Signals, Systems & Inference

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Chapter 1 Solutions

Note from the authors

The book has a total of 350 problems, so it is possible and even likely that there are some omissions and errors in the solutions in this Instructors' Solutions Manual (ISM). It is also possible that an occasional problem in the book is now slightly different from an earlier version for which the solution here was generated. It is therefore important for an instructor to carefully review the solutions to problems of interest, and to modify them as needed.

Many of these solutions have been prepared by the teaching assistants for the course in which this material has been taught at MIT, and their assistance is individually acknowledged in the book. For preparing solutions to the remaining problems, we are particularly grateful to Abubakar Abid (who also constructed the solution template), Leighton Barnes, Fisher Jepsen, Tarek Lahlou, Catherine Medlock, Lucas Nissenbaum, Ehimwenma Nosakhare, Juan Miguel Rodriguez, Andrew Song, and Guolong Su, while they were students at MIT. We would also like to thank Prof. Jeffrey Shapiro of MIT for corrections and improvements to several of the solutions when he used the book in his class in Spring 2016. Finally, Ms. Laura von Bosau provided great assistance in compiling the solutions.

Solution 1.1

(a) Consider a system with input $x(t)$ and output $y(t)$, with input-output relation $y(t) = x^4(t)$ for $-\infty < t < \infty$.

(i) This system is linear:

TRUE **FALSE**

Input $x_1(t)$:

$$y_1(t) = x_1^4(t)$$

Input $x_2(t)$:

$$y_2(t) = x_2^4(t)$$

Input $x_3(t) = x_1(t) + x_2(t)$:

$$\begin{aligned} y_3(t) &= x_3^4(t) \\ &= (x_1(t) + x_2(t))^4 \\ &= x_1^4(t) + 4x_1^3(t)x_2(t) + 6x_1^2(t)x_2^2(t) + 4x_1(t)x_2^3(t) + x_2^4(t) \\ &\neq x_1^4(t) + x_2^4(t) \end{aligned}$$

(ii) This system is time-invariant:

TRUE **FALSE**

Input $x_1(t)$:

$$\begin{aligned} y_1(t) &= x_1^4(t) \\ y_1(t-T) &= x_1^4(t-T) \end{aligned}$$

Input $x_2(t) = x_1(t-T)$:

$$\begin{aligned} y_2(t) &= x_2^4(t) \\ &= x_1^4(t-T) \end{aligned}$$

$y_2(t) = y_1(t-T)$ implies time-invariance

(iii) This system is causal:

TRUE FALSE

Since the output for the system at time t only depends on the input at time t, this system is memoryless, and therefore causal.

(b) Consider a system with input $x[n]$ and output $y[n]$, with input-output relation

$$y[n] = \begin{cases} 0 & n \le 0 \\ y[n-1] + x[n] & n > 0 \end{cases}$$

(i) This system is linear:

TRUE FALSE

The system can be equivalently written as:

$$y[n] = \begin{cases} 0 & n \le 0 \\ \sum_{k=1}^{n} x[k] & n > 0 \end{cases}$$

Input $x_3[n] = \alpha x_1[n] + \beta x_2[n]$:

$$\begin{aligned} y_3[n] = \begin{cases} 0 & n \le 0 \\ \sum_{k=1}^{n} x_3[k] & n > 0 \end{cases} &= \sum_{k=1}^{n} \alpha x_1[n] + \beta x_2[n] \quad \text{for} \quad n > 0 \\ &= \alpha \sum_{k=1}^{n} x_1[n] + \beta \sum_{k=1}^{n} x_2[n] \quad \text{for} \quad n > 0 \\ &= \alpha y_1[n] + \beta y_2[n] \quad \text{for} \quad n > 0 \end{aligned}$$

(ii) This system is time-invariant:

TRUE FALSE

Input $x_1[n] = \delta[n]$:

$$\begin{aligned} y_1[n] &= y_1[n-1] + \delta[n] \quad for\ n > 0 \\ &= u[n] \end{aligned}$$

Input $x_2[n] = \delta[n+T]$ where $T > 0$:

$$\begin{aligned} y_2[n] &= y_2[n-1] + \delta[n+T] \quad for\ n > 0 \\ &= 0 \end{aligned}$$

Since $y_2[n] \neq y_1[n+T]$, the system is not T-I. We can simply see this because there is a fixed location in time, before which the output is always 0.

(iii) This system is causal:

TRUE **FALSE**

Since we know that:

$$y[n] = \begin{cases} 0 & n \leq 0 \\ \sum_{k=1}^{n} x[k] & n > 0 \end{cases}$$

We see that the output at y[n] for $n \leq 0$, do not depend on the input. Also, y[n] for $n > 0$ depends only on the time values of x[k] from k = 1 through n (past inputs). Thus, the system is causal.

(c) Consider a system with input $x(t)$ and output $y(t)$, with input-output relation $y(t) = x(4t+3)$ for $-\infty < t < \infty$. This is similar to Example 1.1, but now in CT.

(i) This system is linear:

TRUE **FALSE**

(ii) This system is time-invariant:

TRUE **FALSE**

(iii) This system is causal:

TRUE **FALSE**

(d) Consider a system with input $x(t)$ and output $y(t)$, with input-output relation

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, d\tau$$

for $-\infty < t < \infty$.

(i) This system is linear:

TRUE **FALSE**

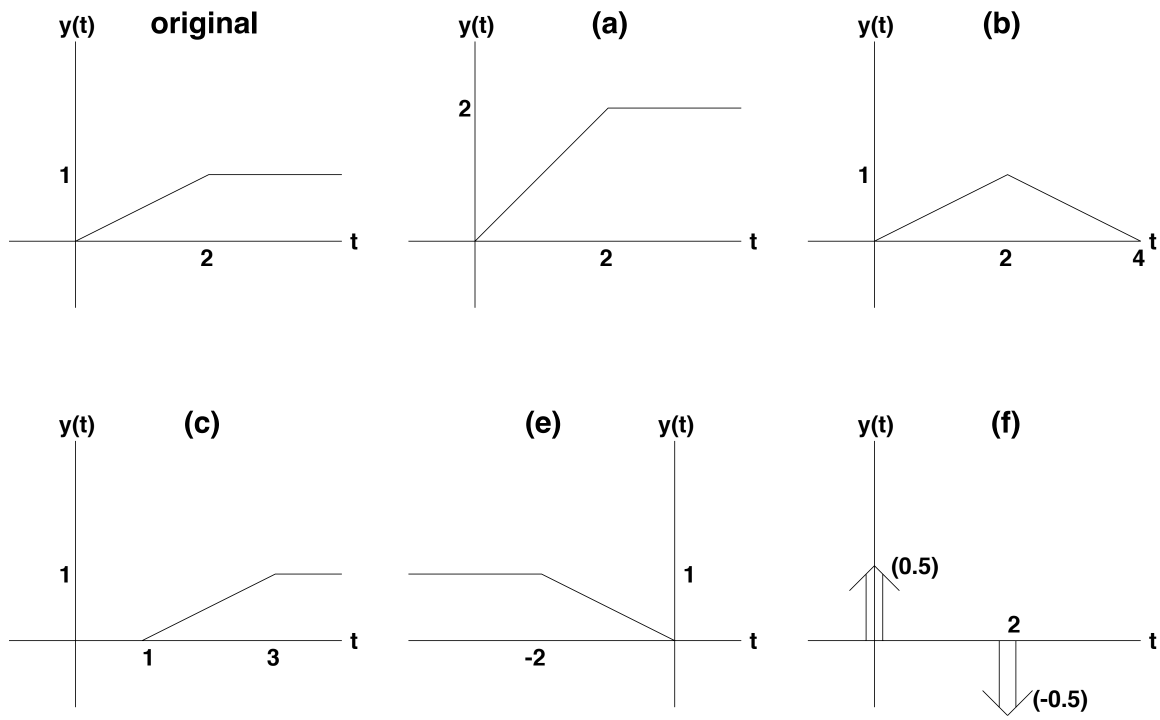
(ii) This system is time-invariant:

TRUE **FALSE**

(iii) This system is causal:

TRUE **FALSE**

Solution 1.2



(a) From the homogeneity property of convolution, doubling the input doubles the output, so $y(t) = 2y_0(t)$.

(b) Time-invariance means that $x_0(t) \rightarrow y_0(t) \Longleftrightarrow x_0(t-2) \rightarrow y_0(t-2)$, and superposition allows $x(t) - x(t-2) \rightarrow y(t) - y(t-2)$, so the result is just the sum of the original response minus the response delayed by 2, so $y(t) = y_0(t) - y_0(t-2)$.

(c) From time invariance, delaying $x(t)$ by 2 and advancing $h_0(t)$ by 1 yields a net delay of 1, so $y(t) = y_0(t-1)$.

(d) In this case $y(t)$ cannot be uniquely determined. For instance, if $x_0(t)$ happened to be even, i.e. $x_0(-t) = x_0(t)$, then $y(t) = y_0(t)$; but if $x_0(t)$ happened to be odd, i.e. $x_0(-t) = -x_0(t)$, then $y(t) = -y_0(t)$. (You can easily construct for yourself examples of even and of odd $x_0(t)$ that can, with an appropriate $h_0(t)$, give rise to the indicated $y_0(t)$.)

(e) Flipping both $h_0(t)$ and $x(t)$ in time is the same as reversing the output $y(t)$ in time, so $y(t) = y_0(-t)$.

(f) The operator $\frac{d}{dt}$ is a linear operator, so taking the derivative of $x(t)$ results in the derivative of the output $y(t)$. Because both $x(t)$ and $h_0(t)$ are differentiated, the result is the second derivative of the original output waveform, so $y(t) = \frac{d^2 y_0(t)}{dt^2}$.

Solution 1.3

(a) Using the definition of convolution $y(t) = \int x(t-s)h(s)\mathrm{d}s$, we see that for $t < 1$, there is no overlap between the support of $x(t-s)$ and $h(s)$, so $y(t) = 0 \quad (t < 1)$.

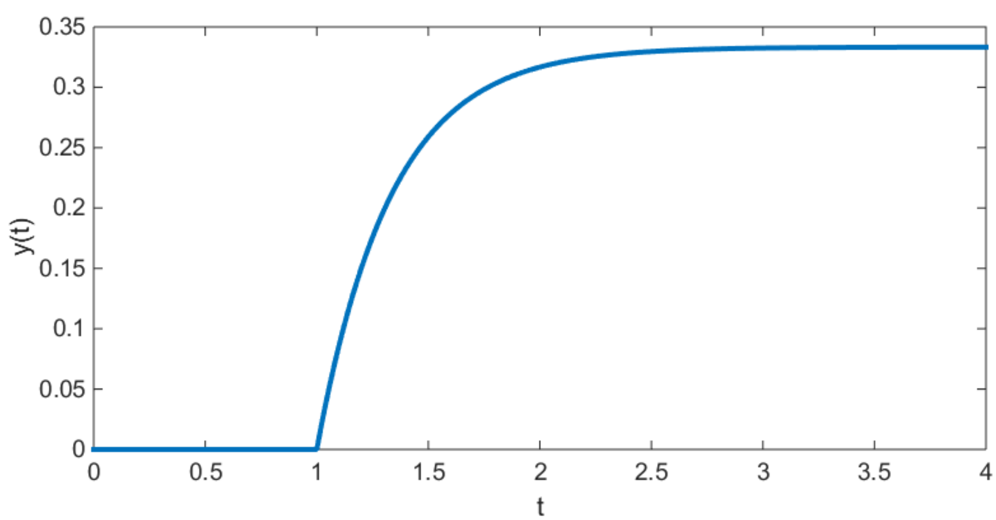
For $t \geq 1$, we see

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-s)h(s)\mathrm{d}s \\ &= \int_{-\infty}^{\infty} e^{-3(t-s)} \cdot u(t-s) \cdot u(s-1)\mathrm{d}s \\ &= \int_{1}^{t} e^{-3(t-s)}\mathrm{d}s \\ &= \frac{1}{3}\left(1 - e^{-3(t-1)}\right). \end{aligned}$$

A combination of the two situations above results in the following solution

$$y(t) = \frac{1}{3}\left(1 - e^{-3(t-1)}\right) \cdot u(t-1),$$

and the plot of $y(t)$ is as follows.



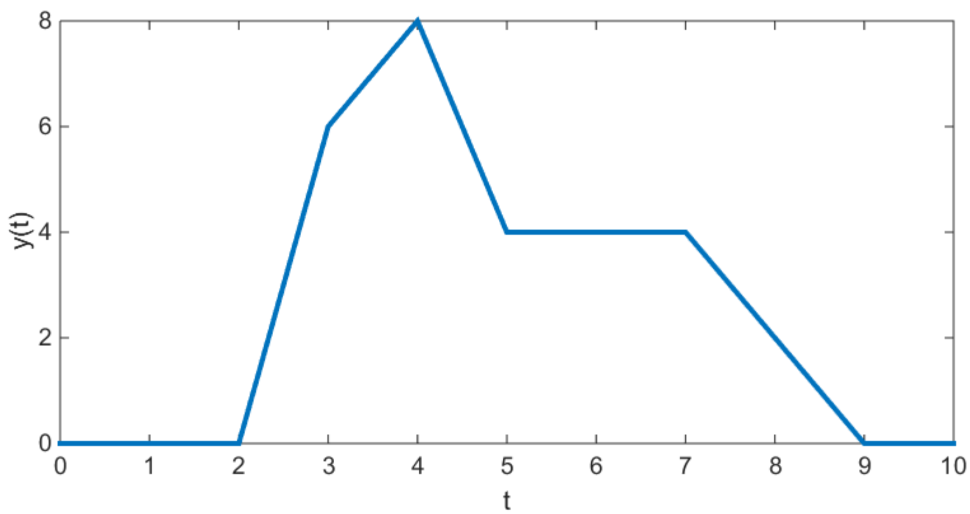
(b) First, the signals in this problem can be expressed as follows

$$\begin{aligned} x(t) &= 2u(t-1) - 2u(t-3), \\ h(t) &= 3u(t-1) - 2u(t-2) - u(t-6). \end{aligned}$$

Then, we utilize two facts about convolution: (i) $u(t)*u(t) = t\cdot u(t)$; (ii) if $f(t)*g(t) = v(t)$, then $f(t-t_1)*g(t-t_2) = v(t-(t_1+t_2))$. With these facts, the convolution result is

$$\begin{aligned}
y(t) &= x(t)*h(t) \\
&= (2u(t-1)-2u(t-3))*(3u(t-1)-2u(t-2)-u(t-6)) \\
&= 6u(t-1)*u(t-1)-4u(t-1)*u(t-2)-2u(t-1)*u(t-6)-6u(t-3)*u(t-1) \\
&\quad +4u(t-3)*u(t-2)+2u(t-3)*u(t-6) \\
&= 6(t-2)\cdot u(t-2)-4(t-3)\cdot u(t-3)-2(t-7)\cdot u(t-7)-6(t-4)\cdot u(t-4) \\
&\quad +4(t-5)\cdot u(t-5)+2(t-9)\cdot u(t-9).
\end{aligned}$$

The plot of $y(t)$ is in the figure below.



Solution 1.4

(a) This can be considered the result of delaying the input $x(t)$ by 2, then feeding the result to a system with impulse response $h'(t) = e^{-t}u(t)$, so $h(t) = e^{-(t-2)}u(t-2)$. The answer can be checked by setting $x(t) = \delta(t)$; the integral then evaluates to $e^{-(t-2)}$ for $t \geq 2$, and to 0 otherwise.

(b) The unit step response of the above system is

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = (1 - e^{-(t-2)}u(t-2) \ ,$$

rising from the value 0 at time $t = 2$ with a time constant of 1, and settling exponentially to the value 1 as $t \to \infty$. Hence the response to the given input, namely $x(t) = u(t+1) - u(t-2)$, is

$$y(t) = s(t+1) - s(t-2) \ .$$

(c) The lower branch results in $x(t-1)$ being applied to the system with impulse response $h(t)$, so $w(t) = y(t) - y(t-1)$, where $y(t)$ is as in part (b).

Solution 1.5

(a) Denote the input and output signals as $x_0(t)$ and $y_0(t)$, respectively. Stability of this LTI system ensures that $y_0(t)$ is bounded. The input signal is $x_0(t) = \alpha = \alpha \cdot e^{0t}$ and thus an eigenfunction of the LTI system with eigenvalue $H(0)$. Thus, the output signal will be $y_0(t) = H(0) \cdot x_0(t) = H(0) \cdot \alpha$.

(b) Denote the output signal as $y_1(t)$ when the input signal is $x_1(t) = t - \alpha$. On one hand, notice $x_1(t) = x(t - \alpha)$, so the time-invariance of the system results in

$$y_1(t) = y(t - \alpha). \tag{1}$$

On the other hand, it can be seen that $x_1(t) = x(t) - x_0(t)$, so the linearity of this system leads to

$$y_1(t) = y(t) - y_0(t) = y(t) - H(0) \cdot \alpha. \tag{2}$$

Thus, there are two distinct expressions in (1) and (2) of the output $y_1(t)$ when the input is $x_1(t) = t - \alpha$.

Fixing $\alpha = t$, (1) and (2) result in the equality below

$$y(t) - H(0) \cdot t = y(0),$$

leading to $y(t) = y(0) + H(0) \cdot t$. Finally, we see that $b = H(0)$.

Solution 1.6

(a) Table 1.2 states that the CTFT for the signal $x_1(t) = e^{-2t} \cdot u(t)$ is $X_1(j\omega) = 1/(2 + j\omega)$. Since $x(t) = x_1(t-1)$, their CTFT satisfy $X(j\omega) = e^{-j\omega} \cdot X_1(j\omega)$. Thus, the CTFT of $x(t)$ is $X(j\omega) = e^{-j\omega}/(2 + j\omega)$.

(b) If we denote $x_2(t) = e^{-t}u(t)$, then $x(t) = x_2(t) + x_2(-t)$. Table 1.2 states that the CTFT for $x_2(t)$ is $X_2(j\omega) = 1/(1+j\omega)$. Furthermore, the time-reverse property of CTFT states that the CTFT for $x_2(-t)$ is $X_2(-j\omega)$, which may be shown by

$$\int_{-\infty}^{\infty} x_2(-t) \cdot e^{-j\omega t} \mathrm{d}t = \int_{-\infty}^{\infty} x_2(t_1) \cdot e^{j\omega t_1} \mathrm{d}t_1 = \int_{-\infty}^{\infty} x_2(t_1) \cdot e^{-j(-\omega)t_1} \mathrm{d}t_1 = X_2(-j\omega),$$

in which we use the change of variable $t_1 = -t$. Thus, the CTFT of $x(t)$ is

$$X(j\omega) = X_2(j\omega) + X_2(-j\omega) = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}.$$

(c) We can observe $x(t) = x_3(t) \cdot x_4(t)$ where $x_3(t) = e^{-\alpha t}u(t)$ and $x_4(t) = \cos(\omega_0 t) = \frac{1}{2} \cdot (e^{j\omega_0 t} + e^{-j\omega_0 t})$, whose CTFT are

$$X_3(j\omega) = \frac{1}{\alpha + j\omega}, \quad X_4(j\omega) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)).$$

With the fact that multiplication in the time domain causes convolution in the frequency domain, the CTFT of $x(t)$ is

$$X(j\omega) = X_3(j\omega) * X_4(j\omega) = \frac{1}{2}(X_3(j(\omega - \omega_0)) + X_3(j(\omega + \omega_0))) = \frac{1}{2} \cdot \left(\frac{1}{\alpha + j(\omega - \omega_0)} + \frac{1}{\alpha + j(\omega + \omega_0)} \right),$$

where we notice that the convolution in the frequency domain has a scaling of $1/(2\pi)$.

Solution 1.7

(a)

$$\begin{aligned} X(e^{j\Omega}) &= \textstyle\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= -3e^{j\Omega 3} - 2e^{j\Omega 2} - e^{j\Omega} + e^{-j\Omega} + 2e^{-j\Omega 2} + 3e^{-j\Omega 3} \\ &= -2j\sin(\Omega) - 4j\sin(2\Omega) - 6j\sin(3\Omega) \\ &= -2j[\sin(\Omega) + 2\sin(2\Omega) + 3\sin(3\Omega)] \end{aligned}$$

(b) Writing the given signal in terms of complex exponentials yields

$$x[n] = \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j} + \frac{e^{jn} + e^{-jn}}{2}$$

Fourier transforming, we find

$$X(e^{j\Omega}) = \frac{\pi}{j}\left[\delta\left(\Omega - \frac{\pi}{2}\right) - \delta\left(\Omega + \frac{\pi}{2}\right)\right] + \pi\left[\delta\left(\Omega - 1\right) + \delta\left(\Omega + 1\right)\right]$$

(c)

$$\begin{aligned} X(e^{j\Omega}) &= \textstyle\sum_{n=-\infty}^{\infty} e^{-2|n|}e^{-j\Omega n} \\ &= \textstyle\sum_{n=-\infty}^{-1} e^{2n-j\Omega n} + \sum_{n=0}^{\infty} e^{-2n-j\Omega n} \\ &= \tfrac{e^{-(2-j\Omega)}}{1-e^{-(2-j\Omega)}} + \tfrac{1}{1-e^{-(2+j\Omega)}} \\ &= \tfrac{1-e^{-4}}{1-2e^{-2}cos\Omega+e^{-4}} \end{aligned}$$

(d) Using the Fourier transform expression for a step function,

$$X(e^{j\Omega}) = \left(\frac{e^{-j\Omega 2}}{1-e^{-j\Omega}} + \pi\delta(\Omega)\right) - \left(\frac{e^{-j\Omega 6}}{1-e^{-j\Omega}} - \pi\delta(\Omega)\right)$$

$$= \frac{e^{-j\Omega 2}}{1-e^{-j\Omega}}(1 - e^{-j\Omega 4})$$

$$= \frac{e^{-j\Omega 4}(e^{j\Omega 2} - e^{-j\Omega 2})}{e^{-j\frac{\Omega}{2}}(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})}$$

$$= e^{-j\frac{7\Omega}{2}}\frac{\sin(2\Omega)}{\sin(\frac{\Omega}{2})} \qquad |\Omega| < \pi$$

or, using geometric series properties, one can write

$$X(e^{j\Omega}) = \sum_{n=2}^{5} e^{-j\Omega n} = \frac{e^{-j\Omega 2} - e^{-j\Omega 6}}{1-e^{-j\Omega}} = \frac{e^{-j\Omega 4}(e^{j\Omega 2} - e^{-j\Omega 2})}{e^{-j\frac{\Omega}{2}}(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})} = e^{-j\Omega\frac{7}{2}}\frac{sin(2\Omega)}{sin(\frac{\Omega}{2})}.$$

(e)

$$x[n] = (\frac{1}{3})^2\delta[n+2] + \frac{1}{3}\delta[n+1] + (\frac{1}{3})^n u[n]$$

Thus,

$$X(e^{j\Omega}) = \frac{1}{9}e^{j2\Omega} + \frac{1}{3}e^{j\Omega} + \frac{1}{1-\frac{1}{3}e^{-j\Omega}}.$$

(f)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty}(n-1)(\frac{1}{3})^{|n|}e^{-j\Omega n} = \left(\sum_{n=-\infty}^{-1}(n-1)(\frac{1}{3})^{-n}e^{-j\Omega n}\right)+\left(\sum_{n=0}^{\infty}(n-1)(\frac{1}{3})^{n}e^{-j\Omega n}\right)$$

$$= \left(\sum_{n=0}^{\infty}(n+1)(\frac{1}{3})^{n}e^{-j\Omega n} - 2\sum_{n=0}^{\infty}(\frac{1}{3})^{n}e^{-j\Omega n}\right) - \left(1-\sum_{n=0}^{\infty}(n+1)(\frac{1}{3})^{n}e^{j\Omega n}\right)$$

Using the Fourier transform table,

$$= \left(\frac{1}{(1-\frac{1}{3}e^{-j\Omega})^2} - \frac{2}{(1-\frac{1}{3}e^{-j\Omega})}\right) + \left(1 - \frac{1}{(1-\frac{1}{3}e^{j\Omega})^2}\right)$$

$$X(e^{j\Omega}) = 1 - \frac{1}{\left(1-\frac{1}{3}e^{j\Omega}\right)^2} + \frac{\frac{2}{3}e^{-j\Omega}-1}{(1-\frac{1}{3}e^{-j\Omega})^2}$$

Solution 1.8

Since the DT LTI system satisfies $H(e^{-j\Omega}) = H^*(e^{j\Omega})$ where the superscript * denotes complex conjugate, the corresponding unit sample response is real, and the result in Eq. (1.33) is applicable to this problem and the output signal $y[n]$ can be solved as

$$y[n] = |H(e^{j\frac{4\pi}{3}})| \cdot \cos\left(\frac{4\pi}{3}n + \frac{\pi}{4} + \angle H(e^{j\frac{4\pi}{3}})\right) = \cos\left(\frac{4\pi}{3} + \frac{5\pi}{4}\right),$$

where we use the 2π periodicity of the frequency response $H(e^{j\frac{4\pi}{3}}) = H(e^{-j\frac{2\pi}{3}}) = e^{j\pi}$.

