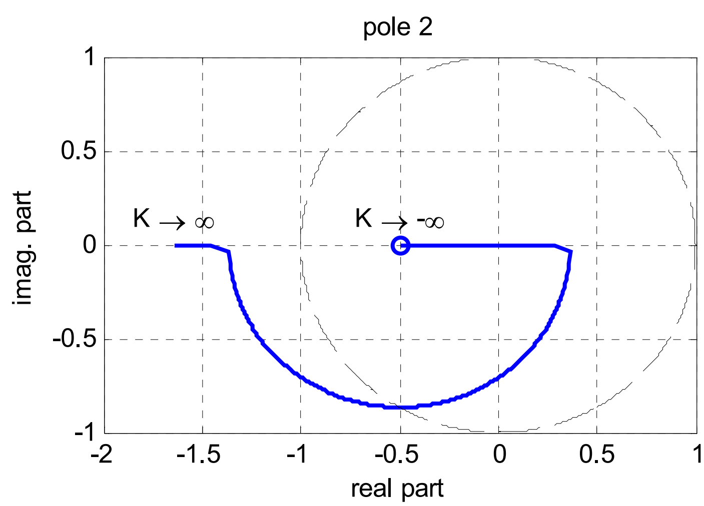
2.79

$$H(z)=\frac{Y(z)}{X(z)}=\frac{Kz^{-1}}{1+(K-1)z^{-1}+\frac{K}{2}z^{-2}}$$

The poles are at $z=\dfrac{1-K\pm\sqrt{(1-K)^2-2K}}{2}$.

The position of the poles in the z-plane (as a function of K) is illustrated below.





Several observations are in order

- The poles are real and distinct for $K<2-\sqrt{3}$ and $K>2+\sqrt{3}$

- The poles are real and repeated for $K = 2 - \sqrt{3}$ and $K = 2 + \sqrt{3}$
- The poles are complex conjugates for $2 - \sqrt{3} < K < 2 + \sqrt{3}$.
- The pole at $z = \dfrac{1 - K + \sqrt{(1-K)^2 - 2K}}{2}$ is inside the unit circle for $0 \le K \le 2$ and $K \ge 4$.
- The pole at $z = \dfrac{1 - K - \sqrt{(1-K)^2 - 2K}}{2}$ is inside the unit circle for $K \le 2$.

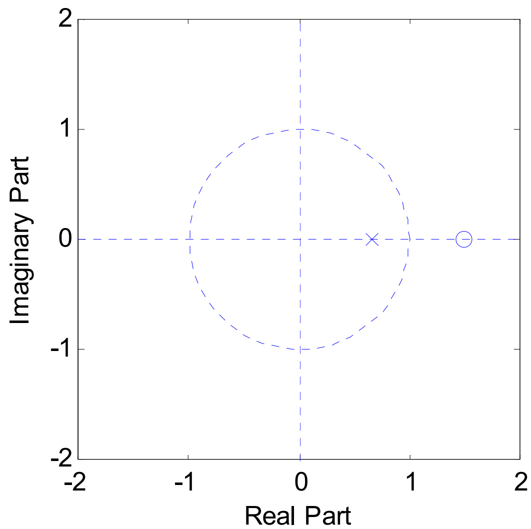
The system is stable for $0 \le K \le 2$

2.80

(a)

$$H(z) = \frac{-\frac{2}{3} + z^{-1}}{1 - \frac{2}{3} z^{-1}}$$

(b)



(c)

$$H\left(e^{j\Omega}\right) = \frac{-\frac{2}{3} + e^{-j\Omega}}{1 - \frac{2}{3} e^{-j\Omega}}$$

(d)

$$\left| H\left(e^{j\Omega}\right)^2 \right| = \frac{-\frac{2}{3} + e^{-j\Omega}}{1 - \frac{2}{3} e^{-j\Omega}} \times \frac{-\frac{2}{3} + e^{j\Omega}}{1 - \frac{2}{3} e^{j\Omega}} = \frac{\frac{4}{9} - \frac{2}{3} e^{j\Omega} - \frac{2}{3} e^{-j\Omega} + 1}{1 \frac{2}{3} e^{j\Omega} - \frac{2}{3} e^{-j\Omega} + \frac{4}{9}} = 1$$

The system is an *all pass* filter!

2.81

(a) $$X\left(e^{j\Omega}\right)=e^{-j\Omega}$$

(b) $$X\left(e^{j\Omega}\right)=\frac{e^{-j\Omega}-e^{-j4\Omega}}{1-e^{-j\Omega}}$$

(c) $$x(n)=\left(\frac{1}{2}\right)^n\times\frac{1}{2}\left[e^{j\frac{\pi}{8}n}e^{-j\frac{\pi}{4}}+e^{-j\frac{\pi}{8}n}e^{j\frac{\pi}{4}}\right]=\frac{e^{-j\frac{\pi}{4}}}{2}\left(\frac{e^{j\frac{\pi}{8}}}{2}\right)^n+\frac{e^{j\frac{\pi}{4}}}{2}\left(\frac{e^{-j\frac{\pi}{8}}}{2}\right)^n$$

$$X\left(e^{j\Omega}\right)=\frac{\frac{e^{-j\frac{\pi}{4}}}{2}}{1-\frac{e^{j\frac{\pi}{8}}}{2}e^{-j\Omega}}+\frac{\frac{e^{j\frac{\pi}{4}}}{2}}{1-\frac{e^{-j\frac{\pi}{8}}}{2}e^{-j\Omega}}=\frac{\cos\left(\frac{\pi}{4}\right)-\frac{1}{2}\cos\left(\frac{2\pi}{8}\right)e^{-j\Omega}}{1-\cos\left(\frac{\pi}{8}\right)e^{-j\Omega}+\frac{1}{4}e^{-j2\Omega}}$$

(d) $$X\left(e^{j\Omega}\right)=\frac{\sin\left(\frac{9\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

(e) $$X\left(e^{j\Omega}\right)=\pi\sum_{l=-\infty}^{\infty}\left[\delta(\Omega-1-2\pi l)+\delta(\Omega+1-2\pi l)\right]$$

(f) $$X\left(e^{j\Omega}\right)=\pi\sum_{l=-\infty}^{\infty}\left[\begin{array}{l}\delta\left(\Omega-\frac{5\pi}{3}-2\pi l\right)+\delta\left(\Omega+\frac{5\pi}{3}-2\pi l\right)\\ \quad -j\delta\left(\Omega-\frac{7\pi}{3}-2\pi l\right)+j\delta\left(\Omega+\frac{7\pi}{3}-2\pi l\right)\end{array}\right]$$

2.82

(a)

$$x(n) = \frac{3}{8}\frac{\sin\left(\frac{3\pi n}{8}\right)}{\frac{3\pi n}{8}}$$

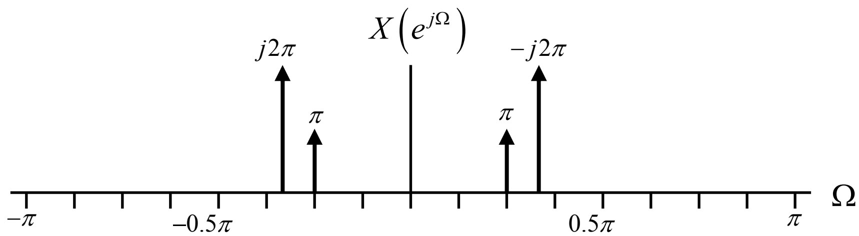
(b)

$$x(n) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)}{\left(n+\frac{1}{2}\right)\pi}$$

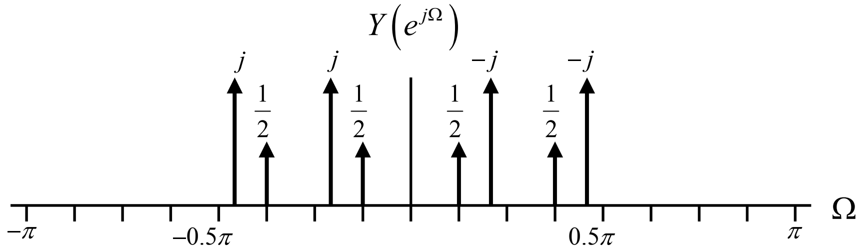
(c) $x(n) = 10\delta(n) - \delta(n-1) + 2\delta(n-2) - 3\delta(n-3)$

2.83

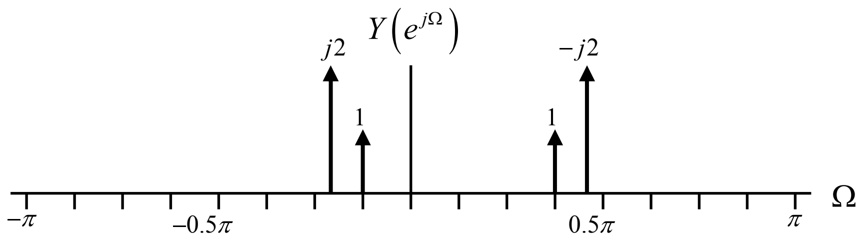
The DTFT of the sequence $x(n)$ is



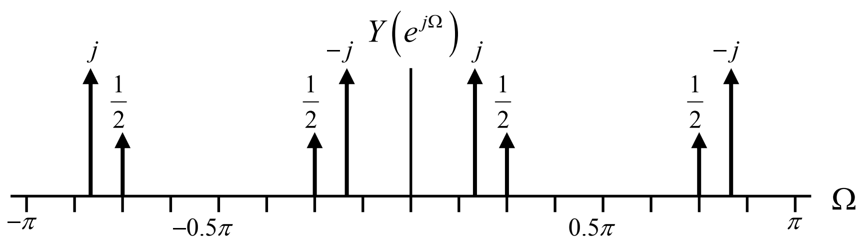
(a)



