PART A

Ordinary Differential Equations (ODEs)

Chap. 1 First-Order ODEs

Sec. 1.1 Basic Concepts. Modeling

To get a good start into this chapter and this section, quickly **review your basic calculus**. Take a look at the front matter of the textbook and see a review of the main differentiation and integration formulas. Also, Appendix 3, pp. A63–A66, has useful formulas for such functions as exponential function, logarithm, sine and cosine, etc. The beauty of ordinary differential equations is that the subject is quite systematic and has different methods for different types of ordinary differential equations, as you shall learn. Let us discuss some Examples of Sec. 1.1, pp. 4–7.

Example 2, p. 5. Solution by Calculus. Solution Curves. To solve the first-order ordinary differential equation (ODE)

$$y' = \cos x$$

means that we are looking for a function whose derivative is $\cos x$. Your first answer might be that the desired function is $\sin x$, because $(\sin x)' = \cos x$. But your answer would be incomplete because also $(\sin x + 2)' = \cos x$, since the derivative of 2 and of any constant is 0. Hence the complete answer is $y = \cos x + c$, where c is an arbitrary constant. As you vary the constants you get an infinite family of solutions. Some of these solutions are shown in **Fig. 3**. The lesson here is that you should never forget your constants!

Example 4, pp. 6–7. Initial Value Problem. In an initial value problem (IVP) for a first-order ODE we are given an ODE, here $y' = 3y$, and an initial value condition $y(0) = 5.7$. For such a problem, the first step is to solve the ODE. Here we obtain $y(x) = ce^{3x}$ as shown in **Example 3**, p. 5. Since we also have an initial condition, we must substitute that condition into our solution and get $y(0) = ce^{3\cdot 0} = ce^0 = c \cdot 1 = c = 5.7$. Hence the complete solution is $y(x) = 5.7e^{3x}$. The lesson here is that for an initial value problem you get a unique solution, also known as a particular solution.

Modeling means that you interpret a physical problem, set up an appropriate mathematical model, and then try to solve the mathematical formula. Finally, you have to interpret your answer. Examples 3 (exponential growth, exponential decay) and 5 (radioactivity) are examples of modeling problems. Take a close look at **Example 5**, p. 7, because it outlines all the steps of modeling.

Problem Set 1.1. Page 8

3. Calculus. From Example 3, replacing the independent variable t by x we know that $y' = 0.2y$ has a solution $y = 0.2ce^{0.2x}$. Thus by analogy, $y' = y$ has a solution

$$1 \cdot ce^{1 \cdot x} = ce^x,$$

where c is an arbitrary constant.

Another approach (to be discussed in details in Sec. 1.3) is to write the ODE as

$$\frac{dy}{dx} = y,$$

and then by algebra obtain

$$dy = y\,dx, \quad \text{so that} \quad \frac{1}{y}dy = dx.$$

Integrate both sides, and then apply exponential functions on both sides to obtain the same solution as above

$$\int \frac{1}{y}dy = \int dx, \qquad \ln|y| = x + c, \qquad e^{\ln|y|} = e^{x+c}, \qquad y = e^x \cdot e^c = c^*e^x,$$

(where $c^* = e^c$ is a constant).

The technique used is called **separation of variables** because we separated the variables, so that y appeared on one side of the equation and x on the other side before we integrated.

7. Solve by integration. Integrating $y' = \cosh 5.13x$ we obtain (chain rule!) $y = \int \cosh 5.13x\,dx = \frac{1}{5.13}(\sinh 5.13x) + c$. Check: Differentiate your answer:

$$\left(\tfrac{1}{5.13}(\sinh 5.13x) + c\right)' = \tfrac{1}{5.13}(\cosh 5.13x) \cdot 5.13 = \cosh 5.13x, \text{ which is correct.}$$

11. Initial value problem (IVP). (a) Differentiation of $y = (x + c)e^x$ by product rule and definition of y gives

$$y' = e^x + (x + c)e^x = e^x + y.$$

But this looks precisely like the given ODE $y' = e^x + y$. Hence we have shown that indeed $y = (x + c)e^x$ is a solution of the given ODE. **(b)** Substitute the initial value condition into the solution to give $y(0) = (0 + c)e^0 = c \cdot 1 = \frac{1}{2}$. Hence $c = \frac{1}{2}$ so that the answer to the IVP is

$$y = (x + \tfrac{1}{2})e^x.$$

(c) The graph intersects the x-axis at $x = 0.5$ and shoots exponentially upward.