Solution 1.9

First, let's interpret these conditions in the time domain, using our DTFT properties and basic definitions.

1. Since $x[n]$ is real for these examples, $\Re e\{X(e^{j\Omega})\}$ is the DTFT of the even component of $x[n]$. Thus, this condition is equivalent to $x[n]$ being an odd signal. Signal (b) has this property.

2. As above, $\Im m\{X(e^{j\Omega})\}$ is the DTFT of the odd component of $x[n]$. This condition is equivalent to $x[n]$ being an even signal. Signals (d) and (g) have this property.

3. Equivalently $x[n]$ is an even signal with a delay (of $-\alpha$). Signals (a), (b), (d), (f), and (g) have this property. Signal (f) might seem tricky; do half-sample delays count? You bet!

$$X(e^{j\Omega}) = e^{-j\Omega} + e^{j\Omega 2}$$

so

$$e^{-j\Omega/2}X(e^{j\Omega}) = e^{-j\Omega\frac{3}{2}} + e^{j\Omega\frac{3}{2}} = 2\cos\left(\frac{3}{2}\Omega\right)$$

which is real.

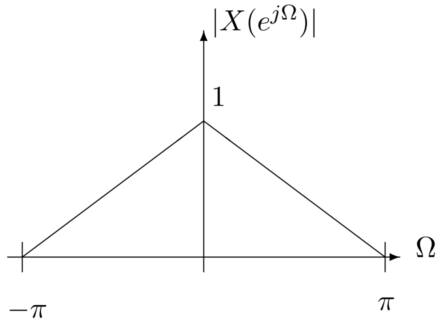
4. From the synthesis equation, $x[0] = 0$. Signals (b), (d), (f), and (g) have this property.

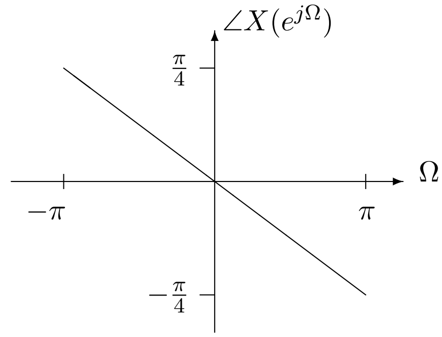
5. $X(e^{j\Omega})$ is always periodic with period 2π. All signals $x[n]$ have this property.

6. From the analysis equation, $\sum_{n=-\infty}^{\infty} x[n] = 0$. Signal (c) has this property. You might think that signal (b) also has this property, but $\sum_{n=-\infty}^{\infty} x[n]$ does not converge.

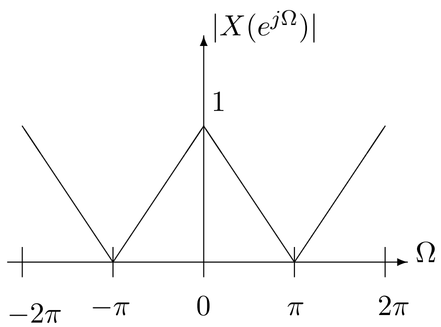
Solution 1.10

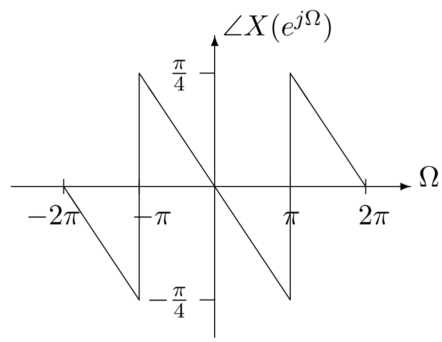
Let $X(e^{j\Omega})$ be the Fourier transform (DTFT) of the discrete time signal $x[n]$. $|X(e^{j\Omega})|$ and $\angle X(e^{j\Omega})$ for $|\Omega| < \pi$ are given as follows:



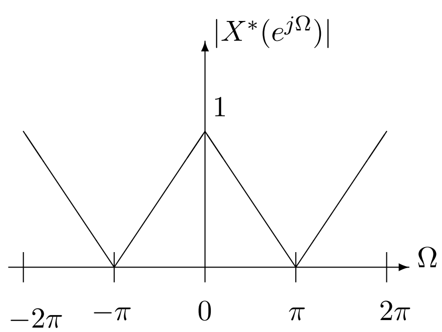


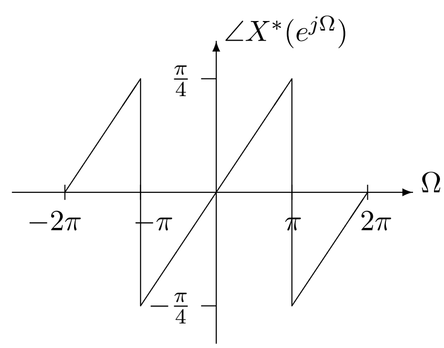
First replot $|X(e^{j\Omega})|$ and $\angle X(e^{j\Omega})$ over the range $|\Omega| < 2\pi$:



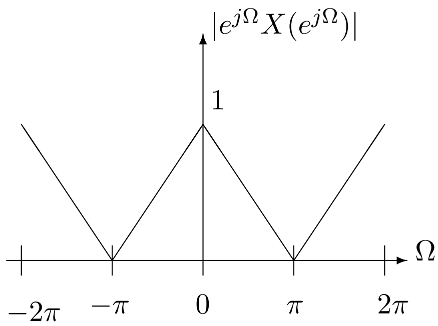


(a) Since the DTFT signal $x[n]$ is even in magnitude and odd in phase, $x[n]$ must be real. Hence the DTFT of $x[-n]$ is $X^*(e^{j\Omega})$.

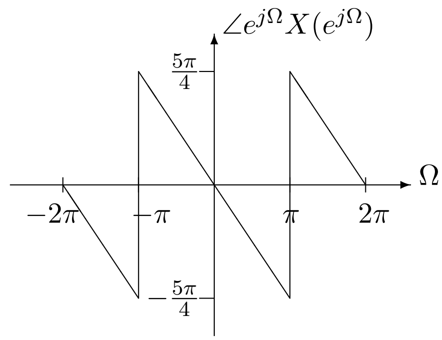




(b) The DTFT of $x[n-1]$ is $e^{j\Omega}X^{(}e^{j\Omega})$.



$|e^{j\Omega}X(e^{j\Omega})|$
1
Ω
-2π
$-\pi$
0
π
2π



$\angle e^{j\Omega}X(e^{j\Omega})$
$\frac{5\pi}{4}$
Ω
-2π
$-\pi$
π
2π
$-\frac{5\pi}{4}$

Solution 1.11

Since the DTFT is

$$X(e^{j\Omega}) = e^{j2\Omega}(1 - e^{-j3\Omega}) = e^{j2\Omega} - e^{-j\Omega} = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\Omega},$$

we see that $x[-2] = 1$, $x[1] = -1$, and other $x[n] = 0$ for $n \neq -2, 1$. Thus, the signal is $x[n] = \delta[n+2] - \delta[n-1]$. This signal is unique due to the uniqueness between DTFT pairs.

Solution 1.12

(a) From the definition of the Fourier transform, we have:

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} \ .$$

Therefore:

$$X(e^{j0}) = \sum_{n=-\infty}^{+\infty} x[n]$$

and

$$X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} (-1)^n x[n] \ .$$

For signal A, we have:

$$\sum_{n=-\infty}^{+\infty} x[n] = 4 = X(e^{j0})$$

and

$$\sum_{n=-\infty}^{+\infty} (-1)^n x[n] = 4 = X(e^{j\pi}) \ .$$

Therefore, **Signal A** corresponds to DTFT number **3**.

For Signal B:

$$\sum_{n=-\infty}^{+\infty} x[n] = 1 = X(e^{j0})$$

and

$$\sum_{n=-\infty}^{+\infty} (-1)^n x[n] = 7 = X(e^{j\pi}) \ .$$

Therefore, **Signal B** corresponds to DTFT number **4**.

For Signal C:

$$\sum_{n=-\infty}^{+\infty} x[n] = 0 = X(e^{j0})$$

and

$$\sum_{n=-\infty}^{+\infty} (-1)^n x[n] = -4 = X(e^{j\pi}) \ .$$

Therefore, **Signal C** corresponds to DTFT number **1**.

For Signal D:

$$\sum_{n=-\infty}^{+\infty} x[n] = 0 = X(e^{j0})$$

and

$$\sum_{n=-\infty}^{+\infty} (-1)^n x[n] = 0 = X(e^{j\pi}) \ .$$

Therefore, **Signal D** corresponds to DTFT number **2**.

(b) From part 1, we can easily observe:

For **DTFT 1:** magnitude at $\Omega = \pi$ is **4**.

For **DTFT 4:** magnitude at $\Omega = \pi$ is **7**.

Solution 1.13

(a) We have:

$$R(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} r[n]e^{-j\Omega n} = \sum_{n=0}^{M} e^{-j\Omega n} \ .$$

This is a complex geometric series, so

$$R(e^{j\Omega}) = \frac{e^{-j\Omega(M+1)} - 1}{e^{-j\Omega} - 1} = \frac{e^{-j\Omega(M+1)/2}}{e^{-j\Omega/2}} \, \frac{e^{-j\Omega(M+1)/2} - e^{j\Omega(M+1)/2}}{e^{-j\Omega/2} - e^{j\Omega/2}}$$

and

$$R(e^{j\Omega}) = \frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)} e^{-j\Omega M/2} \ .$$

The amplitude of $R(e^{j\Omega})$ (see Chapter 2 for the definition) is

$$\frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)} \ ,$$

a "periodic sinc" or "Dirichlet kernel", a sinc-like function, but periodic with period 2π, as any DTFT must be.

(b) Note that

$$w[n] = \frac{1}{2}\left(1 + \cos\frac{2\pi n}{M}\right) r[n] \ .$$

The DTFT of the cosine term at frequency $\Omega_0 = 2\pi/M$ is $\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$, so

$$W(e^{j\Omega}) = \frac{1}{4}[2\delta(\Omega) + \delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \circledast R(e^{j\Omega}) \ ,$$

where $\circledast$ denotes periodic convolution.

Solution 1.14

(a) We simply give the answers here:

(i) ω_c

(ii) $\min(\omega_c, 6000\pi)$

(iii) $2\omega_c$

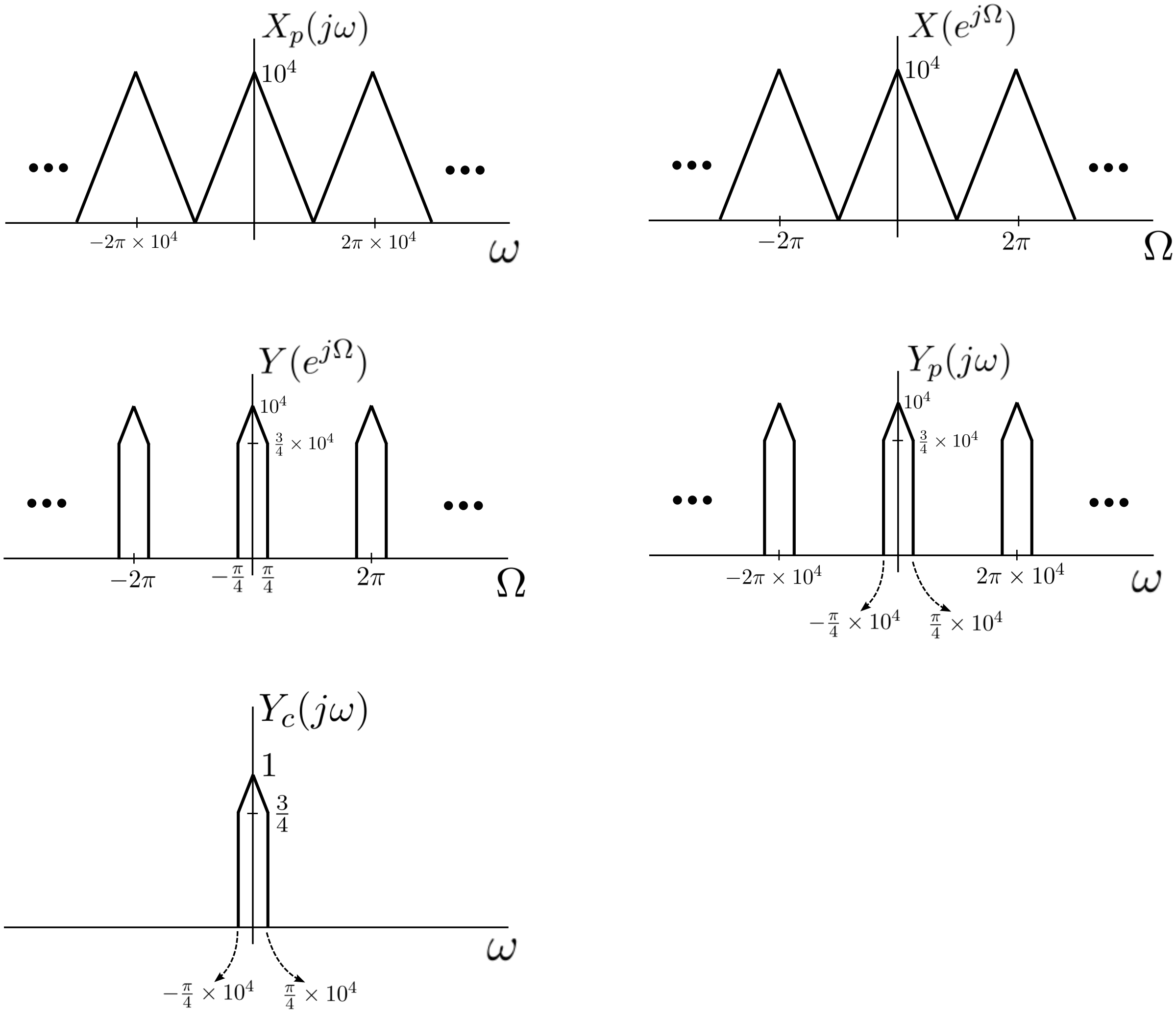
(b) $x_1(t) * x_2(t) \leftrightarrow X_1(j\omega)X_2(j\omega)$. So the highest frequency of $X_1(j\omega)X_2(j\omega)$ is 1000π.

$$
\begin{aligned}
\Rightarrow \quad & \omega_s = \frac{2\pi}{T} > 2(1000\pi) \\
\Rightarrow \quad & T < 10^{-3} \text{ sec.}
\end{aligned}
$$

Solution 1.15

Early printings of the book said sampling was at the "Nyquist frequency" rather than the intended "Nyquist rate"; the Nyquist frequency is half the Nyquist rate.

First, we show the figures for each spectrum before the analysis.



The Nyquist sampling rate is $f = 2 \cdot f_H = 2 \times (\pi \times 10^4)/(2\pi) = 10^4$Hz in which f_H is the highest frequency of the signal. As a result, the corresponding sampling period is $T = 1/f = 10^{-4}$sec. The spectrum after impulse train sampling becomes scaled in magnitude and periodically duplicated

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j \cdot (\omega - \frac{2\pi k}{T})\right).$$

After conversion of impulses into DT samples, the spectrum becomes scaled in the frequency axis

$$X(e^{j\Omega}) = X_p(j\omega)|_{\omega=\Omega/T}.$$

The DT filtering multiplies the spectrum of input with the filter frequency response

$$Y(e^{j\Omega}) = X(e^{j\Omega}) \cdot H(e^{j\Omega}).$$

After the replacement of DT samples with CT impulses, the spectrum has a scaling in the frequency axis again

$$Y_p(j\omega) = Y(e^{j\Omega})|_{\Omega=\omega\cdot T}.$$

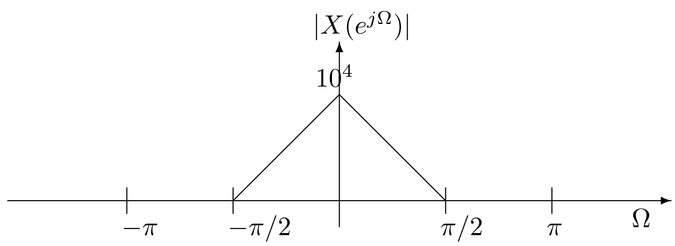
In the last step, the CT filtering multiplies the spectrum with the filter frequency response

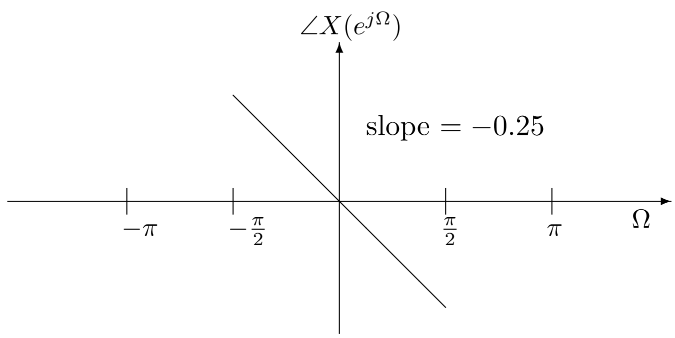
$$Y_c(j\omega) = Y_p(j\omega)L_T(j\omega).$$

With above analysis, we can obtain the figures for each spectrum as shown in the beginning of this solution.

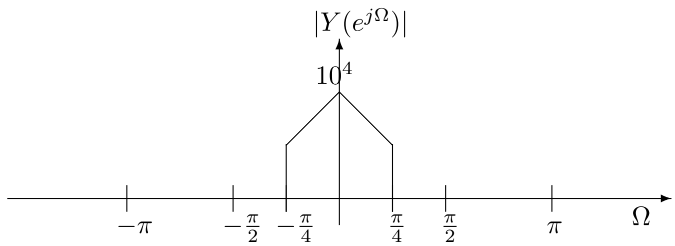
Solution 1.16

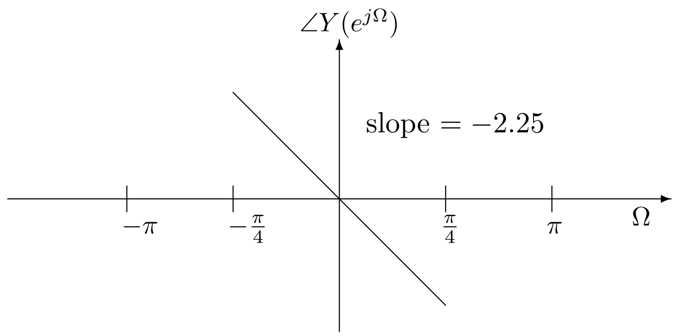
(a) Recall that in general $X(e^{j\Omega})$ is a complex number. So in order to plot $X(e^{j\Omega})$ with respect to Ω, we should plot both the real and imaginary parts with respect to Ω, or we should plot both the magnitude and phase of $X(e^{j\Omega})$ with respect to Ω. We will plot magnitude and phase. Is there aliasing? Noticing that $\frac{\pi}{T} = 10^4\pi\ rad/s$ but that $X_c(j\omega)$ is bandlimited to $0.5 \times 10^4\pi$, we can use Nyquist's theorem to say that there is no aliasing. Using the fact that $X(e^{j\Omega}) = \frac{1}{T}X_c(j\frac{\Omega}{T}), |\Omega| \leq \pi$, we see that the magnitude is scaled by $1/T$ and the phase angle $-m\omega$ becomes the phase angle $-m(\Omega/T) = m(10^4\Omega)$, so we obtain the following figures:



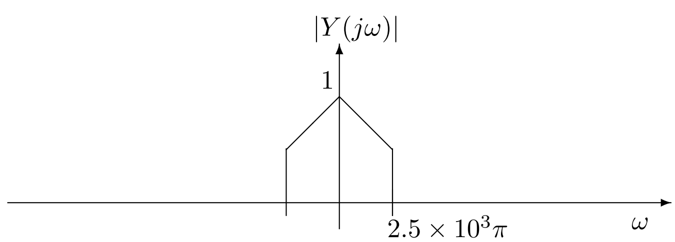


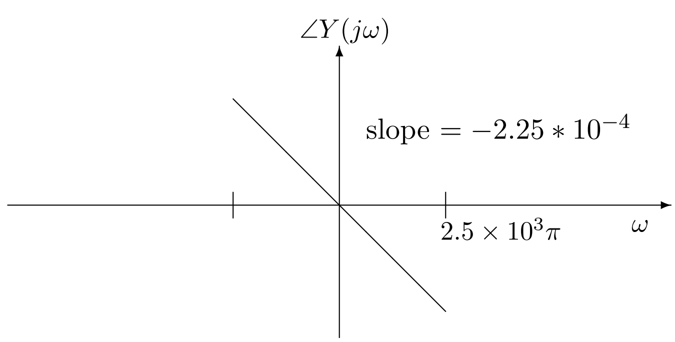
(b) Now let's see what happens when $x[n]$ passes through the discrete time system.