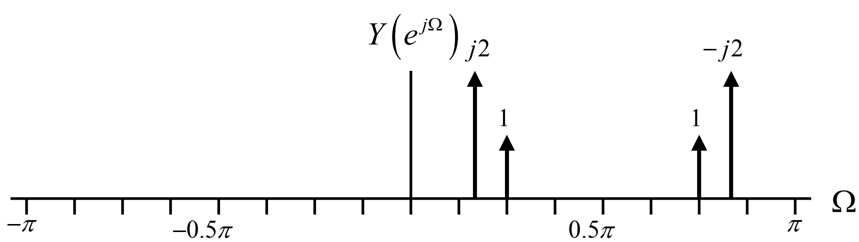
(b)



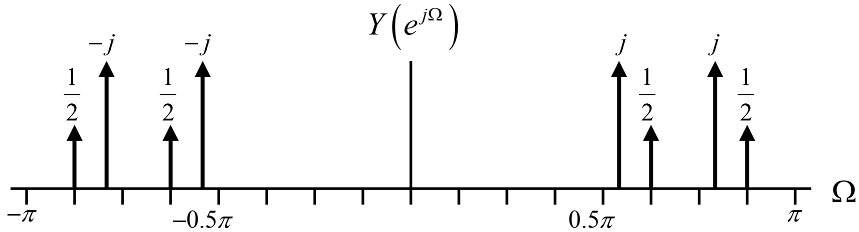
(c)



(d)

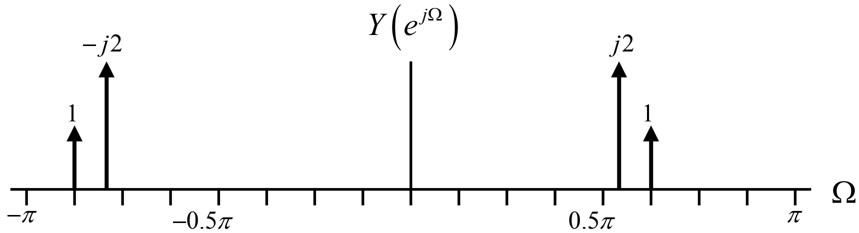


(e)



$Y\left(e^{j\Omega}\right)$
$-j$
$-j$
j
j
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
Ω
$-\pi$
-0.5π
0.5π
π

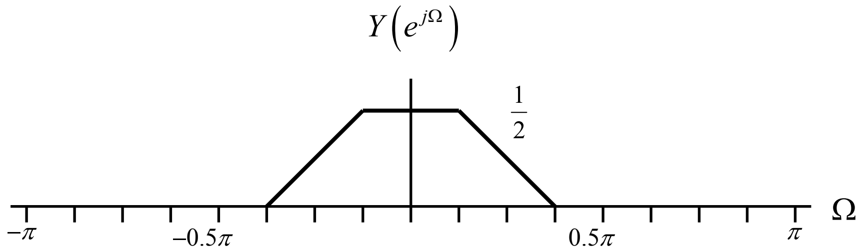
(f)



$Y\left(e^{j\Omega}\right)$
$-j2$
$j2$
1
1
Ω
$-\pi$
-0.5π
0.5π
π

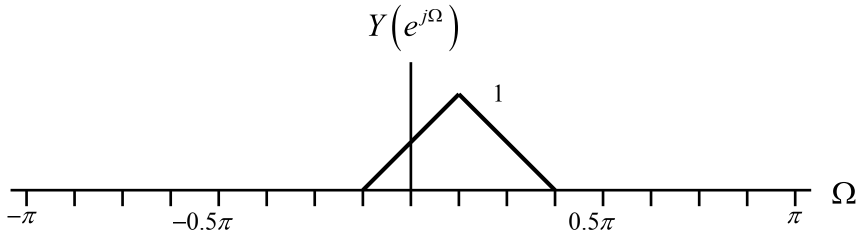
2.84

(a)



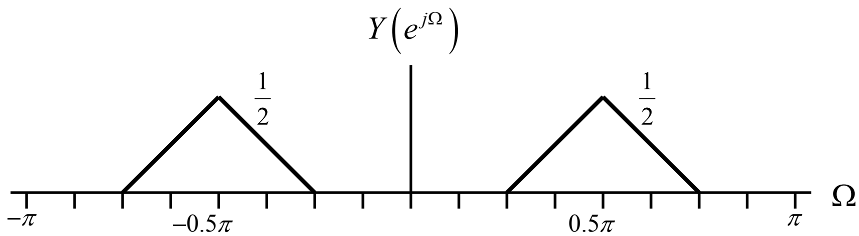
$Y\left(e^{j\Omega}\right)$
$\frac{1}{2}$
Ω
$-\pi$
-0.5π
0.5π
π

(b)



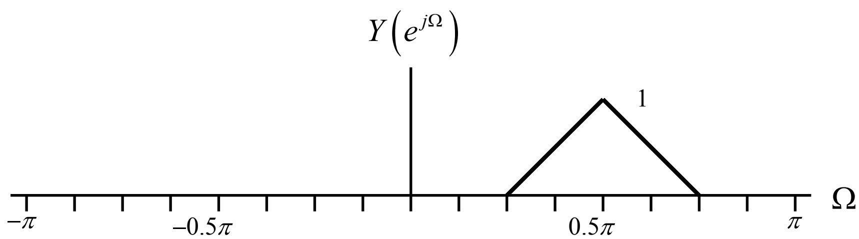
$Y\left(e^{j\Omega}\right)$
1
Ω
$-\pi$
-0.5π
0.5π
π

(c)



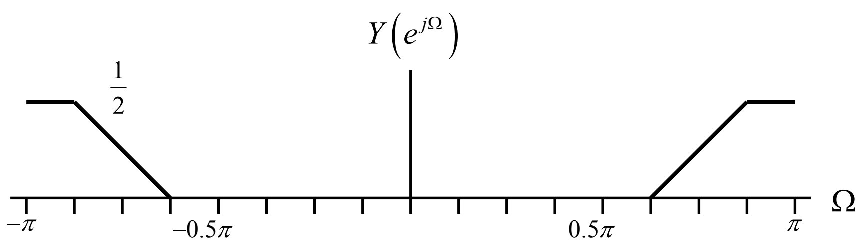
$Y\left(e^{j\Omega}\right)$
$\frac{1}{2}$
$\frac{1}{2}$
Ω
$-\pi$
-0.5π
0.5π
π

(d)



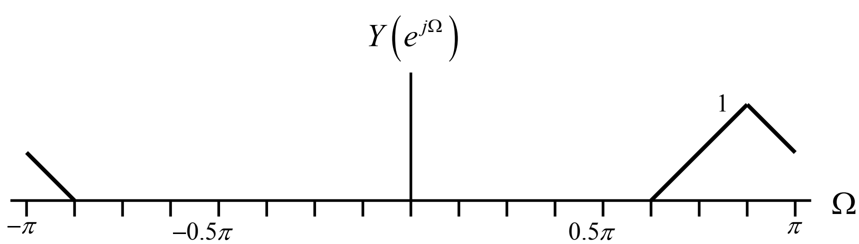
$Y\left(e^{j\Omega}\right)$
1
Ω
$-\pi$
-0.5π
0.5π
π

(e)



$Y\left(e^{j\Omega}\right)$
$\frac{1}{2}$
Ω
$-\pi$
-0.5π
0.5π
π

(f)



$Y\left(e^{j\Omega}\right)$
1
Ω
$-\pi$
-0.5π
0.5π
π

2.85

From the input/output relation we have

$$12Y\left(e^{j\Omega}\right) = 7e^{-j\Omega}Y\left(e^{j\Omega}\right) - e^{-j2\Omega}Y\left(e^{j\Omega}\right) - 12X\left(e^{j\Omega}\right) + 5e^{-j\Omega}X\left(e^{j\Omega}\right)$$

from which we obtain

$$\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}}$$

From the block diagram we have

$$\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = H_1\left(e^{j\Omega}\right) + H_2\left(e^{j\Omega}\right)$$

and we are given $H_1\left(e^{j\Omega}\right) = \dfrac{1}{1 - \frac{1}{3}e^{-j\Omega}} = \dfrac{3}{3 - e^{-j\Omega}}$.