19. Modeling: Free Fall. $y'' = g = \text{const}$ is the model of the problem, an ODE of second order. Integrate on both sides of the ODE with respect to t and obtain the velocity $v = y' = gt + c_1$ (c_1 arbitrary). Integrate once more to obtain the distance fallen $y = \frac{1}{2}gt^2 + c_1 t + c_2$ (c_2 arbitrary). To do these steps, we used calculus. From the last equation we obtain $y = \frac{1}{2}gt^2$ by imposing the initial conditions $y(0) = 0$ and $y'(0) = 0$, arising from the stone starting at rest at our choice of origin, that is the initial position is $y = 0$ with initial velocity 0. From this we have $y(0) = c_2 = 0$ and $v(0) = y'(0) = c_1 = 0$.

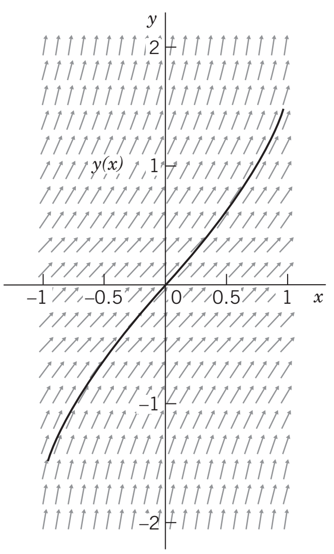
Sec. 1.2 Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method

Problem Set 1.2. Page 11

1. Direction field, verification of solution. You may verify by differentiation that the general solution is $y = \tan(x + c)$ and the particular solution satisfying $y(\frac{1}{4}\pi) = 1$ is $y = \tan x$. Indeed, for the particular solution you obtain

$$y' = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1 + \tan^2 x = 1 + y^2$$

and for the general solution the corresponding formula with x replaced by $x + c$.



Sec. 1.2 Prob. 1. Direction Field

15. Initial value problem. Parachutist. In this section the usual notation is (1), that is, $y' = f(x, y)$, and the direction field lies in the xy-plane. In Prob. 15 the ODE is $v = f(t, v) = g - bv^2/m$, where v suggests velocity. Hence the direction field lies in the tv-plane. With $m = 1$ and $b = 1$ the ODE becomes $v' = g - v^2$. To find the limiting velocity we find the velocity for which the acceleration equals zero. This occurs when $g - v^2 = 9.80 - v^2 = 0$ or $v = 3.13$ (approximately). For $v < 3.13$ you have $v' > 0$ (increasing curves) and for $v > 3.13$ you have $v' < 0$ (decreasing curves). Note that the isoclines are the horizontal parallel straight lines $g - v^2 = \text{const}$, thus $v = \text{const}$.

Sec. 1.3 Separable ODEs. Modeling

Problem Set 1.3. Page 18

1. CAUTION! Constant of integration. It is important to introduce the constant of integration immediately, in order to avoid getting the wrong answer. For instance, let

$$y' = y. \qquad \text{Then} \qquad \ln|y| = x + c, \qquad y = c^* e^x \qquad (c^* = e^c),$$

which is the correct way to do it (the same as in Prob. 3 of Sec. 1.1 above) whereas introducing the constant of integration later yields

$$y' = y, \qquad \ln|y| = x, \qquad y = e^x + C$$

which is not a solution of $y' = y$ when $C \neq 0$.

5. General solution. Separating variables, we have $y\,dy = -36x\,dx$. By integration,

$$\tfrac{1}{2}y^2 = -18x^2 + \tilde{c}, \qquad y^2 = 2\tilde{c} - 36x^2, \qquad y = \pm\sqrt{c - 36x^2} \;\; (c = 2\tilde{c}).$$

With the plus sign of the square root we get the upper half and with the minus sign the lower half of the ellipses in the answer on p. A4 in Appendix 2 of the textbook.