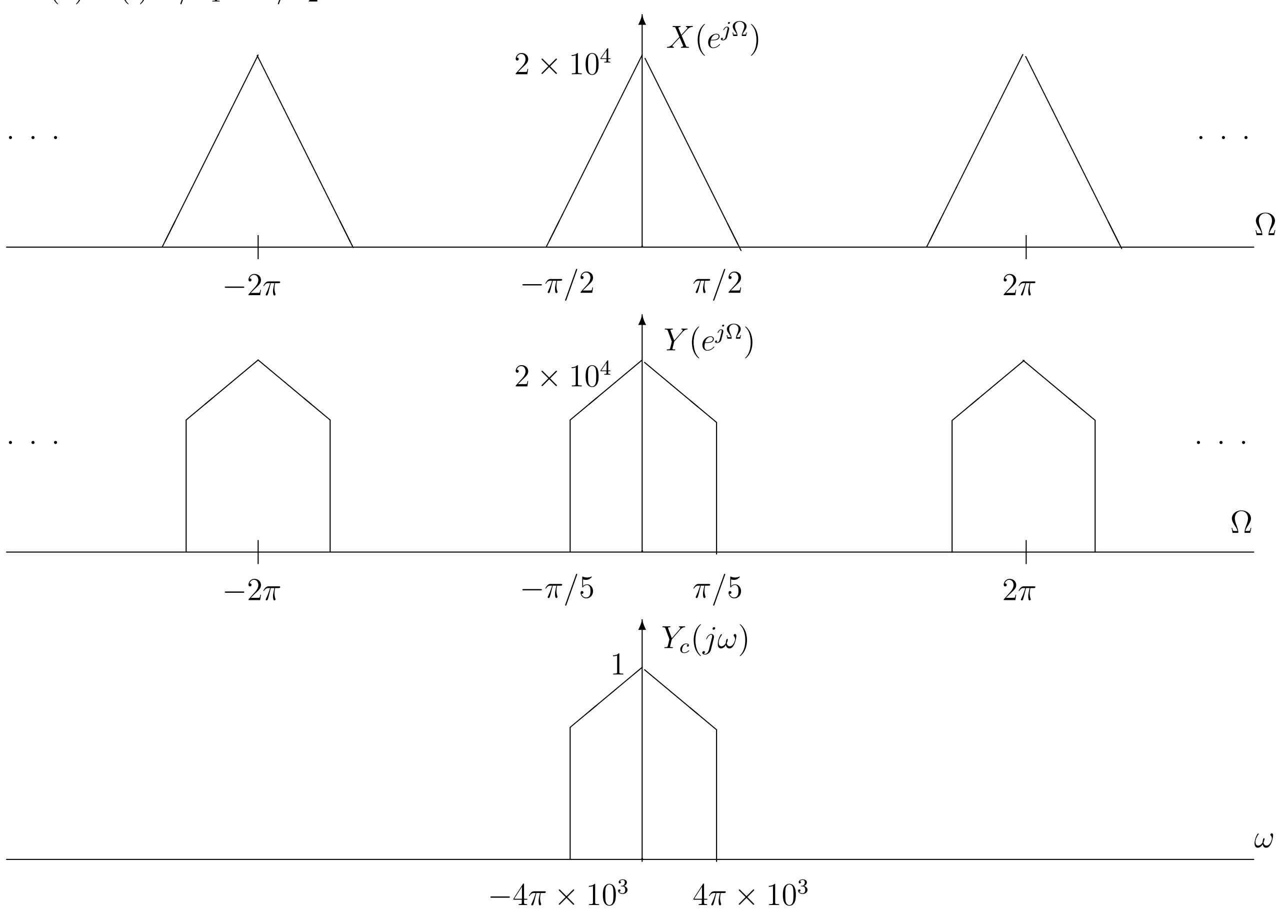
(c) Now we convert everything back to CT:



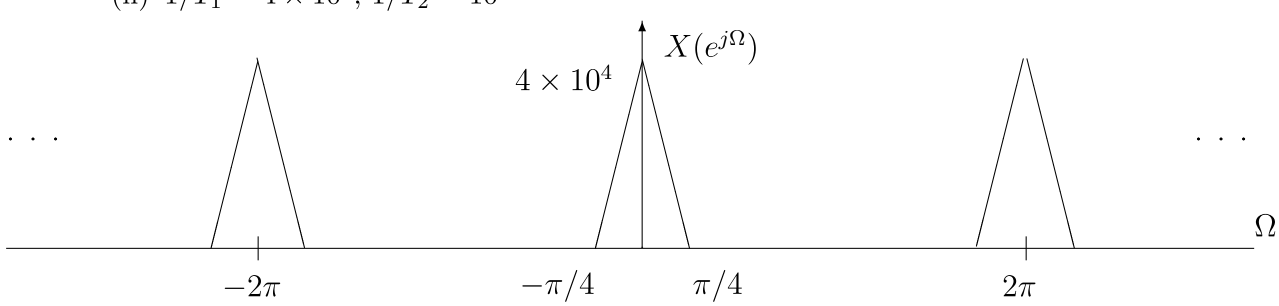


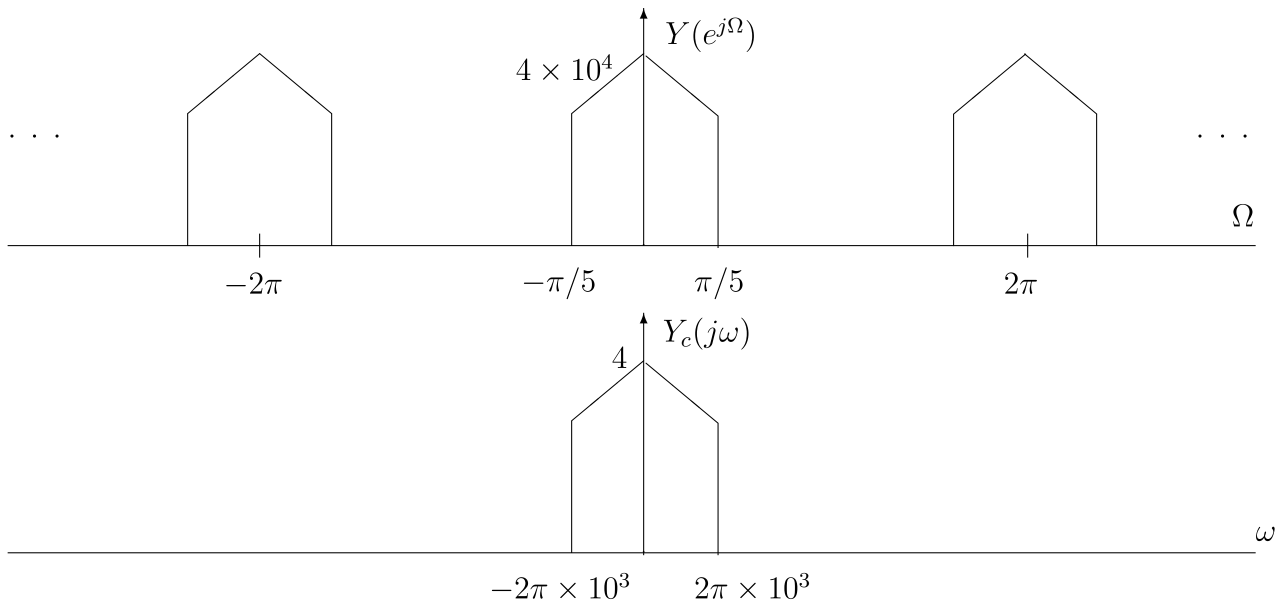
Solution 1.17

(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$

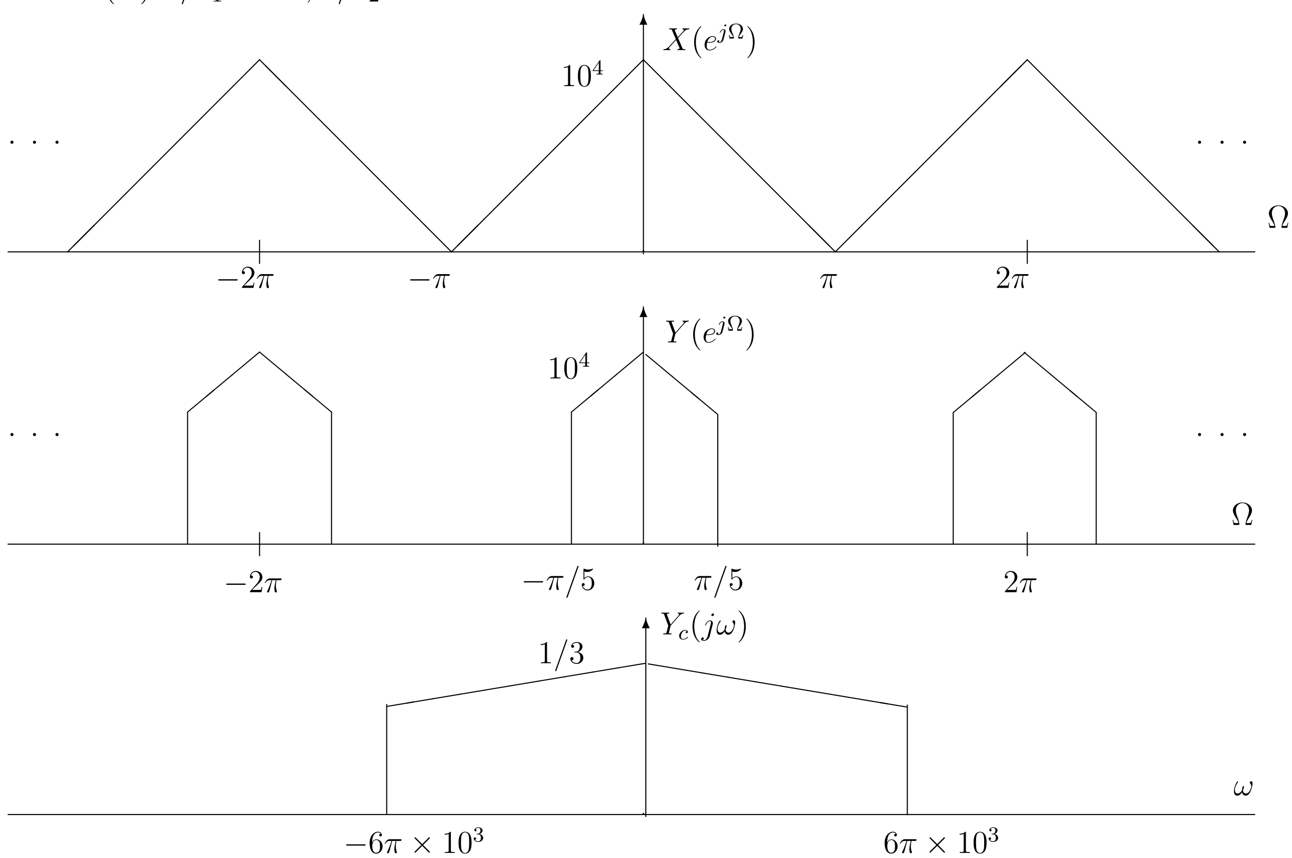


(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$



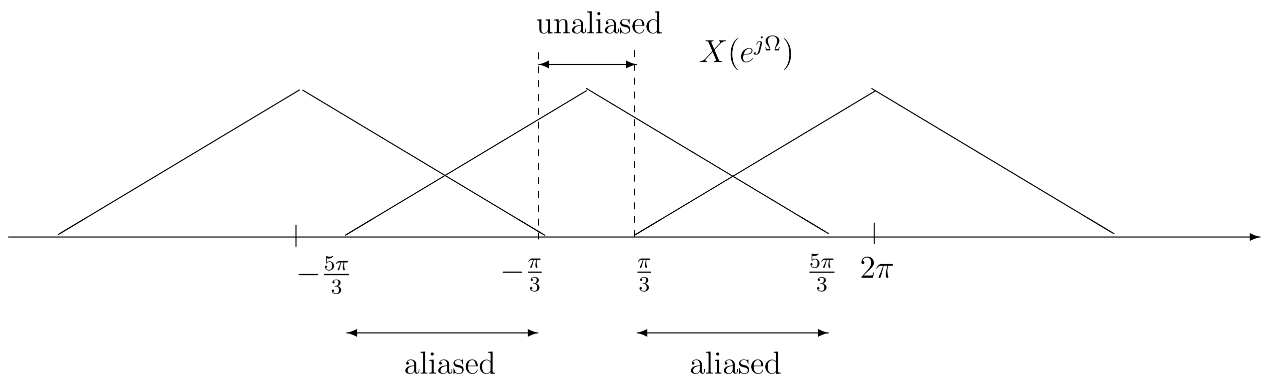


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



(b) From the figure below, it can be seen that the only portion of the spectrum that remains unaffected by the aliasing is $|\Omega| < \pi/3$. So if we choose $\Omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\omega) = \begin{cases} 1 & \text{for } |\omega| < \Omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$



Solution 1.18

Denote Ω_{in} as the frequency of the DT signal sampled from $x_c(t)$, which is within the range $|\Omega_{\text{in}}| \leq \pi$. With the possibility of aliasing, we have the relationship between the two frequencies as

$$\Omega_{\text{in}} = \omega_{\text{in}} \cdot T + 2k\pi,$$

where k is the integer which satisfies $|\omega_{\text{in}} \cdot T + 2k\pi| \leq \pi$.

The output signal of the DT system with input as $x_d[n] = x_c(nT)$ is

$$y_d[n] = |H(e^{j\Omega_{\text{in}}})| \cdot \cos(\Omega_{\text{in}} \cdot n + \theta + \angle H(e^{j\Omega_{\text{in}}})).$$

The final output signal $y_c(t)$ is identically 0 if $y_d[n] = 0$ for all n, resulting in $\Omega_{\text{in}} = \pm\Omega_o$ and therefore

$$\omega_{\text{in}} \cdot T + 2k\pi = \pm\Omega_o.$$

Finally, all values of ω_{in} with $y_c(t) = 0$ have the form

$$\omega_{\text{in}} = \frac{\pm\Omega_o - 2k\pi}{T},$$

in which k is any integer.

Solution 1.19

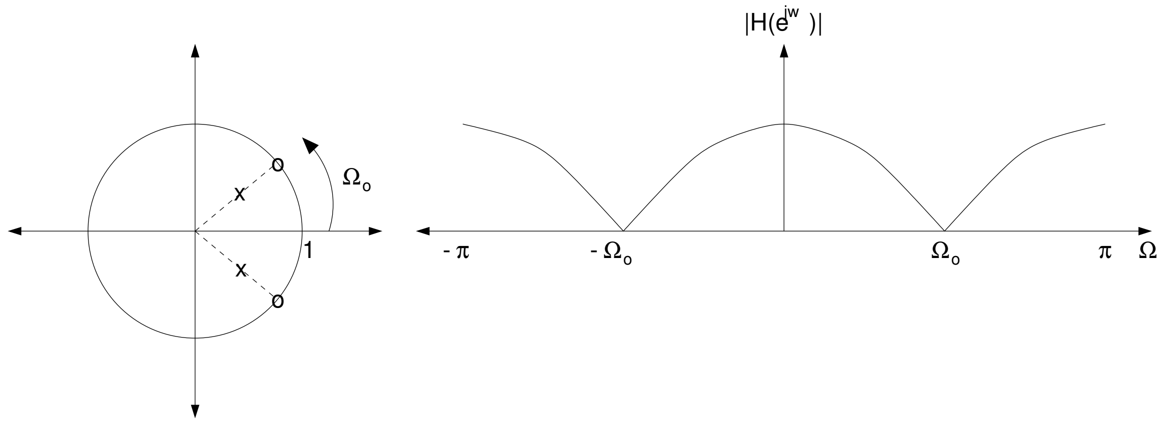
(a) $\omega_{\max} = \frac{\pi}{T} = 5000\pi$ rad/s.

(b) $\Omega_0 = 2\pi(60)T = 0.024\pi$ rads.

(c) To find poles and zeros, replace $e^{j\Omega}$ with z to get

$$H(z) = \frac{(1 - e^{j\Omega_0}z^{-1})(1 - e^{-j\Omega_0}z^{-1})}{(1 - \frac{1}{2}e^{j\Omega_0}z^{-1})(1 - \frac{1}{2}e^{-j\Omega_0}z^{-1})}$$

Poles: $z = \frac{1}{2}e^{\pm j\Omega_0}$. Zeros: $z = e^{\pm j\Omega_0}$



Solution 1.20

(a) The nonaliased band when $T = \frac{1}{2} \times 10^{-6}$ is $|\omega| < (\pi/T) = 2\pi \times 10^6$. Hence the Cases (i) and (ii) are sampled without aliasing. For these two cases, the filter acts effectively as an LTI CT filter that causes a $T/3$ delay for signals in the passband, and zeros out signals outside the passband — for more details, study our treatment of the half-sample delay example. Case (i) falls in the passband of the filter, because $\omega_0 T = (\pi/4) < (\pi/3)$. Case (ii) falls outside the passband, since $\omega_0 T = (\pi/2) > (\pi/3)$. So for Case (i), the output is

$$y_c(t) = \cos\Big(\frac{\pi}{2} \times 10^6 (t - \frac{T}{3})\Big)$$

while for Case (ii) the output is 0 for all time.

In Case (iii), the signal lies outside the nonaliased band, but yields the same samples as a signal in the nonaliased band that differs in frequency by an integer multiple of $(2\pi/T) = 4\pi \times 10^6$. Thus the samples generated by $\cos(\frac{7}{2}\pi \times 10^6 t)$ are the same as those generated by $\cos(-\frac{1}{2}\pi \times 10^6)t$, the latter signal being in the nonaliased band and also in the passband of the filter. The corresponding output is therefore

$$y_c(t) = \cos\Big(-\frac{1}{2}\pi \times 10^6 (t - \frac{T}{3})\Big) .$$

(b) For this, the nonzero part of $X_c(j\omega)$ must map entirely into the interval $|\Omega| < (\pi/3)$ so we require $2\pi \times 10^6 \times T \leq (\pi/3)$ or $T \leq \frac{1}{6} \times 10^{-6}$. The time shift $t_0 = (T/3)$.

(c) In this case, some aliasing is allowed as long as it falls in the stopband of the DT filter. We thus need $2\pi - (2\pi \times 10^6 \times T) \geq (\pi/3)$ so $T \leq \frac{5}{6} \times 10^{-6}$.

Solution 1.21

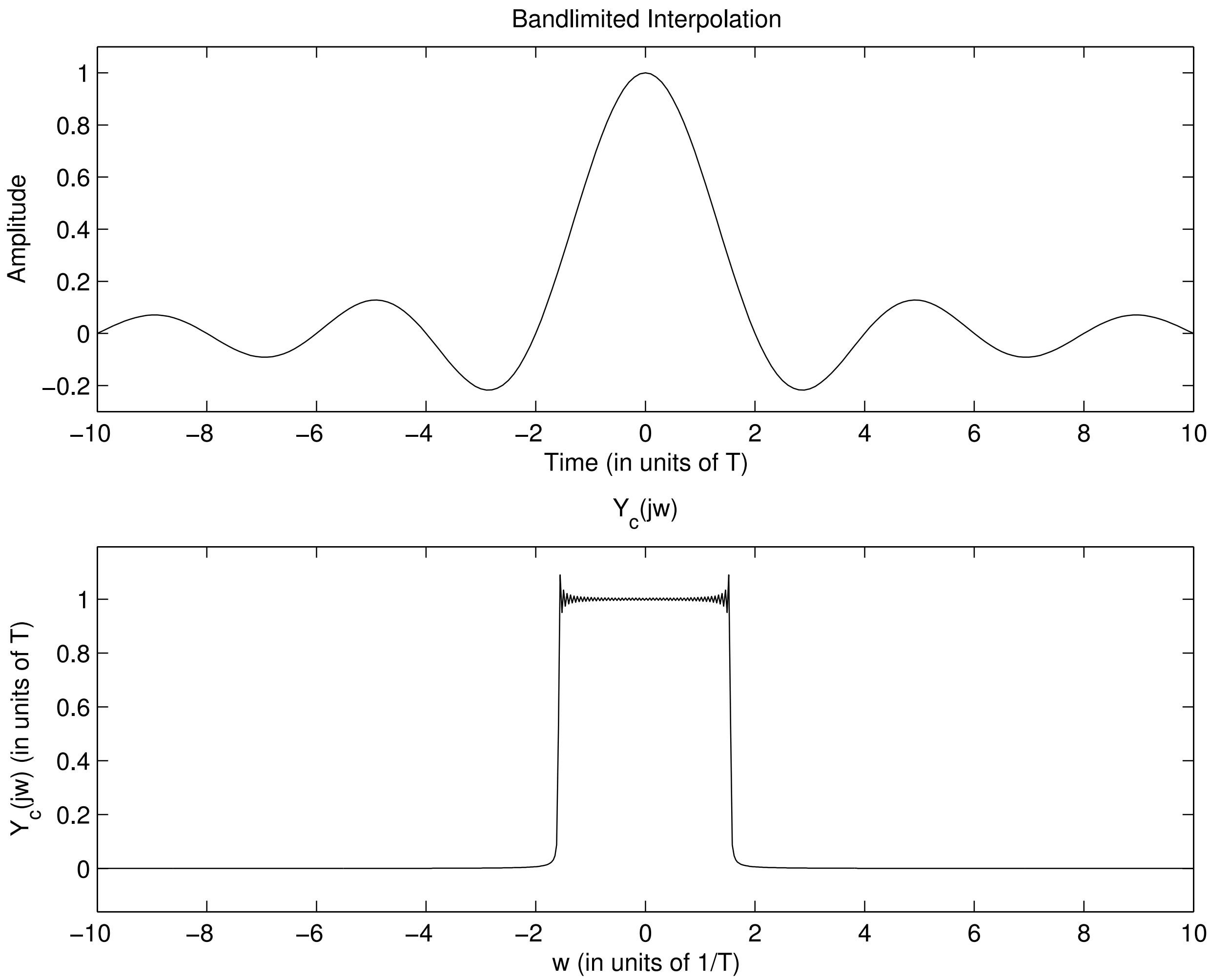
(a) We expect sinc-interpolation to yield the bandlimited signal

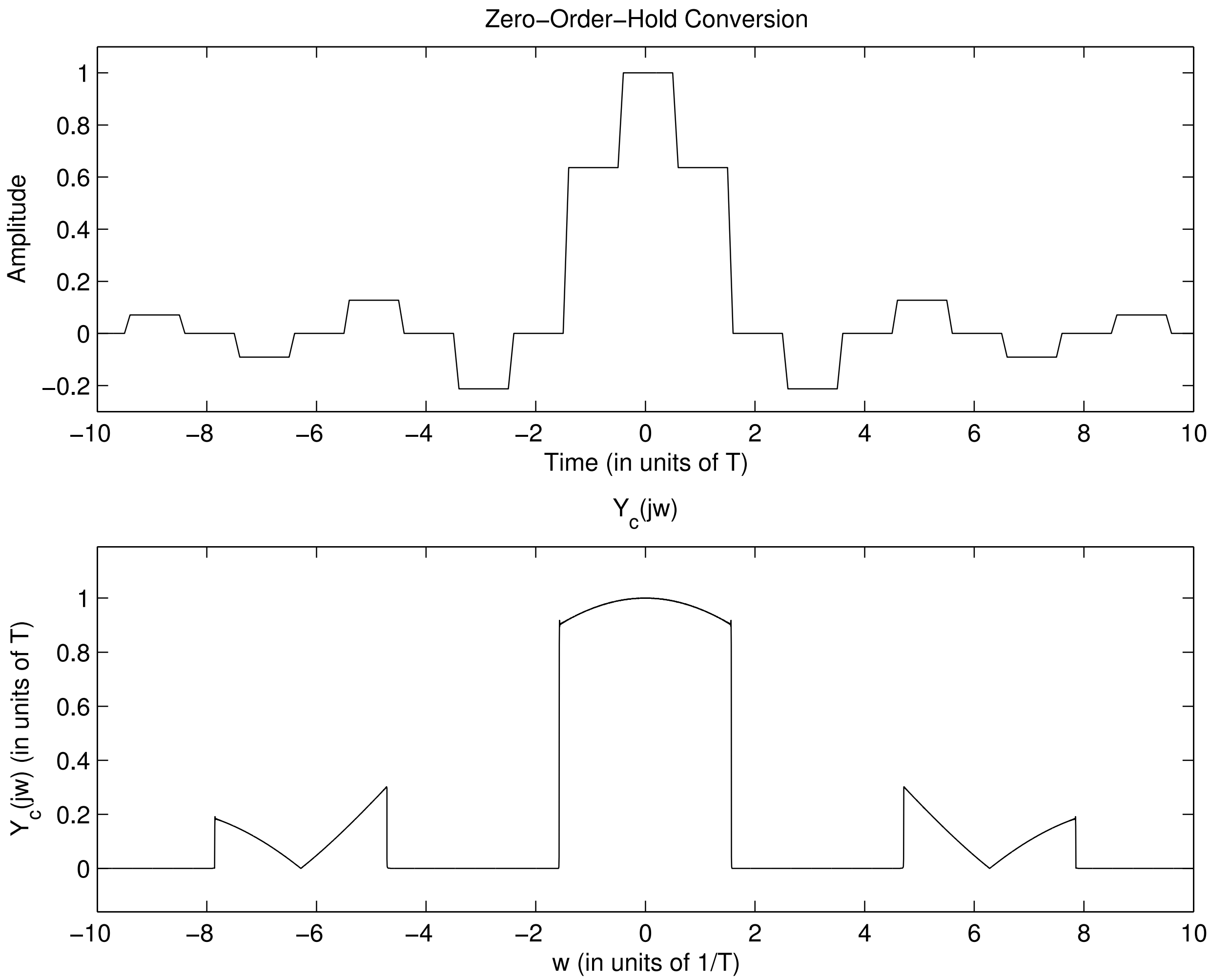
$$y_c(t) = \frac{\sin(\frac{\pi t}{2T})}{\frac{\pi t}{T}} \ ,$$

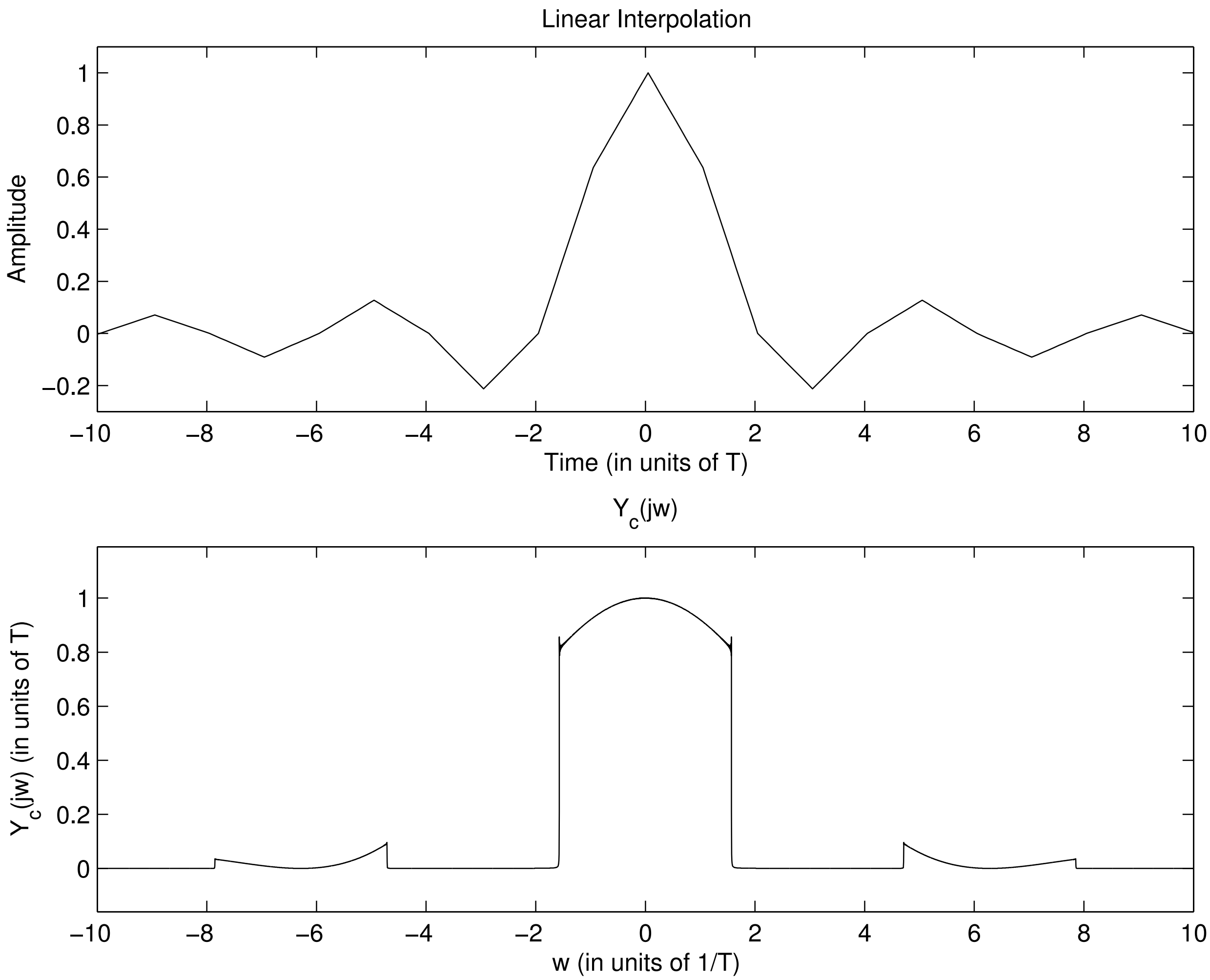
so $Y_c(j\omega)$ should be a "box" of height T for $|\omega| < (\pi/2T)$. This result can also be obtained by multiplication of $Y_d(e^{j\Omega})\Big|_{\Omega=\omega T}$ by the transform of the basic interpolating sinc pulse, namely a box of height T in the frequency domain that extends from $-\pi/T$ to π/T. Either way, our answer is consistent with the figure for this case shown at the end of the solutions, except that because $Y_c(j\omega)$ in the figure was obtained by numerically transforming $y_c(t)$, some computational artifacts are visible.