Putting this all together gives,

$$\frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}} = H_1\left(e^{j\Omega}\right) + H_2\left(e^{j\Omega}\right) = \frac{3}{3 - e^{-j\Omega}} + H_2\left(e^{j\Omega}\right)$$

$$H_2\left(e^{j\Omega}\right) = \frac{-12 + 5e^{-j\Omega}}{12 - 7e^{-j\Omega} + e^{-j2\Omega}} - \frac{3}{3 - e^{-j\Omega}} = -\frac{8}{4 - e^{-j\Omega}} = -\frac{2}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$h_2(n) = -2\left(\frac{1}{4}\right)^n u(n)$$

2.86

$$Y(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

$$X(z) = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2} z^{-1}}{\left(1 - \frac{1}{3} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)} = 1 + \frac{-\frac{2}{3} z^{-1}}{1 - \frac{1}{3} z^{-1}} + \frac{\frac{3}{4} z^{-1}}{1 - \frac{1}{4} z^{-1}}$$

$$h(n) = \delta(n) - \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} u(n-1) + \frac{3}{4}\left(\frac{1}{4}\right)^{n-1} u(n-1)$$

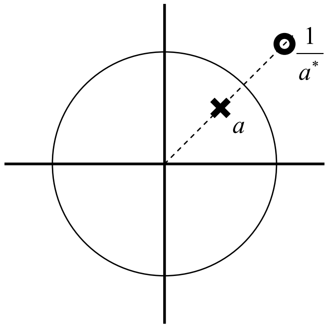
The frequency response is

$$H\left(e^{j\Omega}\right) = \frac{1 - \frac{1}{2} e^{-j\Omega}}{\left(1 - \frac{1}{3} e^{-j\Omega}\right)\left(1 - \frac{1}{4} e^{-j\Omega}\right)}$$

$$\left|H\left(e^{j\Omega}\right)\right|^2 = \frac{\frac{5}{4} - \cos(\Omega)}{\frac{97}{72} - \frac{91}{72}\cos(\Omega) + \frac{1}{6}\cos(2\Omega)}$$

2.87

(a)

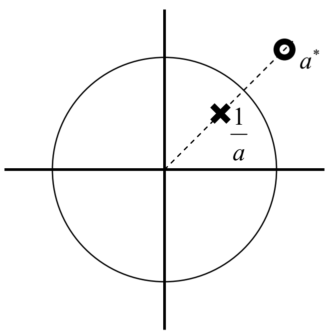


$$a = re^{j\theta} \quad \rightarrow \quad \frac{1}{a^*} = \frac{1}{r} e^{j\theta}$$

$$\left| H\left(e^{j\Omega}\right) \right|^2 = \frac{1 - \frac{1}{a} e^{j\Omega} - \frac{1}{a^*} e^{-j\Omega} + \frac{1}{|a|^2}}{1 - a^* e^{j\Omega} - a e^{-j\Omega} + |a|^2} = \frac{1 + \frac{1}{r^2} - \frac{2}{r} \cos(\Omega - \theta)}{1 + r^2 - 2r\cos(\Omega - \theta)} \neq 1$$

$$H(z) = \frac{1 - \frac{1}{a^*} z^{-1}}{1 - a z^{-1}}$$

(b)

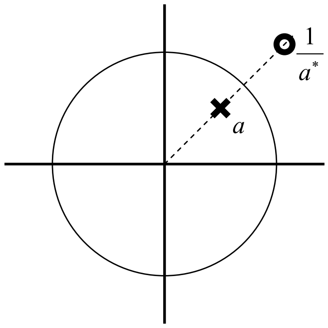


$$a = re^{j\theta} \quad \rightarrow \quad \frac{1}{a^*} = \frac{1}{r} e^{j\theta}$$

$$\left| H\left(e^{j\Omega}\right) \right|^2 = \frac{1 - a^* e^{-j\Omega} - a e^{j\Omega} + |a|^2}{1 - \frac{1}{a} e^{-j\Omega} - \frac{1}{a^*} e^{j\Omega} + \frac{1}{|a|^2}} = \frac{1 + r^2 - 2r\cos(\Omega + \theta)}{1 + \frac{1}{r^2} - \frac{2}{r} \cos(\Omega + \theta)} \neq 1$$

$$H(z) = \frac{1 - a^* z^{-1}}{1 - \frac{1}{a} z^{-1}}$$

(c)

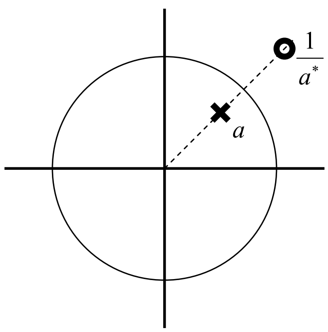


$$a = re^{j\theta} \quad \rightarrow \quad \frac{1}{a^*} = \frac{1}{r}e^{j\theta}$$

$$\left|H\left(e^{j\Omega}\right)\right|^2 = |a|^2 \frac{1 - \frac{1}{a}e^{j\Omega} - \frac{1}{a^*}e^{-j\Omega} + \frac{1}{|a|^2}}{1 - a^* e^{j\Omega} - ae^{-j\Omega} + |a|^2} = \frac{1 + r^2 - 2r\cos(\Omega - \theta)}{1 + r^2 - 2r\cos(\Omega - \theta)} = 1$$

$$H(z) = a\frac{1 - \frac{1}{a^*}z^{-1}}{1 - az^{-1}}$$

(d)



$$a = re^{j\theta} \quad \rightarrow \quad \frac{1}{a^*} = \frac{1}{r}e^{j\theta}$$

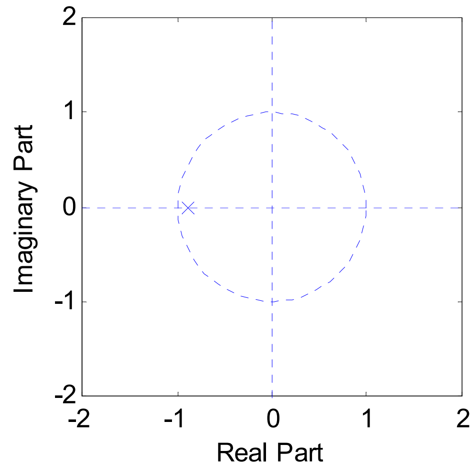
$$\left|H\left(e^{j\Omega}\right)\right|^2 = \frac{|a|^2 - a^* e^{j\Omega} - ae^{-j\Omega} + 1}{1 - a^* e^{j\Omega} - ae^{-j\Omega} + |a|^2} = \frac{1 + r^2 - 2r\cos(\Omega - \theta)}{1 + r^2 - 2r\cos(\Omega - \theta)} = 1$$

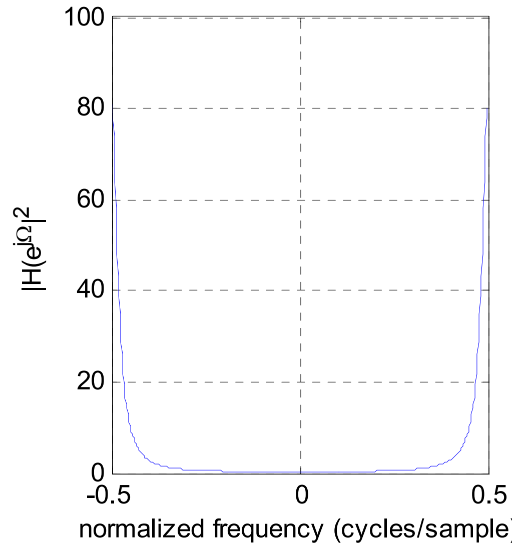
$$H(z) = \frac{a^* - z^{-1}}{1 - az^{-1}}$$

Both the systems of parts (c) and (d) are all-pass systems. However, only the system in part (d) has a z-transform in the standard form.

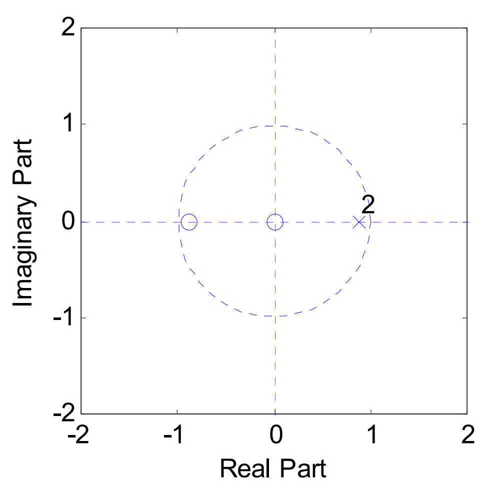
2.88

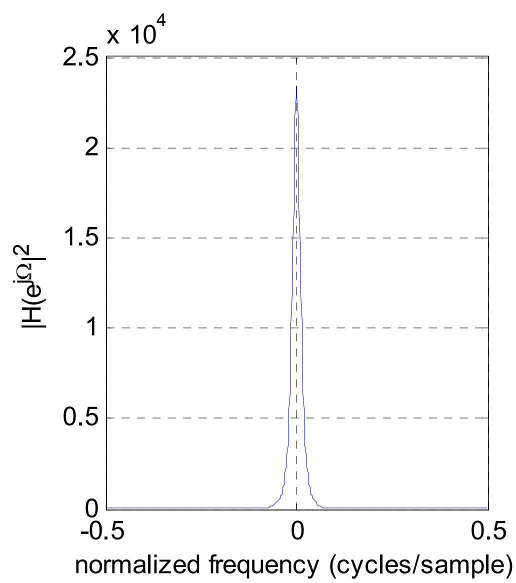
(a) The pole-zero plot (below) shows a pole at $z = -\frac{8}{9}$ which means the frequency response has a large amplitude in the vicinity of $\Omega = \pi$. Hence this is a high-pass filter as shown.