For $y = 0$ (the x-axis) these ellipses have a vertical tangent, so that at points of the x-axis the derivative y' does not exist (is infinite).

17. Initial value problem. Using the extended method (8)–(10), let $u = y/x$. Then by product rule $y' = u + xu'$. Now

$$y' = \frac{y + 3x^4\cos^2(y/x)}{x} = \frac{y}{x} + 3x^3\cos\left(\frac{y}{x}\right) = u + 3x^3\cos^2 u = u + x(3x^2\cos^2 u)$$

so that $u' = 3x^2\cos^2 u$.

Separating variables, the last equation becomes

$$\frac{du}{\cos^2 u} = 3x^2 dx.$$

Integrate both sides, on the left with respect to u and on the right with respect to x, as justified in the text then solve for u and express the intermediate result in terms of x and y

$$\tan u = x^3 + c, \qquad u = \frac{y}{x} = \arctan(x^3 + c), \qquad y = xu = x\arctan(x^3 + c).$$

Substituting the initial condition into the last equation, we have

$$y(1) = 1\arctan(1^3 + c) = 0, \quad \text{hence} \quad c = -1.$$

Together we obtain the answer

$$y = x\arctan(x^3 - 1).$$

23. Modeling. Boyle–Mariotte's law for ideal gases. From the given information on the rate of change of the volume

$$\frac{dV}{dP} = -\frac{V}{P}.$$

Separating variables and integrating gives

$$\frac{dV}{V} = -\frac{dP}{P}, \qquad \int \frac{1}{V} dV = -\int \frac{1}{P} dP, \qquad \ln |V| = -\ln |P| + c.$$

Applying exponents to both sides and simplifying

$$e^{\ln |V|} = e^{-\ln |P| + c} = e^{-\ln |P|} \cdot e^c = \frac{1}{e^{\ln |P|}} \cdot e^c = \frac{1}{|P|} e^c.$$

Hence we obtain for nonnegative V and P the desired law (with $c^* = e^c$, a constant)

$$V \cdot P = c^*.$$

Sec. 1.4 Exact ODEs. Integrating Factors

Use (6) or (6*), on p. 22, only if inspection fails. Use only one of the two formulas, namely, that in which the integration is simpler. For integrating factors try both Theorems 1 and 2, on p. 25. Usually only one of them (or sometimes neither) will work. There is no completely systematic method for integrating factors, but these two theorems will help in many cases. Thus this section is slightly more difficult.

Problem Set 1.4. Page 26

1. Exact ODE. We proceed as in Example 1 of Sec. 1.4. We can write the given ODE as

$$M\,dx + N\,dy = 0 \qquad \text{where } M = 2xy \text{ and } N = x^2.$$

Next we compute $\frac{\partial M}{\partial y} = 2x$ (where, when taking this partial derivative, we treat x as if it were a constant) and $\frac{\partial N}{\partial x} = 2x$ (we treat y as if it were a constant). (See Appendix A3.2 for a review of partial derivatives.) This shows that the ODE is exact by (5) of Sec. 1.4. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int 2xy\,dx + k(y) = x^2 y + k(y).$$

To find $k(y)$ we differentiate this formula with respect to y and use (4b) to obtain

$$\frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy} = N = x^2.$$

From this we see that

$$\frac{dk}{dy} = 0, \qquad k = \text{const.}$$

The last equation was obtained by integration. Insert this into the equation for u, compare with (3) of Sec. 1.4, and obtain $u = x^2 y + c^*$. Because u is a constant, we have

$$x^2 y = c, \qquad \text{hence} \qquad y = c/x^2.$$

5. Nonexact ODE. From the ODE, we see that $P = x^2 + y^2$ and $Q = 2xy$. Taking the partials we have $\dfrac{\partial P}{\partial y} = 2y$ and $\dfrac{\partial Q}{\partial x} = -2y$ and, since they are not equal to each other, the ODE is nonexact. Trying Theorem 1, p. 25, we have

$$R = \frac{(\partial P/\partial y - \partial Q/\partial x)}{Q} = \frac{2y + 2y}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x}$$

which is a function of x only so, by (17), we have $F(x) = \exp \int R(x)\, dx$. Now

$$\int R(x)\, dx = -2 \int \frac{1}{x} dx = -2 \ln x = \ln (x^{-2}) \qquad \text{so that} \qquad F(x) = x^{-2}.$$