(b) We expect interpolation by a centered zero-order-hold (ZOH) to correspond to multiplication of $Y_d(e^{j\Omega})\Big|_{\Omega=\omega T}$ by the transform of the box that comprises the interpolating shape for the ZOH, so multiplication by a sinc in frequency domain. Again, the figure for this case is consistent with our expectations (but the figure labeled $Y_c(j\omega)$ is actually a figure of $|Y_c(j\omega)|$). Note the presence of *images* — at integer multiples of $2\pi/T$ — of the baseband component of interest to us. These images could be essentially removed by further CT filtering. The distortion of the baseband component caused by the sinc multiplication can be compensated for by corrective filtering in CT or, more easily, at the DT processing stage itself.

(c) We expect linear interpolation to correspond to multiplication of $Y_d(e^{j\Omega})\Big|_{\Omega=\omega T}$ by the transform of the triangle that comprises the interpolating shape in this case, so multiplication by sinc^2 in frequency domain. Again, the figure is consistent with this (but again the figure labeled $Y_c(j\omega)$ is actually a figure of $|Y_c(j\omega)|$), and once more shows the presence of images, but attenuated from the ZOH case. These images could similarly be essentially removed by further CT filtering. The distortion of the baseband component caused by the sinc^2 multiplication is more severe than in (b), but can again be compensated for by corrective CT or DT filtering.


Bandlimited Interpolation
Amplitude
1
0.8
0.6
0.4
0.2
0
−0.2
−10
−8
−6
−4
−2
0
2
4
6
8
10
Time (in units of T)
Y_c(jw)
Y_c(jw) (in units of T)
1
0.8
0.6
0.4
0.2
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
w (in units of 1/T)


Zero−Order−Hold Conversion
Amplitude
Time (in units of T)
Y_c(jw)
Y_c(jw) (in units of T)
w (in units of 1/T)


Linear Interpolation
Amplitude
1
0.8
0.6
0.4
0.2
0
−0.2
−10
−8
−6
−4
−2
0
2
4
6
8
10
Time (in units of T)
Y_c(jw)
Y_c(jw) (in units of T)
1
0.8
0.6
0.4
0.2
0
−10
−8
−6
−4
−2
0
2
4
6
8
10
w (in units of 1/T)

Solution 1.22

The first statement is false. The rate of sampling is related to how much priori information that is already known about the signal. For example, if we know a signal is constant at some value α for all time, then a single sample at any time will reconstruct the CT signal without even periodic sampling. For bandlimited signals with a narrow bandwidth, there are other sampling strategies like bandpass sampling theorem, which requires less bandwidth than twice the highest signal frequency.

The second statement is correct. Ideal bandlimited interpolation will reconstruct the signal from the samples.

The third statement is false. There may still be aliasing if the sampling frequency is exactly at twice the highest signal frequency. Consider $x(t) = \sin(2\pi t)$ which has frequency $f_H = 1$Hz, if the sampling frequency is $f = 2$Hz and $T = 1/f = 0.5$sec, then all samples are $x_d[n] = x(nT) = \sin(n\pi) = 0$. As a result, this CT signal cannot be recovered from the samples when we sample exactly at twice the highest signal frequency.

Solution 1.23

(a) The occurrence of the product of the input $x_b(t)$ and the velocity $dy(t)/dt$ causes the model to be nonlinear. However, it is time invariant because the equation has constant parameters; the expression relating the inputs and the response does not change with time. See Chapter 4 for elaboration.

(b) With $x_b(t) \equiv 0$, the nonlinear term drops out, and the model becomes linear (superpositions of solutions are solutions) and remains time invariant (parameters are constant). The equation can also be viewed as defining a mapping from the input $x_a(\cdot)$ to the response $y(\cdot)$, once initial conditions (e.g., $y(t)$ and $dy(t)/dt$ at $t = 0$) are specified; but this mapping is linear only when the initial conditions are zero (otherwise the map is "affine").

Solution 1.24

(a) The spectrum of the output signal is

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) = \begin{cases} X(j\omega), & |\omega \pm \omega_0| \leq \frac{\Delta}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Using Parseval's relation, the energy of the output signal becomes

$$\begin{aligned} \int_{-\infty}^{\infty} |y(t)|^2 \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 \mathrm{d}\omega \\ &= \frac{1}{2\pi} \left(\int_{-\omega_0-\frac{\Delta}{2}}^{-\omega_0+\frac{\Delta}{2}} |X(j\omega)|^2 \mathrm{d}\omega + \int_{\omega_0-\frac{\Delta}{2}}^{\omega_0+\frac{\Delta}{2}} |X(j\omega)|^2 \mathrm{d}\omega \right) \\ &= \frac{1}{\pi} \int_{\omega_0-\frac{\Delta}{2}}^{\omega_0+\frac{\Delta}{2}} |X(j\omega)|^2 \mathrm{d}\omega. \end{aligned}$$

The last step uses the fact that the signal is real-valued so its spectrum has even-symmetric magnitude.

(b) With sufficiently narrow bandwidth Δ, the magnitude of spectrum is approximately equal to the constant $|X(j\omega_0)|$ over the interval $\omega_0 - \Delta/2 \leq \omega \leq \omega_0 + \Delta/2$. Consequently, the energy of the output signal is

$$\int_{-\infty}^{\infty} |y(t)|^2 \mathrm{d}t \approx \frac{1}{\pi} \int_{\omega_0-\frac{\Delta}{2}}^{\omega_0+\frac{\Delta}{2}} |X(j\omega_0)|^2 \mathrm{d}\omega = \frac{\Delta}{\pi} |X(j\omega_0)|^2,$$

which is proportional to $\Delta |X(j\omega_0)|^2$.

(c) We focus on the magnitude of the spectrum of $y[n]$. The multiplication in the frequency domain results in

$$|Y(e^{j\Omega})| = |X(e^{j\Omega})| \cdot |H(e^{j\Omega})| = \begin{cases} 6, & |\Omega| < 0.25\pi \\ 0, & \text{otherwise.} \end{cases}$$

By Parseval's relation, the energy of the output signal is

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\Omega})|^2 \mathrm{d}\Omega = 9.$$

Solution 1.25

Assume the signal $g(t)$ is differentiable, with its derivative denoted as $g'(t)$. Since the impulse $\delta(t)$ is the derivative of the unit step $u(t)$, the impulse response of the LTI system is $g'(t)$. This could be obtained by either the Laplace transformation, or considering the input signal

$$v(t) = \frac{1}{A}(u(t) - u(t-A)), \quad A > 0,$$

whose output is $(g(t) - g(t-A))/A$ from LTI property. Taking the limit of A approaching 0 shows our claim that the impulse response of the LTI system is $g'(t)$.

The response $y(t)$ to the input signal $x(t)$ is the convolution of $x(t)$ with the impulse response, so $y(t) = x(t) * g'(t)$.

For the BIBO stability of the LTI system, a sufficient and necessary condition is that the impulse response is absolutely integrable: $\int_{-\infty}^{\infty} |g'(t)| \mathrm{d}t < \infty$.

Solution 1.26

Denote the Laplace transformation of $x(t)$ and $y(t)$ as $X(s)$ and $Y(s)$, respectively. For the original system, the differential equation implies

$$s^2Y(s) + 6sY(s) + 9Y(s) = s^2X(s) + 3sX(s) + 2X(s),$$

thus the transfer function of the original system is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 3s + 2}{s^2 + 6s + 9}, \quad \text{for } \mathrm{Re}\{s\} > -3.$$

The causal inverse system has the transfer function as

$$G(s) = \frac{1}{H(s)} = \frac{s^2 + 6s + 9}{s^2 + 3s + 2}, \quad \text{for } \mathrm{Re}\{s\} > -1.$$

The differential equation for the inverse system is the same as the given one, except that $y(t)$ is now interpreted as its input, and $x(t)$ as its output.

As for the impulse responses, we can express the transfer functions as

$$\begin{aligned} H(s) &= 1 + \frac{-3}{s+3} + \frac{2}{(s+3)^2}, \quad \text{for } \mathrm{Re}\{s\} > -3 \\ G(s) &= 1 + \frac{4}{s+1} + \frac{-1}{s+2}, \quad \text{for } \mathrm{Re}\{s\} > -1. \end{aligned}$$

Consequently, the impulse responses for the original and the inverse systems are

$$\begin{aligned} h(t) &= \delta(t) - 3e^{-3t}u(t) + 2te^{-3t}u(t), \\ g(t) &= \delta(t) + 4e^{-t}u(t) - e^{-2t}u(t). \end{aligned}$$

Solution 1.27

(a) (i) Simply plug the input/output pairs into the differential equation.

$$\begin{aligned} y(t) &= \frac{3}{4}e^{0.5t}u(t) \\ \frac{dy(t)}{dt} &= \frac{3}{8}e^{0.5t}u(t) + \frac{3}{4}\delta(t) \end{aligned}$$

Plugging into the equation, we get

$$\begin{aligned} 4(\frac{3}{8}e^{0.5t}u(t) + \frac{3}{4}\delta(t)) - 2(\frac{3}{4}e^{0.5t}u(t)) &= 3\delta(t) \\ \frac{3}{2}e^{0.5t}u(t) + 3\delta(t) - \frac{3}{2}e^{0.5t}u(t) &= 3\delta(t) \\ &\downarrow \\ 3\delta(t) &= 3\delta(t) \end{aligned}$$

We also have

$$\begin{aligned} y(t) &= -\frac{3}{4}e^{0.5t}u(-t) \\ \frac{dy(t)}{dt} &= -\frac{3}{8}e^{0.5t}u(-t) + \frac{3}{4}\delta(t) \end{aligned}$$

Plugging into the equation, we get

$$\begin{aligned} 4(-\frac{3}{8}e^{0.5t}u(t) + \frac{3}{4}\delta(t)) - 2(-\frac{3}{4}e^{0.5t}u(t)) &= 3\delta(t) \\ -\frac{3}{2}e^{0.5t}u(t) + 3\delta(t) + \frac{3}{2}e^{0.5t}u(t) &= 3\delta(t) \\ &\downarrow \\ 3\delta(t) &= 3\delta(t) \end{aligned}$$

(ii) *All* possible solutions are obtained by adding all possible *homogeneous* solutions to any *particular* solution — this follows from the basic theory of ordinary linear differential equations, in this case applied to the given differential equation with forcing function $x(t) = \delta(t)$.

One particular solution is the first one in (a)(i), namely $y_p(t) = \frac{3}{4}e^{0.5t}u(t)$, and the set of solutions to the homogeneous equation (i.e., with $x(t) \equiv 0$) is $y_h(t) = Ae^{0.5t}$ for arbitrary A, so the general solution is

$$y(t) = \frac{3}{4}e^{0.5t}u(t) + Ae^{0.5t} \ .$$

Choosing $A = 0$ yields the first solution in (a)(i), while picking $A = -\frac{3}{4}$ yields the second solution in (a)(i).

(iii), (iv) Recall that a *causal* CT system has an impulse response $h(t) = 0$ for $t < 0$, while a *stable* CT system has an impulse response which is absolutely integrable: $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$. Both expressions for $y(t)$ in (a)(i) are impulse responses since $x(t)$ is an impulse (we note in passing that these are also the only two solutions that are Laplace transformable, and therefore the only ones that you would arrive at using Laplace transform calculations). However, the solutions in (a)(ii) constitute *all* the possible impulse responses. This fact makes it easy to see that the only *causal* impulse response is the one obtained for $A = 0$ — i.e., the first impulse response in (a)(i) — while the only *stable* impulse response is that obtained with $A = -\frac{3}{4}$.

(b) Taking the z-transform,

$$\begin{aligned} Y(z) - 3z^{-1}Y(z) &= 2X(z) \\ \Rightarrow \frac{Y(z)}{X(z)} &= \frac{2}{1 - 3z^{-1}} \end{aligned}$$

(i) If the system is causal, then

$$\begin{aligned} h[n] &= 2(3)^n u[n] \\ \sum_{-\infty}^{\infty} |h[n]| &= \sum_{0}^{\infty} 2(3)^n \to \infty \end{aligned}$$

and this system is not BIBO stable.

(ii) If the system is stable, then

$$\begin{aligned} h[n] &= -2(3)^n u[-n-1] \\ \sum_{-\infty}^{\infty} |h[n]| &= \sum_{-\infty}^{-1} 2(3)^n = \frac{2(1/3)}{1 - 1/3} < \infty \end{aligned}$$

confirming that the system is BIBO stable. Since $h[n] \neq 0$ for $n < 0$, this impulse response corresponds to a noncausal system.

Solution 1.28

(a) The z-transform of $h_2[n]$ is $H_2(z) = 1 + z^{-1}$, and the z-transform of $h_2[-n]$ is $H_2(z^{-1}) = 1 + z$. The impulse response of $h[n]$ indicates its z-transform is

$$H(z) = 1 + 5z^{-1} + 10z^{-2} + 11z^{-3} + 8z^{-4} + 4z^{-5} + z^{-6},$$

therefore we have

$$H_1(z) = \frac{H(z)}{H_2(z)H_2(z^{-1})} = z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5},$$

and finally it follows

$$h_1[n] = \delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5].$$

(b) Since the z-transform of $x[n]$ is $X(z) = 1 - z^{-2}$, the z-transform of the output signal is

$$Y(z) = X(z)H(z) = 1 + 5z^{-1} + 9z^{-2} + 6z^{-3} - 2z^{-4} - 7z^{-5} - 7z^{-6} - 4z^{-7} - z^{-8},$$

and thus we have the output

$$y[n] = \delta[n] + 5\delta[n-1] + 9\delta[n-2] + 6\delta[n-3] - 2\delta[n-4] - 7\delta[n-5] - 7\delta[n-6] - 4\delta[n-7] - \delta[n-8].$$

Solution 1.29

(a) Since no explicit output is defined, linearity and time-invariance here are to be considered in the behavioral sense, not in the sense of mappings from input to output.

Linearity in the behavioral sense asks whether arbitrary linear combinations (or superpositions) of system behaviors are themselves (allowed or valid) behaviors of the system. If $r > 0$, then the quadratic term $-rx^2[n]$ results in nonlinearity of the system, because behaviors cannot be superposed.

Time invariance in the behavioral sense asks whether arbitrary time shifts of system behaviors are themselves (allowed or valid) behaviors. The given system, with an r that is constant rather than dependent on n, is time-invariant because the function that produces $x[n+1]$ from $x[n]$ does not depend on time.

(b) For $r \geq 0$, if $x[0] = 1/2$, then the signal $x[n]$ is bounded if and only if $0 \leq r \leq 4$.

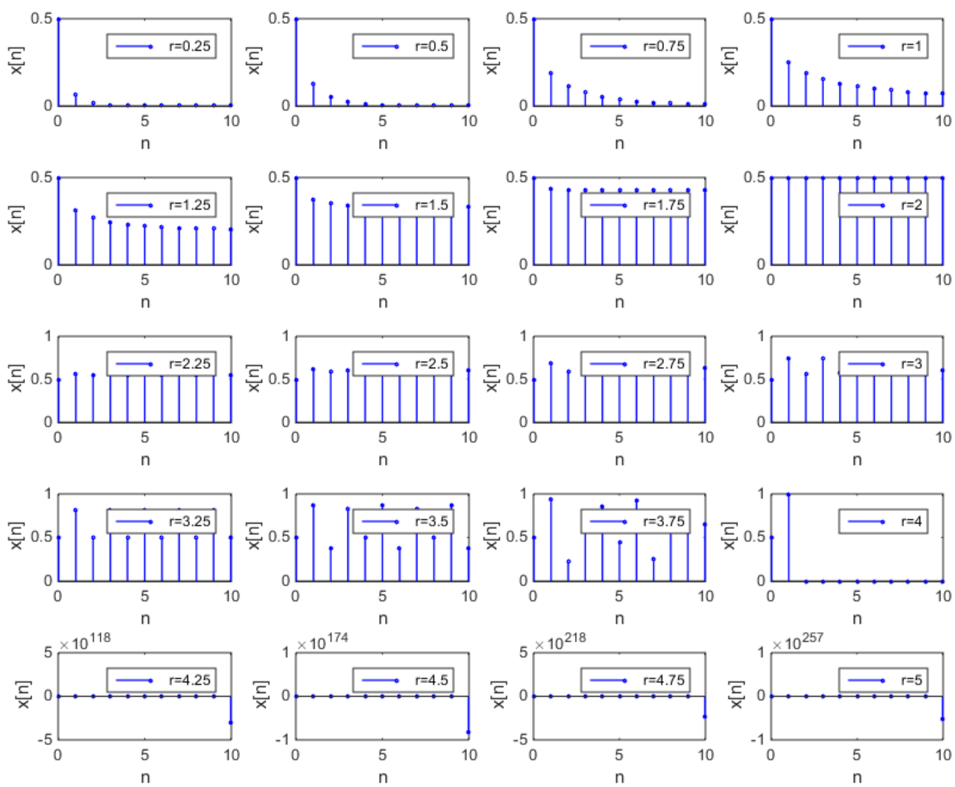
On one hand, if $0 \leq r \leq 4$, then $0 \leq rx(1-x) \leq 4x(1-x) \leq 1$ for any $x \in [0,1]$. Thus, if we know $0 \leq x[n] \leq 1$, then $x[n+1] = rx[n](1 - x[n]) \in [0,1]$. Due to the fact that $x[0] = 1/2 \in [0,1]$, by iteration it follows that $0 \leq x[n] \leq 1$ for all $n \geq 0$, which shows that $x[n]$ is bounded for $0 \leq r \leq 4$.

On the other hand, if $r > 4$, then $x[1] = rx[0](1 - x[0]) = r/4 > 1$. The second iteration shows that $x[2] = rx[1](1 - x[1]) < 0$. Once $x[n] < 0$, the iteration indicates that $x[n+1] < 0$, which says all $x[n] < 0$ with $n \geq 2$. For $n \geq 2$, we have

$$|x[n+1]| = r \cdot |x[n]| \cdot (1 - x[n]) > 4|x[n]| \cdot 1 = 4|x[n]|.$$

Thus, the magnitude of $x[n]$ grows at least exponentially and becomes unbounded.

(c) With MATLAB, the values of $x[n]$ are shown in the figure below, with $r = k/4$ where the integer k is within $1 \leq k \leq 20$. The figures support the result in part (b) that for $r \geq 0$, the signal $x[n]$ is bounded if and only if $0 \leq r \leq 4$.


r=0.25
r=0.5
r=0.75
r=1
r=1.25
r=1.5
r=1.75
r=2
r=2.25
r=2.5
r=2.75
r=3
r=3.25
r=3.5
r=3.75
r=4
r=4.25
r=4.5
r=4.75
r=5
x[n]
n
$\times 10^{118}$
$\times 10^{174}$
$\times 10^{218}$
$\times 10^{257}$

Solution 1.30

For clarity, in this solution we denote $x_1[n] = (-1)^n$ and $x_2[n] = (-1)^{n+1}$. The output of $x_k[n]$ is $y_k[n]$ for $k = 1, 2$.

(a) In this part, $y_1[n] = y_2[n] = 1$.

  (i) The system cannot be linear. Since $x_2[n] = (-1)x_1[n]$, if the system is linear, then the output satisfies $y_2[n] = (-1)y_1[n]$. Since $y_1[n] = y_2[n] = 1$, the statement $y_2[n] = (-1)y_1[n]$ is violated and thus the system is not linear.

  (ii) The system could be time-invariant. An example is $y[n] = |x[n]|$, which satisfies the input-output pairs in this part. The time-invariance of this system can be seen from the fact that if $x_0[n]$ has output $y_0[n]$, then the output for $x'[n] = x_0[n - n_0]$ is $y'[n] = |x'[n]| = |x[n - n_0]| = y[n - n_0]$.

(b) In this part, $y_1[n] = 1$ and $y_2[n] = -1$.

  (i) The system could be linear. As an example, $y[n] = (-1)^n \cdot x[n]$ satisfies both pairs of input and output signals. The linearity of this system can be shown as follows. If $y_a[n]$ and $y_b[n]$ are the output signals of $x_a[n]$ and $x_b[n]$, then the output signal for $x_c[n] = q_1 x_a[n] + q_2 x_b[n]$ is $y_c[n] = (-1)^n x_c[n] = q_1(-1)^n x_a[n] + q_2(-1)^n x_b[n] = q_1 y_a[n] + q_2 y_b[n]$.

  (ii) The system cannot be time-invariant. Since $x_2[n] = x_1[n + 1]$, if the system is time-invariant, then the output signal must satisfy $y_1[n] = y_2[n+1]$. Since the true output signals are against this time-shift relationship, the system cannot be time-invariant.

Solution 1.31

None of these systems *"must"* satisfy a convolutional relationship. A counter example is that each system has the specified output if the input signal happens to be the ones in the experiment, and the output is all 0 for all other input signals. The system above is definitely not linear and thus it does not satisfy a convolutional relationship.

We denote the input and output pairs for system S_k as $x_k[n]$ and $y_k[n]$ ($k = 1,..,4$), respectively.

System S_1 *"definitely cannot"* satisfy a convolution relationship. Assume that the system satisfies a convolution relationship, then it has to be LTI; it can be observed that $\delta[n] = x_1[n] - (1/2)x_1[n-1]$, and thus the impulse response for the LTI system should be $h[n] = y_1[n] - (1/2)y_1[n-1] = 0$. However, such an LTI system contradicts the input-output pair since it should always output 0 for all input signals. This analysis leads to the conclusion that S_1 is not LTI and thus has no convolution relationship.

System S_2 *"possible could"* satisfy a convolution relationship. The z-transform of $y_2[n]$ and $x_2[n]$ are $Y_2(z) = z^{-1}/(1-z^{-1}/2)$ and $X_2(z) = 1/(1-z^{-1}/3)$ with common region of convergence as $|z| > 1/2$. Thus, if we choose $H_2(z) = Y_2(z)/X_2(z) = (1 - z^{-1}/3) \cdot z^{-1}/(1 - z^{-1}/2)$ for $|z| > 1/2$, then we have $y_2[n] = x_2[n] * h_2[n]$ where $h_2[n]$ is the impulse response for the system function $H_2(z)$.

System S_3 *"definitely cannot"* satisfy a convolution relationship. A system with convolution relationship is LTI, which cannot introduce signal component in the output on frequencies that are not included in the input signal. However, $y_3[n]$ has frequency $3\pi/4$ while $x_3[n]$ only has frequency $\pi/2$.