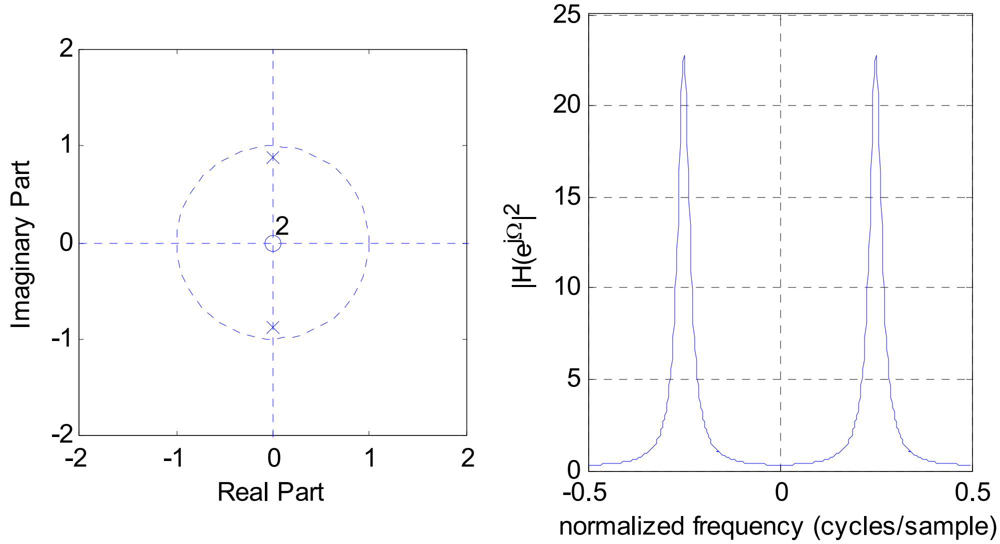


(b) The pole-zero plot (below) shows two poles at $z = \frac{8}{9}$ and a zero at $z = -\frac{8}{9}$ which means the frequency response has a large amplitude in the vicinity of $\Omega = 0$ and a small amplitude at $\Omega = \pi$. Hence this is a low-pass filter as shown.



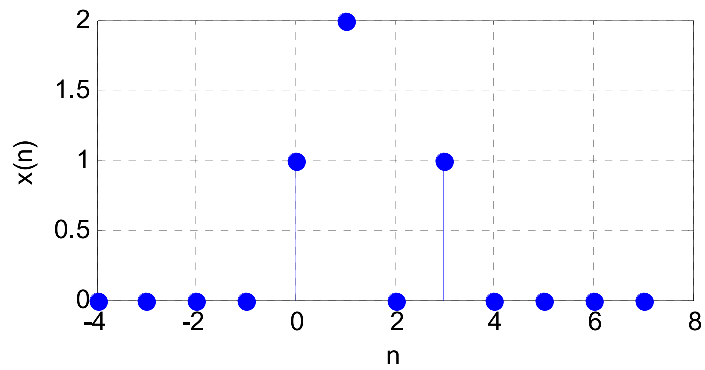


(c) The pole-zero plot (below) shows complex-conjugate poles at $z = \pm j\frac{8}{9}$ and a zero at the origin, which means the frequency response has a large amplitude in the vicinity of $\Omega = \frac{\pi}{2}$. Hence this is a band-pass filter as shown.



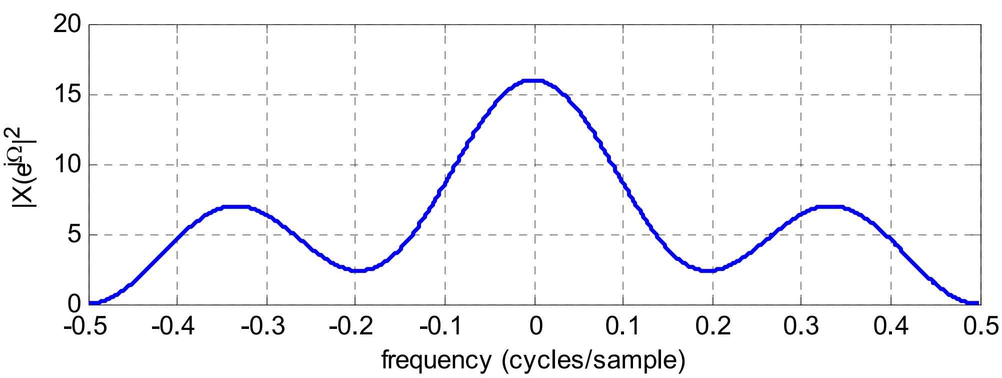
2.89

(a)

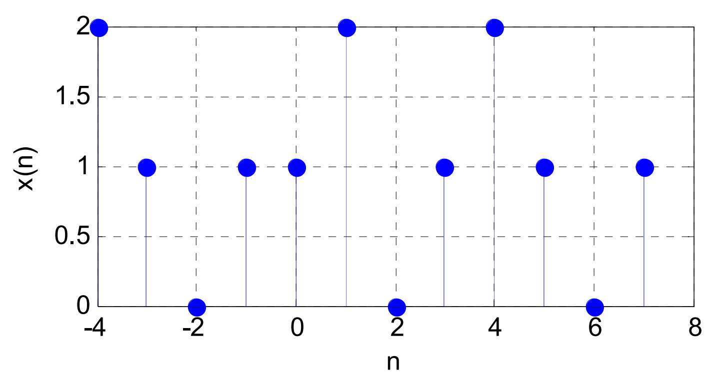


(b) $X\left(e^{j\Omega}\right)=2+e^{-j\Omega}+e^{-j3\Omega}$

$$\left|X\left(e^{j\Omega}\right)\right|^2=6+4\cos(\Omega)+2\cos(2\Omega)+4\cos(3\Omega)$$



(c)



(d) $X[0]=2+1+1=4$

$X[1]=2+e^{-j\pi/2}+e^{-j3\pi/2}=2$

$X[2]=2+e^{-j\pi}+3^{-j3\pi}=0$

$X[3]=2+e^{-j3\pi/2}+e^{-j9\pi/2}=2$

(e) $$X\left(e^{0}\right) = 2 + 1 + 1 = 4$$

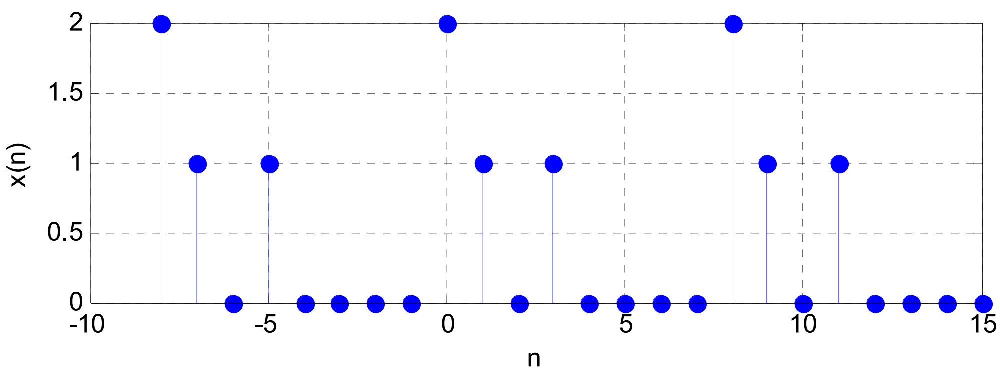
$$X\left(e^{j\pi/2}\right) = 2 + e^{-j\pi/2} + e^{-j3\pi/2} = 2$$

$$X\left(e^{j\pi}\right) = 2 + e^{-j\pi} + 3^{-j3\pi} = 0$$

$$X\left(e^{j3\pi/2}\right) = 2 + e^{-j3\pi/2} + e^{-j9\pi/2} = 2$$

These numbers are the same as those obtained in part (d).

(f)



(g) $$X[0] = 2 + 1 + 1 = 4$$

$$X[1] = 2 + e^{-j\pi/4} + e^{-j3\pi/4} = 2 - j\sqrt{2}$$

$$X[2] = 2 + e^{-j\pi/2} + e^{-j3\pi/2} = 2$$

$$X[3] = 2 + e^{-j3\pi/4} + e^{-j9\pi/4} = 2 - j\sqrt{2}$$

$$X[4] = 2 + e^{-j\pi} + 3^{-j3\pi} = 0$$

$$X[5] = 2 + e^{-j5\pi/4} + e^{-j15\pi/4} = 2 + j\sqrt{2}$$

$$X[6] = 2 + e^{-j3\pi/2} + e^{-j9\pi/2} = 2$$

$$X[7] = 2 + e^{-j7\pi/4} + e^{-j21\pi/4} = 2 + j\sqrt{2}$$

(h) $X\left(e^{j0}\right)=2+1+1=4$

$$X\left(e^{j\pi/4}\right)=2+e^{-j\pi/4}+e^{-j3\pi/4}=2-j\sqrt{2}$$

$$X\left(e^{j2\pi/4}\right)=2+e^{-j\pi/2}+e^{-j3\pi/2}=2$$

$$X\left(e^{j3\pi/4}\right)=2+e^{-j3\pi/4}+e^{-j9\pi/4}=2-j\sqrt{2}$$

$$X\left(e^{j4\pi/4}\right)=2+e^{-j\pi}+3^{-j3\pi}=0$$

$$X\left(e^{j5\pi/4}\right)=2+e^{-j5\pi/4}+e^{-j15\pi/4}=2+j\sqrt{2}$$

$$X\left(e^{j6\pi/4}\right)=2+e^{-j3\pi/2}+e^{-j9\pi/2}=2$$

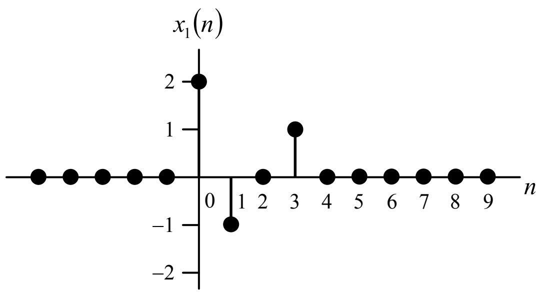
$$X\left(e^{j7\pi/4}\right)=2+e^{-j7\pi/4}+e^{-j21\pi/4}=2+j\sqrt{2}$$

These numbers are the same as those obtained in part (g).