Then

$$M = FP = 1 + x^{-2}y^2 \quad \text{and} \quad N = FQ = -2x^{-1}y. \quad \text{Thus} \quad \frac{\partial M}{\partial y} = 2x^{-2}y = \frac{\partial N}{\partial x}.$$

This shows that multiplying by our integrating factor produced an exact ODE. We solve this equation using 4(b), p. 21. We have

$$u = \int -2x^{-1}y\, dy = -2x^{-1} \int y\, dy = -x^{-1}y^2 + k(x).$$

From this we obtain

$$\frac{\partial u}{\partial x} = x^{-2}y^2 + \frac{dk}{dx} = M = 1 + x^{-2}y^2, \quad \text{so that} \quad \frac{dk}{dx} = 1 \quad \text{and} \quad k = \int dx = x + c^*.$$

Putting k into the equation for u, we obtain

$$u(x,y) = -x^{-1}y^2 + x + c^* \qquad \text{and putting it in the form of (3)} \qquad u = -x^{-1}y^2 + x = c.$$

Solving explicitly for y requires that we multiply both sides of the last equation by x, thereby obtaining (with our constant $= -$constant on p. A5)

$$-y^2 + x^2 = cx, \qquad y = (x^2 - cx)^{1/2}.$$

9. Initial value problem. In this section we usually obtain an implicit rather than an explicit general solution. The point of this problem is to illustrate that in solving initial value problems, one can proceed directly with the implicit solution rather than first converting it to explicit form.

The given ODE is exact because (5) gives

$$M_y = \frac{\partial}{\partial y}(2e^{2x} \cos y) = -2e^{2x} \sin y = N_x.$$

From this and (6) we obtain, by integration,

$$u = \int M\, dx = \int 2\, e^{2x} \cos y\, dx = e^{2x} \cos y + k(y).$$

$u_y = N$ now gives

$$u_y = -e^{2x} \sin y + k'(y) = N, \qquad k'(y) = 0, \qquad k(y) = c^* = \text{const.}$$

Hence an implicit general solution is

$$u = e^{2x} \cos y = c.$$

To obtain the desired particular solution (the solution of the initial value problem), simply insert $x = 0$ and $y = 0$ into the general solution obtained:

$$e^0 \cos 0 = 1 \cdot 1 = c.$$

Hence $c = 1$ and the answer is

$$e^{2x} \cos y = 1.$$

This implies

$$\cos y = e^{-2x}, \qquad \text{thus the explicit form} \qquad y = \arccos(e^{-2x}).$$

15. Exactness. We have $M = ax + by$, $N = kx + ly$. The answer follows from the exactness condition (5), p. 21. The calculation is

$$M_y = b = N_x = k, \qquad M = ax + ky, \qquad u = \int M\,dx = \frac{1}{2}ax^2 + kxy + \kappa(y)$$

with $\kappa(y)$ to be determined from the condition

$$u_y = kx + \kappa'(y) = N = kx + ly, \qquad \text{hence} \qquad \kappa' = ly.$$

Integration gives $\kappa = \frac{1}{2}ly^2$. With this κ, the function u becomes

$$u = \tfrac{1}{2}ax^2 + kxy + \tfrac{1}{2}ly^2 = \text{const.}$$

(If we multiply the equation by a factor 2, for beauty, we obtain the answer on p. A5).