System S_4 *"possible could"* satisfy a convolution relationship. We can equivalently write the output signal in the experiment as $y_4[n] = 4e^{j(9\pi/4)(n-1)} = 4e^{j(\pi/4)(n-1)} = 4x_4[n-1]$, which shows that the output has the same frequency as the input. The system with impulse response $h_4[n] = 4\delta[n-1]$ is an example of such LTI systems that $y_4[n] = x_4[n] * h_4[n]$.

Solution 1.32

(i) This statement is false. A counterexample is the identity system $y(t) = x(t)$, which is LTI. Every input signal is an eigenfunction of this system with eigenvalue 1.

(ii) This statement is false. A stable LTI system has absolutely integrable impulse response. However, if $h(t)$ is periodic and nonzero, then the integration of $|h(t)|$ diverges and thus it is not absolutely integrable.

(iii) This statement is false. For the LTI system $h[n] = \delta[n-1]$ that is causal and stable, if we denote its causal inverse as $g[n]$, then $\delta[n] = h[n] * g[n]$. At $n = 0$, the above equation becomes $1 = \sum_{k=-\infty}^{\infty} h[k]g[-k] = g[-1] = 0$, i.e. $1 = 0$. Thus, $h[n] = \delta[n-1]$ does not have a causal inverse.

Solution 1.33

For a real DT LTI system with frequency response $H(e^{j\Omega})$, if the input signal is $x[n] = \cos(\Omega_0 n)$, then the output signal is $y[n] = |H(e^{j\Omega_0})| \cos(\Omega_0 n + \angle H(e^{j\Omega_0}))$.

From the input/output pairs, equations can be constructed at $\Omega_0 = \pi/2$ and $\Omega_0 = \pi$ as follows

$$\begin{aligned} H(e^{j\pi/2}) &= 1e^{j0} \\ H(e^{j\pi}) &= 2e^{j0}. \end{aligned}$$

Plugging in the form of $H(e^{j\Omega})$, the equations become

$$\begin{aligned} \frac{1-a+2}{1-b} &= 1 \\ \frac{1+a+2}{1+b} &= 2. \end{aligned}$$

Thus, the parameters can be solved as $a = 5, b = 3$.

Solution 1.34

(a) Determination of $H(s)$ at the value $s = -3 + j$ (or $s = -3 - j$) is sufficient to obtain the closed form expression for $y(t)$. Since $h(t)$ is real, the Laplace transformation satisfies $H(s^*) = (H(s))^*$, so the value of $H(s)$ at $s = -3 \pm j$ can be obtained by conjugating the value at $s = -3 \mp j$.

If the output of the system $H(s)$ is well defined for the input signal $x_0(t) = e^{s_0 t}$ where s_0 is a complex number, then the output is $y_0(t) = H(s_0) \cdot x_0(t)$, i.e. $x_0(t)$ is an eigenfunction of $H(s)$ with eigenvalue $H(s_0)$. Since $x(t) = e^{-3t} \cos t = (1/2) \cdot (e^{(-3+j)t} + e^{(-3-j)t})$, the output signal has the form

$$y(t) = \frac{1}{2}\left(H(-3+j)e^{(-3+j)t} + H(-3-j)e^{(-3-j)t}\right) = \mathrm{Re}\{H(-3+j)e^{(-3+j)t}\},$$

in which we used the conjugate symmetric property of $H(s)$ and $\mathrm{Re}\{\cdot\}$ denotes the real part of a complex number.

(b) If we denote $H(-3 + j) = A - jB$, then the output is simplified as

$$y(t) = \mathrm{Re}\{H(-3+j)e^{(-3+j)t}\} = Ae^{-3t}\cos t + Be^{-3t}\sin t = e^{-3t}(A\cos t + B\sin t).$$

From $y(0) = 0$ and $\dot{y}(0) = 1$, the following equations are constructed

$$A = 0, \ -3A + B = 1,$$

and the constants are solved as $A = 0, B = 1$.

Solution 1.35

First, the DTFT for the periodic unit sample "train" $x[n]$ is

$$X(e^{j\Omega}) = \frac{2\pi}{12} \cdot \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi m}{12}\right),$$

which can be verified by the inverse DTFT:

$$\frac{1}{2\pi} \int_{-\epsilon}^{2\pi-\epsilon} X(e^{j\Omega}) e^{j\Omega n} \mathrm{d}\Omega = \frac{1}{12} \cdot \sum_{m=0}^{11} e^{\frac{2\pi mn}{12}} = \begin{cases} 1, & n = 12k \\ 0, & \text{otherwise} \end{cases}$$

which is equivalent to the unit sample "train" $x[n]$. Thus, the signal $x[n]$ has components with frequencies $\Omega = 2\pi m/12$ for $m = 0, \ldots, 11$ (equivalently, $m = -6, \ldots, 5$), each of which has magnitude $1/12$ (in the time domain).

Then, it can be seen that the system $H(e^{j\Omega})$ is a cascade of the ideal lowpass filter with cutoff frequency at $\pi/5$ and the system that is a delay of 3 samples. The output $y[n]$ can be determined by lowpassing $x[n]$ and delaying the result by 3 samples.

Finally, lowpassing $x[n]$ with cutoff frequency $\pi/5$ remains five components with frequencies $\Omega = 2\pi m/12$ for $m = -2, -1, 0, 1, 2$; applying the delay of 3 samples obtains the output signal

$$y[n] = \frac{1}{12} \sum_{m=-2}^{2} e^{j\frac{2\pi m(n-3)}{12}} = \frac{1}{12}\left(1 + 2\cos\left(\frac{\pi(n-3)}{6}\right) + 2\cos\left(\frac{\pi(n-3)}{3}\right)\right).$$

Solution 1.36

(a) The real unit sample response corresponds to a conjugate symmetric frequency response, the phase of which is an odd function. Thus, $g(\Omega) = \Omega^3$.

(b) $\sum_{n=-\infty}^{\infty} h_d[n] = \sum_{n=-\infty}^{\infty} h_d[n]e^{-jn0} = H_d(e^{j0}) = 0$.

(c) By Parseval's identity, the energy of signal is

$$\sum_{n=-\infty}^{\infty} h_d^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{j\Omega})|^2 \mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega^2 \mathrm{d}\Omega = \frac{\pi^2}{3}.$$

Solution 1.37

(a) Given that $H(s)$ is rational and has zeroes at $s = 0$ and $s = 6$ and poles at $s = 1$ and $s = 5$, we know that $H(s) = K\frac{s(s-6)}{(s-1)(s-5)}$.

$$H(3) = K\frac{(3)(-3)}{(2)(-2)} = \frac{9K}{4} .$$

Since we are given that $H(3) = 9$, we know that $K = 4$. Hence,

$$H(s) = \frac{4s(s-6)}{(s-1)(s-5)} .$$

(b) No. In order to satisfy the condition $x(t) = y(t)$, we would need to set

$$G(s) = \frac{(s-1)(s-5)}{4s(s-6)} .$$

In order for $G(s)$ to be causal, its region of convergence needs to be $Re\{s\} > 6$ (right of the rightmost pole). However, in this case, the regions of convergence of the systems $H(s)$ and $G(s)$ would have no intersection, and hence their cascade would not be well-defined.

(c) No. In order for $G(s)$ to be stable, its region of convergence needs to include the $j\Omega$-axis. However, the $j\Omega$-axis cannot be in the region of convergence because there is a pole at the origin.

(d) Yes. We would need to set

$$G(s) = \frac{(s-1)(s-5)}{4s(s-6)} .$$

The region of convergence of $G(s)$ would need to be $0 < Re\{s\} < 6$. Note that the ROC of a two-sided signal consists of a strip in the s-plane.

Solution 1.38

(a)

$$
\begin{aligned}
H(e^{j\Omega}) &= \frac{4-9e^{-j\Omega}}{4\cos\Omega + 2e^{-j\Omega} - 9} \\
&= \frac{4-9e^{-j\Omega}}{2e^{j\Omega} + 4e^{-j\Omega} - 9}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
H(z) &= \frac{4-9z^{-1}}{2z + 4z^{-1} - 9} \\
&= \frac{4z-9}{2z^2 - 9z + 4} \\
&= \frac{4z-9}{(2z-1)(z-4)} \\
&= \frac{2z-\frac{9}{2}}{(z-\frac{1}{2})(z-4)}
\end{aligned}
$$

Using partial fractions, $\alpha_1 = 1$ and $\alpha_2 = 1$.

The ROC is $\frac{1}{2} < |z| < 4$.

(b) The system and its impulse response are not causal because the ROC is not the exterior of a circle outside the outermost pole.

(c)

$$
\begin{aligned}
H(z) &= \frac{1}{z-\frac{1}{2}} + \frac{1}{z-4} \\
&= \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-4z^{-1}}
\end{aligned}
$$

Using the z-transform table,

$$
h[n] = (\frac{1}{2})^{n-1}u[n-1] - 4^{n-1}u[-n]
$$

(d) A signal $x[n] = z_o^n$ is an eigenfunction of a DT LTI system, provided z_o is in the ROC of the system function. Therefore, when we input an "everlasting exponential" z_o^n to the system, what we get out is that exponential weighted by the system response at that (complex) frequency. Specifically, given $h[n]$ is the impulse response of an LTI system:

$$
\begin{aligned}
x[n] &= z_o^n \\
y[n] &= (h*x)[n] \\
&= H(z_o)z_o^n
\end{aligned}
$$

Note this only holds if z_o is in the ROC of $H(z)$, because otherwise $\sum_k h[n]z_o^{-k}$ is not finite.

In our example $x[n] = 3^n$ and $H(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-4z^{-1}}$ with ROC $1/2 < |z| < 4$. Therefore, $y[n]$ is:

$$\begin{aligned} y[n] &= (x * h)[n] \\ &= (3^n * h)[n] \\ &= 3^n H(3)\ , \end{aligned}$$

since $z = 3$ lies in the ROC of $H(z)$, so

$$\begin{aligned} y[n] &= 3^n\left(\frac{3^{-1}}{1-\frac{1}{2}3^{-1}} + \frac{3^{-1}}{1-4(3^{-1})}\right) \\ &= 3^n(-0.6) \end{aligned}$$

(e) The z-transform of the input signal is:

$$X(z) = \frac{1}{1-3z^{-1}}, \quad |z| > 3\ .$$

The z-transform of the output signal will be the product of the z-transforms of the input signal and the system function:

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \left(\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{z^{-1}}{1-4z^{-1}}\right)\frac{1}{1-3z^{-1}} \\ &= \frac{4z-9}{(2z-1)(z-4)}\frac{z}{z-3} \\ &= \frac{-2/5}{2z-1} + \frac{4}{z-4} + \frac{-9/5}{z-3} \\ &= \frac{-\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}} + \frac{4z^{-1}}{1-4z^{-1}} + \frac{-\frac{9}{5}z^{-1}}{1-3z^{-1}} \end{aligned}$$

Using the z-transform table, with $3 < |z| < 4$ to ensure that we are in the regions of convergence for both $H(z)$ and $X(z)$, we get

$$y[n] = -\frac{1}{5}\left(\frac{1}{2}\right)^{n-1}u[n-1] - 4(4)^{n-1}u[-n] - \frac{9}{5}(3)^{n-1}u[n-1]$$

For $n \leq 0$ we have $y[n] = -4(4)^{n-1}$, so $z_0 = 4$.

Solution 1.39

(a) Using the DTFT synthesis equation with $n = 0$,

$$x[0] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\Omega})e^{j\Omega 0}d\Omega = \frac{1}{2\pi}(j\pi + \frac{\pi}{4}(\frac{1}{2}) + \pi + \pi + \frac{\pi}{4}(\frac{1}{2}) - j\pi) = \frac{9}{8}$$

(b) Using the DTFT analysis equation with $\Omega = 0$,

$$0 = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} x[n]e^{-j0n} = X(e^{j0})$$

The DTFTs with $X(e^{j0}) = 0$ are A, C, D, E, F, and G.

(c) Using Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} (x[n])^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{<2\pi>} |X(e^{j\Omega})|^2 d\Omega = \frac{1}{2\pi}\left(\frac{\pi}{4}\right)\left(\frac{1}{4}\right)2 = \frac{1}{16}$$

(Note that the first equality holds because $x[n]$ is real, a fact we can deduce from the symmetry of the DTFT.)

(d)

$$X(e^{j\Omega}) = \frac{1}{2\pi}\mathcal{F}\left\{\frac{\cos(3\pi n/4)}{\pi}\right\} \otimes \mathcal{F}\left\{\frac{\sin(\pi n/2)}{n}\right\} = \frac{1}{2\pi}A(e^{j\Omega}) \otimes B(e^{j\Omega})$$

where $\otimes$ denotes circular convolution, and where in the interval $|\Omega| < \pi$ we have

$$A(e^{j\Omega}) = \delta(\Omega - \frac{3\pi}{4}) + \delta(\Omega + \frac{3\pi}{4})$$

$$B(e^{j\Omega}) = \begin{cases} \pi, & |\Omega| < \frac{\pi}{2} \\ 0, & |\Omega| > \frac{\pi}{2} \end{cases}$$

More explicitly,

$$X(e^{j\Omega}) = \frac{1}{2\pi}\int_{<2\pi>} A(e^{j\theta})B(e^{j(\Omega-\theta)})d\theta$$

Using flip-and-slide convolution with one period of $A(e^{j\theta})$ and the full, periodic $B(e^{j(\Omega-\theta)})$ for shifts Ω in the range $[-\pi, \pi]$, one obtains:

$$X(e^{j\Omega}) = \begin{cases} 2\left(\frac{1}{2}\right), & -\pi < \Omega < -\frac{3\pi}{4} \\ \frac{1}{2}, & -\frac{3\pi}{4} < \Omega < -\frac{\pi}{4} \\ 0, & -\frac{\pi}{4} < \Omega < \frac{\pi}{4} \\ \frac{1}{2}, & \frac{\pi}{4} < \Omega < \frac{3\pi}{4} \\ 2\left(\frac{1}{2}\right), & \frac{3\pi}{4} < \Omega < \pi \end{cases} \Longrightarrow \text{DTFT 'F'}$$

(e) Signals that are even in time have real DTFTs. These are D,F,G,H.

(f) Absolutely summable in time (l_1) means that the DTFT does not have any discontinuities (remember a sinc is not absolutely summable and therefore its DTFT is a box, which has two discontinuities). None of the DTFTs correspond to absolutely summable signals.

(g) $H(e^{j0}) = \sum x[n]$, therefore the sum not equal to zero means that the DC value of the DTFT is non-zero. DTFTs B and H both have non-zero DC gain $= 1$.

(h) $H(e^{j\pi}) = \sum(-1)^n x[n]$, and as we see only DTFT F has a value of 1 at $\Omega = \pi$.

Solution 1.40

(a) The Fourier transform of the input and output is:

$$X(e^{j\Omega}) = \frac{1-\frac{1}{4}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

Thus the frequency response is:

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1-\frac{1}{2}e^{-j\Omega}}{(1-\frac{1}{3}e^{-j\Omega})(1-\frac{1}{4}e^{-j\Omega})} = -\frac{2}{1-\frac{1}{3}e^{-j\Omega}} + \frac{3}{1-\frac{1}{4}e^{-j\Omega}}$$

We get the impulse response by taking the inverse Fourier transform of the frequency response:

$$h[n] = -2(\frac{1}{3})^n u[n] + 3(\frac{1}{4})^n u[n]$$

(b) From the transfer function expression in (a):

$$\frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1-\frac{1}{2}e^{-j\Omega}}{(1-\frac{1}{3}e^{-j\Omega})(1-\frac{1}{4}e^{-j\Omega})} = \frac{1-\frac{1}{2}e^{-j\Omega}}{1-\frac{7}{12}e^{-j\Omega}+\frac{1}{12}e^{-2j\Omega}}$$

Thus:

$$Y(e^{j\Omega}) - \frac{7}{12}e^{-j\Omega}Y(e^{j\Omega}) + \frac{1}{12}e^{-2j\Omega}Y(e^{j\Omega}) = X(e^{j\Omega}) - \frac{1}{2}e^{-j\Omega}X(e^{j\Omega})$$

Taking the inverse Fourier transform, we get a difference equation relating the system input and output:

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = x[n] - \frac{1}{2}x[n-1]\ .$$

Solution 1.41

(a) All that's needed is to apply Parseval's identity to the signal $v[n] = w[n] - y[n]$, and use linearity of the Fourier transform. This yields

$$\mathcal{E} = \frac{1}{2\pi} \int_{|\Omega|<\pi} |W(e^{j\Omega}) - Y(e^{j\Omega})|^2 \, d\Omega.$$

(b) Since it is required that $Y(e^{j\Omega}) = 0$ for $\pi/4 \leq |\Omega| \leq \pi$, we have

$$\mathcal{E} = \frac{1}{2\pi} \int_{|\Omega|<\pi/4} |W(e^{j\Omega}) - Y(e^{j\Omega})|^2 d\Omega + \frac{1}{\pi} \int_{\pi/4\leq\Omega\leq\pi} |W(e^{j\Omega})|^2 d\Omega.$$

where we have invoked the evenness of the magnitude of the fourier transform of a real DT signal to conclude that the integral over $(\pi/4) \leq \Omega \leq \pi$ is identical to the integral over $-\pi \leq \Omega \leq -\pi/4$, allowing the two to be replaced by twice the integral over range $(\pi/4) \leq \Omega \leq \pi$. The second term in the above expression cannot be changed by choice of $Y(e^{j\Omega})$. The first term is nonnegative, and can be made zero by choosing $W(e^{j\Omega}) = Y(e^{j\Omega})$ for $|\Omega| < \pi/4$. This is therefore the optimal choice. The resulting minimal value of $\mathcal{E}$ is

$$\min \mathcal{E} = \frac{1}{\pi} \int_{(\pi/4)\leq\Omega\leq\pi} |W(e^{j\Omega})|^2 d\Omega.$$

(c) Since $Y(e^{j\Omega})$ is essentially a low-pass filtered version of $W(e^{j\Omega})$, cut off for $|\Omega| > \pi/4$, we can use the inverse Fourier transform definition to write

$$y[n] = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} W(e^{j\Omega}) e^{j\Omega n} d\Omega.$$

Solution 1.42

First, the C/D converter has output $x_d[n] = x_c(nT_1) = \cos(\pi n/3)$. Then, the squarer has output

$$z_d[n] = x_d^2[n] = \cos^2\left(\frac{\pi n}{3}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{3}\right).$$

Afterwards, the DC component in $z_d[n]$ is blocked by the DT filter $H_d(e^{j\Omega})$ and the sinusoid with frequency $2\pi/3$ has a gain of 2, and thus the output signal of the DT filter is a pure sinusoid $y_d[n] = \cos(2\pi n/3)$. Finally, the D/C convertor maps the DT frequency $2\pi/3$ to the CT frequency $2\pi/(3T_2)$, with the output CT signal as

$$y_c(t) = \cos\left(\frac{2\pi t}{3T_2}\right) = \cos\left(\frac{\pi t}{3T}\right).$$

Solution 1.43

(a) The discontinuity in $c(t)$ means that its frequency content extends to all frequencies (compare, for instance, with the transform of the unit step – its transform magnitude falls off only as $1/|\omega|$. So $c(t)$ is not bandlimited.

(b) If $x(t)$ is bandlimited and

$$y(t) = x(t - 0.75T),$$

from the time shift property,

$$Y(j\omega) = e^{-j0.75\omega T} X(j\omega)$$

The frequency response, $H_{eff}(j\omega)$, that achieves this is therefore given as

$$H_{eff}(j\omega) = e^{-j0.75\Omega T}$$

We know that

$$H_{eff}(j\omega) = H_d(e^{j\Omega})|_{\Omega=\omega T}$$

So,

$$H_d(e^{j\Omega}) = H_{eff}(j\frac{\Omega}{T}) = e^{-j0.75\Omega}$$

(c) Although the problem only asks for $h_d[0]$, not all of $h_d[n]$, the full calculation is shown here as it's not much harder. The DT impulse response can be found by evaluating the transform

$$\begin{aligned}
h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{J\Omega})e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j0.75\Omega} e^{j\Omega n} d\Omega \\
&= \left.\frac{e^{j\Omega(n-0.75)}}{2\pi j(n-0.75)}\right|_{-\pi}^{\pi} \\
&= \frac{e^{j\pi(n-0.75)} - e^{-j\pi(n-0.75)}}{2\pi j(n-0.75)} \\
&= \frac{\sin(\pi(n-0.75))}{\pi(n-0.75)} \\
h_d[0] &= \frac{\sin(-0.75\pi)}{(-0.75\pi)} = \frac{\sin(0.75\pi)}{0.75\pi} \\
\sum_{n=-\infty}^{\infty} h_d[n] &= H_d(e^{j0}) = e^{-j0.75\Omega}|_{\Omega=0} = 1 \\
\sum_{n=-\infty}^{\infty} (h_d[n])^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{j\Omega})|^2 d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \ d\Omega = 1
\end{aligned}$$

(d) Since $c(t)$ produces the same sample values $x_d[n]$ as $b(t)$ does, the output will be the same in the two cases. This output is easy to write down when the input is the bandlimited input $b(t)$: it will be $y(t) = b(t - 0.75T)$. So this is also the output when the input is $c(t)$.

Solution 1.44

(a) $x[n]$ is just uniform samples of the CT signal $x_c(t)$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \cos\left(2.2\pi n - \frac{\pi}{4}\right) \\ &= \cos\left(2\pi n + 0.2\pi n - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{5}n - \frac{\pi}{4}\right) \end{aligned}$$

Note that $x_c(t)$ is not bandlimited to $\frac{\pi}{T}$, i.e., $X_c(j\omega)$ is non-zero for some frequencies in the range $|\omega| > (\pi/T_1) = 10\pi$. Therefore aliasing occurs during sampling.

We next proceed to find $y_c(t)$. Because $x_c(t)$ is not appropriately bandlimited, the overall system will not be LTI for this input. However, we note that the sampled sequence $x[n]$ would be the same as above if the input were $x_a(t) = \cos(2\pi t - \frac{\pi}{4})$. Because $x[n]$ would be the same for the two input signals $x_c(t)$ and $x_a(t)$, $y[n]$ would be the same, and therefore $y_c(t)$ would be the same as well. Now, because $x_a(t)$ *is* bandlimited to $\frac{\pi}{T}$, the overall system (from $x_a(t)$ to $y_c(t)$) will be LTI with frequency response

$$H_c(j\omega) = \begin{cases} H_d(e^{j\Omega})|_{\Omega=\omega T}, & \text{for} |\omega| < \frac{\pi}{T} \\ \text{anything}, & \text{otherwise.} \end{cases} \tag{3}$$

(Note we can specify $H_c(j\omega)$ arbitrarily for $|\omega| > \frac{\pi}{T}$ because we have restricted the input to be within $|\omega| < \frac{\pi}{T}$.)

Then $H_c(j\omega) = e^{-j\frac{\omega T}{2}}$, $\forall \omega \in \mathbb{R}$ is a valid choice for the continuous time frequency response, which yields a simple impulse response, $h_c(t) = \delta(t - \frac{T}{2})$. Then,

$$\begin{aligned} y_c(t) &= \cos\left(2\pi(t - \frac{T}{2}) - \frac{\pi}{4}\right) \\ &= \cos\left(2\pi t - \frac{7\pi}{20}\right). \end{aligned}$$

Now, because $y_c(t)$ is the band-limited sinc interpolation of $y[n]$, the values it takes at integer multiples of T will be *equal* to the corresponding sample value; i.e.,

$$\begin{aligned} y[n] &= y_c(nT) \\ &= \cos\left(2\pi nT - \frac{7\pi}{20}\right) \\ &= \cos\left(\frac{\pi}{5}n - \frac{7\pi}{20}\right). \end{aligned}$$

(b) Now, note that this system is no longer LTI. However, changing the reconstruction interval T_2 *will not* change the DT signal $y[n]$ we obtained in the last part (although it would change the method we used to find it). Then, by going through the frequency-domain drill of finding $y_c(t)$ from the DTFT of $y[n]$, we get

$$\begin{aligned} y_c(t) &= \cos\left(\frac{\pi}{5T_2}t - \frac{7\pi}{20}\right) \\ &= \cos\left(\pi t - \frac{7\pi}{20}\right) \end{aligned}$$

Solution 1.45

(a) From the CTFT of sinc function and the time delay property, the CTFT of $y_c(t)$ is

$$Y_c(j\omega) = \begin{cases} 2Te^{-j0.7\omega T}, & |\omega| < \frac{\pi}{T} \\ 0, & \text{otherwise.} \end{cases}$$

If the signal $x_c(t)$ is bandlimited to π/T, then the system from $x_c(t)$ to $y_c(t)$ can be equivalent to the CT system with frequency response

$$H_c(j\omega) = \begin{cases} H(e^{j\omega T}), & |\omega| < \frac{\pi}{T} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the bandlimited $x_c(t)$ that has the output $y_c(t)$ must have CTFT as

$$X_c(j\omega) = \frac{Y_c(j\omega)}{H_c(j\omega)} = \begin{cases} 2Te^{-j0.3\omega T}, & |\omega| < \frac{\pi}{T} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Therefore, we know that $x_c(t)$ is the signal obtained by delaying the scaled sinc by $0.3T$ in time

$$x_c(t) = 2\frac{\sin\left(\pi(t-0.3T)/T\right)}{(\pi(t-0.3T)/T)}.$$

(b) If the signal $x_c(t)$ is required to be bandlimited to π/T and correspond to the output $y_c(t)$, then $x_c(t)$ in part (a) is unique. Since the equivalent CT filter $H_c(j\omega)$ is nonzero for $|\omega| < \pi/T$, the relationship $Y_c(j\omega) = H(j\omega)X_c(j\omega)$ for $|\omega| < \pi/T$ leads to a unique solution for $X_c(j\omega)$ over its entire band $|\omega| < \pi/T$, which guarantees the uniqueness of $x_c(t)$ under the bandlimited requirement above. However, if we do not require $x_c(t)$ to be bandlimited to π/T, then all signals that have the same samples at $t = nT$ as $x_c(t)$ in part (a) will produce the same samples $x[n]$ and the same output signal $y_c(t)$; in other words, all signals $x_{all}(t)$ that correspond to $y_c(t)$ are characterized by

$$x_{all}(nT) = x_c(nT) = 2\frac{\sin(\pi(n-0.3))}{\pi(n-0.3)}, \quad \text{for all } n.$$

(c) We know that $y[n]$ is the samples of $y_c(t)$ at $t = nT$, which is unique. The expression for $y[n]$ is $y[n] = y_c(nT) = 2\sin(\pi(n-0.7))/(\pi(n-0.7))$.