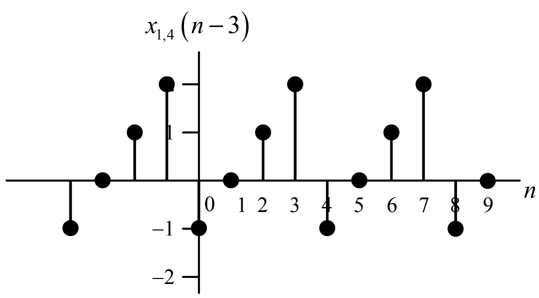
(i) The values obtained in part (d) are a subset of those obtained in part (g). This is because the DFT length of part (g) is divisible by the length of the DFT length of part (d).

2.90

Start with the sequence $x_1(n)$



Draw the length-4 periodic extension of $x_1(n)$ – call it $x_{1,4}(n)$ – and delay it by 3. This produces the sequence



Note that the samples at $n = 0,1,2,3$ are the same as those of $x_2(n)$.

Now the DTFT of $x_{1,4}(n-3)$ is $e^{-j3\frac{2\pi}{4}}$ times the DTFT of $x_1(n)$. Thus we have

$$X_{1,4}[k] = e^{-j\frac{3\pi}{2}k} X_1[k]$$

and

$$e^{-j\frac{3\pi}{2}k} X_1[k] = X_2[k]$$

Thus, $\left|X_2[k]\right| = \left|e^{-j\frac{3\pi}{2}k} X_1[k]\right| = \left|X_1[k]\right|$

2.91

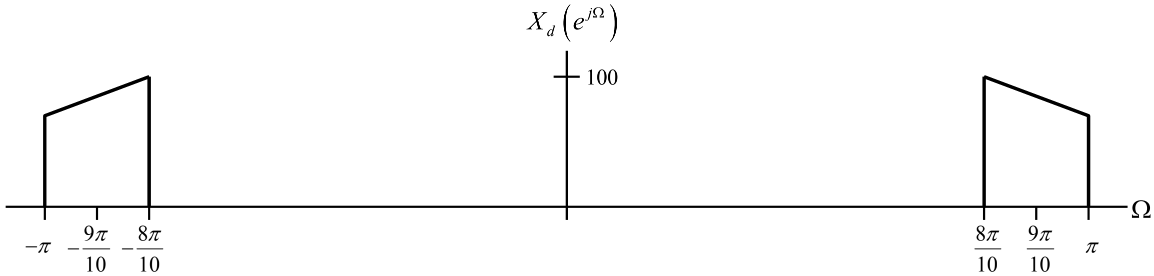
(a)

decimal numbers	3-bit binary equivalent	3-bit binary equivalent index of input
0	000	000
1	001	100
2	010	010
3	011	110
4	100	001
5	101	101
6	110	011
7	111	111

(b) The 3-binary equivalent of the indexes of the inputs are reversed (bit reversed) versions of the indexes associated with the natural (temporal) order.

2.92

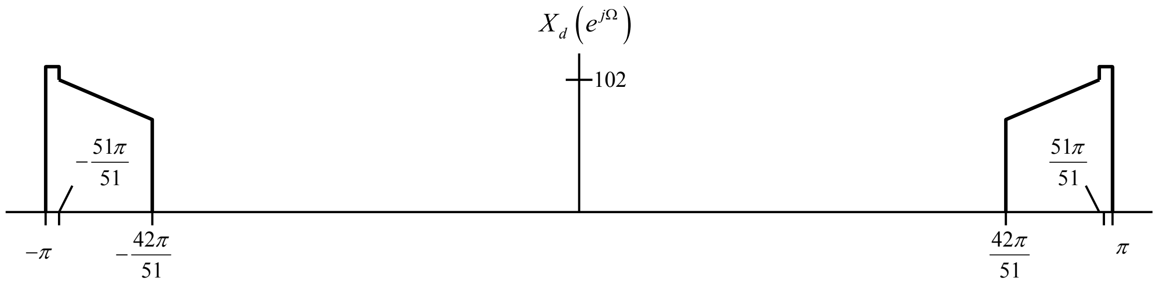
(a)



(b) The negative frequency component of $X_c(f)$ shows up as the positive frequency component of $X_d\left(e^{j\Omega}\right)$, and the positive frequency component of $X_c(f)$ shows up as the negative frequency component of $X_d\left(e^{j\Omega}\right)$.

2.93

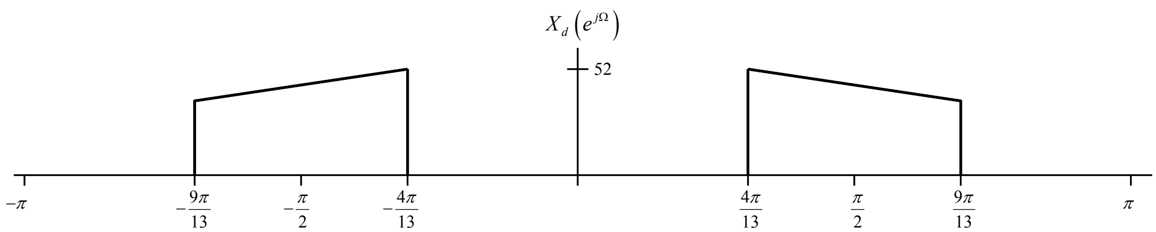
(a)



(b) The negative and positive frequency components of $X_c(f)$ are preserved on the negative and positive frequency components of $X_d\left(e^{j\Omega}\right)$, but there is aliasing.

2.94

(a)



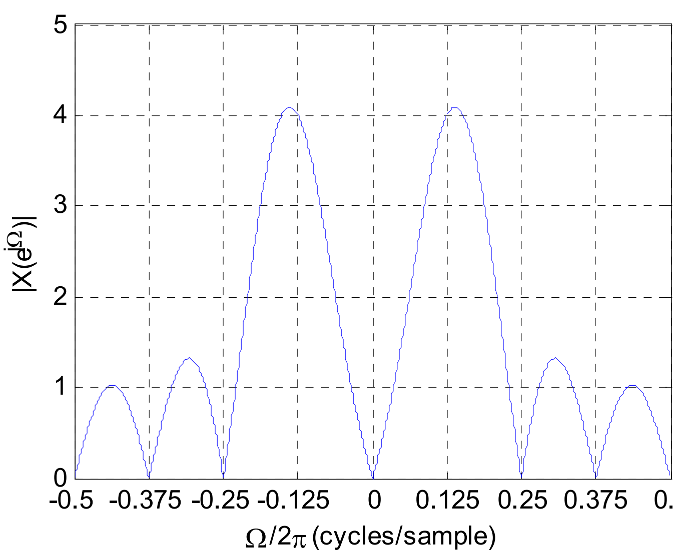
(b) The negative frequency component of $X_c(f)$ shows up as the positive frequency component of $X_d\left(e^{j\Omega}\right)$, and the positive frequency component of $X_c(f)$ shows up as the negative frequency component of $X_d\left(e^{j\Omega}\right)$.

2.95

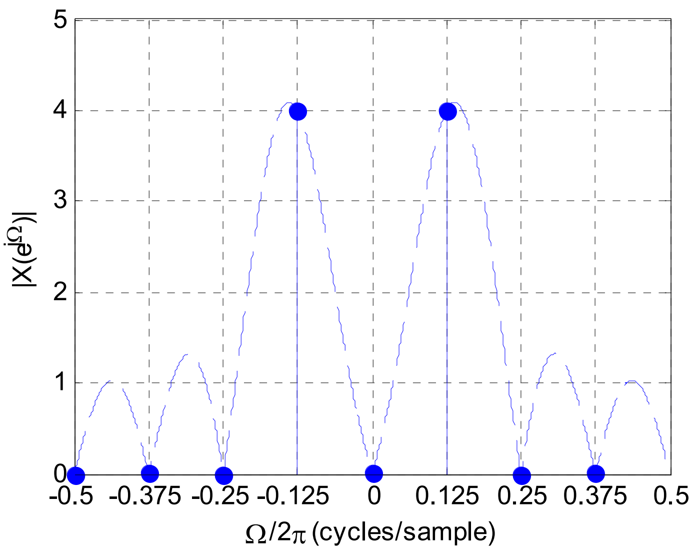
(a) Write $x(n) = w(n)\cos\left(\frac{\pi}{4}n\right)$ where $w(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$.