Sec. 1.5 Linear ODEs. Bernoulli Equation. Population Dynamics

Example 3, pp. 30–31. Hormone level. The integral

$$I = \int e^{Kt} \cos \frac{\pi t}{12}\,dt$$

can be evaluated by integration by parts, as is shown in calculus, or, more simply, by undetermined coefficients, as follows. We start from

$$\int e^{Kt} \cos \frac{\pi t}{12} dt = e^{Kt}\left(a \cos \frac{\pi t}{12} + b \sin \frac{\pi t}{12}\right)$$

with a and b to be determined. Differentiation on both sides and division by e^{Kt} gives

$$\cos \frac{\pi t}{12} = K\left(a \cos \frac{\pi t}{12} + b \sin \frac{\pi t}{12}\right) - \frac{a\pi}{12} \sin \frac{\pi t}{12} + \frac{b\pi}{12} \cos \frac{\pi t}{12}.$$

We now equate the coefficients of sine and cosine on both sides. The sine terms give

$$0 = Kb - \frac{a\pi}{12}, \qquad \text{hence} \qquad a = \frac{12K}{\pi} b.$$

The cosine terms give

$$1 = Ka + \frac{\pi}{12}b = \left(\frac{12K^2}{\pi} + \frac{\pi}{12}\right)b = \frac{144K^2 + \pi^2}{12\pi}b.$$

Hence,

$$b = \frac{12\pi}{144K^2 + \pi^2}, \qquad a = \frac{144K}{144K^2 + \pi^2}.$$

From this we see that the integral has the value

$$e^{Kt}\left(a\cos\frac{\pi t}{12} + b\sin\frac{\pi t}{12}\right) = \frac{12\pi}{144K^2 + \pi^2}e^{Kt}\left(\frac{12K}{\pi}\cos\frac{\pi t}{12} + \sin\frac{\pi t}{12}\right).$$

This value times B (a factor we did not carry along) times e^{-Kt} (the factor in front of the integral on p. 31) is the value of the second term of the general solution and of the particular solution in the example.

Example 4, pp. 32–33. Logistic equation, Verhulst equation. This ODE

$$y' = Ay - By^2 = Ay\left(1 - \frac{B}{A}y\right)$$

is a basic population model. In contrast to the Malthus equation $y' = ky$, which for a positive initial population models a population that grows to infinity (if $k > 0$) or to zero (if $k < 0$), the logistic equation models growth of small initial populations and decreasing populations of large initial populations. You can see directly from the ODE that the dividing line between the two cases is $y = A/B$ because for this value the derivative y' is zero.

Problem Set 1.5. Page 34

5. Linear ODE. Multiplying the given ODE (with $k \neq 0$) by e^{kx}, you obtain

$$(y' + ky)e^{kx} = e^{-kx}e^{ks} = e^0 = 1.$$

The left-hand side of our equation is equal to $(ye^{kx})'$, so that we have

$$(ye^{kx})' = 1.$$

Integration on both sides gives the final answer.

$$ye^{kx} = x + c, \qquad y = (x + c)e^{-kx}.$$

The use of (4), p. 28, is simple, too, namely, $p(x) = k, h = \int p(x)\,dx = \int k\,dx = kx$. Furthermore, $r = e^{-kx}$. This gives

$$\begin{aligned} y &= e^{-kx}\left(\int e^{kx}e^{-kx}dx + c\right) \\ &= e^{-kx}\left(\int 1\,dx + c\right) = e^{-kx}(x + c). \end{aligned}$$

9. Initial value problem. For the given ODE $y' + y \sin x = e^{\cos x}$ we have in (4)

$$p(x) = \sin x$$

so that by integration

$$h = \int \sin x \, dx = -\cos x$$

Furthermore the right-hand side of the ODE $r = e^{\cos x}$. Evaluating (4) gives us the general solution of the ODE. Thus

$$\begin{aligned} y &= e^{\cos x} \left(\int e^{-\cos x} \cdot e^{\cos x} \, dx + c \right) \\ &= e^{\cos x}(x + c). \end{aligned}$$