Since $Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega})$ and $H(e^{j\Omega}) \neq 0$ for $|\Omega| < \pi$, the DTFT $X(e^{j\Omega})$ is unique, leading to the uniqueness of $x[n]$. Sampling $x_c(t)$ at $t = nT$ can obtain $x[n] = 2\sin(\pi(n-0.3))/(\pi(n-0.3))$.

(d) Applying inverse DTFT to $H(e^{j\Omega})$ leads to its impulse response

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})e^{j\Omega n}\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j0.4\Omega}e^{j\Omega n}\mathrm{d}\Omega = \frac{\sin(\pi(n-0.4))}{\pi(n-0.4)}.$$

Solution 1.46

(a) (i) From the property of D/C converter, we know $v[n] = q(nT_1)$. From the property of C/D converter, we know $w[n] = q(nT_2)$. If $T_2 = MT_1$, then the output is related to the input by $w[n] = q(nT_2) = q(nMT_1) = v[nM]$.

If $v[n] = v_c(nT_0)$ is a sampled signal from a CT signal, then $w[n] = v[nM] = v_c(n(MT_0))$. As a result, $w[n]$ corresponds to the periodic sampling of $v_c(t)$ with a sampling period of MT_0.

(ii) After the D/C converter, the spectrum of $Q(j\omega)$ is

$$Q(j\omega) = \begin{cases} T_1 V(e^{j\omega T_1}), & |\omega| < \frac{\pi}{T_1} \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, $v[n]$ is the samples of $q(t)$ with the sampling period of T_1, and the DTFT $V(e^{j\Omega})$ is related to $Q(j\omega)$ by

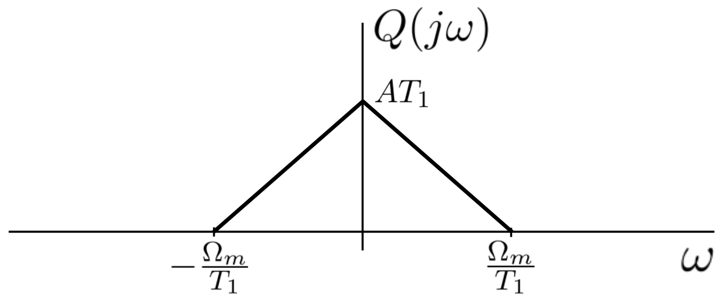
$$V(e^{j\Omega}) = \frac{1}{T_1} \sum_{n=-\infty}^{\infty} Q\left(j \cdot \left(\frac{\Omega}{T_1} - \frac{2\pi \cdot n}{T_1}\right)\right).$$

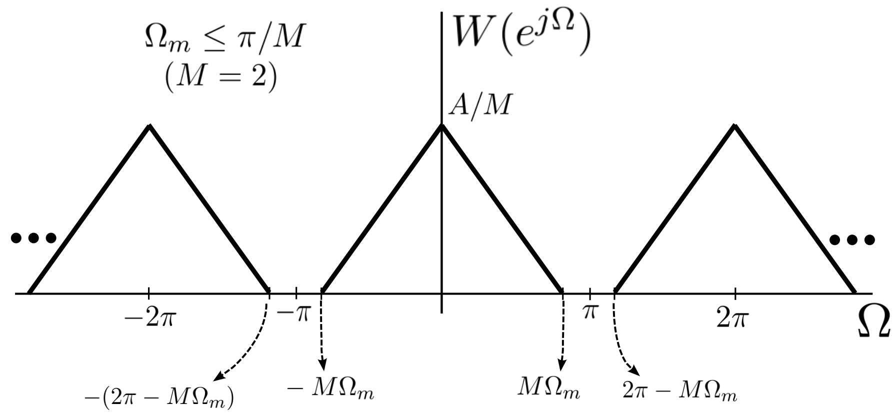
Then, the C/D converter has output spectrum as

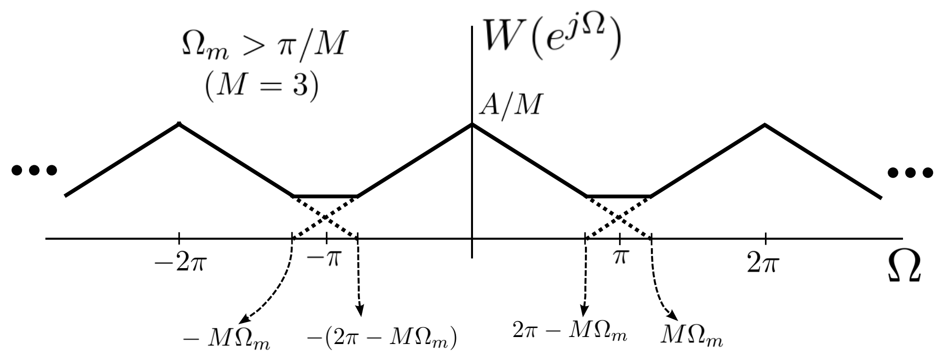
$$\begin{aligned} W(e^{j\Omega}) &= \frac{1}{T_2} \sum_{k=-\infty}^{\infty} Q\left(j \cdot \left(\frac{\Omega}{T_2} - \frac{2\pi \cdot k}{T_2}\right)\right) \\ &= \frac{1}{MT_1} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} Q\left(j \cdot \left(\frac{\Omega}{MT_1} - \frac{2\pi \cdot (nM+m)}{MT_1}\right)\right) \qquad (5) \\ &= \frac{1}{M} \sum_{m=0}^{M-1} \left(\frac{1}{T_1} \sum_{n=-\infty}^{\infty} Q\left(j \cdot \left(\frac{\Omega - 2m\pi}{MT_1} - \frac{2\pi \cdot n}{T_1}\right)\right)\right) \\ &= \frac{1}{M} \sum_{m=0}^{M-1} V(e^{j\frac{\Omega - 2m\pi}{M}}), \end{aligned}$$

where step (5) uses the fact that an integer k can be expressed as $k = nM + m$ uniquely for an integer n and $0 \leq m \leq M-1$.

The sketches of $Q(j\omega)$ and $W(e^{j\Omega})$ are plotted below. The different situations of $\Omega_m \leq \pi/M$ (e.g. $M = 2$) and $\Omega_m > \pi/M$ (e.g. $M = 3$) are shown separately.







A relationship between input and output spectrums is that $W(e^{j\Omega})$ is a magnitude scaled and frequency expanded version of the spectrum that is obtained by “aliasing” $V(e^{j\Omega})$. Precisely, if we “alias” the input spectrum $V(e^{j\Omega})$ with a total of M replicas at spacings of $\Delta\Omega_k = k \cdot 2\pi/M$ $(k = 0, 1, \cdots, M-1)$ in frequency, expand the frequency axis by a factor of M, and scale the magnitude by $1/M$, then it results in the output spectrum $W(e^{j\Omega})$.

(b) (i) The CT signal $q(t)$ is the bandlimited interpolation of $v[n]$ that has the form

$$q(t) = \sum_{m=-\infty}^{\infty} v[m] \cdot \frac{\sin(\pi(t - mT_1)/T_1)}{\pi(t - mT_1)/T_1}.$$

Consequently, the samples $w[n]$ has the form

$$w[n] = q(nT_2) = \sum_{m=-\infty}^{\infty} v[m] \cdot \frac{\sin(\pi(nT_2 - mT_1)/T_1)}{\pi(nT_2 - mT_1)/T_1} = \sum_{m=-\infty}^{\infty} v[m] \cdot \frac{\sin(\pi(n - mM)/M)}{\pi(n - mM)/M},$$

where we apply the relationship $T_2 = T_1/M$.

Since $v[n]$ and $w[n]$ are the samples of the same bandlimited CT signal $q(t)$ at $t = nT_1 = (nM)T_2$ and $t = nT_2$, respectively, we know that $v[n] = w[nM]$ and $w[n]$ is thus an upsampled version of $v[n]$.

(ii) The CTFT $Q(j\omega)$ has the same form as in part (a)(ii), which is

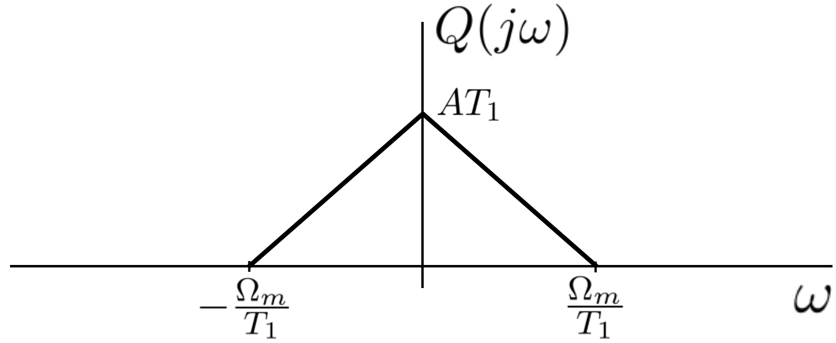
$$Q(j\omega) = \begin{cases} T_1 V(e^{j\omega T_1}), & |\omega| < \frac{\pi}{T_1} \\ 0, & \text{otherwise.} \end{cases}$$

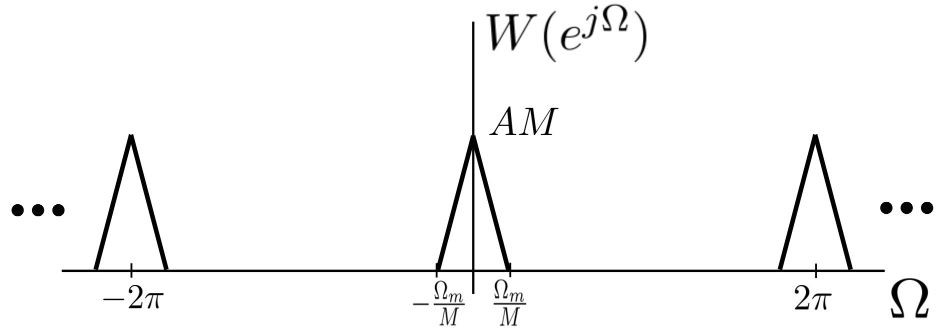
Because $q(t)$ has the highest frequency at π/T_1, sampling $q(t)$ at $T_2 = T_1/M$ has no aliasing effect. In the range $|\Omega| < \pi$, the output spectrum $W(e^{j\Omega})$ has the form

$$\begin{aligned} W(e^{j\Omega}) &= \frac{1}{T_2} \cdot Q\left(j \cdot \frac{\Omega}{T_2}\right) && (6) \\ &= \frac{M}{T_1} \cdot Q\left(j \cdot \frac{M\Omega}{T_1}\right) && (7) \\ &= \begin{cases} MV(e^{j\Omega M}), & |\Omega| < \frac{\pi}{M} \\ 0, & \frac{\pi}{M} \le |\Omega| < \pi \end{cases} && (8) \end{aligned}$$

where step (6) utilizes the fact that no aliasing happens at the C/D converter, (7) applies the relationship $T_2 = T_1/M$, and (8) plugs in the spectrum $Q(j\omega)$ that we obtained. For $|\Omega| > \pi$, the spectrum $W(e^{j\Omega})$ can be determined from the periodicity of DT spectrums.

A sketch of $W(e^{j\Omega})$ is shown as the figure below.





A relationship between input and output spectrums is that $W(e^{j\Omega})$ can be obtained by lowpassing a magnitude scaled and frequency compressed version of $V(e^{j\Omega})$. Precisely, if we increase the magnitude of input spectrum $V(e^{j\Omega})$ by M times, compress the frequency axis by a factor of M, and finally filter the result by an ideal lowpass filter with unit gain and cutoff frequency at π/M, then it results in the output spectrum $W(e^{j\Omega})$.

Solution 1.47

(a) Note that

$$\begin{aligned} \begin{bmatrix} q_1[k+1] \\ q_2[k+1] \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} q_1[k] \\ q_2[k] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[k] \\ y[k] &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q_1[k] \\ q_2[k] \end{bmatrix} \end{aligned}$$

is equivalent to the following three equations

$$q_1[k+1] = \frac{1}{3}q_1[k] + q_2[k] + x[k] \tag{9}$$

$$q_2[k+1] = \frac{1}{2}q_2[k] + x[k] \tag{10}$$

$$y[k] = q_1[k] + q_2[k] \tag{11}$$

The system is linear and time-invariant because each of the above equations is LTI. To see the linearity very concretely and formally for, say, the second equation above, suppose that we perform two experiments, A and B, with the above system, so there are two sets of solutions $\{q_2^A[\,\cdot\,], x^A[\,\cdot\,]\}$ and $\{q_2^B[\,\cdot\,],\ x^B[\,\cdot\,]\}$ for which the second equation above is satisfied:

$$q_2^A[k+1] = \frac{1}{2}q_2^A[k] + x^A[k] \tag{12}$$

and

$$q_2^B[k+1] = \frac{1}{2}q_2^B[k] + x^B[k] \tag{13}$$

The question now is whether an arbitrary superposition of the two sets of solutions, namely a set of the form $\{\alpha q_2^A[\,\cdot\,] + \beta q_2^B[\,\cdot\,], \alpha x^A[\,\cdot\,] + \beta x^B[\,\cdot\,]\}$, where α and β are arbitrary scalars, will satisfy the same equation. And the answer is **yes**, because taking the appropriate linear combination of equations (12) and (13) yields

$$(\alpha q_2^A[k+1] + \beta q_2^B[k+1]) = \frac{1}{2}\Big(\alpha q_2^A[k] + \beta q_2^B[k]\Big) + (\alpha x^A[k] + \beta x^B[k])$$

The system is not memoryless because, as seen in (1) and (2), the values of the states $q_i[\,\cdot\,]$ depend on previous values of the input $x[\,\cdot\,]$.

(b) We can solve this problem either in the time domain or by using z-transforms. We first consider the problem strictly in the time domain.

The equations

$$\begin{aligned} q_1[k+1] &= \frac{1}{3}q_1[k] + q_2[k] + x[k] \\ q_2[k+1] &= \frac{1}{2}q_2[k] + x[k] \end{aligned}$$

are a system of first-order difference equations. We know that $q_1[0] = 2$ and $q_2[0] = 0$ from the initial condition.

We first solve for $q_2[k]$ using the second of the above equations, because this equation does not involve q_1. With $x[k] = \delta[k]$, we find $q_2[1] = \frac{1}{2}(0) + 1 = 1$. For $k > 1$,

$$q_2[k+1] - \frac{1}{2}q_2[k] = 0 \ .$$

The general solution to this equation is $q_2[k] = a(\frac{1}{2})^k$. The fact that $q_2[1] = 1$ gives $a = 2$. Thus $q_2[k] = 2(\frac{1}{2})^k$ for $k > 0$.

Turning to the first equation, note that $q_1[1] = \frac{1}{3}(2) + 0 + 1 = \frac{5}{3}$. For $k > 1$,

$$q_1[k+1] - \frac{1}{3}q_1[k] = 2(\frac{1}{2})^k \ . \tag{14}$$

The general solution to this equation for $k > 1$ is the sum of a particular solution and a homogeneous one and hence is of the form $a(\frac{1}{2})^k + b(\frac{1}{3})^k$. To verify that $b(\frac{1}{3})^k$ is a homogeneous solution, note that

$$(\frac{1}{3})^{k+1} - \frac{1}{3}(\frac{1}{3})^k = 0 \ .$$

To find a, substitute $q_1[k] = a(\frac{1}{2})^k$ into (14):

$$\begin{aligned} a(\frac{1}{2})^{k+1} - \frac{a}{3}(\frac{1}{2})^k &= 2(\frac{1}{2})^k \\ a &= \frac{2}{\frac{1}{2} - \frac{1}{3}} = 12 \end{aligned}$$

To find b, use the fact that $q_1[1] = \frac{5}{3}$ and solve for $b = -13$. It follows that

$$y[k] = 14(\frac{1}{2})^k - 13(\frac{1}{3})^k$$

for $k > 0$.

To solve the problem using z-transforms, we again start with the second equation and take its z-transform. Since initial conditions are specified and we are interested in the solution for positive time, we use the unilateral z-transform. Taking note of equation (10.149) in Oppenheim and Willsky, we get

$$\begin{aligned} zQ_2(z) &= \frac{1}{2}Q_2(z) + 1 + zq_2[0] = \frac{1}{2}Q_2(z) + 1 \\ Q_2(z) &= \frac{1}{z - \frac{1}{2}} \\ &= \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

Taking the inverse z-transform, we find that $q_2[k] = (\frac{1}{2})^{k-1}$ for $k > 0$.

Now we take the z-transform of the first equation, making use of $Q_2(z)$ found above:

$$\begin{aligned}
zQ_1(z) &= \frac{1}{3}Q_1(z) + Q_2(z) + 1 + zq_1[0] = \frac{1}{3}Q_1(z) + \frac{1}{z-\frac{1}{2}} + 1 + 2z \\
Q_1(z) &= \frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})} + \frac{1}{z-\frac{1}{3}} + \frac{2z}{z-\frac{1}{3}} \\
&= \frac{6}{z-\frac{1}{2}} - \frac{5}{z-\frac{1}{3}} + \frac{2z}{z-\frac{1}{3}} \\
&= \frac{6z^{-1}}{1-\frac{1}{2}z^{-1}} - \frac{5z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}
\end{aligned}$$

Taking the inverse z-transform, we find that $q_1[k] = 6(\frac{1}{2})^{k-1} - 5(\frac{1}{3})^{k-1} + 2(\frac{1}{3})^k$ for $k > 0$. Adding $q_1[k]$ and $q_2[k]$, we get $y[k]$ for $k > 0$ as before:

$$\begin{aligned}
y[k] &= 7(\frac{1}{2})^{k-1} - 5(\frac{1}{3})^{k-1} + 2(\frac{1}{3})^k \\
&= 14(\frac{1}{2})^k - 13(\frac{1}{3})^k
\end{aligned}$$

(c) Still linear, still not memoryless, but now time varying.

(d) The parameters (specifically the one time-varying entry of the matrix, and trivially all the constants) are periodic with period 6. The solutions, however, need not be periodic, as is easily seen by example.

Solution 1.48

The answer to all parts of this (trick!) question are based on the fact that $e^{j(\Omega+2\pi k)n} = e^{j\Omega n}$, because $e^{j2\pi kn} = 1$ for integer k, n. Hence, whatever response is obtained in an experiment with input $e^{j\Omega n}$ is also obtained when Ω is incremented by an integer multiple of 2π. Hence $A(\Omega)$ is periodic in Ω with period 2π.

Solution 1.49

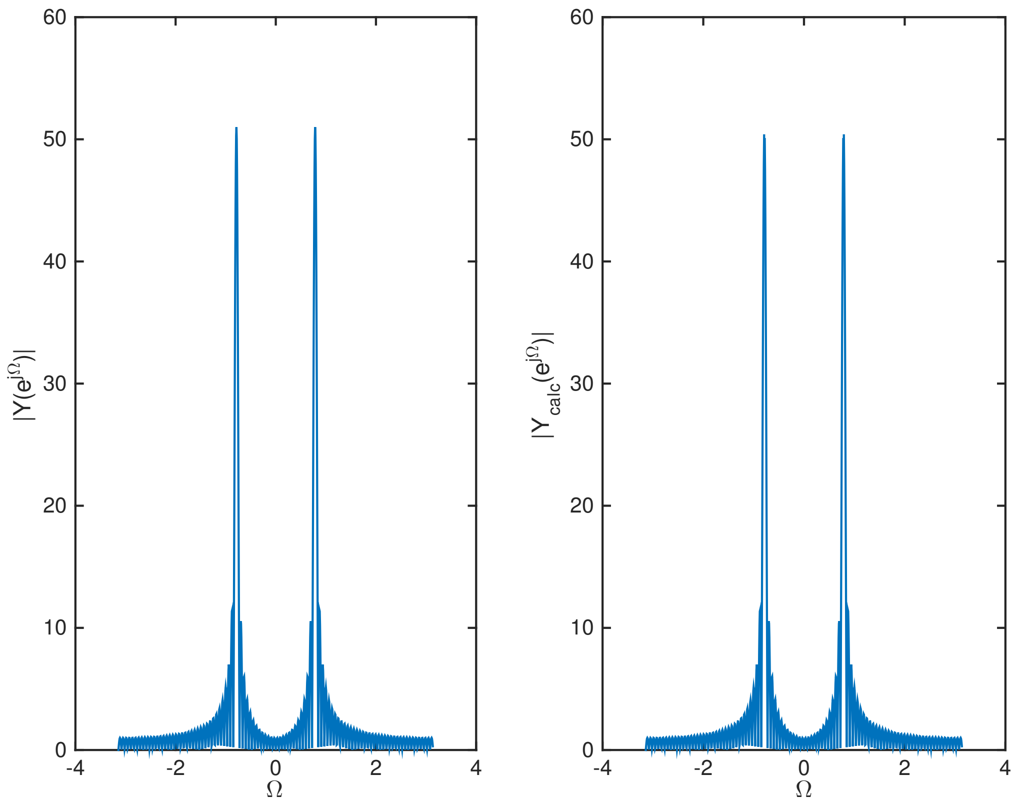
(a) Note that the window extends from 0 to 100 inclusive, so comprises 101 points, and can be thought of as a 50-step delay of a window centered at 0 and extending from -50 to 50. Using the multiplication and time-shifting properties, and using $\circledast$ to denote *periodic* convolution, we obtain

$$\begin{aligned}
Y_1(e^{j\Omega}) &= \frac{1}{2\pi}\mathcal{F}\left\{\cos\left(\frac{\pi}{4}n\right)\right\} \circledast \mathcal{F}\left\{\frac{1}{101}(u[n]-u[n-101])\right\} \\
&= \frac{1}{2}\left[\delta\left(\Omega-\frac{\pi}{4}\right)+\delta\left(\Omega+\frac{\pi}{4}\right)\right] \circledast \frac{\sin(50.5\,\Omega)}{\sin(\Omega/2)}e^{-j50\Omega} \\
&= \frac{1}{2}\frac{\sin(50.5\,(\Omega-\pi/4))}{\sin((\Omega-\pi/4)/2))}e^{-j50(\Omega-\pi/4)} + \frac{1}{2}\frac{\sin(50.5\,(\Omega+\pi/4))}{\sin((\Omega+\pi/4)/2)}e^{-j50(\Omega+\pi/4)} \\
&= \left[\frac{1}{2}\frac{\sin(50.5\,(\Omega-\pi/4))}{\sin((\Omega-\pi/4)/2)}e^{j50\pi/4} + \frac{1}{2}\frac{\sin(50.5\,(\Omega+\pi/4))}{\sin((\Omega+\pi/4)/2)}e^{-j50\pi/4}\right]e^{-j50\Omega} \qquad (15)
\end{aligned}$$

The phase term $e^{-j50\Omega}$ has magnitude 1, and so $|Y_1(e^{j\Omega})|$ is simply the magnitude of the term in square brackets. Each of the two terms in these brackets is a "sinc-like" function: it looks like a sinc for values of Ω near the peak, but the denominator term causes it to repeat with period 2π, as any DTFT has to do. It's also true that the sinc-like peak at $\Omega = \pi/4$ has decayed significantly in magnitude when one examines it in the neighborhood of $\Omega = -\pi/4$, and vice versa. Thus the magnitude of the term in square brackets above looks essentially like the magnitude of a sinc-like term centered at $\Omega = \pi/4$, plus the magnitude of a similar term centered at $\Omega = -\pi/4$.