Then

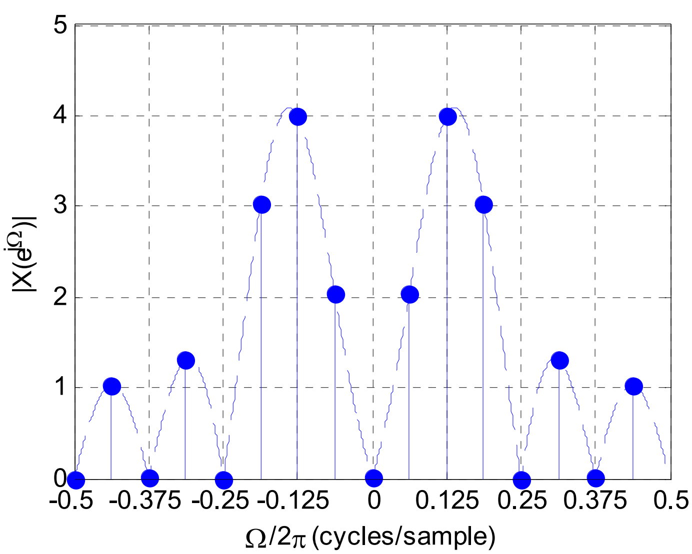
$$\begin{aligned} X\left(e^{j\Omega}\right) &= \frac{1}{2\pi} W\left(e^{j\Omega}\right) * \pi\left[\delta\left(\Omega - \frac{\pi}{4}\right) + \delta\left(\Omega + \frac{\pi}{4}\right)\right] \\ &= \frac{1}{2\pi} \frac{\sin(4\Omega)}{\sin\left(\frac{\Omega}{2}\right)} e^{-j\frac{7}{2}\Omega} * \pi\left[\delta\left(\Omega - \frac{\pi}{4}\right) + \delta\left(\Omega + \frac{\pi}{4}\right)\right] \\ &= \frac{1}{2} \frac{\sin\left(4\left(\Omega - \frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega - \frac{\pi}{4}\right)\right)} e^{-j\frac{7}{2}\left(\Omega - \frac{\pi}{4}\right)} + \frac{1}{2} \frac{\sin\left(4\left(\Omega + \frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega + \frac{\pi}{4}\right)\right)} e^{-j\frac{7}{2}\left(\Omega + \frac{\pi}{4}\right)} \end{aligned}$$



(b)

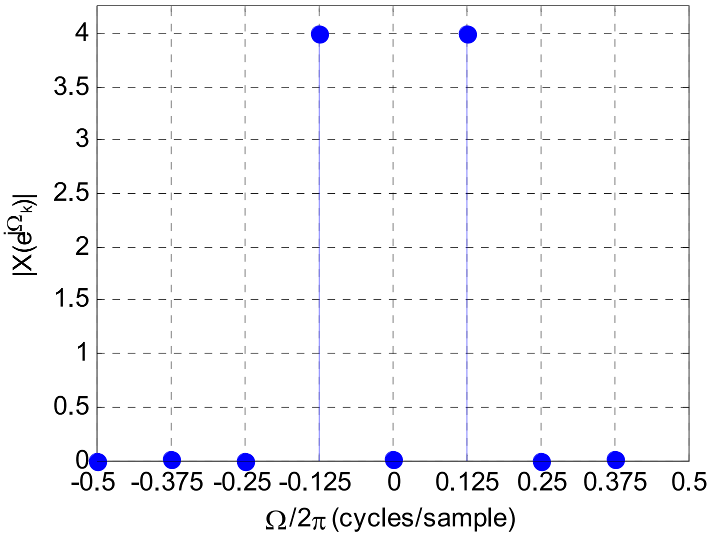

5
4
3
2
1
0
|X(e^{jΩ})|
-0.5
-0.375
-0.25
-0.125
0
0.125
0.25
0.375
0.5
Ω/2π (cycles/sample)

(c)

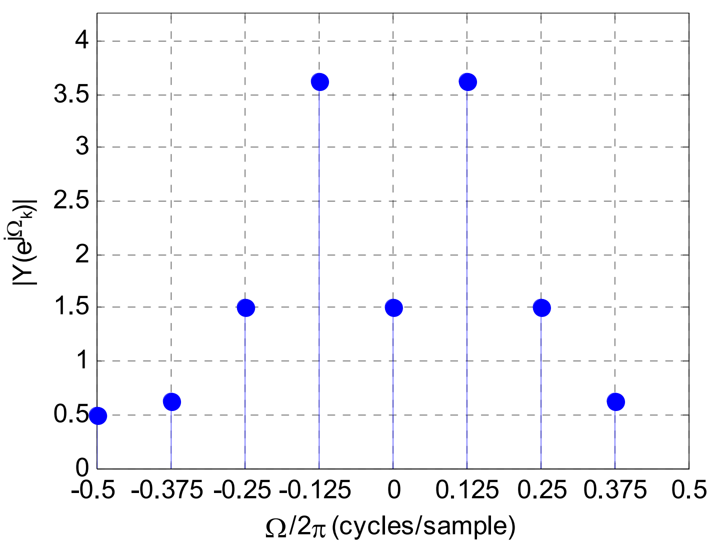

5
4
3
2
1
0
|X(e^{jΩ})|
-0.5
-0.375
-0.25
-0.125
0
0.125
0.25
0.375
0.5
Ω/2π (cycles/sample)

2.96

(a)



(b)



(c) The DTFT of $x(n)$ is

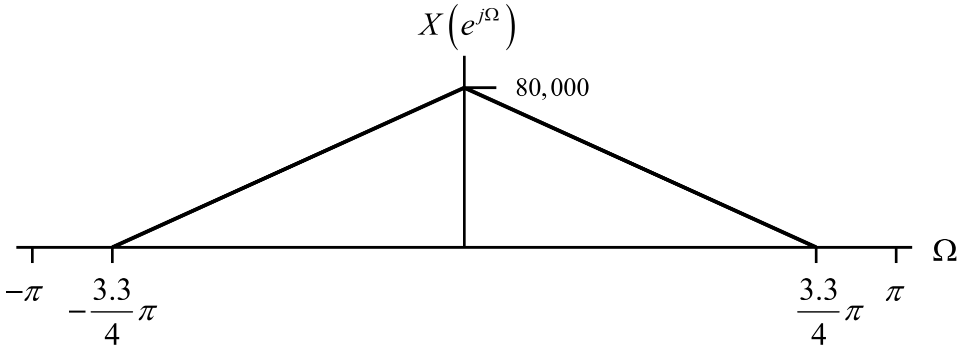
$$X\left(e^{j\Omega}\right)=\frac{1}{2}\frac{\sin\left(4\left(\Omega-\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega-\frac{\pi}{4}\right)\right)}e^{-j\frac{7}{2}\left(\Omega-\frac{\pi}{4}\right)}+\frac{1}{2}\frac{\sin\left(4\left(\Omega+\frac{\pi}{4}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega+\frac{\pi}{4}\right)\right)}e^{-j\frac{7}{2}\left(\Omega+\frac{\pi}{4}\right)}$$

which has zero-crossings at $\Omega=(l\pm 1)\pi$ for $l\neq 0$. The length-8 DFT samples $X\left(e^{j\Omega}\right)$ at $\Omega=\frac{k\pi}{4}$ which corresponds to the zero-crossings. On the other hand, the DTFT of $y(n)$ is

$$Y\left(e^{j\Omega}\right)=\frac{1}{2}\frac{\sin\left(4\left(\Omega-\frac{\pi}{3}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega-\frac{\pi}{3}\right)\right)}e^{-j\frac{7}{2}\left(\Omega-\frac{\pi}{3}\right)}+\frac{1}{2}\frac{\sin\left(4\left(\Omega+\frac{\pi}{3}\right)\right)}{\sin\left(\frac{1}{2}\left(\Omega+\frac{\pi}{3}\right)\right)}e^{-j\frac{7}{2}\left(\Omega+\frac{\pi}{3}\right)}$$

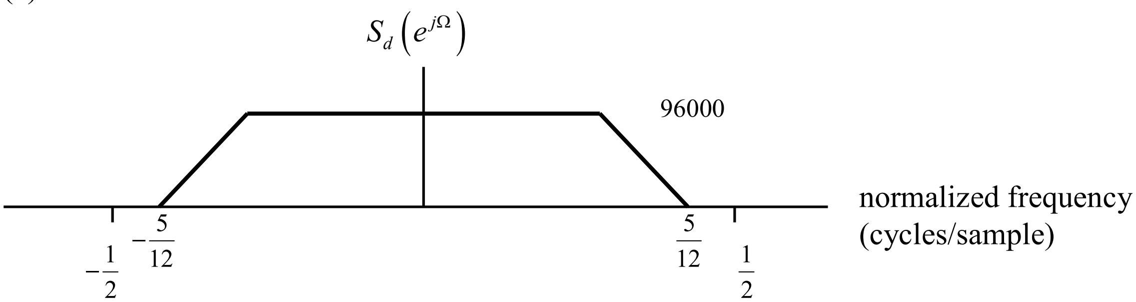
which has zero-crossings at $\Omega=\frac{3l\pm 4}{12}\pi$ for $l\neq 0$. The length-8 DFT samples $X\left(e^{j\Omega}\right)$ at $\Omega=\frac{k\pi}{4}$ which does not correspond to any of the zero-crossings.