We turn to the initial condition and substitute it into our general solution and obtain the value for c

$$y(0) = e^{\cos 0}(0 + c) = -2.5, \qquad c = -\frac{2.5}{e}$$

Together the final solution to the IVP is

$$y = e^{\cos x}\left(x - \frac{2.5}{e} \right).$$

23. Bernoulli equation. In this ODE $y' + xy = xy^{-1}$ we have $p(x) = x, g(x) = x$ and $a = -1$. The new dependent variable is $u(x) = [y(x)]^{1-a} = y^2$. The resulting linear ODE (10) is

$$u' + 2xu = 2x.$$

To this ODE we apply (4) with $p(x) = 2x, r(x) = 2x$ hence

$$h = \int 2x \, dx = x^2, \qquad -h = -x^2$$

so that (4) takes the form

$$u = e^{-x^2} \left(\int e^{x^2}(2x) \, dx + c \right).$$

In the integrand, we notice that $(e^{x^2})' = (e^{x^2}) \cdot 2x$, so that the equation simplifies to

$$u = e^{-x^2}(e^{x^2} + c) = 1 + ce^{-x^2}.$$

Finally, $u(x) = y^2$ so that $y^2 = 1 + ce^{-x^2}$. From the initial condition $[y(0)]^2 = 1 + c = 3^2$. It follows that $c = 8$. The final answer is

$$y = 1 + 8e^{-x^2}.$$

31. Newton's law of cooling. Take a look at Example 6 in Sec. 1.3, pp. 15–16. Newton's law of cooling is given by

$$\frac{dT}{dt} = K(T - T_A).$$

In terms of the given problem, Newton's law of cooling means that the rate of change of the temperature T of the cake at any time t is proportional to the difference of temperature of the cake and the temperature T_A of the room. Example 6 also solves the equation by separation of variables and arrives at

$$T(t) = T_A + ce^{kt}.$$

At time $t = 0$, we have $T(0) = 300 = 60 + c \cdot e^{0 \cdot k} = 60 + c$, which gives that $c = 240$. Insert this into the previous equation with $T_A = 60$ and obtain

$$T(t) = 60 + 240e^{kt}.$$

Ten minutes later is $t = 10$ and we know that the cake has temperature $T(10) = 200$ [°F]. Putting this into the previous equation we have

$$T(10) = 60 + 240e^{10k} = 200, \quad e^k = \left(\tfrac{7}{12}\right)^{1/10}, \qquad k = \tfrac{1}{10}\ln\left(\tfrac{7}{12}\right) = -0.0539.$$

Now we can find out the time t when the cake has temperature of $T(t) = 61$°F. We set up, using the computed value of k from the previous step,

$$60 + 240e^{-0.0539t} = 61, \quad e^{-0.0539t} = \frac{1}{240}, \quad t = \frac{-\ln(240)}{-0.0539} = \frac{-5.48}{-0.0539} = 102\,\text{min}.$$

Sec. 1.6 Orthogonal Trajectories

The method is rather general because one-parameter families of curves can often be represented as general solutions of an ODE of first order. Then replacing $y' = f(x, y)$ by $\tilde{y}' = -1/f(x, \tilde{y})$ gives the ODE of the trajectories to be solved because two curves intersect at a right angle if the product of their slopes at the point of intersection equals -1; in the present case, $y'\tilde{y}' = -1$.

Problem Set 1.6. Page 38

9. Orthogonal trajectories. Bell-shaped curves. Note that the given curves $y = ce^{-x^2}$ are bell-shaped curves centered around the y-axis with the maximum value $(0, c)$ and tangentially approaching the x-axis for increasing $|x|$. For negative c you get the bell-shaped curves reflected about the x-axis. Sketch some of them. The first step in determining orthogonal trajectories usually is to solve the given representation $G(x, y, c) = 0$ of a family of curves for the parameter c. In the present case, $ye^{x^2} = c$. Differentiation with respect to x then gives (chain rule!)

$$y'e^{x^2} + 2xye^{x^2} = 0, \qquad y' + 2xy = 0.$$

where the second equation results from dividing the first by e^{x^2}.

Hence the ODE of the given curves is $y' = -2xy$. Consequently, the trajectories have the ODE $\tilde{y}' = 1/(2x\tilde{y})$. Separating variables gives

$$2\tilde{y}\,d\tilde{y} = dx/x. \quad \text{By integration,} \quad 2\tilde{y}^2/2 = -\ln|x| + c_1, \quad \tilde{y}^2 = -\ln|x| + c_1.$$

Taking exponents gives

$$e^{\tilde{y}^2} = x \cdot c_2. \qquad \text{Thus,} \qquad x = \tilde{c}e^{\tilde{y}^2}$$

where the last equation was obtained by letting $\tilde{c} = 1/c_2$. These are curves that pass through $(\tilde{c}, 0)$ and grow extremely rapidly in the positive x direction for positive $\tilde{c}$ with the x-axis serving as an axis of symmetry. For negative $\tilde{c}$ the curves open sideways in the negative x direction. Sketch some of them for positive and negative $\tilde{c}$ and see for yourself.