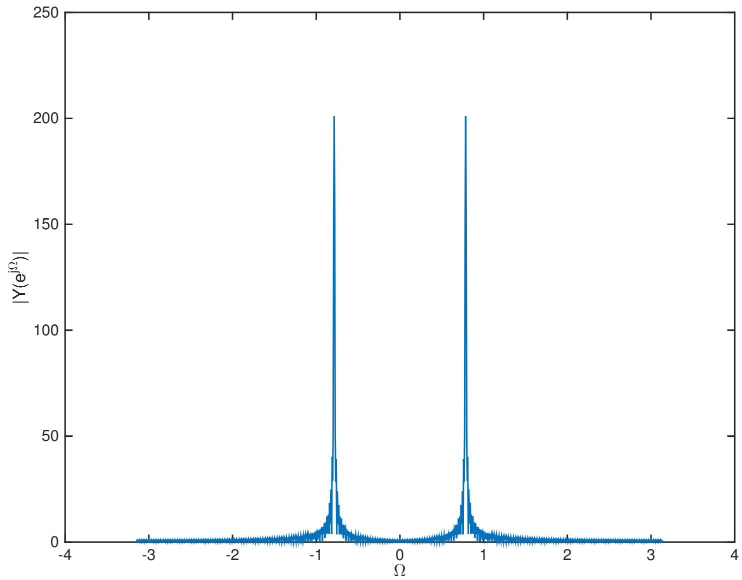
Now we use MATLAB to plot $|Y_1(e^{j\Omega})|$ both using the above formula for $Y_1(e^{j\Omega})$ (we can apply the `abs` function) and also using the `fft` command in MATLAB (but you may have used some other package to evaluate samples of the DTFT at enough points to be able to generate a similar plot). We have to be careful at $\Omega = \pm\pi/4$ since our expression above is an indeterminate form at these points.

The plots, seen below, coincide as expected:



The peaks are at $\pm\pi/4 \approx \pm.785$. The pair of impulses at $\pm\pi/4$ that constitute the spectrum of the cosine is "blurred" (through convolution) by the sinc-like spectrum of the rectangular window. The height of each peak is simply half the height at the origin for the sinc-like transform of the rectangular window, hence half of the window width, i.e., 50.5.

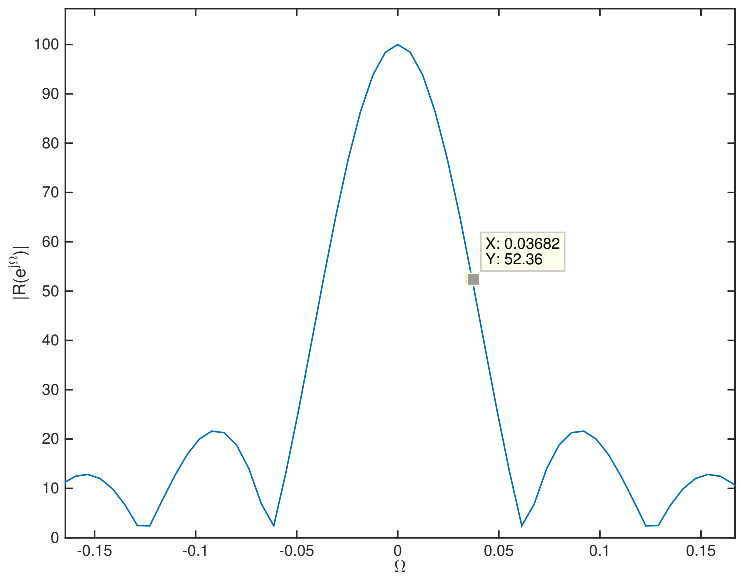
When the window size is increased from 101 points to 401 points (corresponding to the energy of $y_1[n]$ going up by approximately a factor of 4), we get the spectrum below:



As the width of the rectangular window expands, the peaks are reduced in width and increased in height, i.e., the blurring is reduced. (One might say that with more of the waveform available for examination, the DTFT becomes increasingly sure that it's looking at a cosine of a particular frequency.)

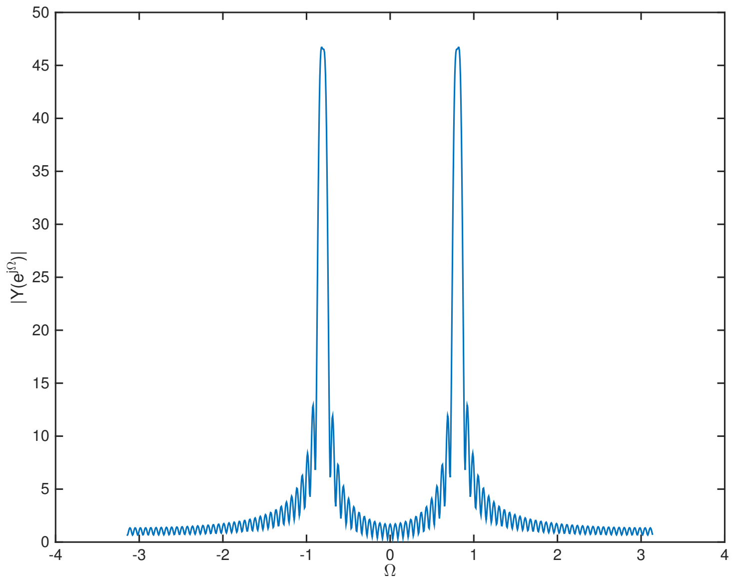
(b) We go back now to a window size of 101. As seen in (a), the spectrum of the windowed cosine is (apart from the $1/2\pi$ scale factor) the convolution of the DTFT of the cosine (i.e., a pair of delta functions) with the DTFT of the rectangular window (i.e., the sinc-like function). So the widths of the peaks in $|Y_1(e^{j\Omega})|$ are determined by the width of this sinc-like function.

We want to assess how far away the windowed transform of another cosine has to be centered before it's quite clearly distinguishable from our reference cosine. One way to do this (traditional in some application areas) is in terms of the "half-width at half-maximum" (HWHM) of the sinc function, which can be estimated from the figure below as around 0.04.

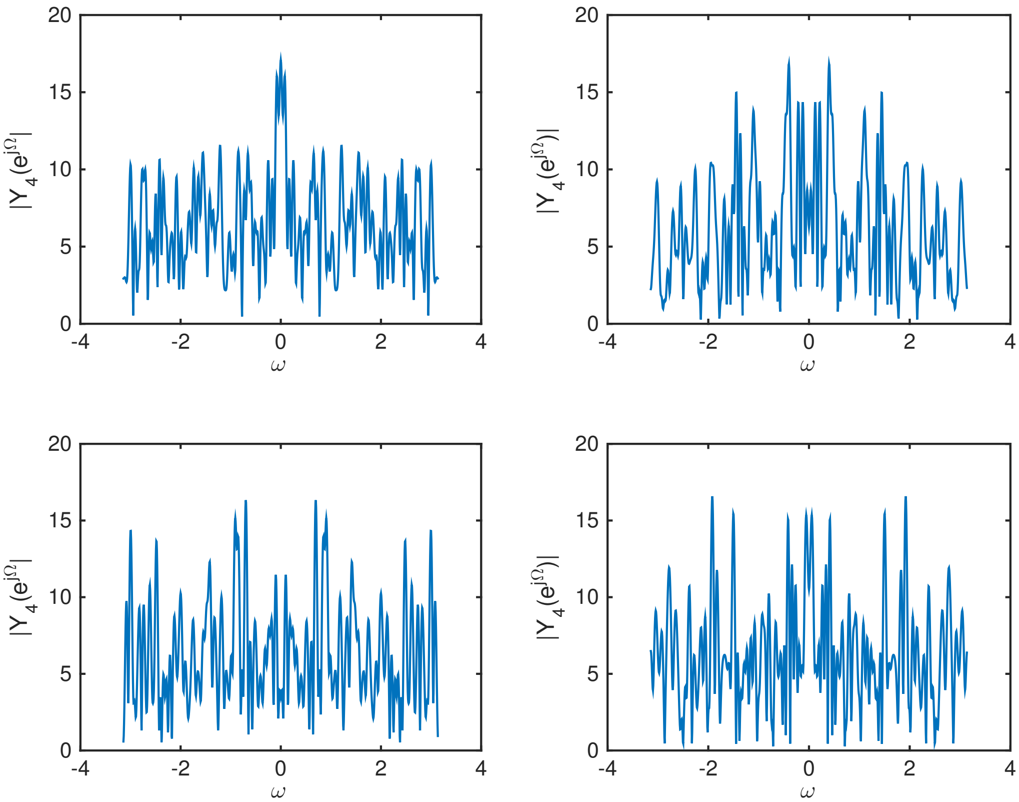


Alternatively, we can take the distance from the peak of our sinc to its first zero — because having another peak at the location of this zero would seem to minimize interference with the peak of our sinc. From the analytical expression in (a), we see that this corresponds to the smallest Ω for which $\sin(50.5\Omega) = 0$, which happens at $\Omega = \pi/50.5 \approx 0.06$.

The figure below shows the DTFT magnitude for the windowed sum of two cosines whose frequency difference is $\Delta = 0.04$, and shows two barely distinct peaks.



(e) We plot $|Y_4(e^{j\Omega})|$ for 4 different realizations of $b[n]$ below:



We notice that $|Y_4(e^{j\Omega})|$ varies from one realization to the next, but in each case has a number of large spikes that are rather uniformly distributed throughout the entire frequency range. We know that multiplication in the time domain corresponds to convolution in the frequency domain, so $Y_1(e^{j\Omega})$ is convolved with the random spectra $B(e^{j\Omega})$ to produce $Y_4(e^{j\Omega})$. The large fluctuations in $B(e^{j\Omega})$ explain why we can no longer make out the cosine signal.

We can recover $y_1[n]$ by multiplying the signal by $b[n]$ so that

$$b[n]y_4[n] = b^2[n]y_1[n] = y_1[n]$$

If we didn't know $b[n]$, however, it would be quite impossible to reconstruct $y_1[n]$.

(f) We leave you to try this. You should find that the sums involving products of distinct $b_i[n]$ are considerably smaller than those involving the squares, so "cross-talk" is small.

Solution 1.50

Solution 1.51

(a) By definition, the spectrum of $h(t)$ is

$$
\begin{aligned}
H(j\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{-(3+j\omega)t}u(t)\mathrm{d}t \\
&= \int_{0}^{\infty} e^{-(3+j\omega)t}\mathrm{d}t \\
&= \frac{1}{3+j\omega}.
\end{aligned}
$$

(b) From the properties of the C/D converter, if the highest frequency of $v_c(t)$ is smaller than π/T, then aliasing is avoided. Since $H(j\omega) \neq 0$ for each frequency, the spectrum of $v_c(t)$ has the same support as that of $x_c(t)$. Consequently, if $x_c(t)$ has its highest frequency smaller than π/T, then aliasing is avoided in the C/D conversion.

(c) We consider only the range of frequency $|\omega| < \pi/T$ in this part. First, we denote the equivalent CT system from $v_c(t)$ to $y_c(t)$ as $G_c(j\omega)$. Then, from the knowledge on DT processing of CT signals, the spectrum of the CT equivalent filter is

$$G_c(j\omega) = G(e^{j\omega T}), \quad |\omega| < \pi/T.$$

If $y_c(t) = x_c(t)$ holds for properly bandlimited $x_c(t)$, then $H(j\omega)G_c(j\omega) = 1$ in the range $|\omega| < \pi/T$. The spectrum $G(e^{j\Omega})$ can then be derived as

$$G(e^{j\Omega}) = G_c(j\Omega/T) = \frac{1}{H(j\Omega/T)} = 3 + j\frac{\Omega}{T}.$$

(d) (i) Applying Parseval's identity, the energy of the impulse response $g[n]$ satisfies

$$\sum_{n=-\infty}^{\infty} g^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\Omega})|^2\mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(9 + \frac{\Omega^2}{T^2}\right)\mathrm{d}\Omega = 9 + \frac{\pi^2}{3T^2}.$$

(ii) The impulse response can be obtained using inverse DTFT

$$
\begin{aligned}
g[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\Omega})e^{j\Omega n}\mathrm{d}\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(3 + j\frac{\Omega}{T}\right)e^{j\Omega n}\mathrm{d}\Omega \\
&= 3\delta[n] + \frac{j}{2\pi T}\int_{-\pi}^{\pi}\Omega e^{j\Omega n}\mathrm{d}\Omega,
\end{aligned}
$$

where the integration in the latter term with $n \neq 0$ can be further simplified as

$$\begin{aligned}\int_{-\pi}^{\pi} \Omega e^{j\Omega n} \mathrm{d}\Omega &= \frac{1}{jn}\Omega e^{j\Omega n}\Big|_{-\pi}^{\pi} - \frac{1}{jn}\int_{-\pi}^{\pi} e^{j\Omega n}\mathrm{d}\Omega \qquad (16)\\ &= \frac{2\pi(-1)^n}{jn} + \frac{1}{n^2}e^{j\Omega n}\Big|_{-\pi}^{\pi}\\ &= \frac{2\pi(-1)^n}{jn}, \quad n \neq 0,\end{aligned}$$

where step (16) uses integration by parts. For the special situation of $n = 0$, the above integration is 0. A combination of the results above leads to the following expression of $g[n]$

$$g[n] = \begin{cases} 3, & n = 0 \\ \frac{(-1)^n}{nT}, & \text{otherwise.} \end{cases}$$

Solution 1.52

(a) The overall system with the specified parameters is an LTI system. Since $2\pi/T_1 = 4\pi \times 10^4$/sec is equal to twice the highest frequency $2\omega_c$, no aliasing happens at the C/D converter. Combining this fact with the observation that $T_1 = T_2$, we know that the subsystem from $v(t)$ to $y_c(t)$ is equivalent to a CT LTI system with the transfer function

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T_1}), & |\omega| < \pi/T_1, \\ 0, & \text{otherwise.} \end{cases}$$

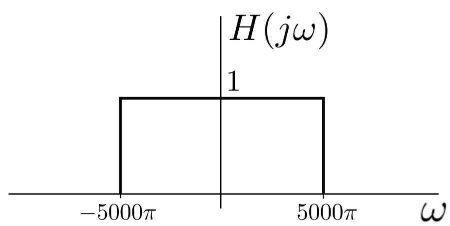
In addition, since the subsystem from $x_c(t)$ to $v(t)$ is also LTI, the system from $x_c(t)$ to $y_c(t)$ is LTI with the transfer function

$$H(j\omega) = L(j\omega)H_c(j\omega) = \begin{cases} 1, & |\omega| < \min\{\omega_c,\ \Omega_c/T_1\} = 5000\pi, \\ 0, & \text{otherwise.} \end{cases}$$

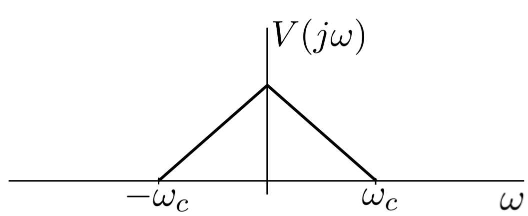
It follows that $H(j\omega)$ goes to zero at the frequency

$$\frac{1}{2\pi}\min\left\{\omega_c, \frac{\Omega_c}{T_1}\right\} = 2500\text{Hz}.$$

We show the plot for $H(j\omega)$ in the figure below.



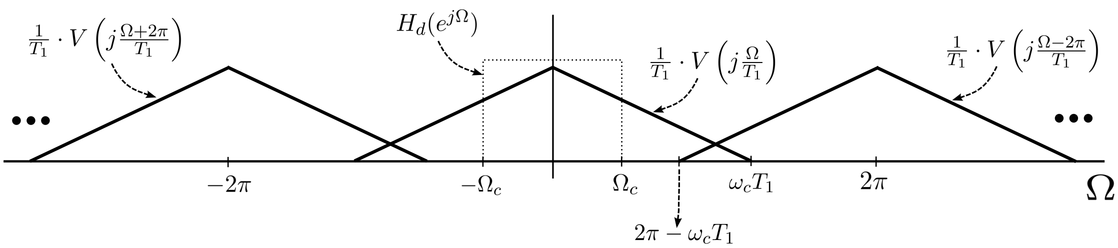
(b) First, we analyze the spectrums of $v(t)$ and $v[n]$. Since $v(t)$ is the output of the anti-aliasing filter, it has cutoff frequency at ω_c. Without loss of generality, an example of the spectrum $V(j\omega)$ of the CT signal $v(t)$ is in the figure below.



After the C/D converter, the spectrum $V_d(e^{j\Omega})$ of the signal $v[n]$ is

$$V_d(e^{j\Omega}) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} V\left(j\frac{\Omega + 2k\pi}{T_1}\right),$$

where aliasing may happen. With the example $V(j\omega)$ in the figure above, the DT spectrum of $v[n]$ is illustrated in the figure below, where we mark the individual components $(1/T_1) \cdot V(j(\Omega + 2k\pi)/T_1)$ that are in the summation above.



Then, we argue that the overall system from $x_c(t)$ to $y_c(t)$ is LTI if and only if all aliasing introduced in the C/D converter (if any) is eliminated by the DT filter $H_d(e^{j\Omega})$. On one hand, if not all aliased frequencies are removed, then there are two different CT frequencies in $x_c(t)$ mapped to one DT frequency in $v[n]$ (and $y[n]$), and finally converted back to a single CT frequency in the output $y_c(t)$, which can never happen for an LTI system. On the other hand, if all aliasing is removed, then we know that for $|\omega| < \Omega_c/T_2$,

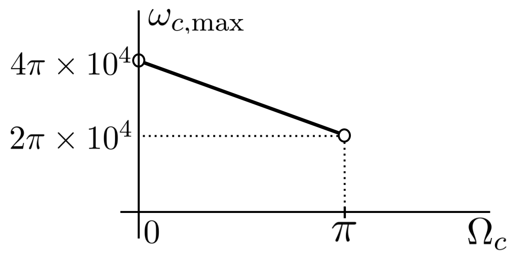
$$Y_c(j\omega) = T_2 Y_d(e^{j\Omega})\big|_{\Omega=\omega T_2} = T_2 V_d(e^{j\Omega}) H_d(e^{j\Omega})\big|_{\Omega=\omega T_2} = V(j\omega) H_d(e^{j\Omega})\big|_{\Omega=\omega T_2},$$

where in the last step we used $T_1 = T_2$ as well as the assumption that all aliasing is removed; in the equation above, $Y_c(j\omega)$ and $Y_d(e^{j\Omega})$ denote the spectrums of $y_c(t)$ and $y[n]$, respectively. The equation above shows that removing all aliased frequencies ensures the LTI property of the subsystem from $v(t)$ to $y_c(t)$, and thus the system from $x_c(t)$ to $y_c(t)$ is LTI.

Finally, the above analysis implies that $\omega_{c,\max}$ with which the system remains LTI is equal to the highest cutoff frequency for the CT filter with which all aliased frequencies at the C/D converter are removed by the DT filter $H_d(e^{j\Omega})$. As ω_c increases from zero, the aliasing starts when $\omega_c > \pi/T_1$ with the lowest aliased frequency at $(2\pi - \omega_c T_1)$, which can be illustrated by the figure above. Therefore, $H_d(e^{j\Omega})$ removes all aliased frequencies if and only if $\Omega_c \leq 2\pi - \omega_c T_1$, which results in

$$\omega_{c,\max} = \frac{2\pi - \Omega_c}{T_1} = 4\pi \times 10^4 - 2 \times 10^4 \Omega_c, \quad 0 < \Omega_c < \pi.$$

The following figure shows $\omega_{c,\max}$ as a function of Ω_c.



(c) With $T_1 = 0.5 \times 10^{-4}$sec and $T_2 = 0.25 \times 10^{-4}$sec, the system is still linear, since each block is linear regardless of the aliasing situation or the sampling/reconstruction period.

However, the overall system is not time-invariant with the new specifications. In particular, the following argument shows that if the input signal is delayed by T_1, then the output signal is always delayed by T_2, which is against the time-invariance [1]. When the input signal is $x_{c1}(t)$, we denote the associated signals in the system as $v_1(t)$, $v_1[n]$, $y_1[n]$, and $y_{c1}(t)$, respectively. From the property of D/C converter, we know that $y_{c1}(t)$ is the bandlimited interpolation

$$y_{c1}(t) = \sum_{n=-\infty}^{\infty} y_1[n] \cdot \frac{\sin(\pi(t - nT_2)/T_2)}{\pi(t - nT_2)/T_2}. \tag{17}$$

If the input signal is changed to a delayed signal $x_{c2}(t)$

$$x_{c2}(t) = x_{c1}(t - T_1),$$

where $T_1 = 0.5 \times 10^{-4}$sec is the sampling period, then the output of the CT filter is $v_2(t) = v_1(t - T_1)$, the sampled signal is $v_2[n] = v_1[n-1]$, the output of the DT filter is $y_2[n] = y_1[n-1]$, and the final output satisfies

$$\begin{aligned}
y_{c2}(t) &= \sum_{n=-\infty}^{\infty} y_2[n] \cdot \frac{\sin(\pi(t - nT_2)/T_2)}{\pi(t - nT_2)/T_2} \\
&= \sum_{n=-\infty}^{\infty} y_1[n-1] \cdot \frac{\sin(\pi(t - nT_2)/T_2)}{\pi(t - nT_2)/T_2} \\
&= \sum_{m=-\infty}^{\infty} y_1[m] \cdot \frac{\sin(\pi((t - T_2) - mT_2)/T_2)}{\pi((t - T_2) - mT_2)/T_2} \qquad (18) \\
&= y_{c1}(t - T_2),
\end{aligned}$$

where in (18) we change the variable $m = n - 1$, and the last step uses (17).