2.97

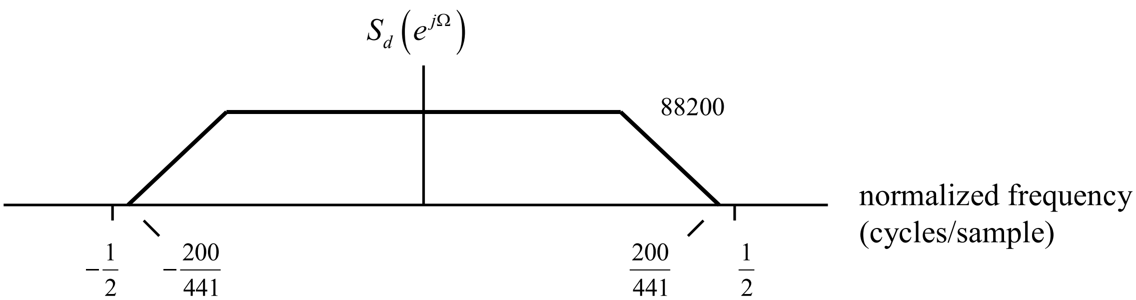


2.98

(a)



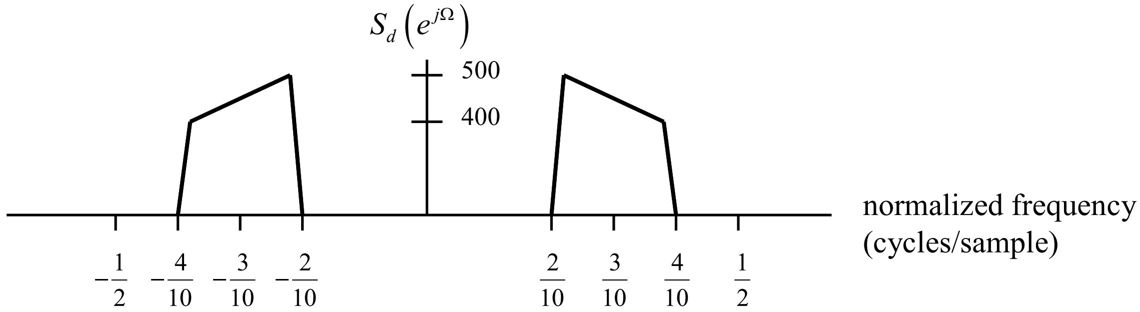
(b)



(c)

The DTFT in part (b) is wider than the DTFT in part (a). This is due to the fact that the sample rate used in part (b) is closer to the minimum sample rate defined in the sampling theorem than the sample rate used in part (a).

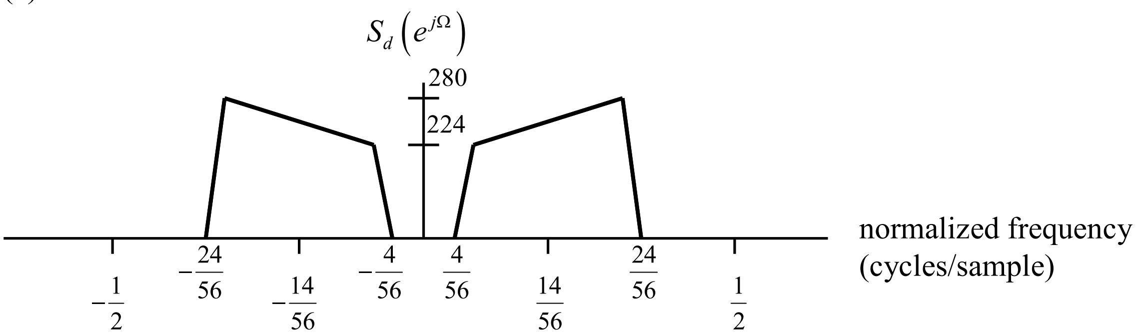
2.99



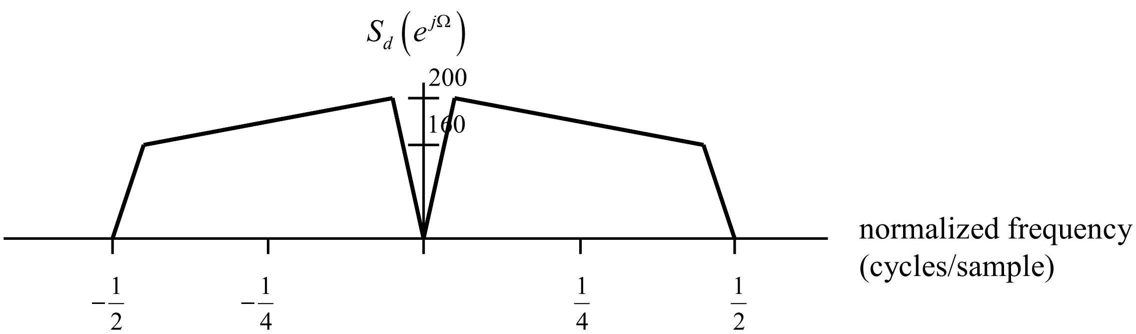
$S_d\left(e^{j\Omega}\right)$
500
400
normalized frequency
(cycles/sample)
$-\frac{1}{2}$
$-\frac{4}{10}$
$-\frac{3}{10}$
$-\frac{2}{10}$
$\frac{2}{10}$
$\frac{3}{10}$
$\frac{4}{10}$
$\frac{1}{2}$

2.100

(a)



(b)

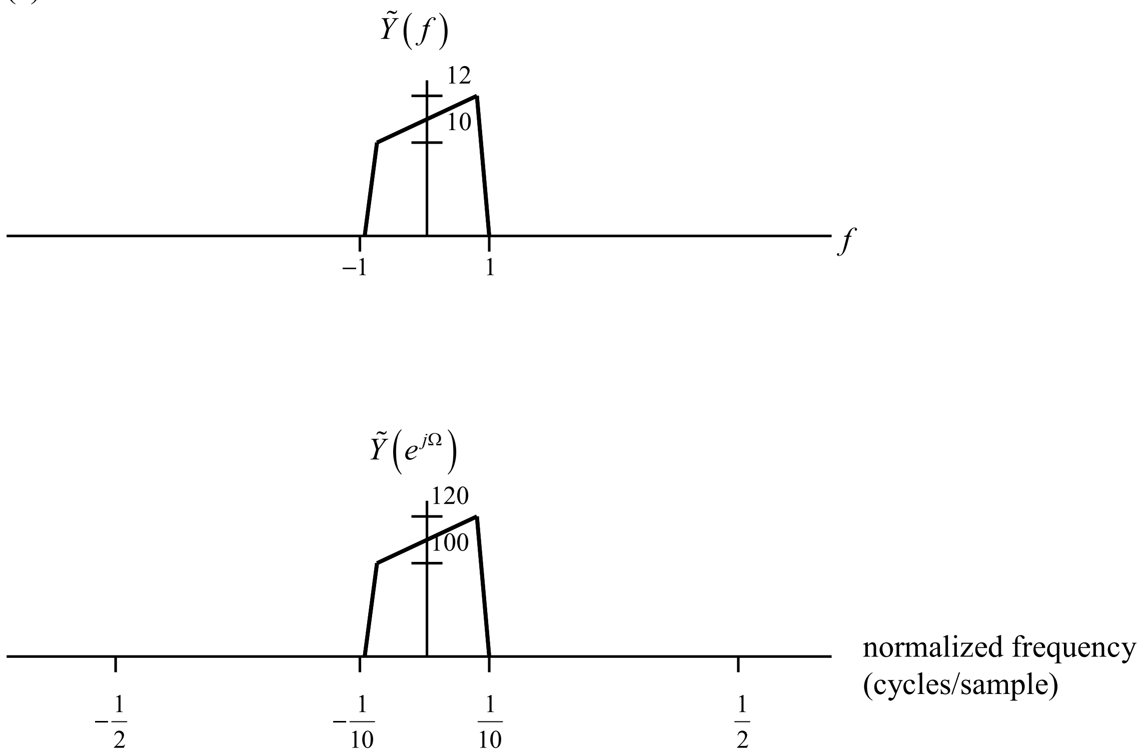


(c)

There are two key differences between the spectra in parts (a) and (b).The first difference is the bandwidth: the spectrum in part (a) is the spectrum of an oversampled signal whereas the spectrum in part (b) is the spectrum of a critically sampled signal (that is, it is sampled at the minimum sample rate defined by the sampling theorem). The second difference is the fact that the two spectra are "flipped" relative to each other. In part (a) the positive frequency component of the continuous-time signal shows up on the positive frequency axis in the DTFT domain. In part (b) the negative frequency component of the continuous-time signals shows up on the positive frequency axis in the DTFT domain.

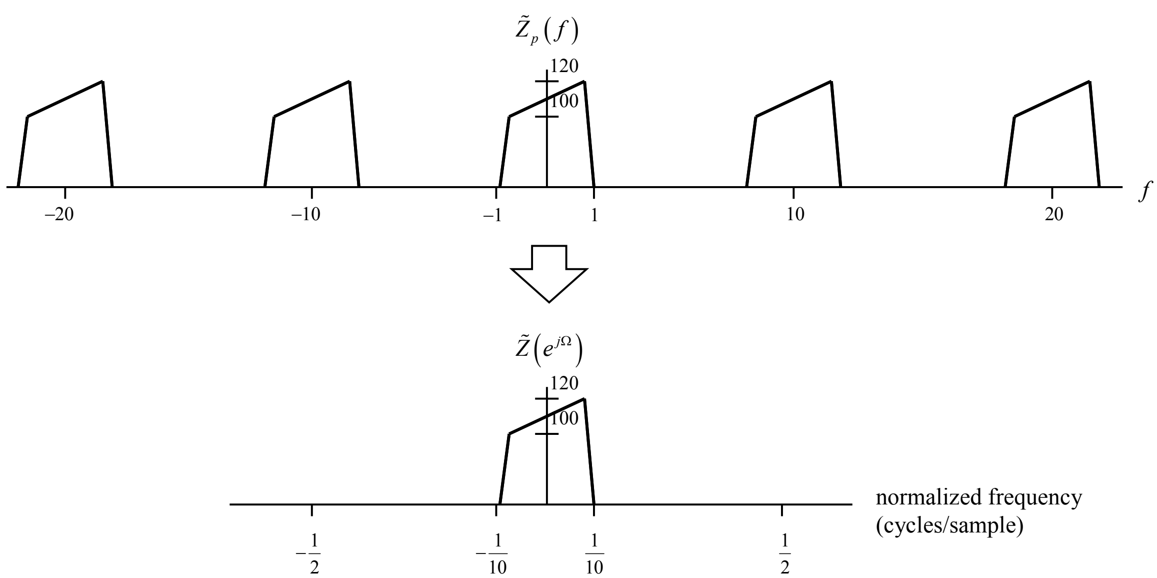
2.101

(a)