12. Electric field. To obtain an ODE for the given curves (circles), you must get rid of c. For this, multiply $(y - c)^2$ out. Then a term c^2 drops out on both sides and you can solve the resulting equation algebraically for c. The next step then is differentiation of the equation just obtained.

13. Temperature field. The given temperature field consists of upper halfs of ellipses (i.e., they do not drop below the x-axis). We write the given equation as

$$G(x, y, c) = 4x^2 + 9y^2 - c = 0 \qquad y > 0.$$

Implicit differentiation with respect to x, using the chain rule, yields

$$8x + 18yy' = 0 \qquad \text{and} \qquad y' = -\frac{4x}{9y}.$$

Using (3) of Sec. 1.6, we get

$$\tilde{y}' = -\frac{1}{4x/9\tilde{y}} = \frac{9\tilde{y}}{4x} \qquad \text{so that} \qquad \frac{d\tilde{y}}{dx} = \frac{9\tilde{y}}{4x} \qquad \text{and} \qquad d\tilde{y}\frac{1}{9\tilde{y}} = dx\frac{1}{4x}.$$

Integrating both sides gives

$$\frac{1}{9}\int \frac{1}{\tilde{y}}d\tilde{y} = \frac{1}{4}\int \frac{1}{x}dx \qquad \text{and} \qquad \frac{1}{9}\ln|\tilde{y}| = \frac{1}{4}\ln|x| + c_1.$$

Applying exponentiation on both sides and using (1) of Appendix 3, p. A63, gives the desired result $y = x^{9/4} \cdot \tilde{c}$, as on p. A5. The curves all go through the origin, stay above the x-axis, and are symmetric to the y-axis.

Sec. 1.7 Existence and Uniqueness of Solutions for Initial Value Problems

Since absolute values are always nonnegative, the only solution of $|y'| + |y| = 0$ is $y = 0$ ($y(x) \equiv 0$ for all x) and this function cannot satisfy the initial condition $y(0) = 1$ or any initial condition $y(0) = y_0$ with $y_0 \neq 0$.

The next ODE in the text $y' = 2x$ has the general solution $y = x^2 + c$ (calculus!), so that $y(0) = c = 1$ for the given initial condition.

The third ODE $xy' = y - 1$ is separable,

$$\frac{dy}{y-1} = \frac{dx}{x}.$$

By integration,

$$\ln|y - 1| = \ln|x| + c_1, \qquad y - 1 = cx, \qquad y = 1 + cx,$$

a general solution which satisfies $y(0) = 1$ with any c because c drops out when $x = 0$. This happens only at $x = 0$. Writing the ODE in standard form, with y' as the first term, you see that

$$y' - \frac{1}{x}y = -\frac{1}{x},$$

showing that the coefficient $1/x$ of y is infinite at $x = 0$.

Theorems 1 and 2, pp. 39–40, concern initial value problems

$$y' = f(x, y), \qquad y(x) = y_0.$$

It is good to remember the two main facts:

1. Continuity of $f(x, y)$ is enough to guarantee the existence of a solution of (1), but is not enough for uniqueness (as is shown in Example 2 on p. 42).
2. Continuity of f and of its partial derivative with respect to y is enough to have uniqueness of the solution of (1), p. 39.

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1. Linear ODE. In this case the solution is given by the integral formula (4) in Sec. 1.5, which replaces the problem of solving an ODE by the simpler task of evaluating integrals – this is the point of (4). Accordingly, we need only conditions under which the integrals in (4) exist. The continuity of f and r are sufficient in this case.

3. Vertical strip as "rectangle." In this case, since a is the smaller of the numbers a and b/K and K is constant and b is no longer restricted, the answer $|x - x_0| < a$ given on p. A6 follows.