In summary, we have shown that delaying the input signal $x_{c1}(t)$ by T_1 will cause a delay in the output by T_2 instead of T_1. If we choose a signal $x_{c1}(t)$ with which the output

[1] It is possible to choose an input signal such that none of the signals in this system is periodic; as a result, delaying the output signal by T_1 and by T_2 corresponds to two different signals.

signal $y_{c1}(t)$ is not periodic, then $y_{c2}(t) = y_{c1}(t - T_2) \neq y_{c1}(t - T_1)$ and thus the system is time-variant. As a concrete example, we can let

$$x_{c1}(t) = \frac{\sin(\pi t/T_1)}{\pi t/T_1},$$

and its output is not periodic

$$y_{c1}(t) = \frac{\sin(\pi t/(4T_2))}{\pi t/T_2}.$$

(d) If we take the Laplace transformation to $x_c(t) = r(t) + \alpha r(t - T_0)$, then it follows that

$$X_c(s) = R(s) + \alpha e^{-sT_0} R(s).$$

As a result, the CT transfer function for the echo cancelation system has the form below

$$H_{ec}(s) = \frac{R(s)}{X_c(s)} = \frac{1}{1 + \alpha e^{-sT_0}}, \quad \text{for } \mathrm{Re}\{s\} > \frac{\ln \alpha}{T_0}.$$

(e) Since the cutoff frequency of $x_c(t)$, the cutoff frequency of the anti-aliasing filter, and half of the sampling rate π/T_1 are all the same at 10kHz, aliasing is avoided and we can consider only the CT frequency range $|\omega| < \pi/T_1 = 2\pi \times 10^4$/sec in the following analysis. Since $T_1 = T_2$ and no aliasing happens at the C/D converter, the equivalent CT filter from $v(t)$ to $y_c(t)$ has the transfer function

$$H_c(j\omega) = H_d(e^{j\Omega})\big|_{\Omega = \omega T_1}, \quad |\omega| < \pi/T_1,$$

and the overall system has the transfer function

$$H(j\omega) = L(j\omega) H_c(j\omega) = H_d(e^{j\Omega})\big|_{\Omega = \omega T_1}, \quad |\omega| < \pi/T_1.$$

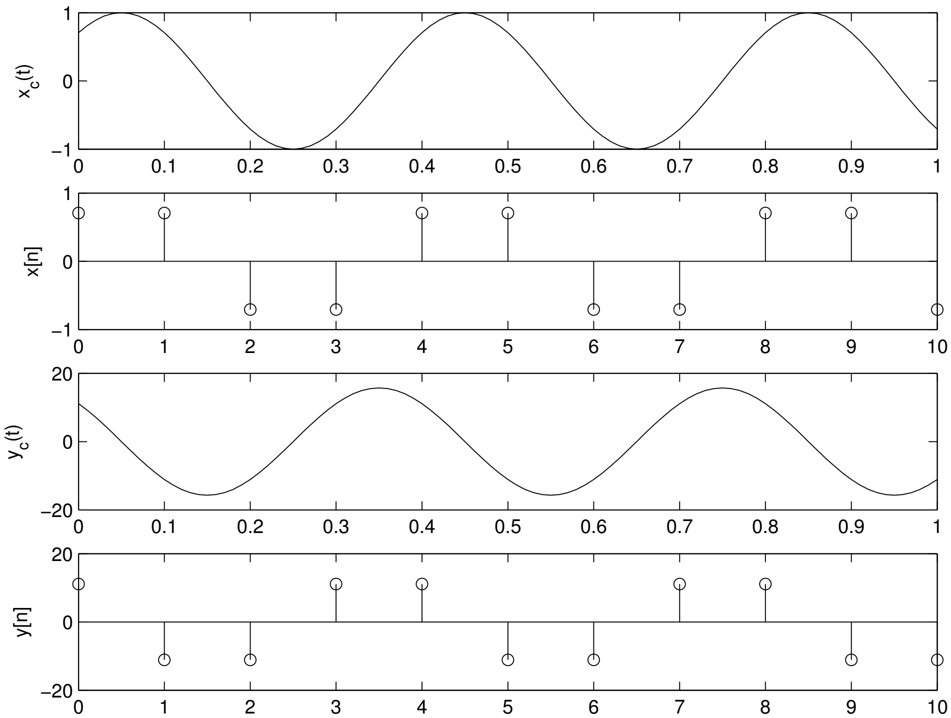
Since the ω-axis is within the convergence region of $H_{ec}(s)$ in part (d), let $s = j\omega$ and the transfer function of the echo canceler is

$$H_{ec}(s)\big|_{s=j\omega} = \frac{1}{1 + \alpha e^{-j\omega T_0}}.$$

Finally, the aimed DT filter is

$$H_d(e^{j\Omega}) = H\left(j\frac{\Omega}{T_1}\right) = H_{ec}\left(j\frac{\Omega}{T_1}\right) = \frac{1}{1 + \alpha e^{-j\frac{\Omega T_0}{T_1}}}, \quad |\Omega| < \pi.$$

Solution 1.53



(a) (i) Note that since $25\pi > (\pi/T_1) = 10\pi$, aliasing occurs:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \cos(2.5\pi n - \frac{\pi}{4}) \\ &= \cos(2\pi n + 0.5\pi n - \frac{\pi}{4}) \\ &= \cos(0.5\pi n - \frac{\pi}{4}) \end{aligned}$$

which is the same sequence we would get by sampling the low-frequency alias

$$x_a(t) = \cos(5\pi t - \frac{\pi}{4}) \ . \tag{19}$$

This waveform is shown at the top of the figure on the next page (although mislabeled as $x_c(t)$!). We can therefore get $y[n]$ and $y_c(t)$ by assuming the input is indeed this low-frequency alias. For input $x_a(t)$, the effective frequency response of the system is

$$H_c(j\omega) \ = \ j\frac{\omega T_1}{T_1} = j\omega$$

which just takes the derivative of $x_a(t)$, so

$$\begin{aligned} y_c(t) &= -5\pi \sin(5\pi t - \frac{\pi}{4}) , \\ y[n] &= -5\pi \sin(0.5\pi n - \frac{\pi}{4}) . \end{aligned}$$

(ii) $y_c(t)$ is clearly not the derivative of $x_c(t)$. Note, however, that it <u>is</u> the derivative of the lowest frequency aliased version of $x_c(t)$.

(iii) The overall system is linear because each subsystem is.

(b) Now the sinusoidal signal at the output drops in frequency by a factor of 2, so $y_c(t) = -5\pi \sin(2.5\pi t - \frac{\pi}{4})$. To see that the phase offset $-\pi/4$ stays the same rather than being scaled by 2, note that $y[0] = y_c(0)$ is unchanged by changing T_2, and so the phase offset must be the same. In the frequency domain, the transform $Y_c(j\omega)$ looks the same as before, except that the impulses are at $\pm 2.5\pi$ rather than $\pm 5\pi$.

Solution 1.54

(a) $dx_1(t)/dt = 9\cos(9t)$.

(b) $h[n] = \frac{1}{2T}(\delta[n+1] - \delta[n-1]$, so $H(e^{j\Omega}) = \frac{1}{2T}(e^{j\Omega} - e^{-j\Omega}) = \frac{j}{T}\sin(\Omega)$.

Although this part of the problem did not ask for an evaluation of the above approximation to a differentiator, note that with this choice we have

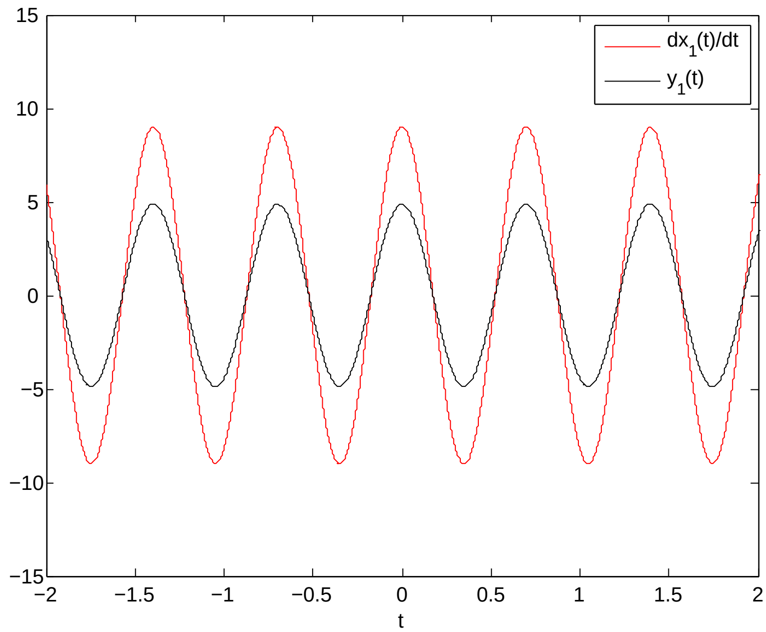
$$H_c(j\omega) = H(e^{j\Omega})\Big|_{\Omega=\omega T} = \frac{j}{T}\sin(\omega T)\ .$$

For signals bandlimited to $[-\frac{\pi}{T}, \frac{\pi}{T}]$, the phase of this frequency response $H_c(j\omega)$ equals that of the frequency response $H(j\omega) = j\omega$ of a CT differentiator, namely π for $\omega > 0$, $-\pi$ for $\omega < 0$, and 0 at $\omega = 0$. However, the magnitude of $H_c(j\omega)$ differs substantially from that of $j\omega$, except when the frequencies of interest are small enough to justify the approximation $\sin(\omega T) \approx \omega T$, i.e., when $|\omega T| << \pi$. In other words, for this system to behave as an approximate differentiator, we require the sampling frequency $f_s = 1/T$ to satisfy $f_s \gg 2f_m$, where f_m is the maximum frequency in input signal. Thus we would need the signal to be sampled much faster than its Nyquist rate, or to be highly "oversampled".

(c) Because the $1/T = 5\,\text{Hz}$ sampling rate exceeds the $9/\pi\,\text{Hz}$ Nyquist rate of $x_1(t) = \sin(9t)$, we know that the DT processor's output when the input is $x_1(t)$ will be

$$y_1(t) = \frac{\sin[9(t+0.2)] - \sin[9(t-0.2)]}{0.4}.$$

The figure below shows that $y_1(t)$ has the expected sinusoidal behavior of $\mathrm{d}x_1(t)/\mathrm{d}t$, and no phase offset from the correct answer, but its amplitude is appreciably lower.



(d) One way to determine $h[n]$ is directly by inverse transformation of the given $H(e^{j\Omega})$:

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{j\Omega}{T} e^{j\Omega n} \, d\Omega \, ,$$

which evaluates to 0 for $n = 0$ and to $\cos(\pi n)/(nT)$ for integer $n \neq 0$, which is the given expression.

Alternatively, for the input

$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t/T} \, ,$$

which is appropriately bandlimited and yields $x[n] = \delta[n]$, the overall system behaves as a differentiator, so $y_c(t)$ is the derivative of this $x_c(t)$, hence 0 at $t = 0$. Since $y[n] = y_c(t)\Big|_{t=nT}$, it follows that $y[0] = 0$. Now for $n \neq 0$ we again have

$$y[n] = y_c(t)\Big|_{t=nT} = \dot{x}_c(t)\Big|_{t=nT} = \cos(\pi n)/(nT) \, ,$$

which brings us back to the answer given earlier.

(e)

(f)

Solution 1.55

(a) (i) For $x[n] = (-1)^n/n^2$ at all $n > 0$, and 0 elsewhere, the ℓ_1 norm is given by

$$\begin{aligned}
\|x[\cdot]\|_1 &= \sum_{n>0} \frac{1}{n^2} \\
&= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ... \\
&= \frac{\pi^2}{6} \\
&\approx 1.645 \ .
\end{aligned}$$

The analytical expression in the third equality is not something we expected you to write down or derive. However, a nice derivation of this is based on Parseval's theorem applied to a well-chosen function, see

`http://en.wikipedia.org/wiki/Basel_problem.`

The sum above is actually the Riemann zeta function $\zeta(s)$, evaluated at $s = 2$, see

`http://en.wikipedia.org/wiki/Riemann_zeta_function.`

An analytical expression for the sum is known for all real, positive, even s, which covers the following case as well.

For the ℓ_2 norm, start by computing

$$\begin{aligned}
(\|x[\cdot]\|_2)^2 &= \sum_{n>0} \left|\frac{(-1)^n}{n^2}\right|^2 = \sum_{n>0} \frac{1}{n^4} \\
&= 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + ... \\
&= \frac{\pi^4}{90} \\
&\approx 1.0823 \ ,
\end{aligned}$$

where the third equation comes from the known analytical expression for $\zeta(4)$. The square root of this then gives the ℓ_2 norm:

$$\|x[\cdot]\|_2 = \frac{\pi^2}{3\sqrt{10}} \approx 1.040 \ .$$

The ℓ_∞ norm of the signal is given as

$$\|x[\cdot]\|_\infty = \sup_n\{|x[n]|\} = |x[1]| = 1$$

(ii) The signal defined by

$$x[n] = \frac{\sin(\pi n/5)}{\pi n}$$

for $n \neq 0$ (with $x[0]$ defined as $1/5$) falls off in magnitude as $1/n$, which is too slow to allow it to be an ℓ_1 signal. If you don't observe this, and instead attempt to approximate the sum of absolute values numerically, you can be badly misled, because the sum grows slowly, essentially as $\log(n)$.

However, the signal **is** ℓ_2:

$$\begin{aligned}
\|x[\cdot]\|_2 &= \sqrt{\sum_{n=-\infty}^{\infty} \left| \frac{\sin(\pi n/5)}{\pi n} \right|^2} \\
&= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| X(j\omega) \right|^2 d\omega} \\
&= \sqrt{\frac{1}{2\pi} \int_{-\pi/5}^{\pi/5} |1|^2 \, d\omega} \\
&= \frac{1}{\sqrt{5}} \,,
\end{aligned}$$

where the second equality follows from Parseval's theorem and the third from the known Fourier transform of the sinc function.

The ℓ_∞ norm of the signal is given as

$$\|x[\cdot]\|_\infty = \sup_n \{|x[n]|\} = x[0] = \frac{1}{5}$$

(iii) The signal $x[n] = ((0.2)^n - 1)\, u[n]$ is neither ℓ_1 nor ℓ_2 because the relevant sums do not converge. However, it **is** ℓ_∞:

$$\sup_n \{ |\, ((0.2)^n - 1)\, u[n] | \} = 1$$

(b) With output $\mathbf{y} = \mathbf{h} * \mathbf{x}$, Young's inequality allows us to deduce the following about the output signal:

i. If the input is bounded, so $\|\mathbf{x}\|_q$ is finite with $q = \infty$, and if the unit sample response is absolutely summable, so $\|\mathbf{h}\|_p$ is finite with $p = 1$, then choosing $r = \infty$ we find from Young's inequality that $\|\mathbf{y}\|_r$ is finite, i.e., the output is bounded or ℓ_∞. Alternatively, if the input is absolutely summable, so $q = 1$, and if the unit sample response is bounded, so $p = \infty$, then again choosing $r = \infty$ we see from Young's inequality that the output is bounded.

ii. If both the input signal and and the unit sample response are square summable, so $p = 2$ and $q = 2$, then choosing $r = \infty$ in Young's inequality shows that the output signal is bounded, i.e. ℓ_∞.

iii. If the unit sample response is absolutely summable, so $p = 1$, and if the input is ℓ_s for some $1 \leq s \leq \infty$, so $q = s$, then choosing $r = s$ in Young's inequality shows that the output is ℓ_s.

Solution 1.56

(a) Note first that the following infinite sum is always nonnegative:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|x[n+k]\pm x[n]|^2 &= \sum_{n=-\infty}^{\infty}|x[n+k]|^2+\sum_{n=-\infty}^{\infty}|x[n]|^2 \pm 2\sum_{n=-\infty}^{\infty}x[n+k]x[n] \\ &= 2(\overline{R}_{xx}[0]\pm\overline{R}_{xx}[k])\ .\end{aligned}$$

It follows that

$$\overline{R}_{xx}[0]\geq \mp\overline{R}_{xx}[k]$$

and consequently

$$\overline{R}_{xx}[0]\geq \left|\overline{R}_{xx}[k]\right| . \tag{20}$$

In other words, $\overline{R}_{xx}[k]$ always takes its maximum value at $k=0$. (Also note that $\overline{R}_{xx}[-k]=\overline{R}_{xx}[k]$, i.e., the deterministic autocorrelation function is even. This is consistent with the fact that its transform, $|X(e^{j\Omega})|^2$, is purely real.)

(b) If $\overline{R}_{xx}[0]=\overline{R}_{xx}[P]$, it follows that

$$\sum_{k=-\infty}^{\infty}|x[k+P]-x[k]|^2=0,$$

implying that $x[k+P]=x[k]$ for all k.

Likewise, if $\overline{R}_{xx}[0]=-\overline{R}_{xx}[P]$, it follows from part (a) that

$$\sum_{k=-\infty}^{\infty}|x[k+P]+x[k]|^2=0,$$

implying that $x[k+P]=-x[k]$ for all k, so $x[k+2P]=x[k]$ for all k.

A periodic nonzero signal necessarily has infinite energy, so our finite-energy signal cannot be periodic, i.e., we cannot have $\overline{R}_{xx}[0]=\pm\overline{R}_{xx}[P]$ for any $P\neq 0$.

To understand the above results in a more intuitive way, note that the deterministic autocorrelation $\overline{R}_{xx}[m]$ can be thought of as the inner product (or "dot product") of a signal "vector" with the vector corresponding to a shifted version of *itself*. (These vectors have an infinite number of components, one for each time instant, so they're not the vectors you're used to dealing with. Their infinite extent is what allows one to think of the vector obtained by shifting a given vector.) The maximum magnitude of the inner product of two vectors is attained precisely when the two vectors are positively or negatively aligned, i.e., when one vector is a positive or negative scalar multiple of the other. Since in the present case the two vectors have the same energy, the scalar multiple has to be 1 or -1, respectively. So one (trivial) way to attain the maximum magnitude is to have the shift of the shifted vector be 0; the inner product is then $\overline{R}_{xx}[0]$. For any other case, i.e., if the shifted signal is shifted by some $P \neq 0$, then having it be 1 or -1 times the unshifted signal would imply that the signal is periodic, with period P or $2P$ respectively, but this is impossible for a finite-energy signal. So we conclude that the maximum magnitude is attained only in the case of zero shift.

(c) The deterministic cross-correlation between $x[\cdot\,]$ and $y[\cdot\,]$ is

$$\overline{R}_{yx}[m] = \sum_{l=-\infty}^{\infty} y[l]x[l-m] = \sum_{l=-\infty}^{\infty} x[l-L]x[l-m] = \overline{R}_{xx}[m-L]$$

This is simply the autocorrelation function delayed by an amount L, i.e., the value at 0 gets shifted to the point L, and similarly for the values at all other times. (Incidentally, this already shows that a cross-correlation function does not in general have the even symmetry that an autocorrelation function has to have.)

From parts (a) and (b), the maximum is achieved at $m = L$ and has the value

$$\overline{R}_{xx}[0] = \sum_{l=-\infty}^{\infty} |x[l]|^2,$$

the scaled energy of $x[\cdot\,]$. The unknown lag can therefore be determined from the location of the maximum value of the deterministic cross-correlation function, $\overline{R}_{yx}[m]$.

(d) We now have

$$\overline{R}_{yx}[m] = \sum_{n=-\infty}^{\infty} y[n]x[n-m] = \sum_{n=-\infty}^{\infty} (x[n-L]+v[n])x[n-m] = \overline{R}_{xx}[m-L]+\sum_{k} v[k]x[k-m];$$

The difference between the noise-free case in (c) and the present case is the term

$$w[m] = \overline{R}_{vx}[m] = \sum_{k} v[k]x[k-m] \ .,$$

which is the deterministic cross-correlation between the signal $x[\cdot]$ and the noise $v[\cdot]$.

To find the mean of $w[m]$, we take an expectation with respect to the random variables $v[k]$, noting that the signal values $x[k-m]$ are deterministic, i.e., are simply scalar weights:

$$\begin{aligned}
\mathbb{E}_{v[\cdot]}\{w[m]\} &= \mathbb{E}_{v[\cdot]}\left\{\sum_{k} v[k]x[k-m]\right\} \\
&= \sum_{k} x[k-m]\mathbb{E}_{v[\cdot]}\{v[k]\} \quad \text{by linearity of expectation} \\
&= \sum_{k} x[k-m]\cdot 0 = 0
\end{aligned}$$

In computing the variance of $w[m]$, we note that the random variables $v[k]$ are independent and hence uncorrelated, so the variance of the term $v[k]x[k-m]$ is $x^2[k-m]\sigma_v^2$, and hence

$$\begin{aligned}\operatorname{var}_{v[\cdot]}(w[m]) &= \operatorname{var}_{v[\cdot]}\left(\sum_k v[k]x[k-m]\right) \\ &= \sum_k x^2[k-m]\operatorname{var}_{v[\cdot]}(v[k]) \\ &= \sum_k x^2[k-m]\cdot\sigma_v^2 = \mathcal{E}_x\sigma_v^2\end{aligned}$$

Hence, the standard deviation of $w[m]$ is given by $\operatorname{stdev}(w[m]) = \sigma_v\sqrt{\mathcal{E}_x}$.

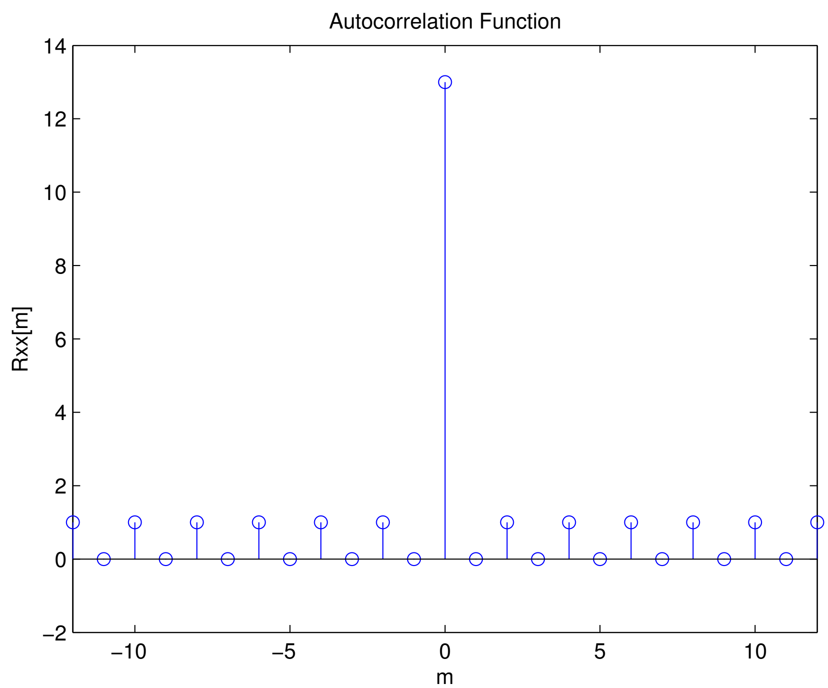
(e) What we want is for the maximum value of $\overline{R}_{yx}[m]$ in the noisy case (d) to occur at the same position as the maximum value of $\overline{R}_{\hat{y}x}[m]$ in the noise-free case (c), namely at $m = L$. The larger the height of the maximum of the noise-free cross correlation $\overline{R}_{\hat{y}x}[m]$ is, relative to the standard deviation of the perturbation $w[m]$, the more likely we are to pick the position of the maximum correctly in the noisy case. In other words, what we want is a large value of

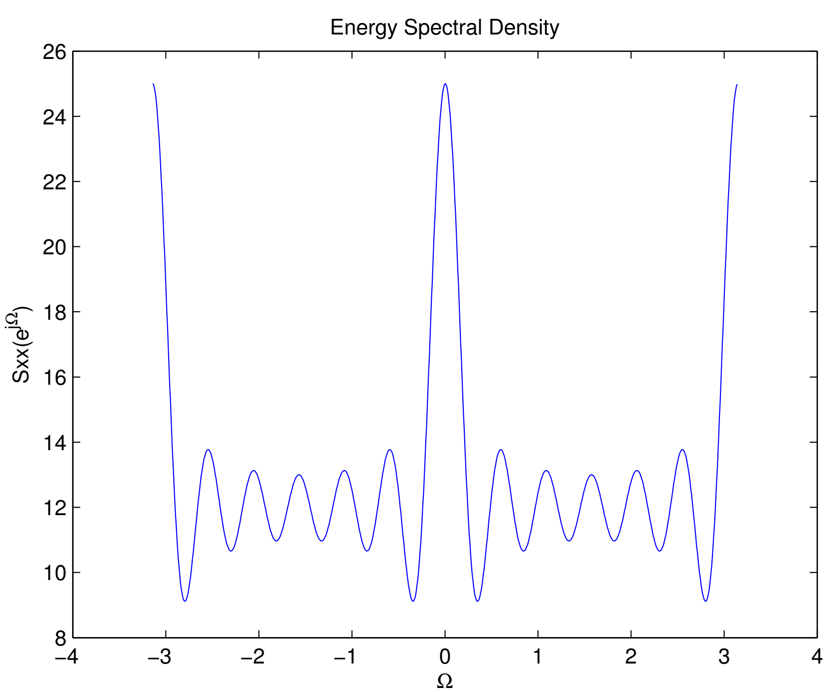
$$\frac{\overline{R}_{xx}[0]}{\operatorname{stdev}(w[m])} = \frac{\mathcal{E}_x}{\sigma_v\sqrt{\mathcal{E}_x}} = \sqrt{\frac{\mathcal{E}_x}{\sigma_v^2}}\,.$$

We therefore expect to do better as the ratio of signal energy to noise variance increases. This makes intuitive sense. It is also helpful for the cross-correlation function in the noise-free case to have a sharply defined peak at L, i.e., that the autocorrelation function has a sharply defined peak at 0. We would want the values of the autocorrelation at nonzero lags to be a few noise standard deviations below the peak value $\mathcal{E}_x$. We'll do a more detailed analysis later in the course.

(f) A plot of $\overline{R}_{xx}[m]$ reveals that this indeed has the value $D = 13$ at $m = 0$, and that its value elsewhere is either 0 or 1. In other words, we have a "sharply defined peak" at 0, which makes this a good signal to use for the kind of application underlying parts (c)-(e).

The corresponding energy spectral density $\overline{S}_{xx}(e^{j\Omega})$ is presented in the second plot, and shows energy broadly distributed in the frequency range $[-\pi, \pi]$. (The plot has not been periodically extended beyond this range, to keep the focus on the principal frequency range.) For a quick check on the plot, note that its value at $\Omega = 0$ should be the sum of the $\overline{R}_{xx}[m]$ values for all m, and that is indeed 25.


Autocorrelation Function
Rxx[m]
m
14
12
10
8
6
4
2
0
−2
−10
−5
0
5
10


Energy Spectral Density
$Sxx(e^{j\Omega})$
Ω
26
24
22
20
18
16
14
12
10
8
−4
−3
−2
−1
0
1
2
3
4