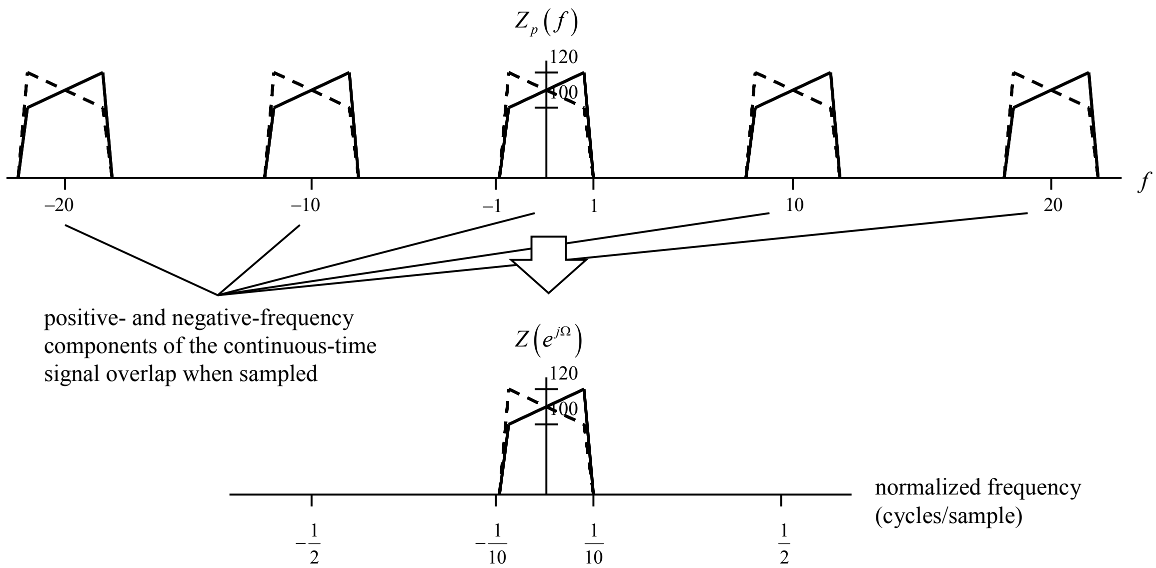
$\tilde{Y}(f)$
12
10
f
-1
1
$\tilde{Y}(e^{j\Omega})$
120
100
normalized frequency
(cycles/sample)
$-\frac{1}{2}$
$-\frac{1}{10}$
$\frac{1}{10}$
$\frac{1}{2}$

(b)



$\tilde{Z}_p(f)$
120
100
f
-20
-10
-1
1
10
20
$\tilde{Z}(e^{j\Omega})$
120
100
normalized frequency
(cycles/sample)
$-\frac{1}{2}$
$-\frac{1}{10}$
$\frac{1}{10}$
$\frac{1}{2}$

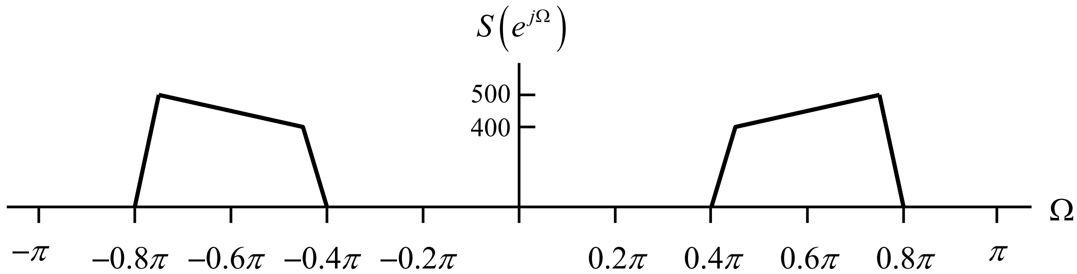
(c)



The fundamental problem with this approach is that the positive- and negative-frequency components of the $z(t)$ overlap when sampled. This was not a problem when using the complex-valued version of the signal. This problem illustrates one of the advantages of signal-processing using complex-valued signals: one does not have to worry about negative- and positive-frequency components aliasing on top of each other. The disadvantage is complexity: a complex-valued signal is a “two-dimensional” signal. Consequently, addition and multiplication require more resources.

2.102

The DTFT of the sampled sequence is



Neglecting scaling constants, the sampled sequence may be written as (see Exericse 2.48)

$$s(nT) = I(nT)\cos(0.6\pi n) + Q(nT)\sin(0.6\pi n).$$

To produce $I(nT)$ and $Q(nT)$ from $s(nT)$, we need $\Omega_0 = 0.6\pi$.

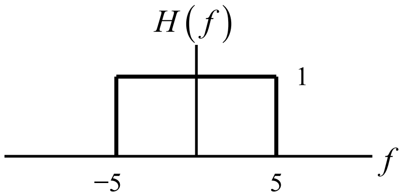
As it is written the system produces

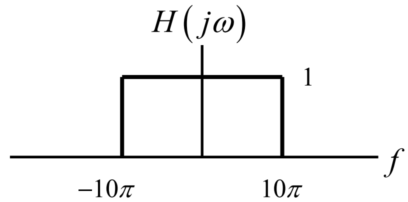
$$x(nT) = \frac{1}{2}I(nT)$$
$$y(nT) = -\frac{1}{2}Q(nT)$$

Hence the system must be modified either by changing the sign on $y(nT)$ or by changing the sign on $\sin(\Omega_0 n)$ -- that is, use $\sin(\Omega_0 n)$ in place of $-\sin(\Omega_0 n)$ on the lower mixer.

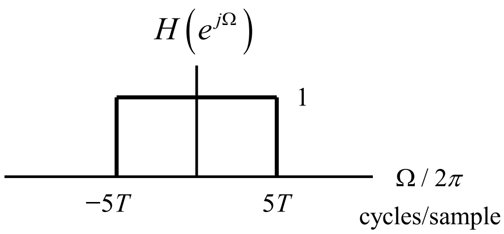
2.103

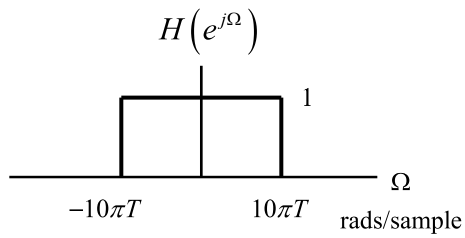
(a)



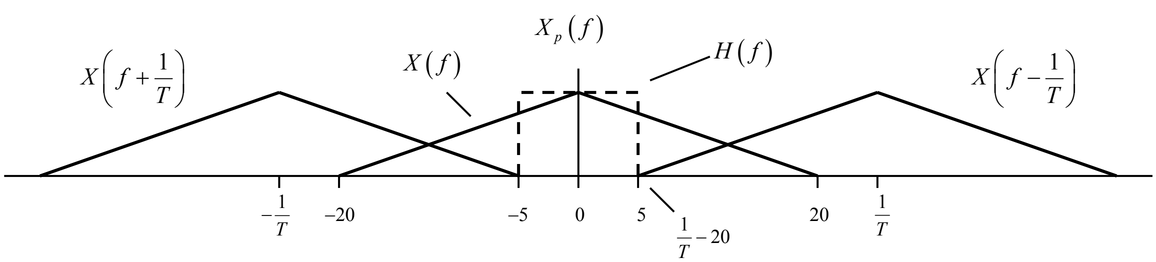


(b)





(c) Construct the diagram of the impulse sampled waveform shown below



From this diagram, we see that the minimum sampling rate is determined from relation

$\dfrac{1}{T}-20>5$ from which we obtain $\dfrac{1}{T}>25$.

2.104

(a) $H_c\left(j\omega\right) = e^{j\omega\Delta T}$

(b) In the interval $-\pi \le \Omega \le \pi$ we have

$$H_d\left(e^{j\Omega}\right) = \begin{cases} e^{-j\frac{\Omega}{T}\Delta T} & -W \le \Omega \le W \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$h_d(n) = \frac{1}{2\pi}\int_{-W}^{W} e^{-j\frac{\Omega}{T}\Delta T} e^{j\Omega n}\, d\Omega = \frac{W}{\pi}\frac{\sin\left(\left(n-\frac{\Delta T}{T}\right)W\right)}{\left(n-\frac{\Delta T}{T}\right)W}$$

(d)

$$W = \pi \quad \rightarrow \quad h_d(n) = \frac{\sin\left(\left(n-\frac{\Delta T}{T}\right)\pi\right)}{\left(n-\frac{\Delta T}{T}\right)\pi}$$

(e)

$$W = \pi, \frac{\Delta T}{T} = \frac{1}{2} \quad \rightarrow \quad h_d(n) = \frac{\sin\left(\left(n-\frac{1}{2}\right)\pi\right)}{\left(n-\frac{1}{2}\right)\pi}$$